On the complexity of active P systems

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Summary. We are going to present a polynomially uniform solution to the Quantified 3SAT decision problem with restricted instances where the quantifiers alternate, based on recognizer P systems with active membranes and no input membrane, having three polarizations using only dissolution and division rules.

1 Introduction

In the twelfth chapter of "The Oxford Handbook on Membrane Computing" [3] the following question can be found: What is the efficiency of P systems with active membranes and electrical charges where evolution and communication rules are forbidden? The answer to this question is that one can give a uniform solution to the **PSPACE**-complete Quantified 3SAT decision problem (having a restricted quantification, which does not alter its complexity class) using such systems. Similar result is obtained by Alberto Leporati et al. in their [1] article. They gave a semi-uniform solution for the Q3SAT decision problem using polarisationless P systems.

In the second section we will recall the definition of the recognizer P systems with active membranes, together with the definition of uniform solution. In the third section, the Q3SAT decision problem will be defined. In the fourth section we will describe the main result of this paper, namely the uniform solution to the restricted Q3SAT decision problem. In the fifth section we are going to draw the conclusions.

2 Recognizer P systems with active membranes

We are going to use P systems with the above mentioned properties through the rest of the paper, so now we give the definition of such systems. For more detailed description see [2].

Definition 1. *A P system with active membranes, having three polarizations using only dissolution and division rules, of degree $q \geq 1$ is a tuple*

$$\Pi = (\Gamma, H, \mu, w_1, \ldots, w_q, h_0, R)$$

where

- *Γ is the finite alphabet of objects,*
- *H is the alphabet of labels for the membranes,*
- *μ is the initial membrane structure of degree q, with all membranes labeled with the elements of H and with electrical charges (positive, negative or neutral) associated with them,*
- *$w_1, \ldots, w_q$ are strings over Γ specifying the multisets of objects present in the compartments of μ,*
- *$h_0 \in \{0, 1, \ldots, q\}$ indicates the region where the result of a computation is obtained (0 represents the environment),*
- *and R is a finite set of rules.*

The rules are of the following types.

- *Dissolution rules of the form*

$$[a]_h^\alpha \to b$$

  where $h \in H$, $\alpha \in \{+, -, 0\}$ and $a, b \in \Gamma$. Here, every membrane in the membrane with label h together with every other object in it goes into the upper neighbor.
- *Division rules for elementary membranes:*

$$[a]_h^{\alpha_1} \to [b]_h^{\alpha_2} [c]_h^{\alpha_3}$$

  where $h \in H$, $\alpha_1, \alpha_2, \alpha_3 \in \{+, -, 0\}$ and $a, b, c \in \Gamma$. Here, every membrane and other objects in the initial h labeled membrane are copied into both newly created h labeled membranes.
- *Division rules for non-elementary membranes:*

$$[[\ \ ]_{h_1}^{\alpha_1} \cdots [\ \ ]_{h_k}^{\alpha_1} [\ \ ]_{h_{k+1}}^{\alpha_2} \cdots [\ \ ]_{h_n}^{\alpha_2}]_h^\alpha$$
$$\to$$
$$[[\ \ ]_{h_1}^{\alpha_3} \cdots [\ \ ]_{h_k}^{\alpha_3}]_h^\beta \quad [[\ \ ]_{h_{k+1}}^{\alpha_4} \cdots [\ \ ]_{h_n}^{\alpha_4}]_h^\gamma$$

  for $k \geq 1$, $n > k$, $h, h_1, \ldots, h_n \in H$, $\alpha, \beta, \gamma, \alpha_1, \ldots, \alpha_4 \in \{+, -, 0\}$ and $\{\alpha_1, \alpha_2\} = \{+, -\}$. Here, every object and the membranes with neutral polarity in the h labeled membrane are copied into both newly created h labeled membranes.

A configuration in a P system can be described by its actual membrane structure together with the multisets of objects present in the regions. A computational step changes the current configuration according to the following principles.

- Each membrane can be subject to at most one rule per computation step. Newly created membranes cannot be the subjects of rules in the actual computational step. The skin membrane should not dissolve or divide.
- The rules are applied in a maximally parallel manner. This means, that every membrane which could be the subject of a rule must be the subject of exactly one rule. When there is more than one rule which we can apply, then the choice should be nondeterministic.
- The rules are applied "from bottom up", so first the rules are applied on the innermost membranes, then on their upper neighbors, and so on until the skin membrane.

We are going to use a recognizer P system. This means, that the Γ alphabet has two distinguished objects representing "yes" and "no", and if one of these objects reach the membrane with label h_0, then the computation halts. The result of the computation is acceptance in the former-, and rejection in the later case.

The computation of such P system is the sequence of its configurations starting from its initial configuration. Every configuration of such computation should be reached from the previous configuration using the principles described above. Such computation can be finite, arriving to a configuration where one of the "yes" or "no" objects enter the h_0 labeled membrane, or it can be infinite if this does not happens. We will only consider confluent recognizer P systems, in which all computations starting from the initial configuration halt and agree on the result.

We are going to build the initial membrane structure according to the given instance of the examined problem. We will show, that this can be done in a polynomial amount of steps which means that our solution is polynomially uniform. In the following definition, I_X denotes the possible instances for the problem X.

Definition 2. *A family $\Pi = \{\Pi(w) | w \in I_X\}$ of recognizer membrane systems without input membrane is polynomially uniform by Turing machines if there exists a deterministic Turing machine working in polynomial time which constructs the system $\Pi(w)$ for the instance $w \in I_X$.*

3 The Quantified 3SAT decision problem

The Boolean satisfiability problem (abbreviated as SAT) can be stated as the following. Lets consider the $x_1, \dots, x_n$ Boolean variables. An instance of SAT consists of conjunctions of clauses, which are disjunctions of literals, occurrences of x_i or $\neg x_i$. An interpretation of the variables is a mapping, which associates a truth value to the variables. The Boolean satisfiability problem asks the following question: is there an interpretation of the given Boolean variables for which interpretation the conjunction of the clauses evaluates to true? For the rest of the paper, we assume that the literals in the clauses are ordered by the indexes of their variables.

The 3SAT decision problem is a variant of the SAT problem, where the clauses contain only three literals. An instance of the Quantified 3SAT decision problem is a well-formed Boolean formula $(Q_1x_1)\dots(Q_nx_n)\,\phi\,(x_1,\dots,x_n)$, where $Q_i \in \{\exists,\forall\}$ and ϕ is an instance of 3SAT over the variables $x_1,\dots,x_n$. This decision problem asks that the given quantified formula is true or false. It can be shown that the Q3SAT decision problem is **PSPACE**-complete, even when restricted to instances where the quantifiers alternate the

$$\exists x_1\forall x_2\exists x_3\forall x_4\dots\exists x_{2i-1}\forall x_{2i}\dots\exists x_{n-1}\forall x_n\phi(x_1,\dots,x_n)$$

way, where n is even.

4 Solving Q3SAT with restricted instances

We are going to describe a recognizer P system without an input membrane, which decides the satisfiability of a given instance of the restricted Q3SAT decision problem. With the following initial membrane structure construction, the number of objects and the number of rules, this will give us a polynomially uniform solution to the restricted Q3SAT decision problem. The required objects, rules and the initial membrane structure together with the initial membrane contents will be given as we describe the system part by part.

4.1 Construction of the initial membrane structure

The initial membrane structure can be seen in figure 1. For the ith universally quantified variable, we introduce the membranes with ε_{t_i} and ε_{f_i} labels having neutral polarity, together with two membranes with δ label having positive polarity, except for the last variable, where we only introduce one such δ labeled membrane. The membranes labeled ε and δ give us $2n-1$ membranes in the initial membrane structure.

The clauses are encoded in the membranes with C_{i_p,j_p,k_p} labels. We are going to call the nested membrane structure of the membranes representing the clauses a clause-chain. The encoding is similar to the one which is used by Porreca et al. in [4] and [5]. We can represent the $C_p = (l_{p,1} \vee l_{p,2} \vee l_{p,3})$ clause with a membrane labeled C_{i_p,j_p,k_p} having neutral polarity, where i_p (resp. j_p and k_p) is the index of the variable in $l_{p,1}$ (resp. $l_{p,2}$ and $l_{p,3}$) with a negative sign if the variable is negated. So for example if our clause is $(x_1 \vee \neg x_2 \vee x_3)$, then the corresponding membrane will be the label $C_{1,-2,3}$. Using this encoding, the upper bound on the number of membranes with C_{i_p,j_p,k_p} labels in our initial membrane structure is $8\binom{n}{3}$. Also, the rules for these membranes can be given in advance (we will do this in section 4.4), so for a restricted Q3SAT instance we only have to construct the initial membrane structure.

The steps required for the generation of the interpretations are $n+1$ and one step is required for the evaluation of the quantified formula. So the c labeled membranes should form a chain of polynomial length greater than $n+2$. We will discuss the reason for this in more details in section 4.4, but the short explanation is that this way, the n object will arrive to the skin membrane right on time.

Summing up the parts, one can see that the size of the initial membrane structure is polynomially bounded.

4.2 Creation of the interpretations

We included the d_1 object in the initial membrane structure. This object initiates the creation of the interpretations. We are going to use the

$$[d_i]_h^0 \to [d_{i+1}]_h^0[e_{n+1-i}]_h^0 \quad (i = 1, \ldots, n) \tag{1}$$

$$[d_{n+1}]_h^0 \to [d]_h^0[d]_h^0 \tag{2}$$

$$[d]_h^0 \to d \tag{3}$$

rules to generate the e objects. These objects will create the variable interpretations with the

$$[e_i]_h^0 \to [t_i]_h^+[f_i]_h^- \quad (i = 1, \ldots, n) \tag{4}$$

$$[t_i]_h^0 \to t_i \quad (i = 1, \ldots, n) \tag{5}$$

$$[f_i]_h^0 \to f_i \quad (i = 1, \ldots, n) \tag{6}$$

rules. The membranes with labels h and b and the membranes representing the clauses are split with the

$$[[\quad]_h^+[\quad]_h^-]_b^0 \to [[\quad]_h^0]_b^+[[\quad]_h^0]_b^- \tag{7}$$

$$[[\quad]_b^+[\quad]_b^-]_{C_{i_1,j_1,k_1}}^0 \to [[\quad]_b^0]_{C_{i_1,j_1,k_1}}^+[[\quad]_b^0]_{C_{i_1,j_1,k_1}}^- \tag{8}$$

and

$$\begin{gathered}[[\quad]_{C_{i_{p-1},j_{p-1},k_{p-1}}}^+[\quad]_{C_{i_{p-1},j_{p-1},k_{p-1}}}^-]_{C_{i_p,j_p,k_p}}^0 \\ \to \\ [[\quad]_{C_{i_{p-1},j_{p-1},k_{p-1}}}^0]_{C_{i_p,j_p,k_p}}^+[[\quad]_{C_{i_{p-1},j_{p-1},k_{p-1}}}^0]_{C_{i_p,j_p,k_p}}^-\end{gathered} \tag{9}$$

$p = 2, \ldots, m$ rules.

Lemma 1. *Starting from the initial membrane structure, applying rules (1)-(9) from step to step, after the (5) and (6) rules are applied on the* $[t_1]_h^0$ *and* $[f_1]_h^0$ *membranes, the membrane structure contains all the possible interpretations of the variables in the leaves.*

Proof. We are going to give a proof by induction. Figure 2 shows the beginning of the creational process. In the general case, we are going to look at a clause-chain with the

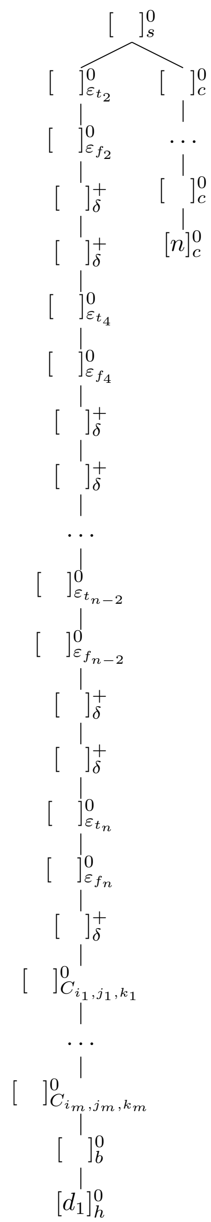

Fig. 1. The initial state. The membrane with the h label is the hatchery, we create the interpretations of the variables here. The membrane with the b label is a boundary membrane. The membranes with the C_{i_p,j_p,k_p} labels are the ones that evaluate the clauses. The δ membranes manage the creation of the quantification tree. The ε membranes check the universal quantifiers. The c contradictory membranes delay the entering of the n symbol into the skin membrane.

$$[d_i]_h^0[e_{n+2-i}]_h^0[t_{n+3-i}]_h^+[f_{n+3-i}]_h^- x_n x_{n-1} \dots x_{n+4-i} \quad (10)$$

membranes in the innermost membrane labeled b, where $i = 4, \dots, n$ and x_p could be either t_p or f_p. The polarity difference induces a non-elementary membrane division according the (7) rule. In the membrane with label b and positive polarity, we will have the

$$[d_i]_h^0[e_{n+2-i}]_h^0[t_{n+3-i}]_h^0 x_n x_{n-1} \dots x_{n+4-i} \quad (11)$$

content and in the membrane with label b and negative polarity we will have the

$$[d_i]_h^0[e_{n+2-i}]_h^0[f_{n+3-i}]_h^0 x_n x_{n-1} \dots x_{n+4-i} \quad (12)$$

content. The non-elementary membrane divisions are propagated upwards in the structure according the rules (8) and (9), forming two clause-chains. After the divisions stopped, a new step begins and (11) becomes

$$[d_{i+1}]_h^0[e_{n+1-i}]_h^0[t_{n+2-i}]_h^+[f_{n+2-i}]_h^- x_n x_{n-1} \dots x_{n+4-i} t_{n+3-i}$$

and (12) becomes

$$[d_{i+1}]_h^0[e_{n+1-i}]_h^0[t_{n+2-i}]_h^+[f_{n+2-i}]_h^- x_n x_{n-1} \dots x_{n+4-i} f_{n+3-i},$$

so in the innermost membrane with label b of both clause-chains, the same state appeared as in (10) just with the additional truth objects t_{n+3-i} and f_{n+3-i}. Note that this way, every possible $x_n x_{n-1} \dots x_{n+4-i}$ variable interpretation is extended with the mentioned truth objects. This holds in every iteration of the recursion.

At the end, the t_1 and f_1 objects are generated, the (5)-(6) rules are applied, and the t_1, f_1 objects enter the b membranes. At this point, every possible interpretation is generated at the bottom of the membrane structure. Because of the (2) rule, there will be no more e objects introduced into the membrane structure. □

Notice that the d objects get into the membranes with label b the same time when the t_1 and f_1 objects get into the mentioned membranes with b label.

4.3 Creation of the quantifier tree

We introduced the membranes with δ label having positive polarity in the initial membrane structure. These membranes delay the non-elementary membrane divisions, so instead of forming chains, we are going to generate a tree structure as the new variable interpretations are created. We add the

$$[[\ ]_{C_{i_1,j_1,k_1}}^+[\ ]_{C_{i_1,j_1,k_1}}^-]_\delta^+ \to [[\ ]_{C_{i_1,j_1,k_1}}^0]_\delta^0[[\ ]_{C_{i_1,j_1,k_1}}^0]_\delta^0 \quad (13)$$

$$[[\ ]_{\varepsilon_{t_i}}^+[\ ]_{\varepsilon_{t_i}}^-]_\delta^+ \to [[\ ]_{\varepsilon_{t_i}}^0]_\delta^0[[\ ]_{\varepsilon_{t_i}}^0]_\delta^0 \quad (i = 2, 4, \dots, n) \quad (14)$$

$$[[\ ]_\delta^+[\ ]_\delta^-]_\delta^+ \to [[\ ]_\delta^0]_\delta^0[[\ ]_\delta^0]_\delta^0 \quad (15)$$

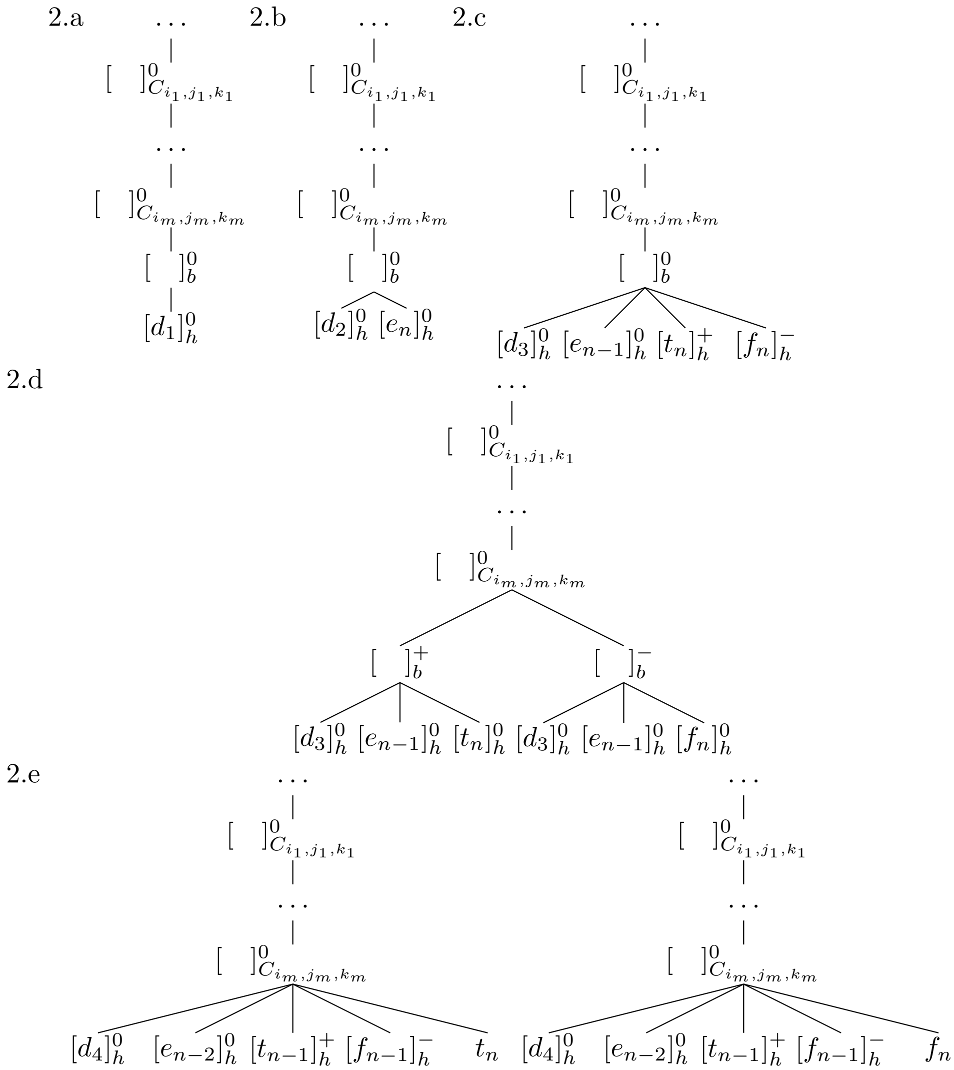

Fig. 2. The beginning of the creation of the variable interpretations. The bottom of the initial membrane structure can be seen on figure (a). The (b) figure shows the structure after one step. We applied the $[d_1]_h^0 \to [d_2]_h^0[e_n]_h^0$ rule. The (c) figure shows the beginning of the second step. Here, the $[d_2]_h^0 \to [d_3]_h^0[e_{n-1}]_h^0$ rule and the $[e_n]_h^0 \to [t_n]_h^+[f_n]_h^-$ rule were applied. The polarity difference induces a non-elementary membrane division, which induces another non-elementary membrane division on the next level, etc. The state after the first division can be seen on figure (d). On figure (e), the non-elementary divisions are finished. The upper membranes are affected by the divisions too, but we are going to describe that in section 4.3. This figure showes the state after the application of the $[d_3]_h^0 \to [d_4]_h^0[e_{n-2}]_h^0$, $[e_{n-1}]_h^0 \to [t_{n-1}]_h^+[f_{n-1}]_h^-$ and $[t_n]_h^0 \to t_n$, $[f_n]_h^0 \to f_n$ rules. One can see how the recursion goes after comparing this figure with figure (c).

rules to the system. These rules change the initial positive polarity of a membrane with label δ to neutral polarity. In this case, we will say that the δ membrane activates. The activation of a δ labeled membrane pair can be seen on figure 3. The activated membrane with label δ will propagate the polarity difference just as the other membranes do with the

$$[[\ \ ]^+_{C_{i_1,j_1,k_1}}[\ \ ]^-_{C_{i_1,j_1,k_1}}]^0_\delta \to [[\ \ ]^0_{C_{i_1,j_1,k_1}}]^+_\delta[[\ \ ]^0_{C_{i_1,j_1,k_1}}]^-_\delta \tag{16}$$

$$[[\ \ ]^+_{\varepsilon_{t_i}}[\ \ ]^-_{\varepsilon_{t_i}}]^0_\delta \to [[\ \ ]^0_{\varepsilon_{t_i}}]^+_\delta[[\ \ ]^0_{\varepsilon_{t_i}}]^-_\delta \quad (i = 2, 4, \ldots, n) \tag{17}$$

$$[[\ \ ]^+_\delta[\ \ ]^-_\delta]^0_\delta \to [[\ \ ]^0_\delta]^+_\delta[[\ \ ]^0_\delta]^-_\delta \tag{18}$$

rules, as it can be seen in figure 4. The ε membranes split by using the

$$[[\ \ ]^+_\delta[\ \ ]^-_\delta]^0_{\varepsilon_{f_i}} \to [[\ \ ]^0_\delta]^+_{\varepsilon_{f_i}}[[\ \ ]^0_\delta]^-_{\varepsilon_{f_i}} \tag{19}$$

$$[[\ \ ]^+_{\varepsilon_{f_i}}[\ \ ]^-_{\varepsilon_{f_i}}]^0_{\varepsilon_{t_i}} \to [[\ \ ]^0_{\varepsilon_{f_i}}]^+_{\varepsilon_{t_i}}[[\ \ ]^0_{\varepsilon_{f_i}}]^-_{\varepsilon_{t_i}} \tag{20}$$

rules ($i = 2, 4, \ldots, n$), so they propagate the polarity difference upwards. Notice that both the activated membranes with δ label and the membranes with label ε propagate the polarity difference upwards, while the not activated δ membranes halt this propagation.

Now we are going to show that when a new variable interpretation is created at the bottom of the tree structure (so the $[t_i]^+_h$ and $[f_i]^-_h$ membranes are introduced), then at the beginning of the next step, after the (5)-(6) rules are applied, the interpretations in the leaves are ordered. By ordered, we mean that when examining the f_i and t_i objects in the leaves of the tree structure from right to left, there is only f_i objects in the rightmost leaf, the object representing the nth variable negate (f_n changes to t_n and t_n changes to f_n) from leaf to leaf, and the object representing the ith variable negate when the object representing the $(i+1)$th variable changes from t_{i+1} to f_{i+1}. For example, the

$$t_1t_2t_3 \quad t_1t_2f_3 \quad t_1f_2t_3 \quad t_1f_2f_3 \quad f_1t_2t_3 \quad f_1t_2f_3 \quad f_1f_2t_3 \quad f_1f_2f_3 \tag{21}$$

sequence of objects in the leaves form an ordered sequence.

Lemma 2. *When we apply the (4) rule for the kth time, in the same step, the kth membrane with label δ from the bottom of the membrane structure activates.*

Proof. We are going to give a proof by induction. Initially, every membrane with label δ have positive polarity, so they are not activated.

- When the $[e_n]^0_h \to [t_n]^+_h[f_n]^-_h$ rule is applied, the lowest membrane with label δ activates. According the (13) rule, this membrane will not propagate the polarity difference.
- In the general case, when the (4) rule is applied for the $(k+1)$th time, the kth membrane with label δ is already activated according the induction, so it will propagate this polarity difference. We have two case here.

– When the $(k+1)$th membrane with label δ is the upper neighbor of the kth membrane with label δ, then the former membrane activates according the (15) rule.
– When there are ε labeled membranes between the $(k+1)$th and the kth membranes labeled δ, then according the (19) and (20) rules, the polarity difference is propagated to the $(k+1)$th membrane with label δ from the kth, so it activates.

□

Beside from the activation, one can see the process how given membranes permute their lower neighbors on figures 3, 4 and 5. By permutation we mean what we give in the descriptions of the mentioned figures: some membranes go to the opposite subtree from their original subtree. We are going to show exactly which membranes permute their lower membranes.

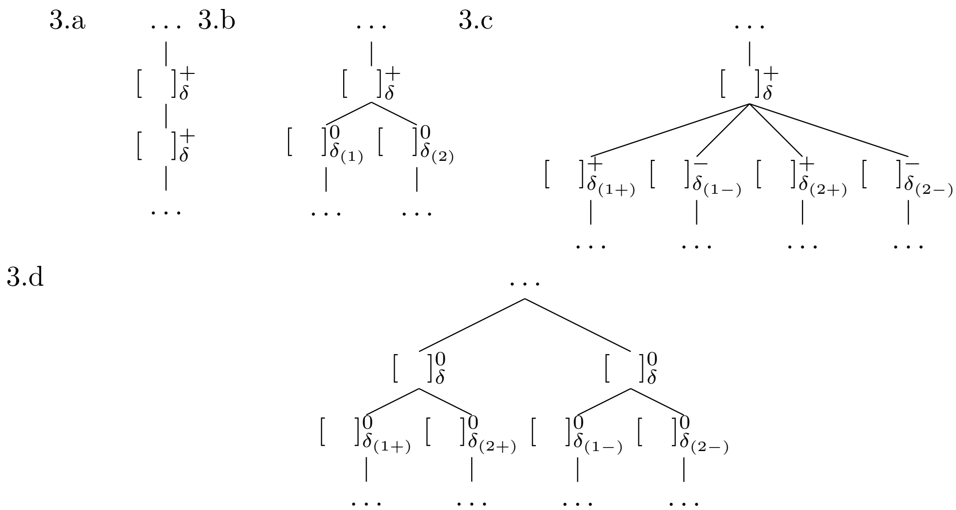

Fig. 3. On the (a) figure, one can see the initial state of two δ labeled membranes. A polarity difference on the lower level splits the bottom δ labeled membrane as one can see in figure (b). Another polarity difference in the bottom membranes splits the activated δ labeled membranes in figure (c). Temporarily we denoted the number and the polarity of the given membranes, so one can trace them. On figure (d) one can see that the membranes with positive polarity go into the left subtree, while the membranes with the negative polarity go into the right subtree. This only influences the $\delta_{(1-)}$ and $\delta_{(2+)}$ membranes: they go to the opposite subtree from their original place.

Lemma 3. *Only the* $(2i+1)$*th membranes with label* δ $(i > 1)$ *and the membranes with label* ε_{f_i} $(i = 2, 4, \ldots, n)$ *perform permutation on non-elementary membrane division.*

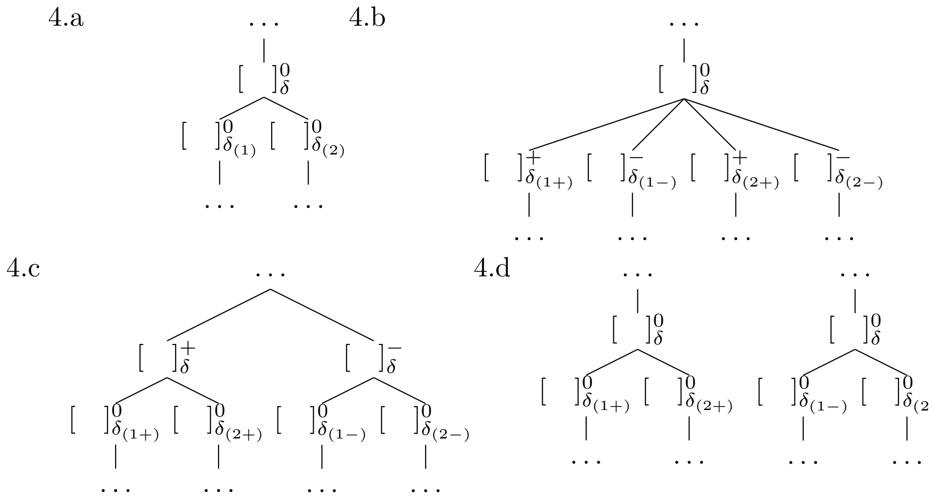

Fig. 4. On the (a) figure, we can see an activated δ labeled formation. We denote the number and the polarity again for better traceability. Only one polarity difference in each bottom δ labeled membrane can cause the whole structure to split in two, see figures (b), (c) and (d). Notice, that the $\delta_{(1-)}$ and $\delta_{(2+)}$ membranes go to the opposite subtree from their original subtree. Furthermore, notice that we arrived at a state where we duplicated the state on figure (a).

Proof. Notice that permutation can occur only in the membranes where more than two membranes are present with polarity difference. This can only happen in the mentioned membranes. These membranes perform permutations as it can be seen in figures 3, 4 and 5. $\square$

Lemma 4. *When we apply the (4) rule for the kth time, in the same step, only the membranes not higher in the membrane structure than the kth membrane with label δ perform permutation.*

Proof. The yet not activated membranes with δ label halt the propagation of the polarity difference, so we should examine the membrane structure from the bottom only until the last activated δ labeled membrane. Applying the (4) rule for the kth time results in the activation of the kth δ labeled membrane according lemma 2, so the membranes lower than this membrane perform their permutation. According to this, we only have to deal with the actually activated membrane with δ label and according lemma 3, we can concentrate on the $(2i + 1)$th membranes with label δ $(i > 1)$. But examining figure 3, one can see that these membranes perform their permutation on activation. $\square$

Lemma 5. *Applying the (4) rule in the tree structure when there is an ordered sequence of interpretations in the leaves, at the beginning of the next step, after the (5)-(6) rules are applied, the interpretations in the leaves will be ordered again.*

Proof. When applying the (4) rule for the kth time, according lemma 2 the kth membrane with δ label activates and according lemma 4 only the membranes not higher in the membrane structure than the kth membrane with label δ perform permutation. According this, we are going to give a proof by induction on the number of activated membranes with δ label.

- No permutation happens when the first membrane with δ label activates. At the beginning of the next step, after the (5)-(6) rules are applied, there is two leaves, having t_n in the left leaf and f_n in the right leaf, so the ordering property holds.
- In general either the $2k$th or the $(2k+1)$th $(k > 0)$ membrane with δ label activates. For the former case see figure (6.a) and (6.b), for the later case see figure (6.c) and (6.d).

□

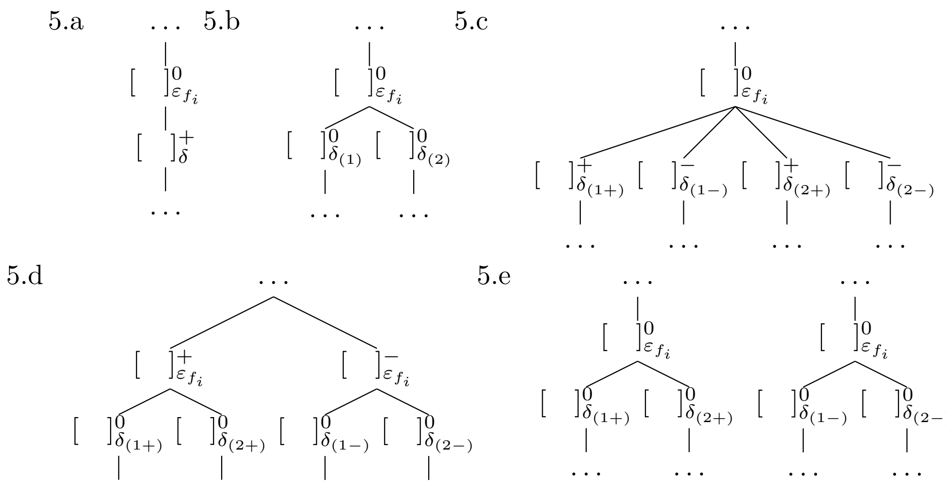

Fig. 5. This figure shows how a membrane with ε_{f_i} permutes the lower neighbors. The membrane with label δ activates in figure (a) and (b). Another polarity difference in the lower neighbors split the membrane with ε_{f_i} label. Notice that the $\delta_{(1-)}$ and $\delta_{(2+)}$ membranes go to the opposite subtree from their original subtree.

Using the result of lemma 1 and lemma 5, we know that after the application of the $[e_1]_h^0 \rightarrow [t_1]_h^+[f_1]_h^-$ rule, at the beginning of the next step, after the (5)-(6) rules are applied, we will have all the possible interpretations in an ordered sequence in the leaves of the membrane structure.

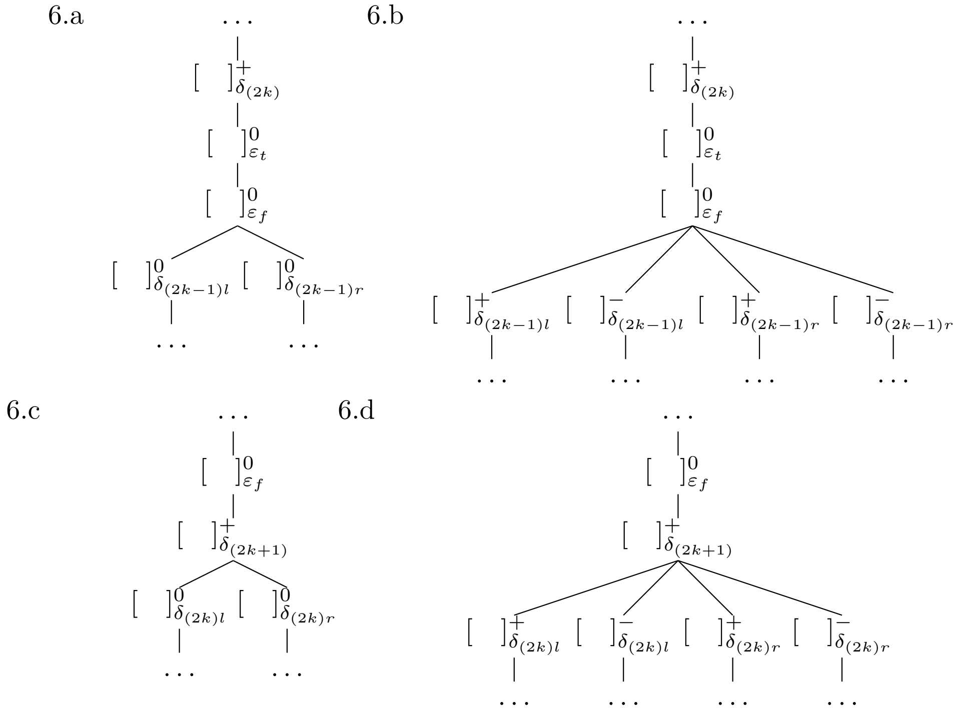

Fig. 6. These figures serve as a part of the proof of lemma 5. We omitted the indexes from the ε_t and ε_f labels for simplicity. In figure (a) we can see the state when the $2k$th membrane with δ label is the next one to be activated, and we have not applied the (4) rule yet. In this case, the interpretations in the leaves of the membrane structure form an ordered sequence. The first half of the ordered sequence is in the subtree under the $\delta_{(2k-1)l}$ labeled membrane, the second half of the ordered sequence is in the subtree under the $\delta_{(2k-1)r}$ labeled membrane. In figure (b), after the application of the (4) rule when the membranes with $\delta_{(2k-1)l}$ and $\delta_{(2k-1)r}$ labels split, according the induction the positively charged membrane with $\delta_{(2k-1)l}$ label contains the first half of the interpretations from the sequence, each one concatenated with $[t_{n+1-2k}]_h^0$ and the negatively charged membrane with $\delta_{(2k-1)l}$ label contains the first half of the interpretations from the sequence, each one concatenated with $[f_{n+1-2k}]_h^0$. The same holds for the membranes with $\delta_{(2k-1)r}$ labels, just with the second half of the sequence. The permutation performed by the membrane with ε_f label exchanges the negatively charged $\delta_{(2k-1)l}$ labeled membrane with the positively charged $\delta_{(2k-1)r}$ labeled membrane. No more permutations are performed after this one in this step. So in the beginning of the next step, after the (5)-(6) rules are applied, the interpretations in the leaves will form an ordered sequence. In figure (c) we can see the state when the $(2k+1)$th membrane with δ label is the next one to be activated, and we have not applied the (4) rule yet. Here we can follow the same reasoning as in the previous case.

4.4 Evaluation

The evaluation stage starts when the d objects get into the b labeled membranes. As we have mentioned, this happens the same time when the t_1 and f_1 enter the b labeled membranes. The evaluation initiates with the use of the

$$[d]_b^0 \to d \tag{22}$$

rule, which sends every object form a b labeled membrane to the upper neighbor.

For evaluating the clauses we introduce the

$$[t_p]^0_{C_{i,j,k}} \to t_p \text{ } \mathit{if}\ p \in \{i, j, k\} \tag{23}$$
$$[f_p]^0_{C_{i,j,k}} \to f_p \text{ } \mathit{if}\ p \in \{-i, -j, -k\} \tag{24}$$

rules where $p = 1, 2, \ldots, n$ and $i, j, k \in \{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$ satisfying $|i| < |j| < |k|$. So for example if our clause membrane is labeled with $C_{1,-2,3}$, then

$$[t_1]^0_{C_{1,-2,3}} \to t_1$$
$$[f_2]^0_{C_{1,-2,3}} \to f_2$$
$$[t_3]^0_{C_{1,-2,3}} \to t_3$$

will be the rules for this membrane. The upper bound on the number of the possible clauses is $8\binom{n}{3}$, and for every clause we introduce 3 rules, so an upper bound on the number or rules introduced with this reasoning is $24\binom{n}{3}$ which is still polynomial.

An interpretation dissolves a clause membrane if one of the truth values (represented by objects) in it evaluates the given clause to a true truth value. The interpretations propagate upward, and they only get into the quantifier tree if they satisfy the formula. For the ε labeled membranes, we introduce the

$$[t_i]^0_{\varepsilon_{t_i}} \to t_i \tag{25}$$
$$[f_i]^0_{\varepsilon_{f_i}} \to f_i \tag{26}$$

rules for $i = 2, 4, \ldots, n$ and for the membranes with δ labels, the

$$[t_i]^0_\delta \to t_i \tag{27}$$
$$[f_i]^0_\delta \to f_i \tag{28}$$

rules where $i = 1, \ldots, n$.

Lemma 6. *Membranes with ε_{t_i} and ε_{f_i} label dissolve during the evaluation stage if and only if $\exists x_1 \dots \exists x_{i-1} \forall x_i \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true. (Here, the variables $x_1, \dots, x_{i-2}$ are existentially quantified, then existentially and universally quantified variables come alternately.)*

Proof. We are going to give a proof by induction.

$\Rightarrow$ Here we assume that the membranes with ε_{t_i} and ε_{f_i} label dissolve during the evaluation stage.

- Let $i = n$. In the membrane labeled ε_{f_n}, there is $x_1 \dots x_{n-1} t_n$ in the leaf on the left side of the branch and $x_1 \dots x_{n-1} f_n$ in the leaf on the right side of the branch. (Here x_p is either t_p or f_p.) Because the membranes with ε_{t_n} and ε_{f_n} dissolve, we know that the interpretations passed through the clause-chain which means that the interpretations satisfy ϕ. From this, we get that $\exists x_1 \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true.
- Let $i = n - 2$. Because the membranes with $\varepsilon_{t_{n-2}}$ and $\varepsilon_{f_{n-2}}$ dissolve, we know that $\exists x_1 \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true (because one have to dissolve membranes with ε_{t_n} and ε_{f_n} labels to achieve this) with t_{n-2} and some x_{n-1}, and it is also true with f_{n-2} and some (probably different) x_{n-1}, which means that $\exists x_1 \dots \exists x_{n-3} \forall x_{n-2} \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true.
- In the general case, lets assume that the statement is true for the membranes with ε_{t_i} and ε_{f_i} labels. Because of this and the induction, we know that $\exists x_1 \dots \exists x_{i+1} \forall x_{i+2} \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true with t_i and some x_{i+1}, and it is also true with f_i and some (probably different) x_{i+1}, which means that $\exists x_1 \dots \exists x_{i-1} \forall x_i \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true.

$\Leftarrow$ Here we assume that $\exists x_1 \dots \exists x_{i-1} \forall x_i \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true.

- Lets assume that $\exists x_1 \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true. This means that two clause-chains under the same membrane with ε_{f_n} label dissolve, because the interpretations in the leaves satisfy ϕ. Both interpretations get into ε_{f_n} which dissolves in the presence of f_n. After this, ε_{t_n} dissolves because of t_n.
- Now lets assume that $\exists x_1 \dots \exists x_{n-3} \forall x_{n-2} \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true. This means, that $\exists x_1 \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true with t_{n-2} and some x_{n-1}, and it is also true with f_{n-2} and some (probably different) x_{n-1}. Because of the structure of the quantifier tree, this means that there exists a membrane pair with $\varepsilon_{f_{n-2}}$ and $\varepsilon_{t_{n-2}}$ labels in the tree which dissolves.
- Lets assume that $\exists x_1 \dots \exists x_{i-1} \forall x_i \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true. This means that $\exists x_1 \dots \exists x_{i+1} \forall x_{i+2} \dots \exists x_{n-1} \forall x_n \phi(x_1, \dots, x_n)$ is true with t_i and some x_{i+1} and it is true with f_i and some (probably different) x_{i+1}. Because of this, plus the induction and the structure of the quantifier tree, we get that there exists a membrane pair with ε_{f_i} and ε_{t_i} labels in the tree which dissolves.

□

Lemma 7. *The $\exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \phi(x_1, \ldots, x_n)$ formula (where n is even) is true (resp. false) if and only if at least one d object (resp. no object) arrives to the skin membrane from the quantifier tree part of the membrane structure.*

Proof. We are going to use the results of lemma 6.

$\Rightarrow$ If the formula is true, then a membrane pair with ε_{f_2} and ε_{t_2} labels dissolves, so d objects get into the skin membrane. If the formula is false, then no objects get into the skin membrane, because no membrane pair with ε_{f_2} and ε_{t_2} labels dissolve.

$\Leftarrow$ If d objects get into the skin membrane, then at least one membrane pair with ε_{f_2} and ε_{t_2} labels dissolved, which means that the formula is true. If no objects enter the skin membrane, then no membrane pair with ε_{f_2} and ε_{t_2} labels dissolved, so the formula is false.

□

If the d objects of at least one interpretation get to the skin membrane, then the formula is satisfiable and we stop. Otherwise, all of the interpretations halt somewhere and the n object gets into the skin using the

$$[n]_c^0 \to n \tag{29}$$

rule. We chose the length of the chain formed by the c membranes to be polynomial and to be longer than $n+2$. This way, if the formula is satisfiable, then the n object would get into the skin later than any other d object. Otherwise, it indicates the unsatisfiability.

Examining the given rules in this section, one can see that the number of objects and the number of rules in the system is polynomially bounded. Together with the polynomial bound on the size of the initial membrane structure, we get that the given solution is polynomially uniform.

5 Conclusions

We have shown that recognizer P systems with active membranes and no input membrane, having three polarizations using only dissolution and division rules are able to solve the Q3SAT decision problem in the restricted case when the quantifiers alternate, which problem - even with the restriction - is **PSPACE**-complete. The presented solution is polynomially uniform.

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References

1. Leporati, A., Feretti, C., Mauri, G., Pérez-Jiménez, M. J., Zandron, C.: Complexity Aspects of Polarizationless Membrane Systems. Natural Computing 8(4), 703–717 (2009)
2. Păun, G.: Membrane computing: An introduction. Springer-Verlag (2002)
3. Păun, G., Rozenberg, G., Salomaa, A. (eds.): The Oxford Handbook of Membrane Computing. Oxford University Press (2010)
4. Porreca, A. E., Leporati, A., Mauri, G., Zandron, C.: P systems with active membranes: Trading time for space. Natural Computing 10(1), 167–182 (2011)
5. Porreca, A. E., Leporati, A., Mauri, G., Zandron, C.: Recent complexity-theoretic results on P systems with active membranes. Journal of Logic and Computation (2013)