How to calculate the slopes of D-modules

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Introduction

The purpose of this paper is to study invariants called the slopes or the critical indices associated with a module over the ring $\mathcal{D}$ of the germs of differential operators at a point x of a complex analytic manifold X.

The notion of a slope of a coherent $\mathcal{D}$-module $\mathcal{M}$, along a smooth hypersurface Y, was introduced by Y. Laurent under the name of a critical index in [4]. On the ring $\mathcal{E}_X$ of microdifferential operators, he defined two filtrations: The filtration F by the order of operators and the filtration V of Malgrange-Kashiwara, along a hypersurface $\Lambda \subset T^*X$. He then considered the intermediate filtration $L_r = pF + qV$ for any rational number $r = p/q \geq 0$. The critical indices are the rationals r for which the characteristic variety of $\mathcal{M}$ associated with $L_r$is not bihomogeneous. Using 2-microdifferential operators, Y. Laurent showed that there is only a finite number of critical indices . C. Sabbah and F. Castro, in the appendix [13], gave another proof of this result by using the notion of the local flattener of a deformation. Z. Mebkhout introduced, in [9], the notion of a transcendental slope of a holonomic $\mathcal{D}$–module, along a hypersurface as being a jump in the Gevrey filtration of the irregularity sheaf. This sheaf is the complex of solutions with values in the quotient $\mathcal{Q}$ by the holomorphic functions of the formal completion, along the hypersurface, of the ring of holomorphic functions. Laurent and Mebkhout proved ([5], see also [10]) a comparison theorem for the slopes of a holonomic $\mathcal{D}$–module asserting that the transcendental slopes are the same as the algebraic ones. They also defined *loc.cit.* the Newton polygon of a holonomic $\mathcal{D}$–module.

Our aim is to study these notions from an effective viewpoint, that is to prove by elementary methods the finiteness of the number of slopes and then to give an algorithm to compute these slopes effectively in the algebraic case. We consider an ideal I of the Weyl algebra A_n and we prove the finiteness of the number of slopes of the quotient A_n/I starting from a system of generators of I. We then develop an algorithm for the computation of the slopes. For this

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purpose we use the technique of standard bases adapted to Weyl algebra, and in particular we give an algorithm for the computation of these standard bases. This algorithm, of Lazard-Mora type (cf. [6], [11], [?]), is in fact valid for any order on the exponents, and thus can be adapted to computing multiplicities. Thanks to the theorem of Laurent and Mebkhout [5] the algoritm for the computation of the slopes provides an effective means to test the regularity of a holonomic $\mathcal{D}$–module along a hypersurface.

Let us give the details of this paper : In the first part we first recall some general facts about the filtrations F, V, and L, about privileged exponents and standard bases and we explain the relationship between standard bases and generator systems of the graded ideal $gr^L I$. We develop then an algorithm for the computation of standard bases for any order adapted to L and compatible with the product of operators. For this purpose we set in $A_n[t]$ by homogenizing with respect to the total order. We use a well ordering on the exponents in $\mathbf{N}^{2n+1}$, which allows us to prove a division lemma and then to obtain standard bases as systems of generators for which the remainders of some elementary divisions are zero modulo $(t-1)A_n[t]$. The difficulty comes from the fact that $A_n[t]$ being non commutative, the remainders of the homogeneous divisions are not necessarily homogeneous. The actual computation of a standard basis is however possible through a technical trick (rehomogenization of remainders and iterated division).

In the second part, we begin by proving the finiteness of the number of slopes by means of two twin lemmas : In the neighbourhood of a form L, the graded ideals associated with a form L', are constant on both sides of L and respectively equal to $gr^F(gr^L(I))$ and to $gr^V(gr^L(I))$. This assertion is also given by Laurent in ([4]). The finitenes of the number of slopes comes then from a compacity argument. The algorithm for the computation of the slopes, is inspired by the method used by Assi [1] to compute the critical tropisms of Lejeune-Teissier [7] : We start from a system of generators $\mathcal{F} = \{P_1, \ldots, P_r\}$ of I which induces a standard basis for $gr^F(I)$ as well as for $gr^V(gr^F(I))$ and we take the first form L for which one of the $\sigma^L(P_i))$ is not bihomogeneous. A finite algorithm allows us to decide whether L is actually a slope and if it is not, to reach a first slope $L^{(1)}$ (or V) in a finite number of steps. Iterating this process starting from $L^{(1)}$ one obtains all the slopes.Remark that, thanks to the first part, all the algorithms are effective in the case of the Weyl algebra. Using a well known lemma of algebraicity, the computation of the slopes of $A_n/I$is also valid for $\mathcal{D}/\mathcal{D}I$. To conclude, in the case of a non necessarily algebraic ideal of $\mathcal{D}$, we can generalise all our results, after making division compatible with the series case, if we admit infinite division processes.

1 Filtrations and construction of standard bases

1.1 Filtrations

We denote $\mathcal{D}_n(\mathbf{C})$ (or $\mathcal{D}_n$) (resp. $A_n(\mathbf{C})$ (or A_n)) the ring of differential operators with coefficients $\mathbf{C}\{x\} = \mathbf{C}\{x_1, \ldots, x_n\}$ (resp. $\mathbf{C}[x] = \mathbf{C}[x_1, \ldots, x_n]$). If $P(x, \partial) \in \mathcal{D}_n$ (or $P(x, \partial) \in A_n$), we write,

$$P(x,\partial)=\sum_{\alpha,\beta} p_{\alpha,\beta}x^\alpha\partial^\beta$$

with $\alpha,\beta\in\mathbf{N}^n$ and $p_{\alpha,\beta}\in\mathbf{C}$.

We call the set $\{(\alpha,\beta)\in\mathbf{N}^{2n}\mid p_{\alpha,\beta}\neq 0\}$ the *Newton diagram* of P (and we denote it by $\mathcal{N}(P)$).

Given a linear form $L(a,b)=pa+qb$ (with (p,q) non negative, relatively prime integers) we define the L-order of $P=P(x,\partial)$, denoted $ord_L(P)$, as being the maximal value of $L(|\beta|,\beta_1-\alpha_1)$ over elements (α,β) of the Newton diagram of P.

We denote by $F_{L,\bullet}(\mathcal{D}_n)$ (resp. $F_{L,\bullet}(A_n)$) the filtration induced by the $L-$ order on $\mathcal{D}_n$ (resp. A_n) i.e. $F_{L,k}$ is the set of operators P with $ord_L(P)\leq k$. We denote F by (resp. by V) the filtration corresponding to the form $L(a,b)=a$ (resp.$L(a,b)=b$). By extension we also write F (resp. V) for the corresponding linear forms. If $L\neq F,V$ then the graded ring associated with this filtration

$$gr^L(\mathcal{D}_n)=\bigoplus_{k\in\mathbf{Z}} F_{L,k}(\mathcal{D}_n)/F_{L,k-1}(\mathcal{D}_n)$$

(resp.

$$gr^L(A_n)=\bigoplus_{k\in\mathbf{Z}} F_{L,k}(A_n)/F_{L,k-1}(A_n))$$

is isomorphic to the commutative graded ring $\mathbf{C}\{x_2,\ldots,x_n\}[x_1,\xi_1,\ldots,\xi_n]$ (resp. to $\mathbf{C}[x,\xi]=\mathbf{C}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n]$) in which the degree of a monomial $x^\alpha\xi^\beta$ is equal to $L(|\beta|,\beta_1-\alpha_1)$.

If $L=F$, the filtration $F_{L,\bullet}$ is the same as the filtration by the order of operators.

If $L=V$, the graded ring $gr^V(\mathcal{D}_n)$ (resp. $gr^V(A_n)$) is isomorphic to $\mathbf{C}\{x_2,\ldots,x_n\}[x_1,\partial_1,\ldots,\partial_n]$ (resp. to the Weyl algebra $A_n=\mathbf{C}[x,\partial]$) in which the degree of a monomial $x^\alpha\partial^\beta$ is $\beta_1-\alpha_1$.

Given an ideal I in $\mathcal{D}_n$ (resp. A_n), we denote by $gr^L(I)$ the graded ideal associated with the filtration induced by $F_{L,\bullet}$ on I. The ideal $gr^L(I)$ is generated by the family $\{\sigma^L(P)|P\in I\}$ where $\sigma^L(P)$ is the principal symbol of P with respect to L. If $L\neq V$,

$$\sigma^L(P)=\sum_{L(|\beta|,\beta_1-\alpha_1)=ord_L(P)} p_{\alpha,\beta}x^\alpha\xi^\beta$$

If L is the form V, the symbol of P with respect to V is the differential operator

$$\sum_{\beta_1-\alpha_1=ord_V(P)} p_{\alpha,\beta}x^\alpha\partial^\beta$$

If L is a form with a non rational slope p/q, one can also define $F_{L,\bullet}$ and the associated graded rings and ideals.

We have the following algebraicity result : Let I be an ideal in A_n and let I' be the ideal $\mathcal{D}\cdot I$. Then $gr^L(I')=gr^L(\mathcal{D}_n)\cdot gr^L(I)$. More precisely, if $\mathcal{F}=\{P_1,\ldots,P_r\}$ is a system of generators of I such that $\mathcal{G}=\{\sigma^L(P_i)\}_{i=1}^r$ generates $gr^L(I)$, then $\mathcal{G}$ generates $gr^L(I')$ over $gr^L(\mathcal{D}_n)$.

Remark. We shall see later that such a family $\mathcal{F}$ can be computed effectively starting from a system of generators of the ideal I.

Proof. In this proof we write σ for σ^L and ord for ord_L. We denote by J' the ideal $gr^L(\mathcal{D}_n) \cdot gr^L(I)$.Only the last part needs to be proved . Such a system $\mathcal{F}$ exists because $gr^L(A_n)$ is noetherian. Let $P \in I'$ and let us write $P = \sum_{i=1}^r Q_i P_i + P'$ with $Q_i \in \mathcal{D}_n$, $P' \in I'$ and $ord(P') < ord(P)$ or $P' = 0$. We set $d = max_{1,\dots,r}\{ord(Q_i P_i)\}$, $\delta = ord(P)$ and $d_i = ord(P_i)$. We can suppose that d is minimal. If $\delta = d$ then $\sigma(P) = \sum_{i=1}^r \sigma^L_{d-d_i}(Q_i)\sigma(P_i) \in J'$. If $\delta < d$ then we have a relation $\sum_{i=1}^r \sigma^L_{d-d_i}(Q_i)\sigma(P_i) = 0$. The ring $gr^L(\mathcal{D}_n)$ being flat[1] over $gr^L(A_n)$, we can write

$$\sigma_{d-d_i}(Q_i) = \sum_{j=1}^m \Lambda_j F_{j,i}$$

where $(F_{j,1}, \dots, F_{j,r})$ is a relation, in $gr^L(A_n)$, between the $\sigma(P_i)$'s and $\Lambda_j \in gr^L(\mathcal{D}_n)$. We denote by $\overline{\Lambda_j}$ (resp. $\overline{F_{j,i}}$) a pre-image in $\mathcal{D}_n$ (resp. A_n) of Λ_j (resp. $F_{j,i}$). We denote by $G_j = \sum_{i=1}^r \overline{F_{j,i}} P_i \in I$. By construction, we have $ord(G_j) < d - ord(\Lambda_j)$. The family $\mathcal{G}$ being a system of generators of $gr^L(I)$, we can write $\sum_{i=1}^r \overline{F_{j,i}} P_i = G_j = \sum_{l=1}^r R_{j,l} P_l + G'_j$ where the $R_{j,l}$ are in A_n, $G'_j \in I$, $ord(R_{j,l}) < d - d_l - ord(\Lambda_j)$ and $ord(G'_j) < \delta - ord(\Lambda_j)$. Thus we have $G'_j = \sum_{i=1}^r (\overline{F_{j,i}} - R_{j,i}) P_i$ and

$$P = P' + \sum_{j=1}^m \overline{\Lambda_j} G'_j + \sum_{i=1}^r (Q_i - \sum_{j=1}^m \overline{\Lambda_j}(\overline{F_{j,i}} - R_{j,i})) P_i.$$

If we denote $Q'_i = Q_i - \sum_{j=1}^m \overline{\Lambda_j}(\overline{F_{j,i}} - R_{j,i})$ we have obtained a new decomposition for P with $max_{i=1,\dots,r}\{ord(Q'_i P_i)\}$ strictly smaller than d, which contradicts the choice of d. $\square$

Remark. To make this more precise, considering the flatness of $\mathbf{C}\{x\}$ over $\mathbf{C}[x]$ we can write

$$\sigma_{d-d_i}(Q_i) = \sum_{j=1}^m \Lambda_j F_{j,i}$$

with $ord_\xi(\sigma_{d-d_i}(Q_i)) \geq ord_\xi(\Lambda_j F_{j,i})$. This allows us to also consider the case $L = V$ and to adapt the proof to the case where L is irrational.

1.2 Privileged exponents

Let $<$ be a well ordering, compatible with the sum, in $\mathbf{N}^{2n}$ (i.e. a good order such that $(\alpha + \alpha'', \beta + \beta'') < (\alpha' + \alpha'', \beta' + \beta'')$ if and only if $(\alpha, \beta) < (\alpha', \beta')$).

[1]If $L = V$ see also the remark after this proof

Let L be a linear form with non negative integer coefficients. We define on $\mathbf{N}^{2n}$ the total ordering (denoted $<_L$) by :

$$(\alpha,\beta) <_L (\alpha',\beta') \Leftrightarrow \left\{ \begin{array}{l} L(|\beta|,\beta_1-\alpha_1) < L(|\beta'|,\beta'_1-\alpha'_1) \\ \text{ou bien } \left\{ \begin{array}{l} L(|\beta|,\beta_1-\alpha_1) = L(|\beta'|,\beta'_1-\alpha'_1) \\ \text{et } (\alpha,\beta) < (\alpha',\beta') \end{array} \right. \end{array} \right.$$

Remark that for any $d \in \mathbf{R}$, the restriction of $<_L$ to the set $\{(\alpha,\beta) \text{ s.t. } L(|\beta|,\beta_1-\alpha_1) = d\}$ is a well ordering.

Let L be a linear form with non negative coefficients. We call the element of $\mathbf{N}^{2n}$ $max_{<_L}\{\mathcal{N}(P)\}$ (where $\mathcal{N}(P)$ is the Newton diagram of P) the privileged L–exponent of $P \in A_n$ (and we denote it by $\exp_L(P)$). We write $\exp(P)$ when no confusion is possible.

Remark. With the previous definition we cannot consider $\exp(P)$ for an operator $P \in \mathcal{D}_n \backslash A_n$.

Let $P = \sum_{\alpha,\beta} p_{\alpha,\beta} x^\alpha \partial^\beta$ be an element of A_n. We call the monomial $p_{\alpha,\beta} x^\alpha \partial^\beta$ where $(\alpha,\beta) = \exp(P)$ the L– initial monomial of P (and we it by denote $In_L(P)$.) The complex number $p_{\alpha,\beta}$ is called the initial coefficient of P with respect to L and is denoted $c_L(P)$. We denote $In(P)$ and $c(P)$ when no confusion is possible.

Let Q, P be elements of A_n. Then we have:

1. $\exp(Q.P) = \exp(Q) + \exp(P)$.
2. If $\exp(Q) \neq \exp(P)$ then we have $\exp(P+Q) = max_{<_L}\{\exp(P), \exp(Q)\}$.
3. If $\exp(Q) = \exp(P)$ et $c(P)+c(Q) \neq 0$ then we have $\exp(P+Q) = \exp(P)$ and $c(P+Q) = c(P)+c(Q)$.
4. f $\exp(Q) = \exp(P)$ and $c(P)+c(Q) = 0$ then we have $\exp(P+Q) <_L \exp(P)$.

1.3 Standard bases

If I is an ideal of A_n we denote the set $\{\exp_L(P) | P \in I\}$ by $\mathrm{Exp}_L(I)$ (or simply $\mathrm{Exp}(I)$ when no confusion is possible). After 1.2 $\mathrm{Exp}_L(I) + \mathbf{N}^{2n} = \mathrm{Exp}_L(I)$.

Let I be an ideal of A_n. A family $\{P_1,\ldots,P_r\}$ of elements of I is called a *standard basis* (relatively to the order $<_L$, or relatively to L) of I if

$$\mathrm{Exp}_L(I) = \bigcup_{i=1}^{r} (\exp_L(P_i) + \mathbf{N}^{2n}).$$

A standard basis of an ideal I of A_n is not necessarily a set of generators of I. It is enough to consider the ideal $I = A_n$. Let L be the linear form $L(i,j) = j$. Let $P = 1 + x$. Then $\{P\}$ is a standard basis of I and P does not generate I.

Let $\mathcal{F} = \{P_1, \dots, P_r\}$ be a system of generators of an ideal I of A_n. If $\mathcal{F}$ is a L-standard basis of I then :

1. $\{\sigma^L(P_1), \dots, \sigma^L(P_r)\}$ is a set of generators of $gr^L(I)$.
2. If in, addition, $E_V(gr^L(I)) = \cup_{i=1}^r(\exp_V(\sigma^L(P_i)) + \mathbf{N}^{2n})$ (resp. $E_F(gr^L(I)) = \cup_{i=1}^r(\exp_F(\sigma^L(P_i)) + \mathbf{N}^{2n})$), then the family $\{\sigma^V(\sigma^L(P_i))\}_{i=1}^r$ (resp. $\{\sigma^F(\sigma^L(P_i))\}_{i=1}^r$) generates $gr^V(gr^L(I))$ (resp. $gr^F(gr^L(I))$).

Proof. 1. Let J be the ideal generated by $\{\sigma^L(P_i)\}_{i=1}^r$. We set $c_i = c(P_i)$.Let $0 \neq P \in I$. We define a family of elements $P^{(s)}$ of I, for all $s \geq 0$, such that

- $P^{(0)} = P$
- $P^{(s+1)} = P^{(s)} - \frac{c(P^{(s)})}{c_{i_s}} x^{\alpha^s}\partial^{\beta^s} P_{i_s}$, where (α^s, β^s) is an element of $\mathbf{N}^{2n}$ such that $(\alpha^s, \beta^s) + \exp(P_{i_s}) = \exp(P^{(s)})$
- $ord_L(P^{(s+1)}) \leq ord_L(P^{(s)})$ et $\exp(P^{(s+1)}) <_L \exp(P^{(s)})$

Thus, after the note 1.2, there is an s such that $ord_L(P^{(s+1)}) < ord_L(P^{(s)})$. Let s be the smallest integer having this property. Then

$$\sigma^L(P) \sum_{j=0}^{s} \sigma^L(\frac{c(P^{(j)})}{c_{i_j}} x^{\alpha^j}\partial^{\beta^j})\sigma^L(P_{i_j}).$$

2. The proof is the same as in 1. We only have to replace $\mathcal{F}$ by $\{\sigma^L(P_1), \dots, \sigma^L(P_r)\}$ and I by $gr^L(I)$. □

1.4 Homogenization, orders in $\mathbf{N}^{2n+1}$

We set $A_n[t] = A_n \otimes_{\mathbf{C}} \mathbf{C}[t]$. If $P = \sum_{\alpha,\beta} p_{\alpha,\beta} x^\alpha \partial^\beta$ is an element of A_n we call the integer $max\{|\alpha| + |\beta| \mid p_{\alpha,\beta} \neq 0\}$ the *total order* of P (and we denote it by $ord^T(P)$.) As in case of A_n, we can define the notion of Newton diagram of an operator in $A_n[t]$.

Let $P = \sum_{\alpha,\beta} p_{\alpha,\beta} x^\alpha \partial^\beta \in A_n$. We call the differential operator

$$h(P) = \sum_{\alpha,\beta} p_{\alpha,\beta} t^{ord^T(P) - |\alpha| - |\beta|} x^\alpha \partial^\beta \in A_n[t].$$

the homogeneization of P.

We denote by $\pi : \mathbf{N}^{2n+1} \longrightarrow \mathbf{N}^{2n}$ the projection defined by $\pi(h, \alpha, \beta) = (\alpha, \beta)$. We consider on $\mathbf{N}^{2n+1}$ the total order denoted $\tilde{<}_L$, thus defined:

$$(k, \alpha, \beta) \tilde{<}_L (k', \alpha', \beta') \Longleftrightarrow \begin{cases} k + |\alpha| + |\beta| < k' + |\alpha'| + |\beta'| \\ \text{or } \begin{cases} k + |\alpha| + |\beta| = k' + |\alpha'| + |\beta'| \text{ and} \\ (\alpha, \beta) <_L (\alpha', \beta') \end{cases} \end{cases}$$

This order on $\mathbf{N}^{2n+1}$ is a well ordering compatible with the sum.

If $H = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} t^k x^\alpha \partial^\beta$ is an element of $A_n[t]$ we call the greatest element with respect to the total order $\prec_L$, of the Newton diagram of H the privileged exponent of H relatively to $\prec_L$ (and we denote it by $\exp_{\prec_L}(H)$). The monomial of H whose exponent is equal to the privileged exponent is called the initial monomial of H and we denote it by $In_{\prec_L}(H)$. The coefficient of the initial monomial of H is called the initial coefficient of H and we denote it by $c_{\prec_L}(H)$. We write $\exp(H), In(H)$ and $c(H)$ when no confusion is possible. It is useful to use the following notation : $\widehat{H} = H - In(H)$.

If $P \in A_n$, we have in general $\exp_L(P) \neq \pi(\exp_{\prec_L}(P))$. The equality happens only if $\exp_L(\sigma^T(P)) = \exp_L(P)$, where $\sigma^T(P)$ is the symbol of P with respect to the total ordering.

We have the following relations with the H_i in $A_n[t]$, P and Q in A_n and where exp denotes the exponent either for $\prec_L$ or for $<_L$.

1. $\exp(H_1 H_2) = \exp(H_1) + \exp(H_2)$.
2. If $\exp(H_1) \neq \exp(H_2)$ then we have $\exp(H_1 + H_2) = max_{\prec_L}\{\exp(H_1), \exp(H_2)\}$.
3. If $\exp(H_1) = \exp(H_2)$ and $c(H_1) + c(H_2) \neq 0$ then we have $\exp(H_1 + H_2) = \exp(H_1)$ and $c(H_1 + H_2) = c(H_1) + c(H_2)$.
4. If $\exp(H_1) = \exp(H_2)$ and $c(H_1) + c(H_2) = 0$ then we have $\exp(H_1 + H_2) \prec_L \exp(H_1)$.
5. $\exp(h(QP)) = \exp(h(Q)h(P))$.
6. $\pi(\exp(h(P)) = \exp(P)$.

1.5 Partition of $\mathbf{N}^{2n+1}$. Division in $A_n[t]$

Given an element $(\mu^1, \ldots, \mu^r)$ of $(\mathbf{N}^{2n+1})^r$, a partition $\{\Delta_1, \ldots, \Delta_r, \overline{\Delta}\}$ of $\mathbf{N}^{2n+1}$ is associated with it in the following way:

$$\Delta_1 = \mu^1 + \mathbf{N}^{2n+1}$$

$$\Delta_i = (\mu^i + \mathbf{N}^{2n+1}) \setminus (\textstyle\bigcup_{j=1}^{i-1} \Delta_j) \text{ if } 2 \leq i \leq r$$

$$\overline{\Delta} = \mathbf{N}^{2n+1} \setminus (\textstyle\bigcup_{j=1}^{r} \Delta_j)$$

Let $(P_1, \ldots, P_r)$ be in A_n^r. We denote $\{\Delta_1, \ldots, \Delta_r, \overline{\Delta}\}$ the partition of $\mathbf{N}^{2n+1}$, associated with $(\exp(h(P_1)), \ldots, \exp(h(P_r)))$. Then, for any $H \in A_n[t]$ there exists a unique element $(Q_1, \ldots, Q_r, R)$ in $A_n[t]^{r+1}$ such that

1. $H = Q_1 h(P_1) + \ldots + Q_r h(P_r) + R$
2. $\exp(h(P_i)) + \mathcal{N}(Q_i) \subset \Delta_i$ et $\exp(Q_i h(P_i)) \preceq_L \exp(H)$ pour $1 \leq i \leq r$.

3. $\mathcal{N}(R) \subset \overline{\Delta}$ et $\exp(R) \preceq_L \exp(H)$.

The proof is classical and left to the reader.

Let $(\alpha, \beta) \in \mathbf{N}^{2n}$. We say that $\mathcal{N}(H)$ is dominated by (α, β) if for any exponent (k', α', β') in $\mathcal{N}(H)$ we have $(\alpha', \beta') \leq_L (\alpha, \beta)$. If in the claim of the division theorem $\mathcal{N}(H)$ is dominated by (α, β) the same is true for $\mathcal{N}(R)$ and for $\mathcal{N}(Q_i h(P_i))$ withr $i = 1, \ldots, r$.

1.6 Semisyzygies

Let G_1, G_2 be two non-zero elements in $A_n[t]$. We denote $\mu^i = \exp_{\prec_L}(G_i)$ and $\mu = ppcm(\mu^1, \mu^2)$. Let us write $\mu = \nu^1 + \mu^1 = \nu^2 + \mu^2$. We consider the operator $S(G_1, G_2) = M_1 G_1 - M_2 G_2$ where M_i is the monomial with exponent ν^i and with coefficient $1/c(G_i)$. We call it the *semisyzygy* relatively to (G_1, G_2). We have a similar definition for the operators in $\mathcal{D}_n$ or A_n.

Let $\mathcal{F} = \{P_1, \ldots, P_r\}$ be a system of generators of the ideal I of A_n such that, for any (i, j), the remainder of the division of $S(h(P_i), h(P_j))$ by $(h(P_1), \ldots, h(P_r))$ is equal to zero, modulo $(t-1)A_n[t]$. Then $\mathcal{F}$ is a standard basis of I.

Proof. We denote $\Delta = \bigcup_{i=1}^r (\exp(P_i) + \mathbf{N}^{2n})$. It is enough to prove that $\mathrm{Exp}_L(I) \subset \Delta$. Let $P \in I$. We write $P = \sum_{i=1}^r Q_i P_i$. We set,

- $d_i = deg^T(Q_i P_i),\ d = max_{i=1,\ldots,r}\{d_i\},\ \delta = deg^T(P) \leq d$,
- $\mu^i = \exp_{\prec_L}(h(P_i)),\ \nu^i = (\nu_0^i, \alpha^i, \beta^i) = \exp_{\prec_L}(h(Q_i)h(P_i))$,
- $(\alpha, \beta) = max_{i=1,\ldots,r}\{\exp_L(Q_i P_i)\}$.

Let $\{i_0, \ldots, i_s\}$ be the set of indices where (α, β) is reached. If $s = 0$ then $\exp(P) = \exp(Q_{i_0} P_{i_0}) \in \Delta$. We suppose then $s \geq 1$. We write

$$t^{d-\delta} h(P) = \sum_{i=1}^r t^{d-d_i} h(Q_i) h(P_i) \text{ mod. } (t-1)A_n[t].$$

Thus we have

$$\begin{array}{ll} \exp_{\prec_L}(t^{d-d_i} h(Q_i)h(P_i)) = (d - |\alpha| - |\beta|, \alpha, \beta) & \text{pour } i = i_0, \ldots, i_s \\ \exp_{\prec_L}(t^{d-d_i} h(Q_i)h(P_i)) \prec_L (d - |\alpha| - |\beta|, \alpha, \beta) & \text{pour } i \notin \{i_0, \ldots, i_s\} \end{array}$$

Let $\mu = ppcm(\mu_{i_0}, \mu_{i_1})$. Let us write $\mu = \mu'_{i_0} + \mu_{i_0} = \mu'_{i_1} + \mu_{i_1}$. Recall that the semisyzygy $S = S(h(P_{i_0}), h(P_{i_1}))$ can be written

$$S = M_{i_0} h(P_{i_0}) - M_{i_1} h(P_{i_1}) =$$

$$= \textstyle\sum_{i=1}^r S_i h(P_i) \text{ mod. } (t-1)A_n[t]$$

where M_{i_k} is a monomial in $A_n[t]$ such that

- $\exp(M_{i_k}) = \mu'_{i_k}, \quad k = 0, 1$
- $\exp(S(h(P_{i_0}), h(P_{i_1}))) \prec_L \mu$

Let us denote $d' = d - |\alpha| - |\beta|$. We have $(d', \alpha, \beta) = \exp(t^{d-d_{i_l}} h(Q_{i_l} P_{i_l}))$ for $l = 0, \ldots, s$ and thus $(d', \alpha, \beta) = \nu + \mu$ for some $\nu \in \mathbf{N}^{2n+1}$. Let M be the unitary monomial such that $\exp(M) = \nu$. Thus we have $\exp(t^{d-d_{i_k}} h(Q_{i_k})) = \exp(MM_{i_k}) = \nu + \mu'_{i_k}$ for $k = 0, 1$. Let us denote by c_k the unique scalar such that

$$In_{\prec_L}(t^{d-d_{i_k}} h(Q_{i_k})) = In_{\prec_L}(c_k M M_{i_k})$$

for $k = 0, 1$. We have

$$t^{d-\delta} h(P) = (c_0 M M_{i_0} - c_0 M \widehat{M}_{i_0} + t^{d-d_{i_0}} h(\widehat{Q}_{i_0})) h(P_{i_0}) + \sum_{i \neq i_0} t^{d-d_i} h(Q_i) h(P_i) =$$

$$= c_0 M (S + M_{i_1} h(P_{i_1})) + (t^{d-d_{i_0}} h(\widehat{Q}_{i_0}) - c_0 M \widehat{M}_{i_0}) h(P_{i_0}) + \sum_{i \neq i_0} t^{d-d_i} h(Q_i) h(P_i) =$$

$$= c_0 M (\sum_{i=1}^{r} S_i h(P_i)) + c_0 M M_{i_1} h(P_{i_1}) + (t^{d-d_{i_0}} h(\widehat{Q}_{i_0}) - c_0 M \widehat{M}_{i_0}) h(P_{i_0}) + \sum_{i \neq i_0} t^{d-d_i} h(Q_i) h(P_i)$$

all these equalities being considered modulo $(t-1)A_n[t]$. Thus we can write

$$t^{d-\delta} h(P) = \sum_{i=1}^{r} H_i h(P_i) \text{ mod. } (t-1)A_n[t]$$

where

- $H_{i_0} = c_0 M S_{i_0} + t^{d-d_{i_0}} h(\widehat{Q}_{i_0}) - c_0 M \widehat{M}_{i_0}$, $H_{i_1} = c_0 M S_{i_1} + c_0 M M_{i_1} + t^{d-d_{i_1}} h(Q_{i_1})$
- $H_i = c_0 M S_i + t^{d-d_i} h(Q_i)$ pour $i \neq i_0, i_1$

with

- $\exp(H_{i_0}) \prec_L \exp(t^{d-d_{i_0}} h(Q_{i_0}))$, $\exp(H_{i_1}) \preceq_L \exp(t^{d-d_{i_1}} h(Q_{i_1}))$
- $\exp(H_i) = \exp(t^{d-d_i} h(Q_i))$ pour $i \neq i_0, i_1$

Thus if we set $Q'_i = H_{i_{|t=1}}$ we get $P = \sum_{i=1}^{r} Q'_i P_i$ with, after 1.5, $max_{<_L}\{\exp_L(Q'_i P_i) \mid i = 1, \ldots, r\} \leq_L (\alpha, \beta)$ et $\exp_L(Q'_{i_0} P_{i_0}) <_L (\alpha, \beta)$. This proves the proposition by induction on s and on (α, β), since on the other side $ord^T(Q'_i P_i)$ remains bounded by d. $\square$

1.7 Construction of a standard basis

Let $P_1, \ldots, P_r$ be operators in A_n. The aim of this section is to build a standard basis for the ideal I, in $\mathcal{D}_n$, generated by $P_1, \ldots, P_r$. Given $P' \in A_n[t]$ and $P' = \sum_{i=1}^r Q_i h(P_i) + R$ a division (see 1.5) in $A_n[t]$, we have by construction :

$$\mathcal{N}(R) \subset \overline{\Delta} = \bigcup_{i=1}^{r} (\exp(h(P_i)) + \mathbf{N}^{2n+1}).$$

If $P' = h(P)$ with $P \in I$ we have in addition $R_{|t=1} \in I$. But $h(R_{|t=1})$ may have a privileged exponent different from $\exp(R)$ and it is even possible that $\pi(\exp(h(R_{|t=1}))) \neq \pi(\exp(R))$ (see 1.4). Thus, in particular, we can have $\exp(h(R_{|t=1})) \notin \overline{\Delta}$. In this situation, a non zero remainder does not necessarily produce a new privileged exponent in $\mathrm{Exp}_L(I)$. In view of solving this difficulty we use the following algorithm : let $P \in A_n[t]$. Let $R^{(p)}$, $p \in \mathbf{N}$, the sequence of operators in $A_n[t]$, defined in the following way :

- $R^{(1)}$ is the remainder of the division of P by $(h(P_1), \ldots, h(P_r))$.
- For $p \geq 2$, $R^{(p)}$is the remainder of the division of $h({R^{(p-1)}}_{|t=1})$ by $(h(P_1), \ldots, h(P_r))$.

Then there is a unique s such that:

- $\mathcal{N}(h({R^{(s)}}_{|t=1})) \subset \overline{\Delta}$, $\mathcal{N}(h({R^{(s-1)}}_{|t=1})) \not\subset \overline{\Delta}$

Furthermore, for any p we can write

$$P = \sum_{i=1}^{r} Q_i^{[p]} h(P_i) + R^{(p)} + (t-1)W^{[p]}$$

where $Q_i^{[p]}$ and $W^{[p]}$ are elements of $A_n[t]$ and for any (k', α', β') in

$$\mathcal{N}(R^{(p)}) \cup (\bigcup_{i=1}^{r} \mathcal{N}(Q_i^{[p]} h(P_i)))$$

we have

- $k' + |\alpha'| + |\beta'| \leq ord^T(P)$
- $(\alpha', \beta') \leq_L (\alpha, \beta)$, if (α, β) dominates $\mathcal{N}(P)$ (voir 1.5).

Proof. After 1.5 we can write

$$P = \sum_{i=1}^{r} Q_i^{(1)} h(P_i) + R^{(1)}.$$

We write also in a unique way, $R^{(1)} = h(R^{(1)}{}_{|t=1}) + (t-1)W^{(1)}$. We write in the same way

$$h(R^{(p-1)}{}_{|t=1}) = \sum_{i=1}^{r} Q_i^{(p)} h(P_i) + R^{(p)}$$

and $R^{(p)} = h(R^{(p)}{}_{|t=1}) + (t-1)W^{(p)}$. Let $\mathcal{N}_p$ be the Newton diagram of $h(R^{(p)}{}_{|t=1})$ et $\overline{\mathcal{N}_p} = \mathcal{N}_p \backslash \overline{\Delta}$. For any p such that $\overline{\mathcal{N}_p} \neq \emptyset$, let $(k_p, \alpha^p, \beta^p) = max_{\prec_L} \overline{\mathcal{N}_p}$ and $d_p = k_p + |\alpha^p| + |\beta^p|$. Let $(k', \alpha', \beta') \in \mathcal{N}(R^{(p)})$, after 1.5 we have : $k' + |\alpha'| + |\beta'| \leq d_{p-1}$ and one of the two following conditions :

- $(k', \alpha', \beta') \in \overline{\Delta}$ and $k' + |\alpha'| + |\beta'| = d_{p-1}$
- $(\alpha', \beta') <_L (\alpha^{p-1}, \beta^{p-1})$

A point (k'', α', β') in the Newton diagram $h(R^{(p)}{}_{|t=1})$, comes necessarily from a point (k', α', β') in the Newton diagram of $R^{(p)}$ and in the first case, the degree condition implies $k'' \leq k'$ hence $(k'', \alpha', \beta') \in \overline{\Delta}$. From this we deduce : $(k_p, \alpha^p, \beta^p) \prec_L (k_{p-1}, \alpha^{p-1}, \beta^{p-1})$. Since $\prec_L$ is a well ordering, there is a unique s such that $\overline{\mathcal{N}_{s-1}} \neq \emptyset$ and $\overline{\mathcal{N}_s} = \emptyset$. By division (voir 1.5) we get $R^{(s')} = h(R^{(s)}{}_{|t=1})$ for any $s' > s$. It remains be written

- $Q_i^{[p]} = \sum_{j=1}^{p} Q_i^{(j)}$, for $i = 1, \ldots, r$.
- $W_i^{[p]} = \sum_{j=1}^{p-1} W_i^{(j)}$

□

We call $R^{(s)}$ the remainder of the iterated division with rehomogenization of the intermediate remainders (or simply the iterated remainder). It is denoted by $R^{it}(P; h(P_1), \ldots, h(P_r))$.

Let $\mathcal{F} = \{P_1, \ldots, P_r\} \subset A_n$ be a system of generators of the ideal I de $\mathcal{D}_n$ such that for any (i,j), the iterated remainder of the division of $S(h(P_i), h(P_j))$ by $(h(P_1), \ldots, h(P_r))$ is equal to zero, modulo $(t-1)A_n[t]$. Then $\mathcal{F}$ is a standard basis of I.

Proof. the proof is similar to the proof of 1.6, by using the properties of quotient and remainders in the iterated division. □

1.8 Algorithm

Let $P_1, \ldots, P_r$ be elements of A_n.We show here how to build a standard basis (relative to a form L) of the ideal I generated by the P_i. If for any (i,j), $i < j$, we have $R^{it}(S(h(P_i), h(P_j)); h(P_1), \ldots, h(P_r)) = 0$ then after 1.7 $\{P_1, \ldots, P_r\}$ is a standard basis (relative to L) of I. If there exists (i,j) with $i < j$ such that $R^{it}(S(h(P_i), h(P_j)); h(P_1), \ldots, h(P_r)) \neq 0$ we set $P^{(r+1)} = R^{it}(S(h(P_i), h(P_j));$

$h(P_1), \ldots, h(P_r))$, $P_{r+1} = (P^{(r+1)})_{|t=1}$ and we repeat this process with $\{P_1, \ldots, P_r, P_{r+1}\}$ as a system of generators of I. In this way we build a family $\{P_1, \ldots, P_r, P_{r+1}, \ldots, P_{r+s}, \ldots\}$ in I such that, if $P_{r+j+1} \neq 0$ then

$$\exp(h(P_{r+j+1})) \notin \cup_{k=1}^{r+j} (\exp(h(P_k)) + \mathbf{N}^{2n+1}).$$

This process stops because $\mathbf{N}^{2n+1}$ is noetherian.

Remark. The algorithm presented above is valid for any ordering $\mathbf{N}^{2n}$ compatible with the sum. It allows us for example to compute the multiplicity at a point of the characteristic variety.

2 The finiteness of the number of slopes. Computing the slopes

2.1

The ring $gr^L(A_n)$ (and even $gr^L(\mathcal{D}_n)$, if $L \neq F, V$) has a graduation with respect to F and another with respect to V. Let I be an ideal of A_n (or $\mathcal{D}_n$) and let $L \neq F, V$ be a linear form. We say that L is a slope of A_n/I (or of $\mathcal{D}_n/\mathcal{D}_n I$) if $gr^L(I)$ is not bihomogeneous with respect to the filtrations F and V. Let P be an operator of $\mathcal{D}_n$. We call the convex hull of the set :

$$\bigcup_{(\alpha,\beta)\in\mathcal{N}(P)} ((|\beta|, \beta_1 - \alpha_1) + (-\mathbf{N})^2).$$

where $\mathcal{N}(P)$ is the Newton diagram of P (voir 1.1) the Newton polygon of P and we denote it by $\mathcal{P}(P)$, .

Remark. If P is an operator in $\mathcal{D}_n$ then the slopes of $\mathcal{D}_n/\mathcal{D}_n P$ are the slopes of the Newton polygon of P.

2.2 Two twin lemmas

In this section $L, L', L'', L^{(1)}, L^{(2)}, \ldots$ are linear forms with non negative coefficients (non necessarily rationnals). The notation $L < L'$ means $slope(L) < slope(L')$. Let I be an ideal of A_n and let $L \neq V$ be a linear form. There exists a linear form $L^{(1)}$ with $L^{(1)} > L$ such that for any form L' such that $L^{(1)} > L' > L$ we have

$$gr^{L'}(I) = gr^V(gr^L(I)).$$

Proof. Let $\{P_1, \ldots, P_r\}$ be a family of elements of I such that

$$Exp_V(gr^L(I)) = \cup_{i=1}^r (\exp_V(\sigma^L(P_i)) + \mathbf{N}^{2n})$$

Let $L^{(1)}$ be a form such that $L^{(1)} > L$ and such that $\sigma^{L^{(1)}}(P_i) = \sigma^V(\sigma^L(P_i))$ for $i = 1, \ldots, r$. In particular, for any form L' such that $L^{(1)} > L' > L$ we also have $\sigma^{L'}(P_i) = \sigma^V(\sigma^L(P_i))$. Thus, after lemma 1.3 we have $gr^V(gr^L(I)) \subset gr^{L'}(I)$. Let us consider the opposite inclusion. After having increased, if necessary, the family $\{P_1, \ldots, P_r\}$, we can suppose that the homogenized elements built from the operators $\sigma^L(P_i)$ constitute a standard basis of the homogenized ideal $h(gr^L(I))$ with respect to the ordering $\prec_{L'}$. Let $P \in I$. We have $\sigma^L(P) = \sum_{i=1}^r \lambda_i \sigma^L(P_i)$, with $ord_{L'}(\lambda_i \sigma^L(P_i)) \leq ord_{L'}(\sigma^L(P))$. We set $P^{(1)} = P - \sum_{i=1}^r \Lambda_i P_i$ where $\Lambda_i \in A_n$ is the obvious preimage of λ_i. We have :

- $ord_L(P^{(1)}) < ord_L(P)$.
- Since we carried out a division with respect to $\prec_{L'}$, $ord_{L'}(P) \geq ord_{L'}(\Lambda_i P_i)$.

If $ord_{L'}(P^{(1)}) < ord_{L'}(P)$ then

$$\sigma^{L'}(P) = \sum_{i=1}^r \sigma^{L',d-d_i}(\Lambda_i)\sigma^{L'}(P_i)$$

where $d = ord_{L'}(P)$ et $d_i = ord_{L'}(P_i)$. If $ord_{L'}(P^{(1)}) = ord_{L'}(P)$ we do the same with $P^{(1)}$. Thus we build a sequence $P^{(s)}$, $s \geq 1$ such that

- $P^{(s+1)} = P - \sum_{i=1}^r \Lambda_i^s P_i$
- $ord_L(P^{(s+1)}) < ord_L(P^{(s)})$
- $ord_{L'}(P) \geq ord_{L'}(\Lambda_i^s P_i)$

If L is rational, and since the sequence $ord_L(P^{(s)})$ strictly decreases, there is an integer s such that $ord_{L'}(P^{(s+1)}) < ord_{L'}(P^{(s)})$ whence

$$\sigma^{L'}(P) = \sum_{i=1}^r \sigma^{L',d-d_i}(\Lambda_i^s)\sigma^{L'}(P_i).$$

We set $(\alpha^i, \beta^i) = \exp_L(P_i)$. If L is rationnal, the only point of $\mathcal{P}(P_i)$ on the lines $L(a,b) = L(|\beta^i|, \beta_1^i - \alpha_1^i)$ and $L'(a,b) = L'(|\beta^i|, \beta_1^i - \alpha_1^i)$ is $(|\beta^i|, \beta_1^i - \alpha_1^i)$. Hence, $\mathcal{P}(P_i)$, without its vertex, is included in the sector $\{L(a,b) < L(|\beta^i|, \beta_1^i - \alpha_1^i)\} \cap \{L'(a,b) < L'(|\beta^i|, \beta_1^i - \alpha_1^i)\}$. This proves that we only have to deal with a finite number of points of $\mathcal{P}(P)$ to obtain $ord_{L'}(P^{(s+1)}) < ord_{L'}(P^{(s)})$. □

Let I be an ideal of A_n and let $L \neq F$ be a linear form. Then there is a linear form $L^{(2)}$ with $L^{(2)} < L$ such that for any form L' with $L^{(2)} < L' < L$ we have

$$gr^{L'}(I) = gr^F(gr^L(I)).$$

Proof. The proof is the same as in 2.2, provided that we write F instead of V. □

2.3 Finiteness of the number of slopes

Let I be an ideal of A_n. Then the number of slopes of the module A_n/I (or of the module $\mathcal{D}_n/\mathcal{D}_nI$) is finite.

Proof. This is a corollary of the twin lemmas, because each form L is in an open set where there is at most one slope, namely the form L itself. We end by a compacity argument. □

2.4 Algorithm

The effective determination of the slopes of A_n/I is based on the following result, analogous to Assi's in [1], in which he computes the critical tropisms of Lejeune-Teissier [7]. Let $L \neq V$ and let $\{P_1, \ldots, P_r\}$ be a system of generators of the ideal I inducing a standard basis of $gr^V(gr^L(I))$. Then, there is a rational linear form L'' and a system of generators $\{P'_1, \ldots, P'_r\}$ such that

- $\sigma^L(P_i) = \sigma^L(P'_i)$
- $L'' > L$ and for any form L' with $L'' > L' > L$, $gr^{L'}(I) = gr^V(gr^L(I))$.
- If $L'' \neq V$, then one of the $\sigma^{L''}(P'_i)$ is not in $gr^V(gr^L(I))$ hence L'' is a slope.

Remark. The proof below shows that the construction of the form L'' and of the system $\{P'_1, \ldots, P'_r\}$ is algorithmic starting from $\{P_1, \ldots, P_r\}$ and from L. On the other hand, we have proved the finiteness of the number of slopes (see 2.3), and in addition, according to 1.8, we can build a family satisfying the hypotheses of the theorem starting from any system of generators of I. Therefore, this proof is an algorithm to compute all the slopes of A_n/I (or of $\mathcal{D}/\mathcal{D}I$.), starting from F.

Proof. Starting from a system of generators of I let us compute a family $\{P_1, \ldots, P_r\}$ in I such that :

- $gr^L(I)$ is generated by $\{\sigma^L(P_1), \ldots, \sigma^L(P_r)\}$.
- $Exp_V(gr^L(I)) = \bigcup_{i=1}^r (\exp_V(\sigma^L(P_i)) + \mathbf{N}^{2n})$

Let $L^{(1)}$ be the smallest linear form $> L$ such that there exists $1 \leq i \leq r$ such that $\sigma^{L^{(1)}}(P_i)$ is not bihomogeneous [2]. If it does not exist we set $L^{(1)} = V$ and the theorem is proved. We suppose

[2] this form is rational

$L^{(1)} \neq V$. Hence for any form Λ, $L^{(1)} > \Lambda > L$, we have $\sigma^{\Lambda}(P_i) = \sigma^V(\sigma^L(P_i))$. After the lemma 2.2 we have $gr^{\Lambda}(I) = gr^V(gr^L(I))$. Let i_0 be the smallest $i \in \{1, \ldots, r\}$ such that $\sigma^{L^{(1)}}(P_{i_0})$ is not bihomogeneous. We suppose $i_0 = 1$. We set $\sigma^{L^{(1)}}(P_1) = \sigma^V(\sigma^L(P_1) + \sum_{k=1}^{s} M(a_k, b_k)$ where $M(a_k, b_k)$is a bihomogeneous element such that $ord_F(M(a_k, b_k) = a_k$ et $ord_V(M(a_k, b_k) = b_k$. We set $a_1 > \cdots > a_s$. We can write, using a division,

$$M(a_1, b_1) = \sum_{j=1}^{s} \gamma_j \sigma^V(\sigma^L(P_j)) + \gamma$$

where the γ_i's and γ are bihomogeneous and $\exp_V(\gamma_j \sigma^V(\sigma^L(P_j)) \leq_V \exp_V(M(a_1, b_1))$. We have :

- If $\gamma \neq 0$ then $M(a_1, b_1)$is not in $gr^{L^{(1)}}(I)$ hence this ideal is not bihomogeneous. The theorem is proved in this case.
- If $\gamma = 0$ we write

  $$P_1^{(1)} = P_1 - \sum_{j=1}^{r} \Gamma_j P_j$$

  where Γ_j is the obvious preimage of γ_j. Since $ord_L(M(a_1, b_1)) < ord_L(P_1)$ we have $\sigma^L(P_1^{(1)}) = \sigma^L(P_1)$ and since we performed a bihomogeneous division, we have $\sigma^{\Lambda}(P_1^{(1)}) = \sigma^V(\sigma^L(P_1))$ for any form Λ such that $L^{(1)} < \Lambda < L$.

In this last case, if $\sigma^{L^{(1)}}(P_1^{(1)})$ is not bihomogeneous it can be written $\sigma^{L^{(1)}}(P_1^{(1)}) = \sigma^V(\sigma^L(P_1^{(1)})) + \sum_{j=1}^{s'} M(a'_j, b'_j)$ where $M(a'_j, b'_j)$ is bihomogeneous with $ord_F(M(a'_j, b'_j)) = a'_j$, $ord_V(M(a'_j, b'_j)) = b'_j$ and $a'_j < a_1$ for any j. We can then repeat this process with $P_1^{(1)}$ instead of P_1. This process stops since the set $\{(a, b) \in \mathbf{N} \times \mathbf{Z} \mid a < a_1 \text{ and } L^{(1)}(a, b) = ord_{L^{(1)}}(P_1)\}$ is finite. Hence we can replace P_1 by P'_1 such that, either $\sigma^{L^{(1)}}(P'_1) \notin gr^V(gr^L(I))$ (in which case $gr^{L^{(1)}}(I)$ is not bihomogeneous) or $\sigma^{L^{(1)}}(P'_1)$ is bihomogeneous. In this last case, we repeat with $\{P'_1, P_2, \ldots, P_r\}$. Let us remark that :

- $gr^L(I)$ is generated by $\{\sigma^L(P'_1), \ldots, \sigma^L(P_r)\}$.
- $Exp_V(gr^L(I)) = (\exp_V(\sigma^L(P'_1)) + \mathbf{N}^{2n}) \cup (\bigcup_{i=2}^{r}(\exp_V(\sigma^L(P_i)) + \mathbf{N}^{2n}))$

This process stops because, for any $1 \leq i \leq r$, the set $\{(a, b) \in \mathbf{N} \times \mathbf{Z} \mid L^{(1)}(a, b) \leq ord_{L^{(1)}}(P_i), b \geq ord_V(\sigma^L(P_i)) \text{ and } a \leq ord_F(\sigma^L(P_i))\}$ is finite. □

References

[1] A. Assi. *Standard bases, critical tropisms and flatness* Journal of Applicable Algebra in Engineering, Communication and Computing, 4, 1993, 197-215.

[2] B. Buchberger. *Ein algorithmisches Kriterium for die Lösbarkeit eines algebraischen Gleichungssystems* Aequationes Math., 374:383–4, 1970.

[3] F. Castro. Thèse de 3ème cycle Univ. de Paris VII, 1984.

[4] Y. Laurent. *Polygone de Newton et b-fonction pour les modules microdifférentiels*, Ann. Scient. Ec. Norm. Sup., 4e série **20** (1987) 391–441.

[5] Y. Laurent et Z. Mebkhout. *Le polygone de Newton d'un $\mathcal{D}_X$–module, à paraître.*

[6] D. Lazard. *Gröbner bases, Gaussian elimination and resolution of systemx of algebraic equations* , Proc. of Eurocal 83, Springer, L.N.C.S. **162**, 146-156 .

[7] M. Lejeune-Jalabert et B.Teissier. *Transversalité, polygone de Newton et installations*, in "Singularités à Cargèse, Astérisque **7-8** (1973) 75-119.

[8] B. Malgrange. *Sur la réduction formelle des équations différentielles à singularités irrégulières*, Prépub. Institut Fourier (1979).

[9] Z. Mebkhout. *Le théorème de positivité de l'irrégularité pour les $\mathcal{D}$–modules*, Grothendieck Festschrift III, Progress in Math. **88** (1990), 84-131.

[10] Z. Mebkhout. *Le polygone de Newton d'un $\mathcal{D}_X$–module*, Actes de la 3e Conférence de la Rábida, *à paraître.*

[11] T. Mora. *Seven variations on standard bases*, Preprint, University of Genova, 1988.

[12] J. P. Ramis. *Théorème d'indice Gevrey pour les équations différentielles ordinaires*, Mem. AMS **296** (1984).

[13] C. Sabbah et F. Castro. *Appendice à "proximité evanescente",I. La structure polaire d'un $\mathcal{D}$–module*, Compositio Math. **62** (1987), 283-328.