Hadamard and Jensen inequalities for s-convex fuzzy processes

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Abstract

We give some inequalities of Hadamard and Jensen type for s-convex fuzzy processes. We also give some applications.

Key words: fuzzy sets, s-convex process, Hadamard and Jensen inequalities

1 Introduction

In [1] the s-convex fuzzy processes were defined and some properties were studied. In this work, we define the s-concave fuzzy processes and we also give some useful inequalities for both, the s-convex and s-concave fuzzy processes.

The paper has the following structure. In Section 2, we fix some basic notation and terminology. In Section 3, we define the s-concave fuzzy process and we give some properties. In Section 4, we establish the Hadamard inequality. In Section 5, we give a generalization of the Jensen inequality.

⋆ Partially financed by the Ministerio de Ciencia y Tecnología, Spain, Grant BFM2003-06579

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[1] Y. Chalco-Cano is supported by Fapesp-Brasil, grant No 00/00055-0.
[2] M.A. Rojas-Medar is partially supported by CNPq-Braszil, grant 300116/93(RN).

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $C \subseteq \mathbb{R}^n$ denote a convex set. Let $s \in (0,1]$ and let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function such that for all $a \in [0,1]$ and for all $x, y \in C$, the following inequality holds

$$f\{ax + (1-a)y\} \leq a^s f(x) + (1-a)^s f(y). \tag{1}$$

These functions are called s-convex and they have been introduced by Breckner [2], where it is also possible to find examples of s-convex functions (see also [3]).

Let $P(\mathbb{R}^n)$ denote the set of all nonempty subsets of $\mathbb{R}^n$. In [4], Breckner generalized the notion of s-convexity for a set-valued mapping $F : C \subseteq \mathbb{R}^m \to P(\mathbb{R}^n)$. F is said to be a s-convex function on C if the following relation is verified

$$(1-a)^s F(x) + a^s F(y) \subseteq F\{(1-a)x + ay\} \tag{2}$$

for all $a \in [0,1]$ and all $x, y \in \mathbb{R}^m$.

We denote by $\mathcal{K}(\mathbb{R}^m)$ the subset of $P(\mathbb{R}^m)$ whose elements are compact and nonempty and by $\mathcal{K}_c(\mathbb{R}^m)$ the subset of $\mathcal{K}(\mathbb{R}^m)$ whose elements are convex. If $A \in \mathcal{K}(\mathbb{R}^m)$, then the support function $\sigma(A, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is defined as

$$\sigma(A, \psi) = \sup_{a \in A} < \psi, a >, \ \forall \psi \in \mathbb{R}^m.$$

It is important to remark that if $A, B \in \mathcal{K}_c(\mathbb{R}^m)$, then, as a direct consequence of the separation Hahn-Banach theorem, we obtain that $\sigma(A, \cdot) = \sigma(B, \cdot) \Leftrightarrow A = B$.

A fuzzy subset of $\mathbb{R}^n$ is a function $u : \mathbb{R}^n \to [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of all fuzzy sets on $\mathbb{R}^n$. We define the addition and the scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by the usual extension principle as follows:

$$(u+v)(y) = \sup_{y_1, y_2 :\, y_1 + y_2 = y} \min\{u(y_1), v(y_2)\}$$

and

$$(\lambda u)(y) = \begin{cases} u(\frac{y}{\lambda}) & if \ \lambda \neq 0, \\ \chi_{\{0\}}(y) & if \ \lambda = 0, \end{cases}$$

where for any subset $A \subseteq \mathbb{R}^n$, χ_A denotes the characteristic function of A.

We can define a partial order $\subseteq$ on $\mathcal{F}(\mathbb{R}^n)$ by setting

$$u \subseteq v \Leftrightarrow u(y) \leq v(y), \quad \forall y \in \mathbb{R}^n.$$

Let $u \in \mathcal{F}(\mathbb{R}^n)$. For $0 < \alpha \leq 1$, we denote by $[u]^\alpha = \{y \in \mathbb{R}^n \quad / \quad u(y) \geq \alpha\}$ the α-level set of u. $[u]^0 = \text{supp}(u) = \overline{\{y \in \mathbb{R}^n \quad / \quad u(y) > 0\}}$ is called the support of u.

A fuzzy set u is called convex if (see [5])

$$u\left\{\lambda y_1 + (1-\lambda) y_2\right\} \geq \min\{u(y_1), u(y_2)\},$$

for all $y_1, y_2 \in \text{supp}(u)$ and $\lambda \in (0,1)$. If $u \in \mathcal{F}(\mathbb{R}^n)$ is convex, then $[u]^\alpha$ is convex for all $\alpha \in [0,1]$.

A fuzzy set $u : \mathbb{R}^n \to [0,1]$ is said to be a fuzzy compact set if $[u]^\alpha$ is compact for all $\alpha \in [0,1]$. We denote by $\mathcal{F}_K(\mathbb{R}^n)$ ($\mathcal{F}_C(\mathbb{R}^n)$) the space of all fuzzy compact (compact convex) sets. Given $u, v \in \mathcal{F}_K(\mathbb{R}^n)$, it is verified that

(a) $u \subseteq v \Leftrightarrow [u]^\alpha \subseteq [v]^\alpha, \quad \forall \alpha \in [0,1]$,
(b) $[\lambda u]^\alpha = \lambda [u]^\alpha, \quad \forall \lambda \in \mathbb{R}, \ \forall \alpha \in [0,1]$,
(c) $[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad \forall \alpha \in [0,1]$.

Any application $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called a fuzzy process. For each $\alpha \in [0,1]$ we define the set-valued mapping $F_\alpha : \mathbb{R}^m \to P(\mathbb{R}^n)$ by

$$F_\alpha(x) = [F(x)]^\alpha.$$

For any $u \in \mathcal{F}_C(\mathbb{R}^n)$ the support function of u, $S(u, (\cdot,\cdot)) : [0,1] \times \mathbb{S}^m \to \mathbb{R}$, where $\mathbb{S}^m = \{\psi \in \mathbb{R}^m / \, \|\psi\| \leq 1\}$, is defined as

$$S(u, (\alpha, \psi)) = \sigma([u]^\alpha, \psi).$$

For details about support functions see for example [6].

A fuzzy process $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called convex if it satisfies the following relation

$$F\left\{(1-a)x_1 + a x_2\right\}(y) \geq \sup_{y_1, y_2 : (1-a)y_1 + a y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\},$$

for all $x_1, x_2 \in \mathbb{R}^m$, $a \in (0,1)$ and $y \in \mathbb{R}^n$. This notion of convex fuzzy processes was recently introduced in [7]. This definition extend the Matloka definition given in [8].

3 S-convex fuzzy processes

In [1] the authors introduced the definition of s-convex fuzzy processes as follows.

Definition 1 *Let $s \in (0,1]$. A fuzzy process $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is said to be a s-convex fuzzy process on C, if for all $a \in (0,1)$ and for all $x, y \in C$ it satisfies the condition*

$$(1-a)^s F(x) + a^s F(y) \subseteq F\{(1-a)x + ay\}.$$

This definition is a generalization of the notion of s-convexity for a set-valued mapping given in (2), since if $\Gamma : C \subseteq \mathbb{R}^m \to P(\mathbb{R}^n)$ is a set-valued mapping, then by putting $F(x) = \chi_{\Gamma(x)}$, we see that Definition 1 coincides with (2).

Usually, 1-convex fuzzy processes are simply called convex fuzzy processes (see [7], [9]).

Example 1 *Let us consider the fuzzy process $F : (0,\infty) \to \mathcal{F}(\mathbb{R})$ that associates to each $x \in (0,\infty)$ the points of the real line "much bigger than $\sqrt{x}$". Now, we define the fuzzy processes $F_1, F_2 : (0,\infty) \to \mathcal{F}(\mathbb{R})$ as follows*

$$F_1(x)(t) = \begin{cases} \frac{t}{\sqrt{x}} - 1 & \text{if } \sqrt{x} \le t \le 2\sqrt{x}, \\ 1 & \text{if } t \ge 2\sqrt{x}, \\ 0 & \text{if } t \le \sqrt{x}, \end{cases}$$

$$F_2(x)(t) = \begin{cases} -\left(\frac{t-2\sqrt{x}}{\sqrt{x}}\right) + 1 & \text{if } \sqrt{x} \le t \le 2\sqrt{x}, \\ 1 & \text{if } t \ge 2\sqrt{x}, \\ 0 & \text{if } t \le \sqrt{x}. \end{cases}$$

For F_1 and $x = 4$, we have that the points of the real line "much bigger than $\sqrt{4} = 2$" is the fuzzy set

$$F_1(4)(t) = \begin{cases} \frac{t}{2} - 1 & \text{if } 2 \le t \le 4, \\ 1 & \text{if } t \ge 4, \\ 0 & \text{if } t \le 2, \end{cases}$$

this means that the points after 4 are "much bigger than 2", while the points in the interval $]2,4[$ are partially "much bigger than 2", i.e., they have a degree of membership to the fuzzy set $F_1(4)$. Similarly, we can see that $F_2(4)$ also models *the fuzzy set of the points of the real line "much bigger than 2". Therefore, both F_1 and F_2 model the fuzzy process F. Thus, we can find diverse fuzzy process that define F. Note that F_1 is $\frac{1}{2}$-convex, but F_2 is not s-convex for all $s \in (0,1]$.*

4 S-concave fuzzy process

In this Section we introduce the concept of s-concave fuzzy process and we establish some properties. This concept generalizes the definition of concave set-valued function given in [4].

Definition 2 *Let $s \in (0,1]$. A fuzzy process $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is said to be a s-concave fuzzy process on C, if for all $a \in (0,1)$ and for all $x, y \in \mathbb{R}^m$ it satisfies the condition*

$$F\left\{(1-a)x + ay\right\} \subseteq (1-a)^s F(x) + a^s F(y).$$

1-concave fuzzy processes will be simply called concave fuzzy processes.

Example 2 *Let us consider the fuzzy process $F : [0,\infty) \to \mathcal{F}(\mathbb{R})$, where $F(x)$ is the isosceles triangular fuzzy set with support $[-f(x), f(x)]$ where $f : [0,\infty) \to \mathbb{R}$ is a s-convex function. It is easy to see that F is s-concave.*

Next we give a characterization for s-concave fuzzy processes by using the membership.

Theorem 1 *Let $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy process on C. Then, F is s-concave if and only if*

$$F((1-a)x_1 + ax_2)(y) \leq \sup_{y_1, y_2 : (1-a)^s y_1 + a^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\},$$

for all $a \in (0,1)$ and for all $x, y \in C$.

Proof The result follows from Definition 2 and the addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$. $\square$

Now, we present another characterization by using the concept of support function of a fuzzy set.

Theorem 2 *Let $F : C \subseteq \mathbb{R}^m \to \mathcal{F}_C(\mathbb{R}^n)$ be a fuzzy process on C. Then, F is s-concave if and only if $S(F(\cdot), (\alpha, \psi))$ is a s-convex function, that is, if and only if $S(F(\cdot), (\alpha, \psi))$ satisfies (1) for all $(\alpha, \psi) \in [0, 1] \times \mathbb{S}^m$.*

Proof Suppose that F is a s-concave fuzzy process. Let $(\alpha, \psi) \in [0, 1] \times \mathbb{S}^m$, $x_1, x_2 \in \mathbb{R}^n$ and $a \in (0, 1)$. Then, from the properties of the support function, we have that

$$\begin{aligned} S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) &\leq S(a^s F(x_1) + (1-a)^s F(x_2), (\alpha.\psi)) \\ &= \sigma(a^s F_\alpha(x_1) + (1-a)^s F_\alpha(x_2), \psi) \\ &= a^s \sigma(F_\alpha(x_1), \psi) + (1-a)^s \sigma(F_\alpha(x_2), \psi). \end{aligned}$$

Consequently,

$$S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) \leq a^s S(F(x_1), (\alpha, \psi)) + (1-a)^s S(F(x_2), (\alpha, \psi)).$$

Therefore, $S(F(\cdot), (\alpha, \psi))$ is s-convex. To prove the converse it suffices to show that

$$S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) \leq S(a^s F(x_1) + (1-a)^s F(x_2), (\alpha, \psi))$$

for all $(\alpha, \psi) \in [0, 1] \times \mathbb{S}^m$, which is a consequence of the properties of the support function of a fuzzy set. □

Example 3 *Let us consider the fuzzy process $F : [0, \infty) \to \mathcal{F}_C(\mathbb{R})$ given by*

$$F(x)(t) = \begin{cases} \frac{t}{x^s} \ si \ 0 \leq t \leq x^s, \\ 0 \quad si \ t \notin [0, x^s], \end{cases}$$

for $x \neq 0$ and $F(0) = \chi_{\{0\}}$. We have that the fuzzy support function $S(F(\cdot), (\alpha, \psi))$, for each $(\alpha, \psi) \in [0, 1] \times S^1$, with $S^1 = \{-1, 1\}$, is given by $S(F(x), (\alpha, 1)) = \alpha x^s$, which is a s-convex function and $S(F(x), (\alpha, -1)) = 0$ which is also s-convex. Then, from Theorem 2 we have that F is a s-concave fuzzy process.

Proposition 1 *Let $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy process on C such that*

(a) $F(x + y) \subseteq F(x) + F(y)$,
(b) $F(tx) = t^s F(x)$.

Then F is a s-concave fuzzy process on C.

Proof ¿From the addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$, and from conditions (a) and (b), we have that

$$\begin{aligned}
& F(ax_1 + (1-a)x_2)(y) \\
\leq & (F(ax_1) + F((1-a)x_2))(y) \\
= & \sup_{y_1,y_2: y_1+y_2=y} \min\{F(ax_1)(y_1), F((1-a)x_2)(y_2)\} \\
= & \sup_{y_1,y_2: a^s y_1+(1-a)^s y_2=y} \min\{F(ax_1)(a^s y_1), F((1-a)x_2)((1-a)^s y_2)\} \\
= & \sup_{y_1,y_2: a^s y_1+(1-a)^s y_2=y} \min\{(a^s F(x_1))(a^s y_1), ((1-a)^s F(x_2))((1-a)^s y_2)\} \\
= & \sup_{y_1,y_2: a^s y_1+(1-a)^s y_2=y} \min\{F(x_1)(y_1), F(x_2)(y_2)\},
\end{aligned}$$

for all $x_1, x_2 \in C$, $a \in (0,1)$ and $y \in \mathbb{R}^n$. Therefore, by Theorem 1, F is a s-concave fuzzy process on C. $\square$

Example 4 *Let $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy quasilinear operator (see [10]), then F satisfies the conditions in Proposition 1 for $s = 1$. Thus, every fuzzy quasilinear operator is a concave fuzzy process.*

5 Hadamard's Inequality

In this Section, we present some inequalities of Hadamard type for s-convex and s-concave fuzzy processes and we give some examples. With this aim, we first recall some basic concepts and properties of fuzzy random variables. A set-valued function $F : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ is called Borel measurable, if its graph, i.e., the set $\{(t,x)/\ x \in F(t)\}$, is a Borel subset of $[0,b] \times \mathbb{R}^n$. Because the Lebesgue measure is complete, the Borel measurability of the set-valued mapping F is equivalent to the following condition: for every Borel set $B \subseteq \mathbb{R}^n$, $F^{-1}(B) = \{t \in [0,b]\ /\ F(t) \cap B \neq \emptyset\} \in \mathbb{L}$, where $\mathbb{L}$ denotes the σ-algebra of all Lebesgue-measurable subsets of interval $[0,b]$. We will say that F is measurable if F is Borel measurable. Also, a measurable set-valued function $F : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ is called a random set.

The integral of a measurable set-valued function $F : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ is defined by

$$\int_0^b F dt = \left\{ \int_0^b f(t)dt/\ f \in S(F) \right\},$$

where $\int_0^b f(t)dt$ is the Bochner-integral and $S(F)$ is the set of all integrable selectors of F, i.e.,

$$S(F) = \left\{ f \in L^1([0,b], \mathbb{R}^n)/\ f(t) \in F(t)\ a.e. \right\}.$$

This definition was introduced by Aumann [11] as a natural generalization of the integration of single-valued functions.

A measurable set-valued function $F : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ is said to be integrably bounded, if there exists a single-valued integrable function $h : [0,b] \to \mathbb{R}^n$ such that $||x|| \leq h(t)$ for all x and t such that $x \in F(t)$.

If $F : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ is an integrably bounded random set, then the Aumann integral of F is a nonempty subset of $\mathbb{R}^n$.

If $\lambda \in \mathbb{R}$ and $F,\ F_1, F_2 : [0,b] \to \mathcal{K}_C(\mathbb{R}^n)$ are integrably bounded random set, then

a) $\int_0^b F dt \in K_C(\mathbb{R}^n)$
b) $\int_0^b (\lambda F_1 + F_2) dt = \lambda \int_0^b F_1 dt + \int_0^b F_2 dt$.

For details see Hiai and Umegaki [12].

Let $F : [0,b] \to \mathcal{F}_K(\mathbb{R}^n)$ be a fuzzy process and define $F_\alpha : [0,b] \to \mathcal{K}(\mathbb{R}^n)$ by $F_\alpha(x) = [F(x)]^\alpha$, $\forall \alpha \in [0,1]$. Then F is called measurable if F_α is measurable for all $\alpha \in [0,1]$. Also, F is called integrably bounded if F_α is an integrably bounded set-valued function for every $\alpha \in [0,1]$. If F is a measurable fuzzy process, then F is called a **fuzzy random variable** (f.r.v.) (see [13]).

Proposition 2 *(Puri and Ralescu [13]) If $F : [0,b] \to \mathcal{F}_K(\mathbb{R}^n)$ is an integrably bounded f.r.v., then there exists a unique fuzzy set $u \in \mathcal{F}_K(\mathbb{R}^n)$ such that $[u]^\alpha = \int_0^b F_\alpha dt\ \forall \alpha \in [0,1]$.*

The element $u \in \mathcal{F}_K(\mathbb{R}^n)$ in Proposition 2 defines the integral of the fuzzy random variable F by $\int_0^b F dt = u \Leftrightarrow [u]^\alpha = \int_0^b F_\alpha dt$, for every $\alpha \in [0,1]$.

Theorem 3 *If $F_1, F_2 : [0,b] \to \mathcal{F}_C(\mathbb{R}^n)$ are integrably bounded f.r.v. and $\lambda \in \mathbb{R}$, then*

$$\int_0^b (\lambda F_1 + F_2) dt = \lambda \int_0^b F_1 dt + \int_0^b F_2 dt.$$

For more details and properties about the integral of f.r.v. see [13].

If $f : [a,b] \to \mathbb{R}$ is a convex function, then following inequalities hold,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

These inequalities are known in the literature as Hadamard's inequalities. Next we extend them. We first prove an inequality of Hadamard type for a s-convex fuzzy process and afterwards for s-concave fuzzy process.

Theorem 4 *Let F be a s-convex integrably bounded fuzzy process on an interval $I \subseteq [0,\infty)$ and let $a, b \in I$, with $a < b$. Then*

$$(s+1)^{-1}\{F(a)+F(b)\} \subseteq \int_a^b F(x)dx/(b-a) \subseteq 2^{s-1}F\left(\frac{a+b}{2}\right). \tag{4}$$

Proof Since F is s-convex on I we have that

$$t^sF(a)+(1-t)^sF(b) \subseteq F\{ta+(1-t)b\}$$

for all $t \in [0,1]$. Integrating this relation we get

$$\begin{aligned}\int_0^1 F\{ta+(1-t)b\}\,dt &\supseteq \int_0^1 \{t^sF(a)+(1-t)^sF(b)\}\,dt \\ &= F(a)\int_0^1 t^s dt + F(b)\int_0^1 (1-t)^s dt \\ &= (s+1)^{-1}\{F(a)+F(b)\}.\end{aligned}$$

Now, making the change of variable $x = tb+(1-t)a$, it follows the first relation in (4).

To prove the second relation in (4), observe that for all $x, y \in I$ we have that

$$F\left(\frac{x+y}{2}\right) \supseteq \frac{1}{2^s}\{F(x)+F(y)\}. \tag{5}$$

Then taking $x = ta+(1-t)b$ and $y = tb+(1-t)a$, from (5) we obtain

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{2^s}\left[F\{ta+(1-t)b\}+F\{tb+(1-t)a\}\right].$$

Integrating this relation we get

$$\begin{aligned}\int_0^1 F\left(\frac{a+b}{2}\right)dt &\supseteq \int_0^1 \frac{1}{2^s}\left[F\{ta+(1-t)b\}+F\{tb+(1-t)a\}\right]dt \\ &= \frac{1}{2^s}\left[\int_0^1 F\{ta+(1-t)b\}\,dt + \int_0^1 F\{tb+(1-t)a\}\,dt\right].\end{aligned}$$

Since

$$\int_0^1 F\{ta+(1-t)b\}\,dt = \int_0^1 F\{tb+(1-t)a\}\,dt = \frac{1}{b-a}\int_a^b F(x)dx,$$

it follows that

$$\int_a^b F(x)dx/(b-a) \subseteq 2^{s-1}F\left(\frac{a+b}{2}\right). \square$$

Theorem 5 *Let F be a s-concave integrably bounded fuzzy process on an interval $I \subseteq [0,\infty)$ and let $a, b \in I$, with $a < b$. Then*

$$2^{s-1}F\left(\frac{a+b}{2}\right) \subseteq \int_a^b F(x)dx/(b-a) \subseteq (s+1)^{-1}\{F(a)+F(b)\}. \quad (6)$$

Proof The proof is analogous to that of Theorem 4. □

Corollary 1 *Let F be a s-concave integrably bounded fuzzy process on an interval $I \subseteq [0,\infty)$ and let $a, b \in I$, with $a < b$. Then*

$$2^{s-1}\Gamma\left(\frac{a+b}{2}\right) \leq \int_a^b \Gamma(x)dx/(b-a) \leq (s+1)^{-1}(\Gamma(a)+\Gamma(b)),$$

where $\Gamma = S(F(\cdot),(\alpha,\psi))$.

Proof The result follows from Theorem 5 and the properties of the fuzzy support function. □

Example 5 *Let us consider $s = 1/2$ and the $1/2$-concave fuzzy process $F : (1/2, 1) \to \mathcal{F}(\mathbb{R})$ as in Example 3. Then*

$$\Gamma(x) = S(F(x),(\alpha,1)) = \alpha\sqrt{x}$$

for each $\alpha \in [0,1]$. Thus, by Corollary 1 we have

$$\frac{\sqrt{6}}{8}\alpha \leq \int_{1/2}^1 \Gamma(x)dx \leq \frac{2+\sqrt{2}}{3}\alpha.$$

5.1 Applications

(a) For an integrably bounded fuzzy process $F : [0, b] \to \mathcal{F}_C(\mathbb{R}^n)$, the fuzzy integral mean of F is a fuzzy process $M_F : (0, b] \to \mathcal{F}(\mathbb{R}^n)$ defined by

$$M_F(x) = \frac{1}{x}\int_0^x F(t)dt\ ,\ \forall x \in (0, b].$$

This concept was introduced in [9], where some properties are also studied. In [1] is studied the s-convexity of the fuzzy integral mean. The following Proposition gives a new relationship for the fuzzy integral mean, which is obtained by using the Hadamard inequality in Theorem 4.

Proposition 3 *Let $F : [0, b] \to \mathcal{F}_C(\mathbb{R}^n)$ be a measurable integrably bounded fuzzy process. If F is s-convex then M_F is s-convex and*

$$(s+1)^{-1}\{F(0) + F(x)\} \subseteq M_F(x) \subseteq 2^{s-1}F\left(\frac{x}{2}\right). \tag{7}$$

Proof As F is s-convex, then from Theorem 4.5 in [1] M_F is s-convex. The relation (7) is an immediate consequence of Theorem 4. □

(b) With the aim of establishing some refinements of (3), Dragomir [14] introduced the mapping

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx,$$

and showed that if $f : [a, b] \to \mathbb{R}$ is a convex function, then $H(t)$ is convex and that

$$f\left(\frac{a+b}{2}\right) \leq H(t) \leq \frac{1}{b-a}\int_a^b f(x)dx, \quad \forall t \in [0, 1].$$

Next, we extend this results for s-convex bounded fuzzy processes. Let $F : [a, b] \to \mathcal{F}_C(\mathbb{R}^n)$ be an integrably bounded fuzzy process and define

$$H_F(t) = \frac{1}{b-a}\int_a^b F\{tx + (1-t)(a+b)/2\}\,dx,$$

for $t \in [0,1]$.

Theorem 6 *Let F be a s-convex integrably bounded fuzzy process on an interval $[a,b]$. Then H_F is s-convex on $[0,1]$ and*

$$H_F(t) \subseteq 2^{s-1} F\left(\frac{a+b}{2}\right), \quad \forall t \in [0,1]. \tag{8}$$

Proof Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
&H_F(\alpha t_1 + \beta t_2) \\
&= \frac{1}{b-a} \int_a^b F\left[(\alpha t_1 + \beta t_2)x + \{1 - (\alpha t_1 + \beta t_2)\}(a+b)/2\right] dx \\
&= \frac{1}{b-a} \int_a^b F\left[\alpha\{t_1 x + (1-t_1)(a+b)/2\} + \beta\{t_2 x + (1-\beta t_2)(a+b)/2\}\right] dx \\
&\supseteq \frac{1}{b-a} \int_a^b \alpha^s F\{t_1 x + (1-t_1)(a+b)/2\} dx + \\
&\quad \frac{1}{b-a} \int_a^b \beta^s F\{t_2 x + (1-\beta t_2)(a+b)/2\} dx \\
&= \alpha^s H_F(t_1) + \beta^s H_F(t_2),
\end{aligned}$$

which shows that H_F is s-convex. Now, let $t \in (0,1]$. Taking $r = tx + (1-t)(a+b)/2$ we obtain

$$H_F(t) = \int_q^p F(r)dr/(p-q)$$

where $p = tb + (1-t)(a+b)/2$ and $q = ta + (1-t)(a+b)/2$. By Theorem 4 we have that

$$\int_q^p F(r)dr/(p-q) \subseteq 2^{s-1} F\left(\frac{p+q}{2}\right) = 2^{s-1} F\left(\frac{a+b}{2}\right),$$

what proves (8).□

Remark 1 *Proceeding as in the proof of Theorem 6, it can be also shown that if F is a s-concave integrably bounded fuzzy process on an interval $[a,b]$, then*

$$2^{s-1} F\left(\frac{a+b}{2}\right) \subseteq H_F(t).$$

6 Jensen's Inequality

In this Section we give a generalization of the Jensen inequality for s-convex and s-concave fuzzy processes.

Theorem 7 *Let $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a s-convex fuzzy process on C and $s > 0$. Then we have the relation*

$$\sum_{i=1}^{n} p_i^s F(x_i) \subseteq F\left(\sum_{i=1}^{n} p_i x_i\right), \tag{9}$$

whenever $p_i \geq 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = 1$. If F is a s-concave fuzzy process on C and $s > 0$. Then we have the relation

$$F\left(\sum_{i=1}^{n} p_i x_i\right) \subseteq \sum_{i=1}^{n} p_i^s F(x_i) \tag{10}$$

whenever $p_i \geq 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = 1$.

Proof We first show (9). To do this, we proceed by induction on n. For $n = 2$, (9) is the definition of s-convexity of F. Now, suppose that (9) holds for $n = k-1$ and given $p_i \geq 0$, $x_i \in C$ and $\sum_{i=1}^{k} p_i = 1$, we may and do assume that all $p_i > 0$. Let $q_j = p_j/(p_1 + ... + p_{k-1})$, $1 \leq j < k$. Then $q_1 + ... + q_{k-1} = 1$ and thus

$$q_1^s F(x_1) + ... + q_{k-1}^s F(x_{k-1}) \subseteq F(q_1 x_1 + ... + q_{k-1} x_{k-1}). \tag{11}$$

Put $P = p_1 + ... + p_{k-1}$, then

$$\begin{aligned} F(p_1 x_1 + ... + p_k x_k) &= F\left\{P\left(\frac{p_1}{P}x_1 + ... + \frac{p_{k-1}}{P}x_{k-1}\right) + p_k x_k\right\} \\ &\supseteq P^s F\left(\frac{p_1}{P}x_1 + ... + \frac{p_{k-1}}{P}x_{k-1}\right) + p_k^s F(x_k) \\ &\supseteq P^s\left(\frac{p_1^s}{P^s}F(x_1) + ... + \frac{p_{k-1}^s}{P^s}F(x_{k-1})\right) + p_k^s F(x_k) \\ &= \sum p_i^s F(x_i), \end{aligned}$$

which establishes (9) for $n = k$, and hence for all $n \in \mathbb{N}$. The proof of (10) follows the same steps and so we omit it. □

Corollary 2 *Let $F : C \subseteq \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ a s-convex fuzzy process on C and $s > 0$. Then*

$$n^{-s} \sum_{i=1}^{n} F(x_i) \subseteq F\left(n^{-1} \sum_{i=1}^{n} x_i\right), \tag{12}$$

whenever $x_i \in C$, $1 \leq i \leq n$.

Example 6 *We consider the 1/2-convex fuzzy process F_1 from Example 1. Thus, from Corollary 2, for each $\alpha \in [0,1]$ we have that*

$$\left[(1+\alpha)\sqrt{n^{-1}\sum_{i=1}^{n} x_i}\,, \infty\right) \supseteq n^{-1/2}(1+\alpha)\sum_{i=1}^{n}[\sqrt{x_i}, \infty)\,.$$

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