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Research Article

Does Kirk's Theorem Hold for Multivalued Nonexpansive Mappings?

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Fixed Point Theory for multivalued mappings has many useful applications in Applied Sciences, in particular, in Game Theory and Mathematical Economics. Thus, it is natural to try of extending the known fixed point results for single-valued mappings to the setting of multivalued mappings. Some theorems of existence of fixed points of single-valued mappings have already been extended to the multivalued case. However, many other questions remain still open, for instance, the possibility of extending the well-known Kirk's Theorem, that is: do Banach spaces with weak normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings? There are many properties of Banach spaces which imply weak normal structure and consequently the FPP for single-valued mappings (for example, uniform convexity, nearly uniform convexity, uniform smoothness,...). Thus, it is natural to consider the following problem: do these properties also imply the FPP for multivalued mappings? In this way, some partial answers to the problem of extending Kirk's Theorem have appeared, proving that those properties imply the existence of fixed point for multivalued nonexpansive mappings. Here we present the main known results and current research directions in this subject. This paper can be considered as a survey, but some new results are also shown.

1. Introduction

The presence or absence of a fixed point (i.e., a point which remains invariant under a map) is an intrinsic property of a map. However, many necessary or sufficient conditions for the existence of such points involve a mixture of algebraic, topological, or metric properties of the mapping or its domain. By Metric Fixed Point Theory, we understand the branch of Fixed Point Theory concerning those results which depend on a metric and which are not preserved when this metric is replaced by another equivalent metric. The first metric fixed point theorem was given by Banach in 1922.

Theorem 1.1 (Banach Contraction Principle, [1]). *Let X be a complete metric space and $T : X \to X$ a contractive mapping, that is, there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for every $x,y \in X$. Then T has a (unique) fixed point x_0. Moreover, $x_0 = \lim_n T^n x$ for every $x \in X$.*

Banach Theorem is a basic tool in Functional Analysis, Nonlinear Analysis and Differential Equations. Thus, it is natural to look for some generalizations under weaker assumptions.

For many years Metric Fixed Point Theory just studied some extensions of Banach Theorem relaxing the contractiveness condition, and the extension of this result for multivalued mappings. In the 1960s, Metric Fixed Point Theory received a strong boost when Kirk [2] proved that every (singlevalued) nonexpansive mapping $T : C \to C$, defined from a convex closed bounded subset C of a reflexive Banach space with normal structure, has a fixed point.

The celebrated Kirk's theorem had a profound impact in the development of Fixed Point Theory and iniciated the search of more general conditions for a Banach space and for a subset C which assure the existence of fixed points.

The result obtained by Kirk is, in some sense, surprising because it uses geometric properties of Banach spaces (commonly used in Linear Functional Analysis, but rarely considered in Nonlinear Analysis until then). Thus, it is the starting point for a new mathematical field: the application of the Geometric Theory of Banach Spaces to Fixed Point Theory. From that moment on, many researchers have tried to exploit this connection, essentially considering some other geometric properties of Banach spaces which can be applied to prove the existence of fixed points for different types of nonlinear operators (e.g., uniform smoothness, Opial property, nearly uniform convexity, nearly uniform smoothness, etc.).

Fixed Point Theory for multivalued mappings has useful applications in Applied Sciences, in particular, in Game Theory and Mathematical Economics. Thus, it is natural to study the problem of the extension of the known fixed point results for singlevalued mappings to the setting of multivalued mappings.

Some theorems of existence of fixed points of single-valued mappings have already been extended to the multivalued case. For example, in 1969 Nadler [3] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. However, many other questions remain open, for instance, the possibility of extending the well-known Kirk's Theorem [2], that is, do Banach spaces with weak normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings?

There are many properties of Banach spaces which imply weak normal structure and consequently the FPP for singlevalued mappings (e.g., uniform convexity, nearly uniform convexity, uniform smoothness, $\ldots$). Thus, it is natural to consider the following problem: Do these properties also imply the FPP for multivalued mappings? As a consequence, some partial answers to the problem of extending Kirk's Theorem have appeared, which are directed to prove that those properties imply the existence of fixed point for multivalued nonexpansive mappings.

Here we present the main known results and current research directions in this subject. This paper can be considered as a survey, but some new results are also included.

2. Preliminaries

In this section we recall the notion of normal structure and some properties of Banach spaces which imply normal structure.

Normal structure plays an essential role in some problems of Metric Fixed Point Theory, especially those concerning nonexpansive mappings. The notion of normal structure was introduced by Brodskiĭ and Mil'man [4] in 1948 in order to study fixed points of isometries. Later, the notion of normal structure was generalized for the weak topology.

Definition 2.1. A Banach space X is said to have normal structure (NS) (resp., weak normal structure (w-NS)) if for every bounded closed (resp., weakly compact) convex subset C of X with $\operatorname{diam}(C) := \sup\{\|x-y\| : x, y \in C\} > 0$, there exists $x \in C$ such that $\sup\{\|x-y\| : y \in C\} < diam(C)$.

In 1965 Kirk [2] obtained a strong connection between normal structure and the FPP for nonexpansive mappings.

Theorem 2.2. *Let C be a bounded closed (resp., weakly compact) convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$). If X is a reflexive Banach space with normal structure (resp., a Banach space with w-NS), then T has a fixed point.*

Bynum [5] defined two coefficients related to normal structure and weak normal structure.

Definition 2.3. The normal structure coefficient of a Banach space X is defined by

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)} : A \subset X \text{ convex closed and bounded with } \operatorname{diam}(A) > 0\right\}, \tag{2.1}$$

where $\operatorname{diam}(A)$ denotes the diameter of A defined by $\operatorname{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ and $r(A)$ denotes the Chebyshev radius of A defined by $r(A) = \inf\{\sup\{\|x-y\| : y \in A\} : x \in A\}$.

The weakly convergent sequence coefficient of X is defined by

$$\operatorname{WCS}(X) = \inf\left\{\frac{\operatorname{diam}_a(\{x_n\})}{r_a(\{x_n\})}\right\}, \tag{2.2}$$

where the infimum is taken over all weakly convergent sequences $\{x_n\}$ which are not norm convergent, where,

$$\begin{aligned} \operatorname{diam}_a(\{x_n\}) &= \lim_{k\to\infty} \sup\{\|x_n - x_m\| : n, m \geq k\}, \\ r_a(\{x_n\}) &= \inf\left\{\limsup_n \|x_n - x\| : x \in co(\{x_n\})\right\} \end{aligned} \tag{2.3}$$

denote the asymptotic diameter and radius of $\{x_n\}$, respectively.

We recall that X is said to have uniform normal structure (UNS) (resp., weak uniform normal structure (w-UNS)) if $N(X) > 1$ (resp., $\operatorname{WCS}(X) > 1$). Notice that this is not the common definition of weak uniform normal structure and is often known as Bynum's condition. It is known that if X has uniform normal structure, then X is reflexive [6].

In the latest fifty years, some geometrical properties implying normal structure have been studied. Here we are going to recall some of these properties and some results which prove that these properties imply the existence of fixed point for multivalued mappings.

First we consider the Opial property. Opial [7] was the first who studied such a property giving applications to Fixed Point Theory. The uniform Opial property was defined in [8] by Prus, and the Opial modulus was introduced in [9] by Lin et al.

Definition 2.4. We will say that a Banach space X satisfies the Opial property if for every weakly null sequence $\{x_n\}$ and every $x \neq 0$ in X, we have

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\|. \tag{2.4}$$

We will say that X satisfies the nonstrict Opial property if

$$\liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x_n + x\| \tag{2.5}$$

under the same conditions.

The Opial modulus of X is defined for $c \geq 0$ as

$$r_X(c) = \inf\left\{ \liminf_{n} \|x_n + x\| - 1 \right\}, \tag{2.6}$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all weakly null sequences $\{x_n\}$ in X with $\liminf_n \|x_n\| \geq 1$.

We will say that X satisfies the uniform Opial property if $r_X(c) > 0$ for all $c > 0$.

There are some relationships between the notions of Opial property and normal structure. If X is a Banach space which satisfies the Opial property, then X has w-NS [10]. On the other hand, $\mathrm{WCS}(X) \geq 1 + r_X(1)$ [9, Theorem 3.2]. Consequently, X has w-UNS if $r_X(1) > 0$.

Next we study the uniform convexity of the space, which is another geometrical property related with normal structure. We recall that a Banach space X is uniformly convex (UC) if and only if

$$\delta_X(\epsilon) := \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_x, \|x - y\| \geq \epsilon \right\} > 0 \tag{2.7}$$

for each $\epsilon \in [0, 2]$, or equivalently

$$\varepsilon_0(X) := \sup\{\epsilon \geq 0 : \delta_X(\epsilon) = 0\} = 0. \tag{2.8}$$

The Clarkson modulus $\delta_X(\epsilon)$ and the coefficient of normal structure $N(X)$ are related by the following inequality: $N(X) \geq (1 - \delta_X(1))^{-1}$. Consequently, the condition $\delta_X(1) > 0$ implies that X is reflexive and has uniform normal structure. In particular, notice that not only do uniformly convex spaces have normal structure, but so do all those spaces which do not have segments of length greater than or equal to 1 near the unit sphere.

In 1980 Huff [11] initiated the study of nearly uniform convexity which is an infinite-dimensional generalization of uniform convexity. Independently of Huff, Goebel and Sękowski [12] also introduced a property which is equivalent to nearly uniform convexity under the name of noncompact uniform convexity. It is known that a Banach space X is nearly uniformly convex (NUC) if and only if

$$\Delta_{X,\phi}(\epsilon) := \inf\{1 - d(0, A) : A \subset B_X \text{ convex}, \ \phi(A) > \epsilon\} > 0 \tag{2.9}$$

for each $\epsilon > 0$, or equivalently

$$\varepsilon_{\phi}(X) := \sup\{\epsilon \geq 0 : \Delta_{X,\phi}(\epsilon) = 0\} = 0, \tag{2.10}$$

where ϕ is a measure of noncompactness. Also we are going to use the following equivalent definition: X is NUC if and only if X is reflexive and

$$\Delta_X(\epsilon) := \inf\left\{1 - \|x\| : \{x_n\} \subset B_X, x_n \rightharpoonup x, \liminf_n \|x_n - x\| \geq \epsilon\right\} > 0 \tag{2.11}$$

for each $\epsilon > 0$, or equivalently

$$\Delta_0(X) := \sup\{\epsilon > 0 : \Delta_X(\epsilon) = 0\} = 0. \tag{2.12}$$

When X is a reflexive Banach space, β is the separation measure and χ is the Hausdorff measure (for definitions see, for instance, [13] or [14]), we have the following relationships among the different moduli:

$$\Delta_{X,\beta}(\epsilon) \leq \Delta_X(\epsilon) \leq \Delta_{X,\chi}(\epsilon), \tag{2.13}$$

and consequently,

$$\varepsilon_{\beta}(X) \geq \Delta_0(X) \geq \varepsilon_{\chi}(X). \tag{2.14}$$

If the space X satisfies the nonstrict Opial property, then $\Delta_0(X)$ coincides with $\varepsilon_{\chi}(X)$.

On the other hand, if $\varepsilon_{\beta}(X) < 1$ (in particular, if X is NUC), then X is reflexive and has weak uniform normal structure (see [13, page 125]).

The dual concept of uniform convexity is uniform smoothness which is also related to normal structure. A Banach space X is said to be uniformly smooth (US) if

$$\rho'_X(0) = \lim_{t \to 0^+} \frac{\rho_X(t)}{t} = 0, \tag{2.15}$$

where ρ_X is the modulus of smoothness of X, defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+ty\| + \|x-ty\|\right) - 1 : \|x\| \leq 1, \|y\| \leq 1\right\} \tag{2.16}$$

for $t \geq 0$.

It is known that $\rho'_X(0) < 1/2$ implies that X is reflexive and has uniform normal structure [15–17]. However, the infinite-dimensional generalization of uniform smoothness, nearly uniform smoothness, does not imply normal structure [13, Example VI.2].

3. Some Properties Implying Weak Normal Structure and the FPP for Multivalued Mappings

In this section we are going to show some results which prove that some properties implying weak normal structure also imply the existence of fixed point for multivalued nonexpansive mappings. As a consequence these results give some partial answers to the problem of extending Kirk's Theorem.

Throughout this section $K(X)$ (resp., $KC(X)$) will denote the family of all nonempty compact (resp., compact convex) subsets of X. We recall that a multivalued mapping $T : X \to K(X)$ is said to be nonexpansive if $H(Tx, Ty) \leq \|x-y\|$ for every $x, y \in X$, where $H(\cdot,\cdot)$ denotes the Hausdorff metric given by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \tag{3.1}$$

for every bounded subsets A and B of X.

In 1973 Lami Dozo gave the following result of existence of fixed point for those spaces which satisfy the Opial property.

Theorem 3.1 (Lami Dozo [18, Theorem 3.2]). *Let X be a Banach space which satisfies the Opial property, let C be a weakly compact convex subset of X, and let $T : C \to K(C)$ be a nonexpansive mapping. Then T has a fixed point, that is, there exists $x \in C$ such that $x \in Tx$.*

In 1974 Lim [19] gave a similar result for uniformly convex spaces using Edelstein's method of asymptotic centers [20].

Theorem 3.2 (Lim [19]). *Let X be a uniformly convex Banach space, let C be a closed bounded convex subset of X and $T : C \to K(C)$ be a nonexpansive mapping. Then T has a fixed point.*

In 1990 Kirk and Massa proved the following partial generalization of Lim's Theorem using asymptotic centers of sequences and nets. We recall that, given a bounded sequence $\{x_n\}$ in a Banach space X and a subset C of X, the asymptotic center of $\{x_n\}$ with respect to C is defined by

$$A(C,\{x_n\}) := \left\{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\right\}, \tag{3.2}$$

where $r(C,\{x_n\})$ denotes the asymptotic radius of $\{x_n\}$ with respect to C defined by

$$r(C,\{x_n\}) := \inf\left\{\limsup_n \|x_n - x\| : x \in C\right\}. \tag{3.3}$$

Theorem 3.3 (Kirk and Massa [21]). *Let C be a closed bounded convex subset of a Banach space X and $T : C \to KC(C)$ a nonexpansive mapping. If the asymptotic center in C of each bounded sequence of X is nonempty and compact, then T has a fixed point.*

We do not know a complete characterization of those spaces in which asymptotic centers of bounded sequences are compact. Nevertheless, there are some partial answers, for example, k-uniformly convex Banach spaces satisfy that condition [22]. However, an example given by Kuczumov and Prus [23] shows that in nearly uniformly convex spaces, the asymptotic center of a bounded sequence with respect to a closed bounded convex subset is not necessarily compact. Therefore, the problem of obtaining fixed point results in nearly uniformly convex spaces remained open. This question (together with the same question for uniformly smooth spaces) explicitly appeared in a survey about Metric Fixed Point Theory for multivalued mappings published by Xu [24] in 2000.

The analysis of the importance of the asymptotic center in Kirk-Massa Theorem led Domínguez Benavides and Lorenzo to study some connections between asymptotic centers and the geometry of certain spaces, including nearly uniformly convex spaces. Thus, in [25] Domínguez and Lorenzo obtained the following relationship between the Chebyshev radius of the asymptotic center of a bounded sequence and the modulus of noncompact convexity with respect to the measures β and χ.

Theorem 3.4 (see [25, Theorem 3.4]). *Let C be a closed convex subset of a reflexive Banach space X and $\{x_n\}$ a bounded sequence in C which is regular with respect to C (i.e., the asymptotic radius is invariant for all subsequences of $\{x_n\}$). Then*

$$r_C(A(C,\{x_n\})) \le \left(1 - \Delta_{X,\beta}(1^-)\right) r(C,\{x_n\}), \tag{3.4}$$

where the Chebyshev radius of a bounded subset D of X relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x - y\| : y \in D\} : x \in C\}. \tag{3.5}$$

Moreover, if X satisfies the nonstrict Opial property, then

$$r_C(A(C,\{x_n\})) \le \left(1 - \Delta_{X,\chi}(1^-)\right) r(C,\{x_n\}). \tag{3.6}$$

The previous inequalities give an iterative method which reduces at each step the value of the Chebyshev radius for a chain of asymptotic centers. Consequently, Domínguez and Lorenzo deduced in [26] the following partial extension of Kirk's Theorem which, in particular, assures that nearly uniformly convex spaces have the fixed point property for multivalued nonexpansive mappings.

Theorem 3.5 (see [26, Theorem 3.5]). *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$. Let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

This result guarantees, in particular, the existence of fixed points in nearly uniformly convex spaces (because $\varepsilon_\beta(X) = 0$ if X is NUC), giving a positive answer to one of the previous open problems proposed by Xu.

Dhompongsa et al. [27] observed that the main tool used in the proofs in [25, 26], in order to obtain fixed point results for multivalued nonexpansive mappings, is a relationship between the Chebyshev radius of the asymptotic center of a bounded sequence and the asymptotic radius of the sequence. This relationship also gives an iterative method which reduces at each step the value of the Chebyshev radius for a chain of asymptotic centers. Consequently, in [27, 28] they introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) and property (D) in the following way.

We recall that a sequence $\{x_n\}$ is regular with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is asymptotically uniform with respect to C if $A(C,\{x_n\}) = A(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 3.6. A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C

$$r_C(A(C,\{x_n\})) \le \lambda r(C,\{x_n\}). \tag{3.7}$$

A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset C of X, any bounded sequence $\{x_n\}$ in C which is regular and asymptotically uniform with respect to C, and any sequence $\{y_n\} \subset A(C,\{x_n\})$ which is regular and asymptotically uniform with respect to C, we have

$$r(C,\{y_n\}) \le \lambda r(C,\{x_n\}). \tag{3.8}$$

From the definition it is easy to deduce that property (D) is weaker than the (DL)-condition. Dhompongsa et al. proved in [28, Theorem 3.2] and [28, Theorem 3.5] that property (D) implies w-NS and the FPP for multivalued nonexpansive mappings.

Theorem 3.7 (see [28, Theorem 3.3]). *Let X be a Banach space satisfying property (D). Then X has w-NS.*

Theorem 3.8 (see [28, Theorem 3.6]). *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies property (D). Let $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

From Theorem 3.5 every Banach space with $\varepsilon_\beta(X) < 1$ satisfies the (DL)-condition. In this paper we present some other properties concerning geometrical constants of Banach spaces which also imply the (DL)-condition or property (D).

Since our goal is to study if properties implying w-NS also imply the FPP for multivalued mappings, a possible approach to that problem is to study if these properties imply either the (DL)-condition or property (D). These results will give only partial answers to the problem of extending Kirk's Theorem for multivalued mappings because we know that uniform normal structure does not imply property (D) ([29, Proposition 5]); therefore, the problem of extending Kirk's Theorem cannot be fully solved by this approach. In this setting the following results have been obtained.

Theorem 3.9 (Dhompongsa et al. [27, Theorem 3.4]). *Let X be a uniformly nonsquare Banach space with property WORTH. Then X satisfies the (DL)-condition.*

We recall that a Banach space X is uniformly nonsquare if there exists $\delta > 0$ such that $\|x + y\| \wedge \|x - y\| \leq 2 - \delta$ for every $x, y \in B_X$ or equivalently $J(X) < 2$, where $J(X)$ denotes the James constant of X defined by

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}. \tag{3.9}$$

X is said to satisfy property WORTH if

$$\limsup_n \|x_n + x\| = \limsup_n \|x_n - x\| \tag{3.10}$$

for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X.

Theorem 3.10 (Dhompongsa et al. [28, Theorem 3.7]). *Let X be Banach space such that*

$$C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}, \tag{3.11}$$

where $C_{NJ}(X)$ denotes the Jordan-von Neumann constant of X defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X \text{ not both zero}\right\}. \tag{3.12}$$

Then X satisfies property (D).

Theorem 3.11 (Domínguez Benavides and Gavira [29, Corollary 1]). *Let X be a Banach space such that*

$$\rho'_X(0) < \frac{1}{2}. \tag{3.13}$$

Then X satisfies the (DL)-condition. In particular, uniformly smooth Banach spaces ($\rho'_X(0) = 0$) satisfy the (DL)-condition.

Theorem 3.12 (Domínguez Benavides and Gavira [29, Corollary 2]). *Let X be a Banach space such that one of the following two equivalent conditions is satisfied:*

(1) $r_X(1) > 0$,

(2) $\Delta_0(X) < 1$.

Then X satisfies the (DL)-condition.

Theorem 3.13 (Saejung [30, Theorem 3]). *A Banach space X has property (D) whenever* $\varepsilon_0(X) < WCS(X)$.

This result improves Theorem 3.10 because it is easy to see that $C_{NJ}(X) \geq 1 + (1/4)(\varepsilon_0(X))^2$.

Theorem 3.14 (Kaewkhao [31, Corollary 3.2]). *Let X be a Banach space such that*

$$J(X) < 1 + \frac{1}{\mu(X)}, \tag{3.14}$$

where $J(X)$ *denotes the James constant of X defined by*

$$J(X) := \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\}, \tag{3.15}$$

and $\mu(X)$ *denotes the coefficient of worthwhileness of X defined as the infimum of the set of real numbers* $r > 0$ *such that*

$$\limsup_{n\to\infty}\|x + x_n\| \leq r \limsup_{n\to\infty}\|x - x_n\| \tag{3.16}$$

for all $x \in X$ *and all weakly null sequences* $\{x_n\}$ *in X. Then X satisfies the (DL)-condition.*

Remark 3.15. This result improves Theorem 3.9 because if X is a uniformly nonsquare Banach space with property WORTH, then

$$\mathrm{J}(X) < 2 = 1 + \frac{1}{\mu(X)}. \tag{3.17}$$

Theorem 3.16 (Kaewkhao [31, Theorem 4.1]). *Let X be a Banach space such that*

$$C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2}. \tag{3.18}$$

Then X satisfies the (DL)-condition.

Theorem 3.17 (Gavira [32, Theorem 4]). *Let X be a Banach space such that*

$$\rho'_X(0) < \frac{1}{\mu(X)}. \tag{3.19}$$

Then X satisfies the (DL)-condition.

Remarks 3.18. (i) This result is a strict generalization of Theorem 3.16 (see [32]).

(ii) Theorem 3.17 applies to the Bynum space $\ell_{2,1}$ while Theorem 3.11 does not (see [32]). However, we do not know if $\rho'_X(0) < 1/2$ implies $\rho'_X(0) < 1/\mu(X)$.

Finally we show a new result which gives a property implying the (DL)-condition in terms of Clarkson modulus and the García-Falset coefficient.

Theorem 3.19. *Let X be a Banach space such that*

$$\delta_X\left(\frac{1}{R(X)}+\sqrt{1-\frac{1}{R(X)}+\frac{1}{(R(X))^2}}\right)>\frac{1}{2}\left(1-\frac{1}{R(X)}\right), \tag{3.20}$$

where $R(X)$ denotes the García-Falset coefficient of X defined by

$$R(X)=\sup\left\{\liminf_{n\to\infty}\|x_n+x\| : x\in B_X, \{x_n\}\subset B_X, x_n\rightharpoonup 0\right\}. \tag{3.21}$$

Then X satisfies the (DL)-condition.

Proof. Let C be a nonempty weakly compact convex subset of X. Let $\{x_n\}$ be a bounded sequence in C which is regular with respect to C. Denote $A=A(C,\{x_n\})$, $r=r(C,\{x_n\})$, and $R=R(X)$. By translation and multiplication we can assume that $\{x_n\}$ is weakly null and $\lim_n\|x_n\|=1$. Let $z\in A$, then $\limsup_n\|x_n-z\|=r\le 1$. Denote $\|z\|$ by a. By the definition of R, we have

$$\liminf_n\left\|x_n+\frac{z}{a}\right\|\le R. \tag{3.22}$$

For every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

(1) $\|x_N-z\|<r+\epsilon$,

(2) $\|x_N+z/a\|<R+\epsilon$,

(3) $\|(1/(r+\epsilon)-1/(R+\epsilon))x_N-(1/(r+\epsilon)+1/a(R+\epsilon))z\|>(1/(r+\epsilon)+1/a(R+\epsilon))a((r-\epsilon)/r)$,

(4) $\|x_N-(1/(r+\epsilon)-1/a(R+\epsilon))/(1/(r+\epsilon)+1/(R+\epsilon))z\|>r-\epsilon$.

Consider $u=(1/(r+\epsilon))(x_N-z)\in B_X$ and $v=(1/(R+\epsilon))(x_N+z/a)\in B_X$. Using the above estimates we obtain

$$\begin{aligned}\|u-v\| &= \left\|\left(\frac{1}{r+\epsilon}-\frac{1}{R+\epsilon}\right)x_N-\left(\frac{1}{r+\epsilon}+\frac{1}{a(R+\epsilon)}\right)z\right\|\\ &> \left(\frac{1}{r+\epsilon}+\frac{1}{a(R+\epsilon)}\right)a\left(\frac{r-\epsilon}{r}\right)=\left(\frac{a}{r+\epsilon}+\frac{1}{R+\epsilon}\right)\left(\frac{r-\epsilon}{r}\right)\\ &> \left(\frac{a}{r}+\frac{1}{R}\right)-o(\epsilon),\end{aligned} \tag{3.23}$$

where $o(\epsilon)$ tends to 0 as $\epsilon \to 0^+$. Furthermore,

$$\begin{aligned}\|u+v\| &= \left\|\left(\frac{1}{r+\epsilon}-\frac{1}{R+\epsilon}\right)x_N-\left(\frac{1}{r+\epsilon}+\frac{1}{a(R+\epsilon)}\right)z\right\| \\ &= \left(\frac{1}{r+\epsilon}+\frac{1}{R+\epsilon}\right)\left\|x_N-\frac{1/(r+\epsilon)+1/a(R+\epsilon)}{1/(r+\epsilon)+1/(R+\epsilon)}z\right\| > \left(\frac{1}{r+\epsilon}+\frac{1}{R+\epsilon}\right)(r-\epsilon) \\ &> \left(\frac{1}{r}+\frac{1}{R}\right)r-o(\epsilon).\end{aligned} \tag{3.24}$$

Also we have

$$\|u+v\| \geq \frac{1}{r+\epsilon}+\frac{1}{R+\epsilon}-\left(\frac{1}{r+\epsilon}-\frac{1}{a(R+\epsilon)}\right)a \geq \frac{2}{R+\epsilon}+\frac{1}{r+\epsilon}-1 > \frac{2}{R}+\frac{1}{r}-1-o(\epsilon). \tag{3.25}$$

Define $f(r)=2/R+1/r-1$ and $g(r)=1+r/R$. Thus, $\|u+v\| \geq \max\{f(r),g(r)\}-o(\epsilon)$. Since $f(r)=g(r)$ for $r=r_0=1-R+\sqrt{R^2+1-R}$, we obtain

$$\|u+v\| \geq g(r_0)=\frac{1}{R}+\sqrt{1-\frac{1}{R}+\frac{1}{R^2}}. \tag{3.26}$$

Consequently, we have

$$\frac{1}{2}\left(\frac{a}{r}+\frac{1}{R}\right)-o(\epsilon) \leq \frac{\|u-v\|}{2} \leq 1-\delta_X\left(\frac{1}{R}+\sqrt{1-\frac{1}{R}+\frac{1}{R^2}}-o(\epsilon)\right). \tag{3.27}$$

Since the last inequality is true for every $\epsilon > 0$ and every $z \in A$, letting $\epsilon \to 0$ and using the continuity of $\delta(\cdot)$, we obtain

$$r_C(A) \leq \left(2-\frac{1}{R}-2\delta_X\left(\frac{1}{R}+\sqrt{1-\frac{1}{R}+\frac{1}{R^2}}\right)\right)r. \tag{3.28}$$

□

In [33] it is proved that X has normal structure under the slightly weaker condition

$$\delta_X\left(1+\frac{1}{R(X)}\right) > \frac{1}{2}\left(1-\frac{1}{R(X)}\right). \tag{3.29}$$

It is an open question if this condition implies the (DL)-condition.

Corollary 3.20. *Let X be a uniformly nonsquare Banach space such that $R(X)=1$. Then X satisfies the (DL)-condition.*

4. Fixed Point Results for Multivalued Nonexpansive Mappings in Modular Function Spaces

The theory of modular spaces was initiated by Nakano [34] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [35] in 1959. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [36–39]). In particular, Dhompongsa et al. [40] have obtained some fixed point results for multivalued mappings in modular functions spaces.

Let us recall some basic concepts about modular function spaces (for more details the reader is referred to [41, 42]).

Let Ω be a nonempty set and Σ a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ-finite measure space). By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, that is, all functions $f : \Omega \to \mathbb{R}$ such that there exist a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Let us recall that a set function $\mu : \Sigma \to [0, \infty]$ is called a σ-subadditive measure if $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ for any $A \subset B$ and $\mu(\bigcup A_n) \le \sum \mu(A_n)$ for any sequence of sets $\{A_n\} \subset \Sigma$. By 1_A, we denote the characteristic function of the set A.

Definition 4.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if:

(1) $\rho(0, E) = 0$ for any $E \in \Sigma$;

(2) $\rho(f, E) \le \rho(g, E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$;

(3) $\rho(f, \cdot) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$;

(4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$;

(5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$;

(6) for any $\alpha > 0$, $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subset \mathcal{P}$ and decreases to $\emptyset$.

A σ-subadditive measure ρ is said to be additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$, whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E) : g \in \mathcal{E}, |g(\omega)| \le |f(\omega)| \text{ for every } \omega \in \Omega\}. \tag{4.1}$$

Definition 4.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. For example, we will say frequently $f_n \to f$ ρ-a.e.

Note that a countable union of ρ-null sets is still ρ-null. In the sequel we will identify sets A and B whose symmetric difference $A \Delta B$ is ρ-null, similarly we will identify measurable functions which differ only on a ρ-null set.

Under the above conditions, we define the function $\rho : \mathcal{M} \to [0,\infty]$ by $\rho(f) = \rho(f,\Omega)$. We know from [41] that ρ satisfies the following properties:

(i) $\rho(f) = 0$ if and only if $f = 0$ ρ-a.e.

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha,\beta \ge 0$ and $f,g \in \mathcal{M}$.

In addition, if the following property is satisfied

(iii)′ $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha,\beta \ge 0$ and $f,g \in \mathcal{M}$,

we say that ρ is a convex modular.

A function modular ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(1_{K_n}) < \infty$ and $\Omega = \bigcup K_n$.

The modular ρ defines a corresponding modular space L_ρ, which is given by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{4.2}$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an F-norm. In fact, when ρ is convex, the formula

$$\|f\|_l = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\} \tag{4.3}$$

defines a norm which is frequently called the Luxemburg norm. The formula

$$\|f\|_a = \inf\left\{\frac{1}{k}(1 + \rho(kf)) : k > 0\right\} \tag{4.4}$$

defines a different norm which is called the Amemiya norm. Moreover, $\|\cdot\|_l$ and $\|\cdot\|_a$ are equivalent norms. We can also consider the space

$$E_\rho = \{f \in \mathcal{M} : \rho(\alpha f, \cdot) \text{ is order continuous for all } \alpha > 0\}. \tag{4.5}$$

Definition 4.3. A function modular ρ is said to satisfy the Δ_2-condition if

$$\begin{gathered}\sup_{n\ge1}\rho(2f_n, D_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty \text{ whenever } \{f_n\} \subset \mathcal{M}, D_k \in \Sigma \\ \text{decreases to } \emptyset \text{ and } \sup_{n\ge1}\rho(f_n, D_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty.\end{gathered} \tag{4.6}$$

It is known that the Δ_2-condition is equivalent to $E_\rho = L_\rho$.

Definition 4.4. A function modular ρ is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \le K\rho(f)$.

In general, the Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that the Δ_2-type condition implies the Δ_2-condition.

Definition 4.5. Let L_ρ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega : f_n(\omega) \nrightarrow f(\omega)\}$ is ρ-null.

(3) A subset C of L_ρ is called ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.

(4) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(5) A subset C of L_ρ is called ρ-bounded if

$$\operatorname{diam}_\rho(C) = \sup\{\rho(f-g) : f, g \in C\} < \infty. \tag{4.7}$$

We know by [41] that under the Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition. Hence, the ρ-convergence defines a topology which is identical to the norm topology.

In the same way as the Hausdorff distance defined on the family of bounded closed subsets of a metric space, we can define the analogue to the Hausdorff distance for modular function spaces. We will speak of ρ-Hausdorff distance even though it is not a metric.

Definition 4.6. Let C be a nonempty subset of L_ρ. We will denote by $F_\rho(C)$ the family of nonempty ρ-closed subsets of C and by $K_\rho(C)$ the family of nonempty ρ-compact subsets of C. Let $H_\rho(\cdot,\cdot)$ be the ρ-Hausdorff distance on $F_\rho(L_\rho)$, that is,

$$H_\rho(A,B) = \max\left\{\sup_{f\in A}\operatorname{dist}_\rho(f,B), \sup_{g\in B}\operatorname{dist}_\rho(g,A)\right\}, \quad A, B \in F_\rho(L_\rho), \tag{4.8}$$

where $\operatorname{dist}_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}$ is the ρ-distance between f and B. A multivalued mapping $T : C \to F_\rho(L_\rho)$ is said to be a ρ-contraction if there exists a constant $k \in [0,1)$ such that

$$H_\rho(Tf, Tg) \le k\rho(f-g), \quad f, g \in C. \tag{4.9}$$

If it is valid when $k = 1$, then T is called ρ-nonexpansive.

A function $f \in C$ is called a fixed point for a multivalued mapping T if $f \in TF$.

Dhompongsa et al. [40] stated the Banach Contraction Principle for multivalued mappings in modular function spaces.

Theorem 4.7 (see [40, Theorem 3.1]). *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty ρ-bounded ρ-closed subset of L_ρ, and $T : C \to F_\rho(C)$ a ρ-contraction mapping, that is, there exists a constant $k \in [0,1)$ such that*

$$H_\rho(Tf, Tg) \le k\rho(f - g), \quad f, g \in C. \tag{4.10}$$

Then T has a fixed point.

By using that result, they proved the existence of fixed points for multivalued ρ-nonexpansive mappings.

Theorem 4.8 (see [40, Theorem 3.4]). *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty ρ-a.e. compact ρ-bounded convex subset of L_ρ, and $T : C \to K_\rho(C)$ a ρ-nonexpansive mapping. Then T has a fixed point.*

They also applied the above theorem to obtain fixed point results in the Banach space L_1 (resp., ℓ_1) for multivalued mappings whose domains are compact in the topology of the convergence locally in measure (resp., w^*-topology).

Consider the space $L_p(\Omega, \mu)$ for a σ-finite measure μ with the usual norm. Let C be a bounded closed convex subset of L_p for $1 < p < \infty$ and $T : C \to K(C)$ a multivalued nonexpansive mapping. Because of uniform convexity of L_p, it is known that T has a fixed point. For $p = 1$, T can fail to have a fixed point even in the singlevalued case for a weakly compact convex set C (see [43]). However, since L_1 is a modular space where $\rho(f) = \int_\Omega |f| d\mu = \|f\|$ for all $f \in L_1$, Theorem 4.8 implies the existence of a fixed point when we define mappings on a ρ-a.e. compact ρ-bounded convex subset of L_1. Thus the following can be stated.

Corollary 4.9 (see [40, Corollary 3.5]). *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a nonempty bounded convex set which is compact for the topology of the convergence locally in measure, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

In the case of the space ℓ_1, we also can obtain a bounded closed convex set C and a nonexpansive mapping $T : C \to C$ which is fixed point free. Indeed, consider the following easy and well-known example.

Let

$$C = \left\{ \{x_n\} \in \ell_1 : 0 \le x_n \le 1, \ \sum_{n=1}^{\infty} x_n = 1 \right\}. \tag{4.11}$$

Define a nonexpansive mapping $T : C \to C$ by

$$T(x) = (0, x_1, x_2, x_3, \ldots), \quad \text{where } x = \{x_n\}, \tag{4.12}$$

then T is a fixed point free map. However, if we consider $L_\rho = \ell_1$, where $\rho(x) = \|x\|$, for all $x \in \ell_1$, then ρ-a.e. convergence and ω^*-convergence are identical on bounded subsets of ℓ_1 (see [36]). This fact leads to the following corollary.

Corollary 4.10 (see [40, Corollary 3.6]). *Let C be a nonempty ω^*-compact convex subset of ℓ_1 and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Next we will give a property of closed convex bounded subsets of ℓ_1 more general than weak star compactness which implies the fixed point property for nonexpansive mappings.

Domínguez et al. introduced in [44] some compactness conditions concerning proximinal subsets called Property (P). Following this idea we will use the following similar notion for modular function spaces.

Definition 4.11. Let C be a nonempty ρ-closed convex ρ-bounded subset of L_ρ. It is said that C has Property (P_ρ) if for every $f \in L_\rho$, which is the ρ-a.e. limit of a sequence in C, the set $P_{\rho,C}(f)$ is a nonempty and ρ-compact subset of C, where $P_{\rho,C}(f) = \{g \in C : \rho(g-f) = \operatorname{dist}_\rho(f,C)\}$.

Using that notion and the following two lemmas, we obtain a new fixed point result for multivalued ρ-nonexpansive mappings.

Lemma 4.12 (see [40, Lemma 3.3]). *Let ρ be a convex function modular satisfying the Δ_2-type condition, $f \in L_\rho$, and K a nonempty ρ-compact subset of L_ρ. Then there exists $g_0 \in K$ such that*

$$\rho(f-g_0) = \operatorname{dist}_\rho(f,K). \tag{4.13}$$

Lemma 4.13 (see [37, Lemma 1.3]). *Let ρ be a function modular satisfying the Δ_2-type condition, and $\{f_n\}_n$ be a sequence in L_ρ such that $f_n \overset{\rho\text{-a.e.}}{\to} f \in L_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f-g) \quad \forall g \in L_\rho. \tag{4.14}$$

Theorem 4.14. *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty ρ-closed ρ-bounded convex subset of L_ρ satisfying Property (P_ρ) such that every sequence in C has a ρ-a.e. convergent subsequence in L_ρ, and $T : C \to K_\rho C(C)$ a ρ-nonexpansive mapping. Then T has a fixed point.*

Proof. Fix $f_0 \in C$. For each $n \in \mathbb{N}$, the ρ-contraction $T_n : C \to F_\rho(C)$ is defined by

$$T_n(f) = \frac{1}{n} f_0 + \left(1 - \frac{1}{n}\right) Tf, \quad f \in C. \tag{4.15}$$

By Theorem 4.7, we can conclude that T_n has a fixed point, say f_n. It is easy to see that

$$\operatorname{dist}_\rho(f_n, Tf_n) \le \frac{1}{n} \operatorname{diam}_\rho(C) \longrightarrow 0. \tag{4.16}$$

By our assumptions, we can assume, by passing through a subsequence, that $f_n \overset{\rho\text{-a.e.}}{\to} f$ for some $f \in L_\rho$. By Lemma 4.12, for each $n \in \mathbb{N}$ there exists $g_n \in Tf_n$ such that

$$\rho(f_n - g_n) = \operatorname{dist}_\rho(f_n, Tf_n). \tag{4.17}$$

Now we are going to show that $P_{\rho,C}(f) \cap Th \neq \emptyset$ for each $h \in P_{\rho,C}(f)$. Taking any $h \in P_{\rho,C}(f)$, from the ρ-compactness of Th and Lemma 4.12, we can find $h_n \in Th$ such that

$$\rho(g_n - h_n) = \operatorname{dist}_\rho(g_n, Th) \leq H_\rho(Tf_n, Th) \leq \rho(f_n - h), \tag{4.18}$$

and we can assume, by passing through a subsequence, that $h_n \xrightarrow{\rho} h_0$ for some $h_0 \in Th$. From above and using Lemma 4.13, it follows that

$$\begin{aligned} \liminf_n \rho(f_n - h_0) &= \liminf_n \rho(g_n - h_0) = \liminf_n \rho(g_n - h_n) \leq \liminf_n \rho(f_n - h) \\ &= \liminf_n \rho(f_n - f) + \rho(f - h). \end{aligned} \tag{4.19}$$

On the other hand, by Lemma 4.13 we also have

$$\liminf_n \rho(f_n - h_0) = \liminf_n \rho(f_n - f) + \rho(f - h_0). \tag{4.20}$$

Thus, we deduce $\rho(f - h_0) \leq \rho(f - h)$, which implies that $h_0 \in P_{\rho,C}(f)$ and so $P_{\rho,C}(f) \cap Th \neq \emptyset$.

Now we define the mapping $\widetilde{T} : P_{\rho,C}(f) \to KC(P_{\rho,C}(f))$ by $\widetilde{T}(h) = P_{\rho,C}(f) \cap Th$. From [45, Proposition 2.45] we know that the mapping $\widetilde{T}$ is upper semicontinuous. Since $P_{\rho,C}(f) \cap Th$ is a nonempty ρ-compact convex set and the ρ-topology is a norm-topology, we can apply the Kakutani-Bohnenblust-Karlin Theorem (see [14]) to obtain a fixed point for $\widetilde{T}$ and hence for T. □

If we apply the previous theorem in the particular case of the space $L_1(\Omega, \mu)$ for a σ-finite measure μ with the usual norm, we obtain the following result, which can be also deduced from [44, Theorem 4.9].

Corollary 4.15. *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a nonempty closed bounded convex set which satisfies Property (P). Suppose, in addition, that every sequence in C has a convergent locally in measure subsequence in L_1. If $T : C \to KC(C)$ is a nonexpansive mapping, then T has a fixed point.*

If we consider now the space ℓ_1, then the assumption of existence of a w^*-convergent subsequence for every sequence in C can be removed and we can state the following result.

Corollary 4.16. *Let C be a nonempty closed bounded convex subset of ℓ_1 which satisfies Property (P). If $T : C \to KC(C)$ is a nonexpansive mapping, then T has a fixed point.*

Notice that in ℓ_1 there exists a subset with Property (P) which is not w^*-compact.

Example 4.17 (see [44, Example 4.8]). Let (a_n) be a bounded sequence of nonnegative real numbers and let (e_n) be the standard Schauder basis of ℓ_1. It is clear that the set $C := \overline{\operatorname{conv}}(x_n)$, where $x_n := (1 + a_n)e_n$, is never weakly star compact. Nevertheless, by using [46, Example 1] it is easy to show that C has Property (P) if and only if $N_0 := \{n \in \mathbb{N} : a_n = \inf_{m \in \mathbb{N}} a_m\}$ is nonempty and finite.

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