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On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p–spaces with different weights

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1 Introduction and main results

Let M be the Hardy–Littlewood maximal operator defined for locally integrable functions f by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all the cubes containing x, and let $1 < p < \infty$. B. Muckenhoupt [10] characterized the weights w satisfying the weighted norm inequality

$$\int_{\mathbb{R}^n} (w(y)Mf(y))^p \, dy \leq c \int_{\mathbb{R}^n} (w(y)f(y))^p \, dy \tag{1}$$

for all nonnegative functions f, as those weights satisfying the A_p condition

$$\left(\frac{1}{|Q|} \int_Q w(y)^p \, dy\right)^{1/p} \left(\frac{1}{|Q|} \int_Q w(y)^{-p'} \, dy\right)^{1/p'} \leq c \tag{2}$$

for all cubes Q. It is natural to consider a similar problem for a couple of weights (w, v). However, simple examples show (cf. [6] p. 395) that the analogous necessary condition for (w, v)

$$\left(\frac{1}{|Q|} \int_Q w(y)^p \, dy\right)^{1/p} \left(\frac{1}{|Q|} \int_Q v(y)^{-p'} \, dy\right)^{1/p'} \leq c, \tag{3}$$

for all cubes Q is not sufficient for the boundedness of M from $L^p(v^p)$ to $L^p(w^p)$. E. Sawyer has shown in [13], that the correct necessary and sufficient condition is given by

$$\int_Q (w(y)M(v^{-p'}\chi_Q)(y))^p\,dy \le c\int_Q v(y)^{-p'}\,dy, \tag{4}$$

for all cubes Q. E. Sawyer's condition involves the operator M itself, and it is interesting to obtain sufficient conditions close in form to the necessary and simpler one (3). The first result in that direction was obtained by C. Neugebauer in [11]. He noticed that if (w,v) is a couple of weights such that for some $r>1$

$$\left(\frac{1}{|Q|}\int_Q w(y)^{pr}\,dy\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q v(y)^{-p'r}\,dy\right)^{1/p'r} \le c \tag{5}$$

for all cubes Q, then

$$\int_{\mathbb{R}^n} (w(y)Mf(y))^p\,dy \le c\int_{\mathbb{R}^n} (v(y)f(y))^p\,dy \tag{6}$$

for all nonnegative functions f.

In this paper we take up this problem and show with a different approach that (6) holds assuming very weak conditions on the weights. We shall see that it is enough to replace the average norm associated to the weight v^{-1} in (3) by a stronger norm defined in terms of any Banach function space whose associated space satisfies certain mapping property.

To be precise we let X be a Banach function space over $\mathbb{R}^n$ with respect to the Lebesgue measure dx (cf. next section). Given a measurable function f and any cube Q we define the X–average of f over Q by

$$\|f\|_{X,Q} = \left\|\tau_{\ell(Q)}(f\,\chi_Q)\right\|_X, \tag{7}$$

where τ_δ, $\delta>0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$, χ_E is the characteristic function of E and $\ell(Q)$ is the sidelength of the cube Q.

We define a natural maximal operator associated to the space X.

Definition 1.1 *For each locally integrable function f the maximal operator M_X is defined by*

$$M_X f(x) = \sup_{x\in Q}\|f\|_{X,Q},$$

where the supremum is taken over all the cubes containing x.

Let $X = L^B$ be the Orlicz space defined by the Young function B (cf. section 2 or [7] [8]). Then the maximal operator $M_X = M_B$ is defined in terms of the average

$$\|f\|_{X,Q} = \|f\|_{B,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}$$

(cf. [1]). If X is the Lorentz space $X = L^{s,q}$, then the maximal operator is

$$M_X f(x) = M_{s,q} f(x) = \sup_{x \in Q} \frac{1}{|Q|^{1/s}} \|f \chi_Q\|_{s,q}$$

(cf. [12], [14] and [3]).

Given a Banach function space X, X' will denote its associate space, which is another Banach function space (cf. next section).

Theorem 1.2 *Let $1 < p < \infty$, and let X be a Banach function space such that $M_{X'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Suppose that (w, v) is a couple of weights such that there is a positive constant K for which*

$$\left(\frac{1}{|Q|} \int_Q w(y)^p \, dy\right)^{1/p} \|v^{-1}\|_{X,Q} \leq K, \tag{8}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n} (w(y) M f(y))^p \, dy \leq c \int_{\mathbb{R}^n} (v(y) f(y))^p \, dy \tag{9}$$

for all nonnegative functions f.

A particular example is when $X = L^{p'r}$, with $r > 1$. In this case the associate space is $X' = L^{(p'r)'}$ whose corresponding maximal operator is given by

$$M_{X'} f(x) = M_{(p'r)'} f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^{(p'r)'} \, dy\right)^{1/(p'r)'},$$

which is bounded on $L^p(\mathbb{R}^n)$. On the other hand, observe that when $r = 1$ $X' = L^p$, whose corresponding maximal function M_p is not bounded on $L^p(\mathbb{R}^n)$ since M itself fails to be bounded on $L^1(\mathbb{R}^n)$.

Corollary 1.3 *Let $1 < p < \infty$, and suppose that (w, v) is a couple of weights such that for some $r > 1$, there is a positive constant K for which*

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v(y)^{-p'r}\,dy\right)^{1/p'r} \leq K, \tag{10}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n}(w(y)Mf(y))^p\,dy \leq c\int_{\mathbb{R}^n}(v(y)f(y))^p\,dy \tag{11}$$

for all nonnegative functions f.

We can deduce a better result using the scale of Lorentz spaces: if $X = L^{p'r,\infty}$, then $X' = L^{(p'r)',1}$ and $M_{X'}$ is bounded on $L^p(\mathbb{R}^n)$ (cf. section 6). Hence

Corollary 1.4 *Let $1 < p < \infty$, and $1 < r < \infty$. Suppose that (w, v) is a couple of weights such that there is a positive constant K for which*

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p}\frac{1}{|Q|^{1/rp'}}\left\|\chi_Q v^{-1}\right\|_{L^{rp'},\infty} \leq K, \tag{12}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n}(w(y)Mf(y))^p\,dy \leq c\int_{\mathbb{R}^n}(v(y)f(y))^p\,dy \tag{13}$$

for all nonnegative functions f.

More interesting examples are provided by the theory of Orlicz spaces.

Theorem 1.5 *Let $1 < p < \infty$, and let B be a doubling Young function such that*

$$\int_c^\infty \left(\frac{t^{p'}}{B(t)}\right)^{p-1}\frac{dt}{t} < \infty, \tag{14}$$

for some positive constant c.
i)Let (w, v) be a couple of weights such that there is a positive constant K for which

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p}\left\|v^{-1}\right\|_{B,Q} \leq K, \tag{15}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n} (w(y)Mf(y))^p\,dy \le c \int_{\mathbb{R}^n} (v(y)f(y))^p\,dy \tag{16}$$

for all nonnegative functions f.
ii) Condition (14) is also a necessary condition. That is, suppose that B has the property that

$$\int_{\mathbb{R}^n} (w(y)Mf(y))^p\,dy \le c \int_{\mathbb{R}^n} (v(y)f(y))^p\,dy$$

for all nonnegative functions f, whenever the couple of weights (w, v) satisfies

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p} \|v^{-1}\|_{B,Q} \le K,$$

for all cubes Q. Then B satisfies (14).

Particular examples are given by

$$B(t) \approx t^{p'} \log^{p'-1+\delta}(1+t),$$

or the weaker one

$$B(t) \approx t^{p'} \log^{p'-1}(1+t)[\log\log(1+t)]^{p'-1+\delta},$$

with $\delta > 0$.

The key fact is the boundedness of $M_{\bar{B}}$ on $L^p(\mathbb{R}^n)$, and the relevant class of Young functions is the following.

Definition 1.6 *Let $1 < p < \infty$. We say that a doubling Young function B satisfies the B_p condition if there is a positive constant c such that*

$$\int_c^\infty \frac{B(t)}{t^p}\,\frac{dt}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\bar{B}(t)}\right)^{p-1} \frac{dt}{t} < \infty.$$

Then we have the following characterization.

Theorem 1.7 *Let $1 < p < \infty$. Suppose that B is a Young function. Then the following are equivalent.*
i)

$$B \in B_p\,; \tag{17}$$

ii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_B f(y)^p \, dy \leq c \int_{\mathbb{R}^n} f(y)^p \, dy \tag{18}$$

for all nonnegative functions f;
iii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_B f(y)^p \, w(y) dy \leq c \int_{\mathbb{R}^n} f(y)^p \, Mw(y) dy \tag{19}$$

for all nonnegative functions f and w;
iv) there is a constant c such that

$$\int_{\mathbb{R}^n} Mf(y)^p \, \frac{w(y)}{[M_{\bar{B}}(u^{1/p})(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, \frac{Mw(y)}{u(y)} dy, \tag{20}$$

for all nonnegative functions f, w and u.

A consequence is the following inequality:

Corollary 1.8 *Let $1 < p < \infty$. Suppose that w is a weight. Then*

$$\int_{\mathbb{R}^n} Mf(y)^p \, M^{[p']+1} w(y)^{1-p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, w(y)^{1-p} dy \tag{21}$$

for all nonnegative functions f.

As usual $[r]$ denotes the integer part of r.

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2 Preliminaries

In this section we provide the necessary background from the theory of function spaces that will be used later. We begin by recalling some basic facts about the theory of Banach function spaces introduced by W.A.J. Luxemburg in [9], and we shall refer the reader to [2] for a complete account. Let (R, μ) be a measure space, and let $M^+(R)$ be the cone of μ–measurable functions on R whose values lie in $[0, \infty]$. A mapping $\rho : M^+(R) \to [0, \infty]$ is called a Banach function norm if, for all $f, g, f_n, (n = 1, 2, 3, \ldots)$ in $M^+(R)$, for all constants $a \geq 0$, and for all μ–measurable subsets E of R, the following properties hold:

i) $\rho(f) = 0$ iff $f = 0$ μ–a.e.; $\rho(af) = a\rho(f)$;
$\rho(f + g) \leq \rho(f) + \rho(g)$

ii) $0 \leq g \leq f$ μ–a.e. implies $\rho(g) \leq \rho(f)$

iii) $0 \leq f_n \uparrow f$ μ–a.e. implies $\rho(f_n) \uparrow \rho(f)$

iv) $\mu(E) < \infty$ implies $\rho(\chi_E) < \infty$

v) $\mu(E) < \infty$ implies $\int_E f\, d\mu \leq C_E\, \rho(f)$,

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f. Let $M(R)$ denote the collection of all μ–measurable functions on R. The collection $X = X(\rho)$ of all functions $f \in M(R)$ for which $\rho(|f|) = \|f\|_X < \infty$ is called a Banach function space. The most important property of the Banach function spaces is the generalized Hölder inequality

$$\int_R |f(y)g(y)|\, d\mu(y) \leq \|f\|_X \|g\|_{X'}, \tag{22}$$

where X' is the associate space to X.

A Banach function space X is said to be rearrangement–invariant if whenever $f, g \in X$ are equimeasurable, then $\|f\|_X = \|g\|_X$. Recall that two functions are equimeasurable if $\mu_f(t) = \mu_g(t)$, $t > 0$, where $\mu_f(t) = \mu\{x \in R : |f(x)| > t\}$, is the distribution of f. Most of the properties of the rearrangement–invariant spaces can be formulated in terms of the fundamental function of X, φ_X, given by

$$\varphi_X(t) = \|\chi_E\|_X,$$

where $\mu(E) = t$. Observe that the particular choice of the set E with $\mu(E) = t$ is immaterial by the rearrangement–invariance of X. φ_X is quasiconcave and continuous, except perhaps at the origin. Furthermore, if X' is the associated space of X the following identity holds

$$\varphi_X(t)\varphi_{X'}(t) = t, \qquad t > 0. \tag{23}$$

Examples of rearrangement–invariant spaces include the Lebesgue L^p spaces, the minimal and maximal Lorentz spaces Λ, M, (cf. [2]). Also, the Orlicz and $L^{s,q}$ spaces that we are going to describe briefly next.

A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. We shall assume that B is normalized so that $B(1) = 1$. Also, we shall require that B satisfies the doubling condition

$$B(2t) \le C\,B(t), \qquad t > k \tag{24}$$

for some constants $C > 0$, $k \ge 0$. We shall make use of the following property

$$B(t) \approx t\,B'(t), \qquad t > 0, \tag{25}$$

and that $t \to \frac{B(t)}{t}$ is increasing.

Each Young function B has associated a complementary Young function $\bar{B}$ that satisfies

$$t \le B^{-1}(t)\bar{B}^{-1}(t) \le 2t, \qquad t > 0. \tag{26}$$

Let (X, μ) be a measure space and let B be a Young function. The Orlicz space $L^B(\mu)$ consists of all μ–measurable functions f such that

$$\int_X B\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) < \infty,$$

for some $\lambda > 0$. $L^B(\mu)$ can be normed by the Luxemburg norm defined by

$$\|f\|_{B,\mu} = \inf\{\lambda > 0 : \int_X B\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) \le 1\}.$$

$L^B(\mu)$ is a rearrangement–invariant space with fundamental function given by

$$\varphi_B(t) = \varphi_{L^B(\mu)}(t) = \frac{1}{B^{-1}(\frac{1}{t})}. \tag{27}$$

In particular if E is a measurable subset of X, then

$$\|\chi_E\|_{B,\mu} = \frac{1}{B^{-1}(\frac{1}{\mu(E)})}. \tag{28}$$

Finally, the associated space to $L^B(\mu)$ is $L^{\bar{B}}(\mu)$.

A function f belongs to the Lorentz space $L^{s,q}$, $0 < s, q \le \infty$, if

$$\|f\|_{L^{s,q}(\mu)} = \left[q \int_0^\infty \left(t\,\mu\{x \in \mathbb{R}^n : |f(x)| > t\}^{1/s}\right)^q \frac{dt}{t}\right]^{1/q} < \infty,$$

whenever $q < \infty$, and

$$\sup_{0<t<\infty} t\,\mu\{x \in \mathbb{R}^n : |f(x)| > t\}^{1/s} < \infty,$$

if $q = \infty$. For each $1 < s, q \leq \infty$ $L^{s,q}$ is a rearrangement–invariant Banach function space with fundamental function

$$\varphi(t) = t^{1/s},$$

and associated space $L^{s',q'}$.

3 The general case

Let X be a Banach function space over $\mathbb{R}^n$ with respect to the Lebesgue measure. Recall that for any measurable function f and arbitrary cube Q we defined the X–average of f over Q by

$$\|f\|_{X,Q} = \left\|\tau_{\ell(Q)}(f\,\chi_Q)\right\|_X, \tag{29}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$, χ_E is characteristic function of E and $\ell(Q)$ is the sidelength of the cube Q.

Note that Hölder's inequality for Banach function spaces (22) yields after the change of variable $y = \ell(Q)z$

$$\frac{1}{|Q|}\int_Q f(y)g(y)\,dy \leq \|f\|_{X,Q}\|g\|_{X',Q}. \tag{30}$$

We also introduce the following maximal operator associated to the space X. For each locally integrable function f we have also defined M_X by

$$M_X f(x) = \sup_{x\in Q}\|f\|_{X,Q},$$

where the supremum is taken over all the cubes containing x.

Proof of Theorem 1.2: Since the set of bounded functions with compact support is dense in $L^p(v^p)$ it is enough to show that there is a constant c such that

$$\int_{\mathbb{R}^n}(w(y)Mf(y))^p\,dy \leq c\int_{\mathbb{R}^n}(v(y)f(y))^p\,dy, \tag{31}$$

for each nonnegative bounded function with compact support f.

For each integer k, and for any constant $a > 2^n$ we let Ω_k and D_k be the sets

$$\Omega_k = \{x \in \mathbb{R}^n : a^k < Mf(x)\},$$

$$D_k = \{x \in \mathbb{R}^n : M^d f(x) > \frac{a^k}{4^n}\}.$$

Here M^d denotes the dyadic Hardy–Littlewood maximal operator. By Lemma 4.1 below with $t = a^k$ (note that the lemma also holds for the degenerate Young function $B(t) = t$, the classical Calderón–Zygmund decomposition) there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ for which $\Omega_k \subset \cup_j 3Q_{k,j}$, $D_k = \cup_j Q_{k,j}$, and

$$\frac{a^k}{4^n} < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)\, dy \leq \frac{a^k}{2^n}. \tag{32}$$

We can now estimate the left side of (31) as follows

$$\int_{\mathbb{R}^n} Mf(y)^p\, w(y)^p dy = \sum_k \int_{\Omega_k - \Omega_{k+1}} Mf(y)^p\, w(y)^p dy \leq \tag{33}$$

$$\leq a^p \sum_k a^{kp} w^p(\Omega_k) \leq C \sum_{k,j} a^{kp} w^p(3Q_{k,j}) \leq$$

$$\leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)\, dy \right)^p w^p(3Q_{k,j}) =$$

$$= C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f(y) v(y)\, v(y)^{-1}\, dy \right)^p w^p(3Q_{k,j}) \leq$$

$$\leq C \sum_{k,j} \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) v(y)\, v(y)^{-1}\, dy \right)^p w^p(3Q_{k,j}).$$

For each integer k, j we let we let $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets, and by Lemma 4.2 (as above the lemma is also valid for the degenerate Young function $B(t) = t$) there is a positive constant β such that for each k, j $|Q_{k,j}| < \beta\, |E_{k,j}|$. This together with (30), and (8) allows to dominate last sum by

$$C \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} \|v^{-1}\|^p_{X,3Q_{k,j}} w^p(3Q_{k,j}) =$$

$$= C \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} \|v^{-1}\|^p_{X,3Q_{k,j}} \frac{w^p(3Q_{k,j})}{|3Q_{k,j}|} |Q_{k,j}| \leq$$

$$\leq CK^p \sum_{k,j} \|fv\|^p_{X',3Q_{k,j}} |E_{k,j}| \leq$$

$$\leq C\sum_{k,j}\int_{E_{k,j}} M_{X'}(fv)(y)^p\,dy \leq C\int_{\mathbb{R}^n} M_{X'}(fv)(y)^p\,dy \leq$$

$$\leq C\int_{\mathbb{R}^n}(f(y)v(y))^p\,dy,$$

since the sets $\{E_{k,j}\}$ are pairwise disjoint, and because we are assuming that $M_{X'}: L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$. This concludes the proof of the theorem.

$\square$

It is a simple consequence of this Theorem that condition (8) is stronger than A_p since the later is necessary for the boundedness of M from $L^p(v^p)$ to $L^p(w^p)$. However, there is a direct argument that we shall outline. It is enough to show that there is a positive constant c such that

$$\left(\frac{1}{|Q|}\int_Q f(y)^{p'}\,dy\right)^{1/p'} \leq c\,\|f\|_{X,Q}, \tag{34}$$

for all cubes, and for all nonnegative functions f. Let us assume that $f\geq 0$, and Q is fixed. By assumption on $M_{X'}$, there is a constant c such that

$$\int_{\mathbb{R}^n} M_{X'}g(y)^p\,dy \leq c\int_{\mathbb{R}^n} g(y)^p\,dy \tag{35}$$

for all nonnegative functions g. Consider $g=f^{p'-1}\chi_Q$. Then

$$\|f^{p'-1}\|^p_{X',Q}|Q| \leq \int_Q M_{X'}(f^{p'-1}\chi_Q)(y)^p\,dy \leq c\int_Q f(y)^{p'}\,dy.$$

This together with (30) gives

$$\frac{1}{|Q|}\int_Q f(y)^{p'}\,dy = \frac{1}{|Q|}\int_Q f(y)\,f(y)^{p'-1}\,dy \leq \|f\|_{X,Q}\|f^{p'-1}\|_{X',Q} \leq$$

$$\leq c\|f\|_{X,Q}\left(\frac{1}{|Q|}\int_Q f(y)^{p'}\,dy\right)^{1/p},$$

which readily gives (34).

We conclude this section by proving a weighted inequality "dual" to the classical Fefferman–Stein inequality

$$\int_{\mathbb{R}^n} Mf(y)^p\,w(y)dy \leq c\int_{\mathbb{R}^n} f(y)^p\,Mw(y)dy.$$

The main interest follows from the fact that its "dual" inequality, namely

$$\int_{\mathbb{R}^n} Mf(y)^{p'}\, Mw(y)^{1-p'} dy \leq c \int_{\mathbb{R}^n} f(y)^{p'}\, w(y)^{1-p'} dy$$

is false in general (consider $f = w$ positive and integrable).

Theorem 3.1 *Let $1 < p < \infty$, and let X be a Banach function space such that $M_{X'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.*
i) There is a constant c such that

$$\int_{\mathbb{R}^n} Mf(y)^p \,\frac{1}{[M_X(u^{1/p})(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \,\frac{1}{u(y)} dy, \tag{36}$$

for all nonnegative functions f and u.
ii) If furthermore X is rearrangement–invariant with fundamental function φ_X, then there is a positive constant c such that

$$\int_0^c \frac{\varphi_{X'}(t)^p}{t} \frac{dt}{t} < \infty. \tag{37}$$

We do not know whether condition (37) it is also sufficient for the boundedness of $M_{X'}$ on $L^p(\mathbb{R}^n)$ as in the Orlicz or Lorentz cases (see Theorems 1.7 6.1 respectively).

Proof: We notice first that (36) is equivalent to

$$\int_{\mathbb{R}^n} M(fg)(y)^p \,\frac{1}{[M_X(g)(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, dy,$$

for all nonnegative functions f and g. Now, it follows from the generalized Hölder's inequality (30) that

$$\int_{\mathbb{R}^n} M(fg)(y)^p \,\frac{1}{[M_X(g)(y)]^p} dy \leq \int_{\mathbb{R}^n} M_{X'}(f)(y)^p M_X(g)(y)^p \,\frac{1}{[M_X(g)(y)]^p} dy \leq$$
$$\leq C \int_{\mathbb{R}^n} f(y)^p \, dy,$$

since $M_{X'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. This gives i).

As for ii) we denote by $Q(x, r)$ the cube centered at $x \in \mathbb{R}^n$ and with sidelength equal to r. Taking $f = u = \chi_{Q(0,1)}$ in (36) we have,

$$\int_{\mathbb{R}^n} Mf(y)^p \,\frac{1}{[M_X(f)(y)]^p} dy \leq C. \tag{38}$$

On the other hand the rearrangement–invariance of X yields

$$M_X(f)(y) = \sup_{y\in Q} \left\|\tau_{\ell(Q)}(\chi_{Q\cap Q(0,1)})\right\|_X = \sup_{y\in Q} \left\|\chi_{\ell(Q)^{-1}(Q\cap Q(0,1))}\right\|$$

$$= \sup_{y\in Q} \varphi_X\left(|\ell(Q)^{-1}(Q\cap Q(0,1))|\right) = \sup_{y\in Q} \varphi_X\left(\frac{|Q\cap Q(0,1)|}{|Q|}\right).$$

Now, since φ_X is increasing it is easy to see that there exist positive constants a, b such that whenever $|y| > a$

$$M_X(f)(y) = \varphi_X\left(\frac{b}{|y|^n}\right). \tag{39}$$

Hence, by using polar coordinates and (23) we get

$$\int_{\mathbb{R}^n} Mf(y)^p \frac{1}{[M_X(f)(y)]^p} dy \geq C \int_{|y|>a} \frac{1}{|y|^{np}} \varphi_X\left(\frac{b}{|y|^n}\right)^{-p} dy =$$

$$= C \int_a^\infty \frac{1}{r^{np}} \varphi_X\left(\frac{b}{r^n}\right)^{-p} r^n \frac{dr}{r} = C \int_{a_1}^\infty \frac{1}{r^p} \varphi_X\left(\frac{1}{r}\right)^{-p} r \frac{dr}{r} =$$

$$= C \int_0^c \frac{t^p}{\varphi_X(t)^p} \frac{1}{t} \frac{dt}{t} = C \int_0^c \frac{\varphi_{X'}(t)^p}{t} \frac{dt}{t}.$$

This estimate together with (38) concludes the proof of the Theorem.

□

We shall conclude this section showing that if X is rearrangement–invariant, condition

$$\int_0^c \frac{\varphi_{X'}(t)^p}{t} \frac{dt}{t} < \infty,$$

in Theorem 3.1 is also necessary for the statement in Theorem 1.2.

Proposition 3.2 *Let $1 < p < \infty$, and let X be a rearrangement–invariant Banach function space with the property that*

$$\int_{\mathbb{R}^n} (w(y)Mf(y))^p \, dy \leq c \int_{\mathbb{R}^n} (v(y)f(y))^p \, dy$$

for all nonnegative functions f, whenever the couple of weights (w, v) satisfies

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p}\|v^{-1}\|_{X,Q}\le K, \tag{40}$$

for all cubes Q. Then

$$\int_0^c \frac{\varphi_{X'}(t)^p}{t}\,\frac{dt}{t}<\infty.$$

Proof: For the proof take any nonnegative locally integrable function g, and consider the couple of weights $(w,v)=(M_X(g^{1/p})^{-1},g^{-1/p})$. Since

$$\left(\frac{1}{|Q|}\int_Q M_X(g^{1/p})(y)^{-p}\,dy\right)^{1/p}\|v^{-1}\|_{X,Q}\le$$

$$\le\left(\frac{1}{|Q|}\int_Q \left\|g^{1/p}\right\|_{X,Q}^{-p}\,dy\right)^{1/p}\left\|g^{1/p}\right\|_{X,Q}=1,$$

(w,v) satisfies condition (40). Hence, by hypothesis on X there is a constant c such that

$$\int_{\mathbb{R}^n} Mf(y)^p\,\frac{1}{[M_X(g^{1/p})(y)]^p}dy\le c\int_{\mathbb{R}^n} f(y)^p\,\frac{1}{g(y)}dy, \tag{41}$$

for all nonnegative functions f. Finally, by Theorem 3.1 X must satisfy

$$\int_0^c \frac{\varphi_{X'}(t)^p}{t}\,\frac{dt}{t}<\infty.$$

$\square$

4 The B_p condition

Recall that a doubling Young function B satisfies the B_p condition if there is a positive constant c for which

$$\int_c^\infty \frac{B(t)}{t^p}\,\frac{dt}{t}<\infty.$$

Since sometimes is more convenient to deal with the complementary function $\bar{B}$ of B, it can be checked using (26) and (25) that the Young function B satisfies the

B_p condition if there is a positive constant c for which

$$\int_c^\infty \left(\frac{t^{p'}}{\bar{B}(t)}\right)^{p-1} \frac{dt}{t} < \infty. \tag{42}$$

We now give the proof of Theorem 1.5 which relates the B_p condition, the boundedness of M_B on L^p and with dual weighted estimates for M.

4.1 Proof of Theorem 1.7

For the proof that i) implies ii) we need the following lemma.

Lemma 4.1 *Suppose that B is a Young function, and that f is a nonnegative bounded function with compact support. For each $t > 0$, let $\Omega_t = \{y \in \mathbb{R}^n : M_B f(y) > t\}$. Then, if Ω_t is not empty, we have*

$$\Omega_t \subset \cup_j 3Q_j, \tag{43}$$

where Q_j is the family of nonoverlapping maximal dyadic cubes satisfying

$$\frac{t}{4^n} < \|f\|_{B,Q_j} \le \frac{t}{2^n} \tag{44}$$

for each integer j.

Furthermore it follows that

$$|\Omega_t| \le C \int_{\{y\in\mathbb{R}^n : f(y) > t/2\}} B\left(\frac{f(y)}{t}\right) dy. \tag{45}$$

and

$$\{y \in \mathbb{R}^n : M_B^d f(y) > \frac{t}{4^n}\} = \cup_j Q_j. \tag{46}$$

We defer the proof of the lemma for the moment, and assume i). To prove ii) we shall use the classical approach (cf. for instance [6] Ch. 2.) Hence, (45) and the change of variable $t = \frac{f(y)}{s}$ yield

$$\int_{\mathbb{R}^n} M_B f(y)^p\, dy = p \int_0^\infty t^p |\{y \in \mathbb{R}^n : M_B f(y) > t\}| \frac{dt}{t} \le$$

$$\le C \int_0^\infty t^p \int_{\{y\in\mathbb{R}^n : f(y) > t/2\}} B\left(\frac{f(y)}{t}\right) dy \frac{dt}{t} = C \int_{\mathbb{R}^n} \int_0^{2f(y)} t^p B\left(\frac{f(y)}{t}\right) \frac{dt}{t}\, dy =$$

$$= C \int_{\mathbb{R}^n} f(y)^p \, dy \int_{1/2}^{\infty} \frac{B(t)}{t^p} \frac{dt}{t} = C \int_{\mathbb{R}^n} f(y)^p \, dy,$$

since $B \in B_p$. This proves that i) implies ii).

For the proof that ii) implies iii) we discretize as in Theorem 1.2. We fix a constant $a > 2^n$, and for each integer k we let Ω_k, and D_k be the sets

$$\Omega_k = \{x \in \mathbb{R}^n : M_{_B} f(x) > a^k\},$$

$$D_k = \{x \in \mathbb{R}^n : M_{_B}^d f(x) > \frac{a^k}{4^n}\}.$$

Here $M_{_B}^d$ denotes the dyadic version of $M_{_B}$. Hence, by Lemma 4.1 with $t = a^k$ there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ for which $\Omega_k \subset \cup_j 3Q_{k,j}$, $D_k = \cup_j Q_{k,j}$, and

$$\frac{a^k}{4^n} < \|f\|_{_{B,Q_{k,j}}} \leq \frac{a^k}{2^n}. \tag{47}$$

We shall need the following lemma.

Lemma 4.2 *Suppose $a > 2^n$. For all integers k, j we let $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfy*

$$|Q_{k,j} \cap D_{k+1}| < \frac{2^n}{a} \, |Q_{k,j}|, \tag{48}$$

and

$$|Q_{k,j}| < \frac{1}{1 - \frac{2^n}{a}} \, |E_{k,j}|. \tag{49}$$

We postpone the proof of this also until the end of the proof of the theorem.

Now, using (47), and (49) we estimate the left side of (19) as in the proof of Theorem 1.2 by

$$\int_{\mathbb{R}^n} M_{_B} f(y)^p \, w(y) dy = \sum_k \int_{\Omega_k - \Omega_{k+1}} M_{_B} f(y)^p \, w(y) dy \leq \tag{50}$$

$$\leq a^p \sum_k a^{kp} w(\Omega_k) \leq C \sum_{k,j} a^{kp} w(3Q_{k,j}) \leq$$

$$\leq C \sum_{k,j} \|f\|_{_{B,Q_{k,j}}}^p w(3Q_{k,j}) = C \sum_{k,j} \|f\|_{_{B,Q_{k,j}}}^p \frac{w(3Q_{k,j})}{|3Q_{k,j}|} |Q_{k,j}| \leq$$

$$\leq C \sum_{k,j} \left\| f \left(\frac{w(3Q_{k,j})}{|3Q_{k,j}|} \right)^{1/p} \right\|_{B,Q_{k,j}}^p |E_{k,j}| \leq$$

$$\leq C \sum_{k,j} \int_{E_{k,j}} M_B(f(Mw)^{1/p})(y)^p \, dy \leq C \int_{\mathbb{R}^n} M_B(f(Mw)^{1/p})(y)^p \, dy \leq$$

$$\leq C \int_{\mathbb{R}^n} f(y)^p \, Mw(y) dy,$$

since we are assuming ii). This proves iii).

Let us assume that iii) holds. Observing that (20) is equivalent with

$$\int_{\mathbb{R}^n} M(fg)(y)^p \, \frac{w(y)}{[M_{\bar{B}}(g)(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, Mw(y) dy,$$

for all nonnegative functions f, g, and w, iv) follows immediately from (19) after an application of the inequality

$$M(fg)(y) \leq M_B f(y) M_{\bar{B}} g(y), \qquad\qquad y \in \mathbb{R}^n,$$

which is a consequence of the generalized Hölder's inequality (30).

To prove that iv) implies i) we let $w = 1$ in (20) obtaining

$$\int_{\mathbb{R}^n} Mf(y)^p \, \frac{1}{[M_{\bar{B}}(u^{1/p})(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, \frac{1}{u(y)} dy,$$

for all nonnegative functions f, and u. Since this is (36) in Theorem 3.1 with $X = L^{\bar{B}}$, we can apply that proposition to get a constant $c > 0$ for which

$$\int_0^c \frac{\varphi_B(t)^p}{t} \, \frac{dt}{t} < \infty. \tag{51}$$

Here $\varphi_B = \varphi_{L^B}$ is the fundamental function of L^B. We claim that (51) is equivalent with $B \in B_p$. Indeed, by (27) and (25) it readily follows that

$$\int_0^c \frac{\varphi_B(t)^p}{t} \, \frac{dt}{t} = \int_0^c \frac{1}{B^{-1}(\frac{1}{t})^p} \frac{1}{t} \, \frac{dt}{t} \approx \tag{52}$$

$$\approx \int_c^\infty \frac{B(t)}{t^p} \, \frac{dt}{t},$$

from which we obtain the claim. This concludes the proof of the Theorem save for the proofs of Lemmas 4.1 and 4.2.

Proof of Lemma 4.1: The proof is a simple adaptation of arguments in [6] Ch. 2. Since f is bounded with compact support, say supp$f \subset K$,

$$\|f\|_{B,Q} \leq \|f\|_{L^\infty} \|\chi_K\|_{B,Q} =$$

$$= \|f\|_{L^\infty} \frac{1}{B^{-1}\left(\dfrac{|Q|}{|Q \cap K|}\right)},$$

and it follows that

$$\|f\|_{B,Q} \to 0$$

as $Q \uparrow \mathbb{R}^n$. Hence, if there are any dyadic cubes Q with $\|f\|_{B,Q} > t$, they are contained in cubes of this type which are maximal with respect to inclusion. We let $C_t = \{P_j\}$ be the family of the dyadic maximal nonoverlapping cubes satisfying

$$t < \|f\|_{B,P_j}.$$

Let P_j' be the only dyadic cube containing P_j with sidelength twice that of P_j. Then

$$t < \|f\|_{B,P_j} \leq 2^n \|f\|_{B,P_j'}.$$

The last inequality can easily be deduced from the definition of the Luxemburg norm using the fact that $t \to \frac{B(t)}{t}$ is increasing. Hence by the maximality of the cubes $\{P_j\}$ we get

$$t < \|f\|_{B,P_j} \leq 2^n t. \tag{53}$$

Observe that from this discussion it is clear that

$$\{y \in \mathbb{R}^n : M_B^d f(y) > t\} = \cup_j P_j. \tag{54}$$

Let $x \in \Omega_t$. By definition, there is a cube R containing x such that

$$t < \|f\|_{B,R}. \tag{55}$$

Let k be the unique integer such that $2^{-(k+1)n} < |R| \leq 2^{-kn}$. There is some dyadic cube with side length 2^{-k}, and at most 2^n of them, $\{J_i : i = 1, \ldots, n\}$, meeting the interior of R. It is easy to see that for one of these cubes, say J_1,

$$\frac{t}{2^n} < \|\chi_{J_1} f\|_{B,R}. \tag{56}$$

This can be seen as follows. If for each $i = 1, \dots, 2^n$ we had

$$\left\|\chi_{J_i} f\right\|_{B,R} \leq \frac{t}{2^n},$$

we would get since $R \subset \cup_{i=1}^{2^n} J_i$ that

$$\|f\|_{B,R} = \left\|\chi_{\cup_{i=1}^{2^n} J_i} f\right\|_{B,R} \leq$$

$$\leq \sum_{i=1}^{2^n} \left\|\chi_{J_i} f\right\|_{B,R} \leq 2^n \frac{t}{2^n} = t,$$

contradicting (55). Using that $|R| \leq |J_1| < 2^n |R|$ one can also show

$$\frac{t}{4^n} < \|f\|_{B,J_1}. \tag{57}$$

By letting $C_{t/(4)^n} = \{Q_j\}$, we have by (53) that

$$\frac{t}{4^n} < \|f\|_{B,Q_j} \leq \frac{t}{2^n}, \tag{58}$$

for each j, yielding (44). (46) also follows since $\{y \in \mathbb{R}^n : M_B^d f(y) > \frac{t}{4^n}\} = \cup_j Q_j$. Also, we see from (57) that $J_1 \subset Q_k$, for some k, and then $R \subset 3J_1 \subset 3Q_k$. This gives

$$\Omega_t \subset \cup_j 3Q_j,$$

which is (43). Now, by the left side of the inequality (58), and the definition of $\|f\|_{B,Q}$ we get

$$|\Omega_t| \leq C \sum_j |Q_j| \leq$$

$$\leq C \sum_j \int_{Q_j} B\left(\frac{4^n f(y)}{t}\right) dy \leq C \int_{\mathbb{R}^n} B\left(\frac{f(y)}{t}\right) dy. \tag{59}$$

To obtain (45) we just use the standard idea of writing f as $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $f(x) > \frac{t}{2}$, and $f_1(x) = 0$ otherwise. Then $M_B f(x) \leq M_B f_1(x) + M_B f_2(x) \leq M_B f_1(x) + \frac{t}{2}$. Finally, since (59) holds for each $f \geq 0$, $t > 0$ we have

$$|\Omega_t| \leq \left|\{y \in \mathbb{R}^n : M_B f_1(y) > \tfrac{t}{2}\}\right| \leq C \int_{\mathbb{R}^n} B\left(\frac{f_1(y)}{t}\right) dy =$$

$$= C \int_{\{y \in \mathbb{R}^n : f(y) > t/2\}} B\left(\frac{f(y)}{t}\right) dy,$$

concluding the proof of Lemma 4.1.

□

We now conclude the proof of the Theorem by proving Lemma 4.2.

Proof of Lemma 4.2: The family $E_{k,j}$ is clearly disjoint. We note that (47) and the definition of the Luxemburg norm implies that

$$1 < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} B\left(\frac{4^n}{a^k} f(y)\right) dy,$$

and

$$\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} B\left(\frac{2^n}{a^k} f(y)\right) dy \leq 1.$$

Hence by standard properties of the dyadic cubes we can estimate what portion of $Q_{k,j}$ is covered by D_{k+1} as in [4] (cf. [6] p. 398)

$$\frac{|Q_{k,j} \cap D_{k+1}|}{|Q_{k,j}|} = \sum_i \frac{|Q_{k,j} \cap Q_{k+1,i}|}{|Q_{k,j}|} =$$

$$= \sum_{i : Q_{k+1,i} \subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} <$$

$$< \sum_{i : Q_{k+1,i} \subset Q_{k,j}} \frac{1}{|Q_{k,j}|} \int_{Q_{k+1,i}} B\left(\frac{4^n}{a^{k+1}} f(y)\right) dy \leq$$

$$\leq \frac{2^n}{a} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j} \cap \cup_i Q_{k+1,i}} B\left(\frac{2^n}{a^k} f(y)\right) dy \leq$$

$$\leq \frac{2^n}{a}.$$

Here we have used that $B(\frac{2^n}{a} t) \leq \frac{2^n}{a} B(t)$, $t > 0$, since $\frac{2^n}{a} < 1$, and because $t \to \frac{B(t)}{t}$ is increasing. This gives (48). Finally

$$\frac{|E_{k,j}|}{|Q_{k,j}|} > 1 - \frac{2^n}{a} > 0,$$

completing the proof of the Lemma and hence that of Theorem 1.7.

□

4.2 Proof of Corollary 1.8

If we let $w = 1$ and u is replaced by w^{p-1} in (20) we have the weighted inequality

$$\int_{\mathbb{R}^n} Mf(y)^p \frac{1}{[M_{\bar{B}}(w^{(p-1)/p})(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \frac{1}{w(y)^{p-1}} dy,$$

when $B \in B_p$. Let $\delta = [p'] - p' + 1 > 0$, and take $B(t) \approx \frac{t^p}{\log^{1+\delta}(1+t)}$. Then $\bar{B}(t) \approx t^{p'} \log^{[p']}(1+t)$ and $[M_{\bar{B}}(w^{(p-1)/p})(y)]^p = [M_A(w)(y)]^{p-1}$, where $A(t) \approx t \log^{[p']}(1+t)$.

Then Corollary 1.8 will follow if we prove the pointwise inequality

$$M_A w(x) \leq C\, M^{[p']+1} w(x).$$

It is enough to prove that there is a constant C such that for each cube Q

$$\|f\|_{A,Q} \leq \frac{C}{|Q|} \int_Q M^{[p']} f(y)\, dy.$$

By homogeneity we can assume that the right hand side is equal to C. Then, by the definition of the Luxemburg norm we need to prove

$$\frac{1}{|Q|} \int_Q A(w(y))\, dy = \frac{1}{|Q|} \int_Q w(y) \log^{[p']}(1 + w(y))\, dy \leq C.$$

But this is a consequence of iterating the following inequality of E.M. Stein [15]

$$\int_Q w(y) \log^k (1 + w(y))\, dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y))\, dy, \qquad (60)$$

with $k = 1, 2, 3, \cdots$.

$\square$

4.3 Some further considerations about the class B_p

We observe that $1 < p < q < \infty$ implies that

$$B_p \subset B_q.$$

A typical Young function that belongs to the class B_p is $B(t) = t^s$ with $1 \leq s < p$. Another more interesting example is the function B given by

$$B(t) \approx \frac{t^p}{\log^{1+\delta}(1+t)},$$

or

$$B(t) \approx \frac{t^p}{\log(1+t)\,[\log\log(1+t)]^{1+\delta}},$$

with $\delta > 0$.

Since the function $B(t) = t^s$ belongs to B_p we have $1 < s < p$ and so implies that $B \in B_{p-\epsilon}$ with $0 < \epsilon < p - s$, one could think that the same property would hold for any Young function in B_p. However, this is false as the following example shows. For $\delta > 0$, consider the example mentioned above

$$B(t) \approx \frac{t^p}{\log^{1+\delta}(1+t)}.$$

Then, $B \in B_p$, but it can be easily shown that there is no $\epsilon > 0$ for which $B \in B_{p-\epsilon}$. We can remedy this situation if we restrict attention to those Young functions that are submultiplicative. We say that the Young function B is submultiplicative if

$$B(ts) \leq B(t)B(s)$$

for each $t, s > 0$.

Lemma 4.3 *Let $1 < p < \infty$. Assume that B is a submultiplicative Young function such that $B \in B_p$. Then there exists $\epsilon > 0$ for which*

$$B \in B_{p-\epsilon}.$$

Proof: This is a simple consequence of the fact that

$$B \in B_p \text{ if and only } \bar{\alpha}(B) < p.$$

(Cf. for instance [2] Ch. 5.) Here $\bar{\alpha}(B)$ denotes

$$\bar{\alpha}(B) = \lim_{t\to\infty} \frac{logB(t)}{logt} = inf_{t>1}\frac{logB(t)}{logt},$$

and it can be shown that the limit exists, is finite, and strictly positive.

$\square$

Let us make the following observation concerning a particular case of (20). Taking the weight $w = 1$, inequality (20) becomes

$$\int_{\mathbb{R}^n} Mf(y)^p \frac{1}{[M_{\bar{B}}(u^{1/p})(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p \frac{1}{u(y)} dy, \tag{61}$$

for all nonnegative functions f, and u. Let $1 < r < \infty$, and consider $B(t) = t^{(p'r)'}$. Then $B \in B_p$, and (61) is

$$\int_{\mathbb{R}^n} Mf(y)^p \frac{1}{[M(u^{r(p'-1)})(y)]^{(p-1)/r}} dy \leq c \int_{\mathbb{R}^n} f(y)^p \frac{1}{u(y)} dy.$$

However, this estimate follows from well–known results. Indeed, it is enough to show that $[M_{\bar{B}}(u^{1/p})(y)]^{-p}$ is an A_p weight by the theorem of Muckenhoupt and the Lebesgue differentiation theorem. Now, recall that a weight w belongs to A_p if and only if $w = w_1 \, w_2^{1-p}$ where w_1 and w_2 are A_1 weights, and that $(Mg)^\delta \in A_1$ $0 < \delta < 1$ (see [6] p. 436). Then it is clear that

$$[M_{\bar{B}}(u^{1/p})(y)]^{-p} = [M(u^{r(p'-1)})(y)^{1/r}]^{(1-p)}$$

is an A_p weight.

This argument may suggest that $[M_{\bar{B}}(u^{1/p})(y)]^{-p}$ satisfies the A_p condition for each $B \in B_p$. However, the following example indicates that this is not true in general, and thus above argument is not sharp enough to get (61).

Let $1 < p < \infty$, $\delta > 0$, and let B be the Young function such that $\bar{B}(t) \approx t^{p'} \log^{p'-1+\delta}(1+t)$. Then $B \in B_p$ but $w = M_{\bar{B}}(\chi_{Q(0,1)})^{-p} \notin A_p$. Otherwise there would exist $\epsilon > 0$ such that $w \in A_{p-\epsilon}$ (cf. [6] p. 399.) Hence for each $M > 0$ we would have

$$\int_{|y|>M} \frac{w(y)}{|y|^{n(p-\epsilon)}} \, dy < \infty. \tag{62}$$

(cf. [6] p. 412.) However (39), and (27) yield

$$w(y) \approx \bar{B}^{-1}(b\,|y|^n)^p, \qquad\qquad |y| > a,$$

for some positive dimensional constant a, b. Thus using polar coordinates and (25)

$$\int_{|y|>a} \frac{w(y)}{|y|^{n(p-\epsilon)}} \, dy \approx \int_{|y|>a} \frac{\bar{B}^{-1}(b\,|y|^n)^p}{|y|^{n(p-\epsilon)}} \, dy \approx \int_{a_1}^{\infty} \frac{\bar{B}^{-1}(t)^p}{t^{p-1-\epsilon}} \, \frac{dt}{t}$$

$$\approx \int_{a_2}^{\infty} \frac{t^p}{\bar{B}(t)^{p-1-\epsilon}} \, \frac{dt}{t} \approx \int_{a_2}^{\infty} \frac{t^{\epsilon p'}}{\log^{1+\delta(p-1)-\epsilon(p'-1+\delta)}(t)} \, \frac{dt}{t} = \infty,$$

contradicting (62).

5 Orlicz spaces and two-weight inequalities

Proof of Theorem 1.5 Part i) follows immediately from Theorem 1.2 together with Theorem 1.7. Indeed, let $X = L^B$ with associate space $X' = L^{\bar{B}}$. Then the hypothesis $M_{X'} = M_{\bar{B}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ in Theorem 1.2 is equivalent with $\bar{B} \in B_p$ by Theorem 1.7.

Part ii) follows from Proposition 3.2 and the computation in (52).

□

As an easy consequence of this theorem we can obtain sufficient conditions much in the spirit of [5].

Corollary 5.1 *Let $1 < p < \infty$. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is increasing, that $\varphi(2t) \le C\varphi(t), t > 0$, and that for some positive constant c*

$$\int_c^\infty \frac{1}{\varphi(t)^{p-1}} \frac{dt}{t} < \infty. \tag{63}$$

Assume that (w, v) is a couple of weights which satisfies for some positive constant K

$$\left(\frac{1}{|Q|} \int_Q w(y)^p \, dy \right)^{1/p} \left[\frac{1}{|Q|} \int_Q v(y)^{-p'} \varphi \left(\left(\frac{w^p(Q)}{|Q|} \right)^{1/p} v(y)^{-1} \right) dy \right]^{1/p'} \le K, \tag{64}$$

for all cubes Q. Then

$$\int_{\mathbb{R}^n} (w(y) M f(y))^p \, dy \le c \int_{\mathbb{R}^n} (v(y) f(y))^p \, dy, \tag{65}$$

for all nonnegative functions f.

Proof: Consider the Young function defined by $B(t) \approx \frac{t^{p'}\varphi(t)}{K^{p'}}$ which satisfies (14) (i.e. that $\bar{B} \in B_p$). Now, (64) implies

$$\frac{1}{|Q|} \int_Q B \left(\left(\frac{w^p(Q)}{|Q|} \right)^{1/p} v(y)^{-1} \right) dy \le 1,$$

or equivalently

$$\left\| \left(\frac{w^p(Q)}{|Q|} \right)^{1/p} v^{-1} \right\|_{B,Q} \le 1.$$

From this and by homogeneity we get (15). Then Theorem 1.5 applies.

□

6 Lorentz spaces and two-weight inequalities

In this section we study two weighted norm inequalities for the Hardy–Littlewood maximal operator whenever the weights satisfy (8) with X being a Lorentz space. We begin by recalling that the maximal operator associated to the Lorentz space $L^{s,q}$ is given by

$$M_{s,q}f(x) = \sup_{x\in Q} \frac{1}{|Q|^{1/s}} \left\| f\,\chi_Q \right\|_{s,q}.$$

We now state a result similar to Theorem 1.7.

Theorem 6.1 *Let $1 < p, s < \infty$, and $1 \le q < \infty$. Then the following are equivalent.*
i)

$$s < p\,; \tag{66}$$

ii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_{s,q}f(y)^p\,dy \le c \int_{\mathbb{R}^n} f(y)^p\,dy \tag{67}$$

for all nonnegative functions f ;
iii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_{s,q}f(y)^p\,w(y)dy \le c \int_{\mathbb{R}^n} f(y)^p\,Mw(y)dy \tag{68}$$

for all nonnegative functions f, and w ;
iv) there is a constant c such that

$$\int_{\mathbb{R}^n} Mf(y)^p\,\frac{w(y)}{[M_{s',q'}(u^{1/p})(y)]^p}dy \le c \int_{\mathbb{R}^n} f(y)^p\,\frac{Mw(y)}{u(y)}dy, \tag{69}$$

for all nonnegative functions f, w and u.

Proof: Let us assume i). Since $L^{s,1} \subset L^{s,q}$ the proof of ii) can be reduced to showing that

$$\int_{\mathbb{R}^n} M_{s,1}f(y)^p\,dy \le c \int_{\mathbb{R}^n} f(y)^p\,dy.$$

We use the following weak-type inequality for the operator $M_{s,1}$ established in [14]

$$|x \in \mathbb{R}^n : M_{s,1}f(x) > t| \le \frac{C}{t^s}\|f\|_{s,1}^s \qquad\qquad t > 0,$$

namely,

$$M_{s,1} : L^{s,1}(\mathbb{R}^n) \to L^{s,\infty}(\mathbb{R}^n).$$

On the other hand we always have

$$M_{s,1} : L^{\infty}(\mathbb{R}^n) \to L^{\infty}(\mathbb{R}^n).$$

Hence by the Calderón version for Lorentz spaces of the Marcinkiewicz interpolation theorem (cf. [2] p. 225), we have

$$M_{s,1} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

since $p > s$. This gives ii).

Assume now ii). The proof of iii) follows the same line as that of Theorem 1.7, and we shall outline it. Again it is enough to prove

$$\int_{\mathbb{R}^n} M_{s,1}f(y)^p\,w(y)dy \le c \int_{\mathbb{R}^n} f(y)^p\,Mw(y)dy,$$

for each nonnegative bounded function f with compact support. We discretize the left hand part of the inequality as in Theorem 1.7. We fix a constant $a > 2^n$, and for each integer k we let Ω_k and D_k be the sets

$$\Omega_k = \{x \in \mathbb{R}^n : M_{s,1}f(x) > a^k\},$$

$$D_k = \{x \in \mathbb{R}^n : M^d_{s,1}f(x) > \frac{a^k}{4^n}\},$$

where $M^d_{s,1}$ is as above the dyadic version of $M_{s,1}$. Hence, arguing as in Lemma 4.1 we can find for each k a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}_{j\in\mathbb{Z}}$ for which $\Omega_k \subset \cup_j 3Q_{k,j}$, $D_k = \cup_j Q_{k,j}$, and

$$\frac{a^k}{4^n} < \frac{1}{|Q_{k,j}|^{1/s}}\left\|f\,\chi_{Q_{k,j}}\right\|_{s,1} \le \frac{a^k}{2^n}. \tag{70}$$

We let $\{E_{k,j}\}$ be the disjoint family $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. We claim that

$$|Q_{k,j}| < \frac{1}{1 - \frac{2^{ns}}{a^s}} |E_{k,j}|. \tag{71}$$

The proof requires a simple modification of the argument given in the proof of Lemma 4.2. Indeed, (70) and Minkowski's inequality yield

$$\frac{|Q_{k,j} \cap D_{k+1}|}{|Q_{k,j}|} = \sum_i \frac{|Q_{k,j} \cap Q_{k+1,i}|}{|Q_{k,j}|} = \sum_{i: Q_{k+1,i} \subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} <$$

$$< \frac{1}{|Q_{k,j}|} \left(\frac{4^n}{a^{k+1}}\right)^s \sum_{i: Q_{k+1,i} \subset Q_{k,j}} \left\| f \, \chi_{Q_{k+1,i}} \right\|_{s,1}^s =$$

$$= \frac{1}{|Q_{k,j}|} \left(\frac{4^n}{a^{k+1}}\right)^s \left[\sum_{i: Q_{k+1,i} \subset Q_{k,j}} \left(\int_0^\infty |\{x \in Q_{k+1,i} : f(x) > t\}|^{1/s} \, dt \right)^s \right]^{\frac{1}{s} s} \leq$$

$$\leq \frac{1}{|Q_{k,j}|} \left(\frac{4^n}{a^{k+1}}\right)^s \left[\int_0^\infty \left(\sum_{i: Q_{k+1,i} \subset Q_{k,j}} |\{x \in Q_{k+1,i} : f(x) > t\}| \right)^{1/s} dt \right]^s =$$

$$= \frac{1}{|Q_{k,j}|} \left(\frac{4^n}{a^{k+1}}\right)^s \left[\int_0^\infty |\{x \in Q_{k,j} \cap \cup_i Q_{k+1,i} : f(x) > t\}|^{1/s} \, dt \right]^s \leq$$

$$\leq \frac{1}{|Q_{k,j}|} \left(\frac{4^n}{a^{k+1}}\right)^s \left\| f \, \chi_{Q_{k,j}} \right\|_{s,1}^s \leq \left(\frac{2^n}{a}\right)^s < 1,$$

from which the claim readily follows.

Now, as in the proof of Theorem 1.7 part iii) we have the following chain of inequalities

$$\int_{\mathbb{R}^n} M_{s,1} f(y)^p \, w(y) dy = \sum_k \int_{\Omega_k - \Omega_{k+1}} M_{s,1} f(y)^p \, w(y) dy \leq a^p \sum_k a^{kp} w(\Omega_k) \leq$$

$$\leq C \sum_{k,j} a^{kp} w(3Q_{k,j}) \leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1/s}} \left\| f \, \chi_{Q_{k,j}} \right\|_{s,1} \right)^p w(3Q_{k,j}) =$$

$$= C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1/s}} \left\| f \, \chi_{Q_{k,j}} \left(\frac{w(3Q_{k,j})}{|3Q_{k,j}|} \right)^{1/p} \right\|_{s,1} \right)^p |Q_{k,j}| \leq$$

$$\leq C \sum_{k,j} \left(\frac{1}{|Q_{k,j}|^{1/s}} \left\| f \chi_{Q_{k,j}} (Mw)^{1/p} \right\|_{s,1} \right)^p |E_{k,j}| \leq$$

$$\leq C \sum_{k,j} \int_{E_{k,j}} M_{s,1}(f(Mw)^{1/p})(y)^p \, dy \leq C \int_{\mathbb{R}^n} M_{s,1}(f(Mw)^{1/p})(y)^p \, dy \leq$$

$$\leq C \int_{\mathbb{R}^n} f(y)^p \, Mw(y) dy,$$

since we are assuming that $M_{s,1} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. This proves iii).

That iv) follows from iii) is as in Theorem 1.7 a simple consequence of

$$M(fg)(y) \leq M_{s,q} f(y) M_{s',q'} g(y), \qquad y \in \mathbb{R}^n.$$

To prove that iv) implies i) we set $w = 1$ in (69), obtaining the same inequality as in (36) with $X = L^{s',q'}$. Now, recalling that $\varphi(t) = t^{1/s}$ is the fundamental function of $L^{s,q}$ we can apply Theorem 3.1 to deduce that φ must satisfy

$$\int_0^c \frac{\varphi(t)^p}{t} \frac{dt}{t} < \infty,$$

for some $c > 0$. This is equivalent with $p > s$, concluding the proof of the theorem.

□

We finish with the proof of Corollary 1.4:

Proof: The associated space of $L^{rp',\infty}$ is $L^{(rp')',1}$. Hence by Theorem 6.1 with $s = (rp')' < p$, and $q = 1$

$$M_{(rp')',1} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

Thus the corary follows from Theorem 1.2.

□

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