



Programa de Doctorado “Matemáticas”

PHD DISSERTATION

Derived Homotopy Algebras

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Abstract

Derived A_∞ -algebras are derived and homotopy invariant versions of differential graded algebras. They were introduced by Steffen Sagave in 2010 in order to construct minimal models for differential graded algebras over arbitrary commutative rings. Muriel Livernet, Constanze Roitzheim, and Sarah Whitehouse showed in 2013 how they can be viewed as algebras over the minimal model of the operad encoding bicomplexes with a compatible associative multiplication. We extend their work for the associative operad to a general quadratic Koszul operad \mathcal{O} satisfying standard projectivity assumptions. This leads to the new notion of derived homotopy \mathcal{O} -algebra, where minimal models for \mathcal{O} -algebras are defined. We explicitly compute generating operations and relations when \mathcal{O} is the associative operad, the commutative operad, and the operad encoding Lie algebras.

*In loving memory of grandparents,
Piet van de Ven (1923 - 2013) and
Jeanne van de Ven - Smits (1928 - 2014),
who were not able to see my final thesis,
but were always in my mind when I was writing it.*

- The tree which moves some to tears of joy is in the eyes of others only a green thing that stands in the way. Some see nature all ridicule and deformity... and some scarce see nature at all. But to the eyes of the man of imagination, nature is imagination itself. -

William Blake in a letter to Trusler.

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- *Zing, vecht, huil, bid, lach, werk, en bewonder.* -
Ramses Shaffy

- *Life is what happens to you while you're busy making other plans.* -
John Lennon

Since writing this PhD thesis spanned a considerable part of my lifetime so far, it seems unnatural to look at it as a product of academic accomplishment alone. In fact, it is not. My social environment and general feeling of happiness have had a great influence on it.

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Contents

Introduction	10
1 Operads	15
1.1 Monoidal categories	15
1.2 Monoids and distributive laws	24
1.3 Sequences and collections	29
1.4 Operads and cooperads	35
1.5 Algebras over operads	43
2 Koszul duality for operads	45
2.1 Free operads	45
2.2 Cofree cooperads	56
2.3 The bar and cobar constructions	61
2.4 Quadratic operads and cooperads	69
2.5 The Koszul dual cooperad	72
3 Derived operads and their algebras	76
3.1 Derived operads	76
3.2 Derived algebras	83
3.3 A presentation for quadratic derived operads	85
4 Koszul duality of derived operads	93
4.1 The trivial distributive law	94
4.2 Mock derived ns-operads and their bar construction	99
4.3 The Koszul dual ns-cooperad of a derived ns-operad	114
4.4 An alternative chain homotopy	128
4.5 The Koszul dual cooperad of a derived operad	135
5 Derived homotopy algebras, applications and examples	141
5.1 Derived homotopy algebras	141
5.2 Twisted complexes	143

5.3	Minimal derived homotopy models	145
5.4	Derived homotopy associative algebras	148
5.5	Derived homotopy commutative algebras	152
5.6	Derived homotopy Lie algebras	155
	Future Directions	159
	Bibliography	161
	Index	163
	List of Notations	167

Introduction

In the early sixties, J. Stasheff introduced the notion of an A_∞ -algebra, also known as (strongly) homotopy associative algebra, in his paper entitled ‘Homotopy associativity of H -spaces’ [23]. An A_∞ -algebra over a commutative ring k is a (\mathbb{Z}) -graded k -module A with structure maps $m_n: A^{\otimes n} \rightarrow A$, $n \geq 1$, of degree $n - 2$ satisfying certain relations. These relations say, in particular, that m_1 is a differential for A , yielding a chain complex (A, m_1) , m_2 is a binary multiplication on A , m_1 satisfies the Leibniz rule with respect to m_2 , and despite m_2 need not be strictly associative, it is so up to the explicit chain homotopy m_3 . Therefore the homology $H_*(A)$ is a graded associative algebra. Differential graded algebras are special cases of A_∞ -algebras, those for which $m_n = 0$ for $n \geq 3$.

One motivation for the introduction of A_∞ -algebras is the naive but natural question whether, up to quasi-isomorphism, we can reconstruct a differential graded algebra A from its homology $H_*(A)$. Simple examples show that this is clearly impossible, but they naturally lead to wonder which additional structure is needed to reconstruct A from $H_*(A)$. The answer was given by Kadeishvili in 1980 in his article ‘On The Homology Theory of Fibre Spaces’ [14]. There he proved that, if k is a field, every differential graded algebra A admits a quasi-isomorphism of A_∞ -algebras from a minimal A_∞ -algebra, where minimality means that $m_1 = 0$. The underlying graded module of such a *minimal model* for A is its homology algebra H_*A . Therefore the minimal A_∞ -structure on $H_*(A)$ precisely specifies the additional structure needed to reconstruct A from $H_*(A)$, up to quasi-isomorphism.

Over a general commutative ring k , Kadeishvili’s theorem only works for differential graded algebras A with projective homology $H_*(A)$. This great inconvenience motivated Sagave to address the reconstruction of a differential graded algebra from its homology in a different way [21]. In this 2010 paper, Sagave introduced the notion of a derived A_∞ -algebra, which simultaneously generalizes projective resolutions and A_∞ -algebras. More precisely, a derived A_∞ -algebra is a bigraded k -module with structure maps $m_{in}: E^{\otimes n} \rightarrow E$, $i \geq 0$, $n \geq 1$, of bidegree $(-i, i + n - 2)$ satisfying appropriate relations.

Ordinary A_∞ -algebras can be regarded as derived A_∞ -algebras concentrated in horizontal degree 0 with $m_n = m_{0n}$. Sagave considered the notion of E^2 -equivalences, which generalizes quasi-isomorphisms, and showed that any differential graded algebra A over a general commutative ring k , even with non-projective homology, has an E^2 -equivalent minimal (i.e. $m_{01} = 0$) derived A_∞ -model E such that the graded chain complex with horizontal differentials (E, m_{11}) is a projective resolution of $H_*(A)$.

With this, Sagave found the additional structure needed to reconstruct the quasi-isomorphism type of *any* differential graded algebra A over an arbitrary commutative ground ring k : a k -projective resolution of its homology $H_*(A)$ in the direction of the new horizontal grading equipped with a minimal dA_∞ -algebra structure, minimal meaning that $m_{01} = 0$.

In [23], J. Stasheff furthermore laid the origin for the notion of operad, which was later coined by J. P. May in ‘The Geometry of Iterated Loop Spaces’ [18]. Operads are algebraic devices that serve to study almost all kinds of algebras. Each operad \mathcal{O} has an associated category of algebras, called \mathcal{O} -algebras, and there is an operad for any class of algebras whose structure maps are multilinear operations (associative, commutative, Lie, A -infinity, etc.). Operads are 40 years old in algebraic topology, with a trend of appearance in several other areas, such as algebraic and differential geometry, combinatorics, and mathematical physics.

Kadeishvili’s proved his aforementioned result by using ad-hoc techniques. Nevertheless, we nowadays know a more conceptual proof using Koszul duality theory which in addition yields explicit formulas for the A_∞ -algebra structure on the minimal model.

Koszul duality theory is a modern approach in homological algebra to provide small resolutions for certain operads (e.g. quadratic ones). The method of construction splits into two steps. First, from the data defining the quadratic operad \mathcal{O} , we construct a quadratic cooperad \mathcal{O}^i , called Koszul dual. Second, we apply to this cooperad the cobar construction Ω in order to get a (differential graded) operad $\Omega\mathcal{O}^i$. When a certain acyclicity condition is fulfilled (and we then say that \mathcal{O} is Koszul), $\mathcal{O}_\infty = \Omega\mathcal{O}^i$ is the desired small resolution (incidentally also called minimal model) for \mathcal{O} . As an example, the minimal model for the associative operad \mathcal{A} , encoding differential graded algebras, is the operad \mathcal{A}_∞ encoding A_∞ -algebras. The operadic Koszul duality theory works perfectly over a field [11, 16] and also over a commutative ring [8] under appropriate projectivity conditions *on the operad* (not on its algebras).

Minimal models are also known to exist for algebras over quadratic Koszul operads \mathcal{O} over a field k . Also over a general commutative ground ring k if the algebra has projective homology, as in the associative case. This

leads to the following natural question: Can we define minimal models for arbitrary \mathcal{O} -algebras over a general commutative ground rings k as Sagave did in the associative case ($\mathcal{O} = \mathcal{A}$)? In this thesis, we affirmatively answer this question under quite general hypotheses on \mathcal{O} by means of what we call derived homotopy \mathcal{O} -algebras.

In our quest for an algebraic framework where minimal models can be constructed, we noted that, in their 2013 paper entitled ‘Derived A_∞ -algebras in an operadic context’ [15], Livernet, Roitzheim, and Whitehouse placed derived A_∞ -algebras in the operadic framework of Koszul duality theory. More precisely, they considered the operad $d\mathcal{A}$ parametrizing bicomplexes with a compatible associative multiplication. They proved that $d\mathcal{A}$ is a Koszul operad in the category of graded complexes and that its minimal model $d\mathcal{A}_\infty$ is the operad determining derived A_∞ -algebras. The relevant operad $d\mathcal{A}$ can also be constructed in terms of the associative operad \mathcal{A} (concentrated in horizontal degree 0), the ring of dual numbers $\mathcal{D} = k[\Delta]/(\Delta^2)$ (concentrated in vertical degree 0), and a distributive law. This somehow is why derived A_∞ -algebras extend ordinary A_∞ -algebras and projective resolutions at the same time. Moreover, all this allows to place Sagave’s result (which in principle used the same techniques as Kadeishvili) in the more conceptual context of Koszul duality theory.

In this thesis, we follow the approach of Livernet, Roitzheim, and Whitehouse, replacing the associative operad \mathcal{A} with a general quadratic Koszul operad \mathcal{O} satisfying mild projectivity hypotheses shared with \mathcal{A} . More concretely, we assume that both \mathcal{O} and its Koszul dual cooperad \mathcal{O}^i are projective over the ground ring. We want to stress that we, like Sagave, do *not* require *any* projectivity hypothesis on (the homology of) \mathcal{O} -algebras. Hence, our results apply not only to (differential graded) associative algebras, but also to commutative and Lie algebras, as we illustrate in the final chapter. For symmetric operads (i.e. such that the laws of \mathcal{O} -algebras involve permutations of variables, e.g. commutative or Lie algebras), we must require in addition that the ground ring contains the rationals $k \supset \mathbb{Q}$. This condition is not surprising, it is actually often necessary in homotopical contexts when dealing with symmetric operads.

Under the conditions of the previous paragraph, we consider the operad $d\mathcal{O}$ of graded chain complexes whose algebras are bicomplexes endowed with a compatible \mathcal{O} -algebra structure. As in the associative case, this operad can be built from \mathcal{O} (concentrated in horizontal degree 0) and from the ring of dual numbers \mathcal{D} (concentrated in vertical degree 0) by means of a distributive law. We prove that $d\mathcal{O}$ is Koszul and compute its Koszul dual cooperad $d\mathcal{O}^i$ in terms of the Koszul dual \mathcal{O}^i of \mathcal{O} (concentrated in horizontal degree 0), the polynomial coalgebra $\mathcal{D}^i = k[s\Delta]$, which is the Koszul dual of

\mathcal{D} (concentrated in vertical degree 0), and a certain codistributive law. This directly leads to the minimal model $d\mathcal{O}_\infty = \Omega d\mathcal{O}^i$ of $d\mathcal{O}$. Derived homotopy \mathcal{O} -algebras are precisely algebras over this operad $d\mathcal{O}_\infty$. This is precisely where minimal models for general \mathcal{O} -algebras live, the existence of minimal models being essentially a formal consequence of Koszul duality theory. As in the associative case, derived \mathcal{O} -algebras are special examples of derived homotopy \mathcal{O} -algebras.

We therefore generalize the table of Livernet, Roitzheim, and Whitehouse in the introduction of [15], presenting old and new algebraic structures and their homotopy invariant counterparts, in the following way:

Underlying category	Operad	Corresponding algebra
Complexes	\mathcal{A}	differential graded algebra
	\mathcal{A}_∞	A_∞ -algebra
	\mathcal{O}	\mathcal{O} -algebra
	\mathcal{O}_∞	homotopy \mathcal{O} -algebra
Graded complexes	$d\mathcal{A}$	bidifferential graded algebra
	$d\mathcal{A}_\infty$	derived A_∞ -algebra
	$d\mathcal{O}$	\mathcal{O} -algebra in bicomplexes
	$d\mathcal{O}_\infty$	derived homotopy \mathcal{O} -algebra

Organization

This thesis is organized in the following way.

In the first two chapters, which provide essential background for the rest of the thesis, Koszul duality theory of operads is spelled out in the closed symmetric monoidal category of graded complexes. To this end, monoidal categories and functors are recalled and the relevant monoidal structures are described in detail, such as those for graded complexes, sequences, and collections. Koszul duality theory is developed separately for both quadratic nonsymmetric and (symmetric) operads in graded complexes. Furthermore, algebras and (co)distributive laws are introduced, and we provide an important tool for the main chapters, which is a universal property for the tensor product of monoids twisted by a distributive law. In the section on cooperads, the coradical filtration is presented in a novel way (equivalence with the definition given in [16, section 5.8.5] is proven). In the section on the bar and cobar constructions, the claim that the differential of the bar construction squares to zero is proved in more detail than usual, in particular paying attention to the behaviour of signs.

In the third and fourth chapters we present the main theory of the thesis. We first define derived operads $d\mathcal{O}$ and describe algebras over them. We prove that $d\mathcal{O}$ is quadratic if \mathcal{O} is, explicitly giving a quadratic presentation

for $d\mathcal{O}$. This puts us in the position to develop Koszul duality theory for derived operads $d\mathcal{O}$, in particular to find their minimal model $d\mathcal{O}_\infty$, using our knowledge of Koszul duality theory for ordinary quadratic operads \mathcal{O} and the ring of dual numbers. This is first done for nonsymmetric operads and afterwards for (symmetric) operads, building on the former.

In [15], for the proof that $d\mathcal{A}$ is Koszul the authors refer to [16, Theorem 8.6.4], whose proof contains a gap (see Remark 4.3.3 below). We take a completely different approach based on homological perturbation theory. We regard this as one of our main contributions.

In the fifth chapter, we consider derived homotopy \mathcal{O} -algebras and explicitly calculate their generating operations and relations in the associative, commutative, and Lie cases. We also deduce the existence of minimal models for \mathcal{O} -algebras.

We conclude with some paragraphs considering some possible research lines for the future, continuing with the work started here.

Chapter 1

Operads

Operads are monoids in certain monoidal categories. In this chapter we review some basic material on monoidal categories, we introduce the specific monoidal categories we will work with, and we define operads and their algebras, as well as the dual notion of cooperad. We made certain emphasis on distributive laws, which are tools for building new monoids from old ones. More precisely, we characterize the tensor product of two monoids with a distributive law in terms of a universal property. This is relevant for this thesis since derived operads arise in this way.

1.1 Monoidal categories

This background section is mostly based on [17, Chapter VII].

Definition 1.1.1. A *monoidal category*, denoted $(\mathcal{C}, \otimes, 1)$, is a category \mathcal{C} equipped with:

- A functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the *tensor product*,
- a unit object $1 \in \mathcal{C}$, called the *tensor unit*,
- a natural isomorphism

$$a = a_{A,B,C}: A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C,$$

called the *associator*,

- and natural isomorphisms $l = l_A: 1 \otimes A \rightarrow A$ and $r = r_A: A \otimes 1 \rightarrow A$,

such that the associator, l , and r are subject to certain coherence conditions, which are equivalent to the commutative diagrams given in [17, chapter VII, section 1].

A monoidal category is called *strict* if $a_{A,B,C}$, l_A , and r_A are identity morphisms.

The *opposite* \mathcal{C}^{op} of a monoidal category is again monoidal. It has the same tensor product and unit $(\mathcal{C}^{op}, \otimes, 1)$ and the inverse structure isomorphisms.

As it is customary in category theory, several notion below will be called co-whatever when applied to the opposite monoidal category (e.g. comonoids, colax functors, etc.). These notions will be used later (e.g. cooperads). However, in order to keep these preliminaries short, we will neither explicitly spell out the definitions of these *co-notions* nor the results about them.

Definition 1.1.2. A *lax monoidal functor* between two monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ endowed with:

- a natural transformation

$$\phi = \phi_{A,B}: F(A) \otimes_{\mathcal{D}} F(B) \rightarrow F(A \otimes_{\mathcal{C}} B),$$

- and a morphism $\Psi: 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$ in \mathcal{D} ,

such that the properties stated in [16, B.3.3] hold.

A lax monoidal functor is called *strong* if ϕ and Ψ are isomorphisms. It is called a *strict* monoidal functor if they are identities.

Strong, and in particular strict, monoidal functors are also colax.

Definition 1.1.3. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, 1)$ equipped with a natural isomorphism

$$s = s_{A,B}: A \otimes B \rightarrow B \otimes A, \tag{1.1.4}$$

called the *symmetry isomorphism*, such that this symmetry isomorphism, the associator, l , and r are subject to certain extra coherence conditions, which are equivalent to the commutative diagrams given in [17, chapter VII, section 7].

The opposite of a symmetric monoidal category is also symmetric.

Definition 1.1.5. A *closed symmetric monoidal category* is a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ equipped with a functor

$$[-, -]: \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}, \tag{1.1.6}$$

called the *inner Hom*, such that $- \otimes B: \mathcal{C} \rightarrow \mathcal{C}$ is left adjoint to

$$[B, -]: \mathcal{C} \rightarrow \mathcal{C},$$

i.e. there is a bijection

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, [B, C]),$$

natural in A , B , and C , objects in \mathcal{C} . We refer to elements in $[A, B]$ as *internal morphisms*.

A category \mathcal{C} equipped with such a closed symmetric monoidal structure induces an enrichment of \mathcal{C} over itself, see [17, chapter VII, section 7]. The dual of a closed symmetric monoidal category is also closed.

In the rest of this section we discuss examples of closed symmetric monoidal categories which are important for the purposes of this thesis.

Example 1.1.7 (Modules). Let k be a commutative ring. We will provide the conventional data for the category Mod_k of modules over k as a closed symmetric monoidal category (see also e.g. [1, Proposition 2.14]). They are given by:

- the tensor product $\otimes = \otimes_k$ of modules over k ,
- the tensor unit k ,
- the associator

$$\begin{aligned} a = a_{X,Y,Z}: X \otimes (Y \otimes Z) &\rightarrow (X \otimes Y) \otimes Z \\ x \otimes (y \otimes z) &\mapsto (x \otimes y) \otimes z, \end{aligned}$$

- the isomorphisms

$$\begin{aligned} l = l_X: k \otimes X &\rightarrow X, & r = r_X: X \otimes k &\rightarrow X, \\ \alpha \otimes x &\mapsto \alpha x, & x \otimes \alpha &\mapsto \alpha x, \end{aligned}$$

- the symmetry isomorphism

$$\begin{aligned} s = s_{X,Y}: X \otimes Y &\rightarrow Y \otimes X \\ x \otimes y &\mapsto y \otimes x, \end{aligned}$$

- the inner Hom

$$[-, -]: \text{Mod}_k^{\text{op}} \times \text{Mod}_k \rightarrow \text{Mod}_k, \quad [X, Y] = \text{Hom}_k(X, Y),$$

which is given by the usual k -module structure on the set of morphisms between two modules.

Example 1.1.8 (Graded modules). A *graded module* X is a module which decomposes as a direct sum of modules indexed by the integers,

$$X = \bigoplus_{n \in \mathbb{Z}} X_n.$$

Elements x in a direct factor X_n are called *homogeneous elements* of degree n , denoted $|x| = n$.

A *graded map* $f: X \rightarrow Y$ is a homomorphism of underlying modules that respects the grading, i.e.

$$f(X_n) \subset Y_n \text{ for all } n \in \mathbb{Z}.$$

Equivalently, such a map is a sequence of module homomorphisms,

$$f = \{f_n\}_{n \in \mathbb{Z}}, \quad f_n: X_n \rightarrow Y_n.$$

Composition is defined by composition of homomorphisms.

We denote the resulting category by GrMod_k . Modules will be considered as graded modules concentrated in degree 0, i.e. a module X is a direct sum consisting of summands

$$X_n = \begin{cases} X, & n = 0; \\ 0, & n \neq 0; \end{cases}$$

for $n \in \mathbb{Z}$.

The category GrMod_k can be equipped with a closed symmetric monoidal structure, where

- the tensor product

$$\begin{aligned} \otimes: \text{GrMod}_k \times \text{GrMod}_k &\rightarrow \text{GrMod}_k, \\ X \otimes Y &= \bigoplus_{n \in \mathbb{Z}} (X \otimes Y)_n \\ &= \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i+j=n} X_i \otimes Y_j, \end{aligned}$$

- the tensor unit, the associator, l , and r are as in Mod_k ,
- the symmetry isomorphism

$$\begin{aligned} s = s_{X,Y}: \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i+j=n} X_i \otimes Y_j &\rightarrow \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i+j=n} Y_j \otimes X_i \\ x_i \otimes y_j &\mapsto (-1)^{ij} y_j \otimes x_i, \end{aligned}$$

using the *Koszul sign convention* when exchanging x_i and y_j . That is, when we interchange two symbols of degree i and j we multiply by $(-1)^{ij}$.

- The inner Hom

$$\begin{aligned} [-, -]: \text{GrMod}_k^{\text{op}} \times \text{GrMod}_k &\rightarrow \text{GrMod}_k, \\ [X, Y] &= \bigoplus_{n \in \mathbb{Z}} [X, Y]_n \\ &= \bigoplus_{n \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_k(X_j, Y_{j+n}). \end{aligned}$$

In other words, $[X, Y]_n$ consists of module homomorphisms $f: X \rightarrow Y$ satisfying

$$f(X_j) \subset Y_{j+n} \text{ for all } j \in \mathbb{Z}.$$

Equivalently, f is a sequence of module homomorphisms

$$f = \{f_j\}_{j \in \mathbb{Z}}, \quad f_j: X_j \rightarrow Y_{j+n}.$$

Notice that graded maps belong to $[X, Y]_0$. The composition of inner morphisms is defined as follows. For $f: X \rightarrow Y$ with $|f| = n$ and $g: Y \rightarrow Z$, the composite $gf: X \rightarrow Z$ is given by $(gf)_j = g_{j+n}f_j$.

Example 1.1.9 (Complexes). A *complex* (X, d^X) of k -modules is a graded module X equipped with a degree -1 map of graded modules

$$d^X = d \in [X, X]_{-1},$$

called the *differential*, such that $d^2 = 0$. We can depict a complex as

$$\cdots \xrightarrow{d} X_{n+1} \xrightarrow{d} X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \cdots$$

A *map of complexes* $f: (X, d^X) \rightarrow (Y, d^Y)$ is a graded map $f: X \rightarrow Y$ commuting with d in the sense that $f d^X = d^Y f$. That is, such that the following diagram commutes,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & X_{n+1} & \xrightarrow{d} & X_n & \xrightarrow{d} & X_{n-1} \xrightarrow{d} \cdots \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ \cdots & \xrightarrow{d} & Y_{n+1} & \xrightarrow{d} & Y_n & \xrightarrow{d} & Y_{n-1} \xrightarrow{d} \cdots \end{array}$$

Composition is defined as for graded modules. We denote the resulting category by Ch_k . See also [27, 1.1]. The differential will often be dropped from notation, writing just X instead of (X, d^X) .

Graded modules will be considered as complexes with trivial differential. Similarly, a graded map $f: X \rightarrow Y$ is a map of complexes $f: (X, 0) \rightarrow (Y, 0)$.

The category Ch_k can be equipped with a closed symmetric monoidal structure, where:

- the tensor product $\otimes: \text{Ch}_k \times \text{Ch}_k \rightarrow \text{Ch}_k$,

$$(X, d^X) \otimes (Y, d^Y) = (X \otimes Y, d^{X \otimes Y})$$

is as in GrMod_k in the first variable, equipped with the differential

$$d^{X \otimes Y} = d^X \otimes \mathbb{1}_Y + \mathbb{1}_X \otimes d^Y,$$

where $\mathbb{1}$ is the symbol we use for identity maps,

- the tensor unit and the structure isomorphisms are as in GrMod_k ,
- the inner Hom, $[(X, d^X), (Y, d^Y)] = ([X, Y], d^{[X, Y]})$, is again as in the category GrMod_k in the first variable, equipped with the differential

$$d^{[X, Y]}(f) = d^Y \circ f - (-1)^n f \circ d^X,$$

when evaluated in $f: X \rightarrow Y$ with $|f| = n$. Note that maps of complexes are 0-cycles in this inner Hom.

We finally consider two not-so-used examples of closed symmetric monoidal categories, bigraded modules and graded complexes, which will be very relevant in this thesis. See e.g. also [15, section 2.1].

Example 1.1.10 (Bigraded modules). An *bigraded module* X is a module which decomposes as a direct sum of modules indexed by $\mathbb{Z} \times \mathbb{Z}$,

$$X = \bigoplus_{i, j \in \mathbb{Z}} X_{ij}.$$

Elements x in any factor X_{ij} of the decomposition are called *homogeneous elements of bidegree* (i, j) , denoted $|x| = (i, j)$.

An bigraded map $f: X \rightarrow Y$ is a homomorphism of underlying modules that respects the bigrading, i.e.

$$f(X_{ij}) \subset Y_{ij} \text{ for all } i, j.$$

Equivalently, such a map is a sequence of module homomorphisms,

$$f = \{f_{ij}\}_{i, j \in \mathbb{Z}}, \quad f_{ij}: X_{ij} \rightarrow Y_{ij}.$$

Composition is defined by composition of homomorphisms. We denote the resulting category by BiGrMod_k .

For X_{ij} a direct summand of X , the i -grading will be referred to as the *horizontal degree* and the j -grading as the *vertical degree*. The sum of the horizontal and vertical degrees will be referred to as the *total degree*. For an $x \in X_{ij}$ we use the notation $\|x\| = i + j$. Graded modules will be considered as bigraded modules concentrated in horizontal degree 0, i.e. a graded module X is a direct sum consisting of summands

$$X_{ij} = \begin{cases} X_j, & i = 0; \\ 0, & i \neq 0; \end{cases}$$

for $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. In particular, modules can be considered as bigraded modules concentrated in bidegree $(0, 0)$.

We can equip the category BiGrMod_k with a closed symmetric monoidal structure where,

- the tensor product

$$\otimes: \text{BiGrMod}_k \times \text{BiGrMod}_k \rightarrow \text{BiGrMod}_k,$$

$$\begin{aligned} X \otimes Y &= \bigoplus_{i,j \in \mathbb{Z}} (X \otimes Y)_{ij} \\ &= \bigoplus_{i,j \in \mathbb{Z}} \bigoplus_{\substack{u+k=i \\ v+l=j}} X_{uv} \otimes Y_{kl}, \end{aligned}$$

- the tensor unit, the associator, l and r as in Mod_k ,
- the symmetry isomorphism

$$\begin{aligned} s = s_{X,Y}: \bigoplus_{\substack{i,j \in \mathbb{Z} \\ u+k=i \\ v+l=j}} X_{uv} \otimes Y_{kl} &\rightarrow \bigoplus_{\substack{i,j \in \mathbb{Z} \\ u+k=i \\ v+l=j}} Y_{kl} \otimes X_{uv} \\ x_{uv} \otimes y_{kl} &\mapsto (-1)^{(u+v)(k+l)} y_{kl} \otimes x_{uv}, \end{aligned}$$

using the Koszul sign convention with respect to the total degree,

- the inner Hom

$$[-, -]: \text{BiGrMod}_k^{\text{op}} \times \text{BiGrMod}_k \rightarrow \text{BiGrMod}_k,$$

$$\begin{aligned} [X, Y] &= \bigoplus_{i,j \in \mathbb{Z}} [X, Y]_{ij} \\ &= \bigoplus_{i,j \in \mathbb{Z}} \prod_{u,v \in \mathbb{Z}} \text{Hom}_k(X_{uv}, Y_{u+i, v+j}). \end{aligned}$$

In other words, $[X, Y]_{ij}$ consists of module homomorphisms $f: X \rightarrow Y$ satisfying

$$f(X_{uv}) \subset Y_{u+i, v+j} \text{ for all } u, v \in \mathbb{Z}.$$

Equivalently, such a map is a sequence of module homomorphisms,

$$f = \{f_{uv}\}_{u, v \in \mathbb{Z}}, \quad f_{uv}: X_{u, v} \rightarrow Y_{u+i, v+j}.$$

Notice that bigraded maps belong to $[X, Y]_{00}$. For $f: X \rightarrow Y$ with $|f| = (i, j)$ and $g: Y \rightarrow Z$, composition $gf: X \rightarrow Z$ is given by $(gf)_{uv} = g_{u+i, v+j}f_{uv}$.

Example 1.1.11 (Graded complexes). A *graded complex* (X, d^X) is a bigraded module X together with a map of bidegree $(0, -1)$

$$d^X = d \in [X, X]_{0, -1},$$

called the *vertical differential*, such that $d^2 = 0$, i.e. $d_{i, j-1}d_{i, j} = 0$ for all i, j . Note that the vertical differential keeps the horizontal degree constant and lowers the vertical degree by 1, hence its name. Actually, a graded complex can be depicted as

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 \cdots & & X_{0,1} & \cdots & X_{i-1,1} & & X_{i,1} & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 \cdots & & X_{0,0} & \cdots & X_{i-1,0} & & X_{i,0} & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 \cdots & & X_{0,-1} & \cdots & X_{i-1,-1} & & X_{i,-1} & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

A *map of graded complexes* $f: (X, d^X) \rightarrow (Y, d^Y)$ is a bigraded map $f: X \rightarrow Y$ commuting with the vertical differential $f d^X = d^Y f$. Composition is defined as for bigraded modules. We denote the resulting category by GrCh_k . Quite often, we drop the differential from notation and simply write X for (X, d^X) .

Complexes will be considered as graded complexes concentrated in horizontal degree 0, i.e. in the vertical axis. Also, bigraded modules will be considered as graded complexes with trivial differential, and an bigraded map $f: X \rightarrow Y$ is a map of graded complexes $f: (X, 0) \rightarrow (Y, 0)$.

The category GrCh_k can be equipped with a closed symmetric monoidal structure, where:

- The tensor product $\otimes: \text{GrCh}_k \times \text{GrCh}_k \rightarrow \text{GrCh}_k$ is as in BiGrMod_k in the first coordinate, $(X, d^X) \otimes (Y, d^Y) = (X \otimes Y, d^{X \otimes Y})$, equipped with the differential

$$d^{X \otimes Y} = d^X \otimes \mathbf{1}_Y + \mathbf{1}_X \otimes d^Y.$$

- The tensor unit and the structure isomorphisms are as in BiGrMod_k .
- The inner Hom is once again as in BiGrMod_k in the first coordinate, $[(X, d^X), (Y, d^Y)] = ([X, Y], d^{[X, Y]})$, equipped with the differential defined as follows on $f: X \rightarrow Y$,

$$d^{[X, Y]}(f) = d^Y \circ f - (-1)^{\|f\|} f \circ d^X.$$

Maps of graded complexes are cycles of bidegree $(0, 0)$ here.

Example 1.1.12 (Totalization of graded complexes). The *total complex* of a graded complex (X, d^X) is the plain complex $(\text{Tot}(X), d^{\text{Tot}(X)})$ with

$$\text{Tot}(X)_n = \bigoplus_{i+j=n} X_{i,j}$$

and

$$d^{\text{Tot}(X)} = \bigoplus_{i+j=n} d_{i,j}^X: \bigoplus_{i+j=n} X_{i,j} \rightarrow \bigoplus_{i+j=n} X_{i,j-1}.$$

Up to a naive degree shift, this is the plain direct sum of the vertical complexes in the picture of Example 1.1.11. It is also a particular case of the well-known totalization of bicomplexes. This construction clearly defines a strong monoidal functor

$$\text{Tot}: \text{GrCh}_k \rightarrow \text{Ch}_k$$

between these two closed symmetric monoidal categories introduced above.

1.2 Monoids and distributive laws

Distributive laws, usually considered for monads, serve to define a monad structure on the composition of two monads [25]. They have also been studied in the operadic context, see [16, 8.6]. We here consider distributive laws for monoids in a possibly nonsymmetric monoidal category $(\mathcal{C}, \otimes, 1)$. They are used to endow the tensor product of two monoids with a monoid structure. We characterize such a tensor product monoid in terms of a universal property for which we have not found a reference in the literature, although it may be known to experts.

Definition 1.2.1. A *monoid* (A, μ_A, η_A) is an object $A \in \mathcal{C}$ together with maps:

- $\mu_A: A \otimes A \rightarrow A$, called the *multiplication*, and
- $\eta_A: 1 \rightarrow A$, called the *unit*,

which satisfy the associativity and unitary conditions corresponding with the commutative diagrams given in [17, chapter VII, section 3]. We often drop the structure maps from notation and simply write A for the monoid (A, μ_A, η_A) .

A *morphism of monoids* $f: (A, \mu_A, \eta_A) \rightarrow (B, \mu_B, \eta_B)$ is a map $f: A \rightarrow B$ that is compatible with the multiplication and unit. We denote the resulting category by $\text{Mon}(\mathcal{C})$.

The following proposition is completely obvious.

Proposition 1.2.2. *The image $F(A)$ of a monoid (A, μ_A, η_A) in \mathcal{C} under a lax monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoid in \mathcal{D} with multiplication*

$$F(A) \otimes_{\mathcal{D}} F(A) \xrightarrow{\phi_{A,A}} F(A \otimes_{\mathcal{C}} A) \xrightarrow{F(\mu_A)} F(A)$$

and unit

$$1_{\mathcal{D}} \xrightarrow{\Psi} F(1_{\mathcal{C}}) \xrightarrow{F(\eta_A)} F(A).$$

The tensor unit 1 , equipped with the identity map $\eta_1: 1 = 1$ and with the natural isomorphism $\mu_1: 1 \otimes 1 \cong 1$ which is part of the monoidal structure, is the initial monoid. For any monoid (A, μ_A, η_A) , the unit η_A defines the unique monoid morphism $(1, \mu_1, \eta_1) \rightarrow (A, \mu_A, \eta_A)$.

Definition 1.2.3. A monoid (A, μ_A, η_A) is *augmented* if it is equipped with a monoid morphism $\epsilon_A: (A, \mu_A, \eta_A) \rightarrow (1, \mu_1, \eta_1)$, called *augmentation*. A morphism of augmented monoids $(A, \mu_A, \eta_A, \epsilon_A) \rightarrow (B, \mu_B, \eta_B, \epsilon_B)$ is a morphism between the underlying monoids which is compatible with the augmentations $\epsilon_B f = \epsilon_A$. The category of augmented monoids will be denoted by $\text{AugMon}(\mathcal{C})$.

The augmentation is necessarily a retraction of the unit $\epsilon_A \eta_A = \mathbb{1}_A$. Proposition 1.2.2 has an obvious extension to the augmented setting.

Definition 1.2.4. A *distributive law* consists of two monoids, (A, μ_A, η_A) and $(B, \mu_B, \eta_B) \in \text{Mon}(\mathcal{C})$, and a map

$$\varphi : B \otimes A \rightarrow A \otimes B$$

in \mathcal{C} such that the following diagrams commute:

$$1. \quad \begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{\varphi \otimes \mathbb{1}_A} & A \otimes B \otimes A & \xrightarrow{\mathbb{1}_A \otimes \varphi} & A \otimes A \otimes B \\ \mathbb{1}_B \otimes \mu_A \downarrow & & & & \downarrow \mu_A \otimes \mathbb{1}_B \\ B \otimes A & \xrightarrow{\varphi} & A \otimes B, & & \end{array}$$

$$2. \quad \begin{array}{ccc} B \otimes B \otimes A & \xrightarrow{\mathbb{1}_B \otimes \varphi} & B \otimes A \otimes B & \xrightarrow{\varphi \otimes \mathbb{1}_B} & A \otimes B \otimes B \\ \mu_B \otimes \mathbb{1}_A \downarrow & & & & \downarrow \mathbb{1}_A \otimes \mu_B \\ B \otimes A & \xrightarrow{\varphi} & A \otimes B, & & \end{array}$$

$$3. \quad \begin{array}{ccc} B \otimes 1 & \xrightarrow{r_B} & B & \xrightarrow{l_B^{-1}} & 1 \otimes B \\ \mathbb{1}_B \otimes \eta_A \downarrow & & & & \eta_A \otimes \mathbb{1}_B \downarrow \\ B \otimes A & \xrightarrow{\varphi} & A \otimes B, & & \end{array}$$

$$4. \quad \begin{array}{ccc} 1 \otimes A & \xrightarrow{l_A} & A & \xrightarrow{r_A^{-1}} & A \otimes 1 \\ \eta_B \otimes \mathbb{1}_A \downarrow & & & & \mathbb{1}_A \otimes \eta_B \downarrow \\ B \otimes A & \xrightarrow{\varphi} & A \otimes B. & & \end{array}$$

If A and B are augmented, then φ is augmented provided the following extra diagram commutes:

$$5. \quad \begin{array}{ccc} B \otimes A & \xrightarrow{\varphi} & A \otimes B \\ \epsilon_B \otimes \epsilon_A \searrow & & \swarrow \epsilon_A \otimes \epsilon_B \\ & \mathbb{1} \otimes \mathbb{1}. & \end{array}$$

Proposition 1.2.5. *If $\varphi : B \otimes A \rightarrow A \otimes B$ is a distributive law, then $A \otimes B$ is a monoid for the multiplication*

$$\mu_\varphi = (\mu_A \otimes \mu_B)(\mathbb{1}_A \otimes \varphi \otimes \mathbb{1}_B) : (A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B$$

and for the unit

$$\eta_\varphi = \eta_A \otimes \eta_B : I \cong I \otimes I \rightarrow A \otimes B.$$

Moreover, if A , B , and φ are augmented, then so is this monoid structure on $A \otimes B$ with augmentation

$$\epsilon_\varphi : A \otimes B \xrightarrow{\epsilon_A \otimes \epsilon_B} 1 \otimes 1 \cong 1.$$

The proof is given in [16, Proposition 8.6.1] in the operadic context, except for the augmentation part. The argument for the general case is the same, and the augmentation part is almost obvious. We will denote the, possibly augmented, monoid $(A \otimes B, \mu_\varphi, \eta_\varphi)$ by

$$A \otimes_\varphi B.$$

This monoid (without the augmentation) is characterized by the universal property in Theorem 1.2.6 below.

We define $f : A \rightarrow A \otimes_\varphi B$ as the composition

$$A \cong A \otimes I \xrightarrow{\mathbb{1}_A \otimes \eta_B} A \otimes_\varphi B.$$

The map f is a morphism of monoids. Indeed, the following commutative diagram proves that f is compatible with the multiplications:

$$\begin{array}{ccc}
A \otimes I \otimes A \otimes I & \xrightarrow{\mathbb{1}_A \otimes \eta_B \otimes \mathbb{1}_A \otimes \eta_B} & A \otimes B \otimes A \otimes B \\
\mathbb{1}_A \otimes l_A \otimes \mathbb{1}_I \downarrow & & \downarrow \mathbb{1}_A \otimes \varphi \otimes \mathbb{1}_B \\
A \otimes A \otimes I & \text{commutes since } \varphi \text{ is a distributive law} & \\
\mathbb{1}_A \otimes r_A^{-1} \otimes \mathbb{1}_I \downarrow & & \\
A \otimes A \otimes I \otimes I & \xrightarrow{\mathbb{1}_A \otimes \mathbb{1}_A \otimes \eta_B \otimes \eta_B} & A \otimes A \otimes B \otimes B \\
\mu_A \otimes \mu_I \downarrow & & \downarrow \mu_A \otimes \mu_B \\
A \otimes I & \xrightarrow{\mathbb{1}_A \otimes \eta_B} & A \otimes B.
\end{array}$$

Moreover, f preserves units since the following triangle commutes,

$$\begin{array}{ccc}
& I \otimes I & \\
\eta_A \otimes \mathbb{1}_I \swarrow & & \searrow \eta_A \otimes \eta_B \\
A \otimes I & \xrightarrow{\mathbb{1}_A \otimes \eta_B} & A \otimes_\varphi B.
\end{array}$$

We define $g: B \rightarrow A \otimes_{\varphi} B$ as the composition

$$B \cong I \otimes B \xrightarrow{\eta_A \otimes \mathbf{1}_B} A \otimes_{\varphi} B.$$

By similar arguments, it follows that g is a morphism of monoids.

Theorem 1.2.6. *Given two morphisms of monoids $\xi: A \rightarrow C$ and $\psi: B \rightarrow C$ such that the following pentagon commutes,*

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\xi \otimes \psi} & C \otimes C \\ \uparrow \varphi & & \downarrow \mu_C \\ B \otimes A & & C \\ & \searrow \psi \otimes \xi & \nearrow \mu_C \\ & C \otimes C & \end{array}$$

there exists a unique morphism of monoids $\zeta: A \otimes_{\varphi} B \rightarrow C$ fitting into the following two commutative triangles,

$$\begin{array}{ccc} A & \xrightarrow{\xi} & C \\ \searrow f & & \nearrow \zeta \\ & A \otimes_{\varphi} B & \\ \nearrow g & & \searrow \psi \\ B & & C \end{array}$$

Conversely, given a morphism of monoids $\zeta: A \otimes_{\varphi} B \rightarrow C$, the morphisms $\xi := \zeta f$ and $\psi := \zeta g$ fit into the previous commutative pentagon.

Proof. Given morphisms of monoids $\xi: A \rightarrow C$ and $\psi: B \rightarrow C$, we define

$$\zeta := \mu_C(\xi \otimes \psi).$$

The second diagram in the statement commutes, since

$$\begin{aligned} \zeta f &= \mu_C(\xi \otimes \psi)(\mathbf{1}_A \otimes \eta_B) \\ &= \mu_C(\xi \otimes \psi \eta_B) \\ &= \mu_C(\xi \otimes \eta_C) \\ &= \mu_C(\mathbf{1}_C \otimes \eta_C) \xi \\ &= \xi, \end{aligned}$$

and

$$\begin{aligned}
\zeta g &= \mu_C(\xi \otimes \psi)(\eta_A \otimes \mathbf{1}_B) \\
&= \mu_C(\xi \eta_A \otimes \psi) \\
&= \mu_C(\eta_C \otimes \psi) \\
&= \mu_C(\eta_C \otimes \mathbf{1}_C)\psi \\
&= \psi.
\end{aligned}$$

Moreover, ζ preserves units since

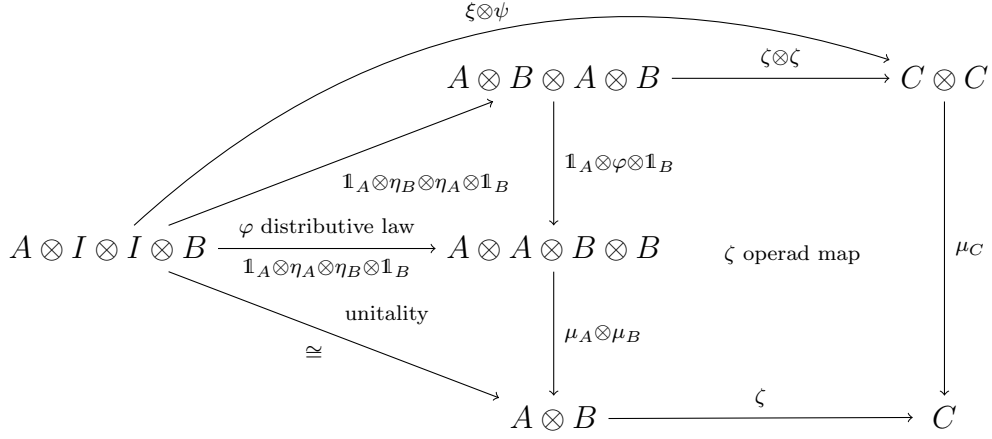
$$\begin{aligned}
\zeta \eta_{A \otimes_\varphi B} &= \zeta(\eta_A \otimes \eta_B) \\
&= \mu_C(\xi \otimes \psi)(\eta_A \otimes \eta_B) \\
&= \mu_C(\xi \eta_A \otimes \psi \eta_B) \\
&= \mu_C(\eta_C \otimes \eta_C) \\
&= \eta_C.
\end{aligned}$$

Furthermore, ζ is compatible with the multiplications since the following diagram commutes,

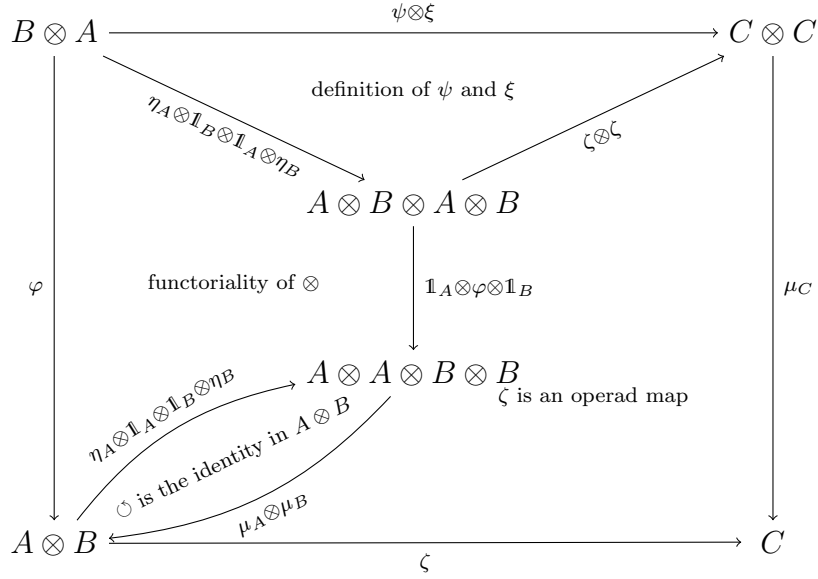
$$\begin{array}{ccccc}
A \otimes B \otimes A \otimes B & \xrightarrow{\mathbf{1}_A \otimes \psi \otimes \xi \otimes \mathbf{1}_B} & A \otimes C \otimes C \otimes B & \xrightarrow{\xi \otimes \mathbf{1}_C \otimes \mathbf{1}_C \otimes \psi} & C \otimes C \otimes C \otimes C & \xrightarrow{\mu_C \otimes \mu_C} & C \otimes C \\
\downarrow \mathbf{1}_A \otimes \varphi \otimes \mathbf{1}_B & & \downarrow \mathbf{1}_A \otimes \mu_C \otimes \mathbf{1}_B & \text{functoriality of } \otimes & \downarrow \mathbf{1}_C \otimes \mu_C \otimes \mathbf{1}_C & & \downarrow \mu_C \\
A \otimes A \otimes B \otimes B & \xrightarrow{\mathbf{1}_A \otimes \xi \otimes \psi \otimes \mathbf{1}_B} & A \otimes C \otimes C \otimes B & \xrightarrow{\xi \otimes \mathbf{1}_C \otimes \psi} & C \otimes C \otimes C & \xrightarrow{\mu_C(\mathbf{1}_C \otimes \mu_C)} & C \\
\downarrow \mu_A \otimes \mu_B & & \uparrow \mathbf{1}_A \otimes \mu_C \otimes \mathbf{1}_B & \text{functoriality of } \otimes & \uparrow \mathbf{1}_C \otimes \mu_C \otimes \mathbf{1}_C & \text{associativity of } \mu_C & \\
A \otimes B & \xrightarrow{\xi \otimes \psi} & C \otimes C & \xrightarrow{\mu_C} & C & & C \\
& & \text{functoriality of } \otimes \text{ and operad morphisms } \xi \text{ and } \psi & & \text{associativity of } \mu_C & &
\end{array}$$

Uniqueness is a consequence of the following diagram, where ζ is an op-

erad morphism fitting into the second commutative diagram of the statement,



Moreover, the converse is a consequence of the commutativity of the following diagram



This diagram yields the commutative pentagon since, as we have just checked, ζ is necessarily $\mu_C(\xi \otimes \psi)$. \square

1.3 Sequences and collections

We here review some nonsymmetric monoidal categories which are relevant for operads. For the sake of simplicity, we work over the base of the closed

symmetric monoidal category of graded complexes introduced in Example 1.1.11 above. Nevertheless, almost everything generalizes to a (co)complete closed symmetric monoidal base category.

Definition 1.3.1. A *sequence* in GrCh_k is a family

$$X = (X(0), X(1), \dots, X(n), \dots)$$

of graded complexes $X(n)$, $n \geq 0$.

A *morphism of sequences* $f: X \rightarrow Y$ is a family of maps

$$f(n): X(n) \rightarrow Y(n)$$

of graded complexes, $n \geq 0$.

We denote the resulting category by Seq_k . For $x \in X(n)$ we say that x is an *operation of arity n* . When $X(0) = 0$ we say that the sequence is *reduced*.

The category Seq_k is equipped with a monoidal structure, where:

- The tensor product is the *circle product*,

$$\circ: \text{Seq}_k \times \text{Seq}_k \rightarrow \text{Seq}_k,$$

$$(X \circ Y)(n) = \bigoplus_{\substack{k \geq 0, \\ i_1 + \dots + i_k = n}} X(k) \otimes Y(i_1) \otimes \dots \otimes Y(i_k).$$

- The tensor unit is $I = (0, k, 0, \dots)$, and the associated structure isomorphisms r and l are

$$\begin{aligned} (X \circ I)(n) &= X(n) \otimes I(1) \otimes \dots \otimes I(1) \\ &= X(n) \otimes k \otimes \dots \otimes k \\ &\cong X(n), \end{aligned}$$

and

$$\begin{aligned} (I \circ X)(n) &= I(1) \otimes X(n) \\ &= k \otimes X(n) \\ &\cong X(n). \end{aligned}$$

- The associator for \circ uses the symmetry isomorphism for the tensor

product of graded complexes \otimes (in addition to the associator) since

$$\begin{aligned}
& ((X \circ Y) \circ Z)(n) \\
&= \bigoplus_{\substack{k \geq 0 \\ i_1, \dots, i_k \geq 0 \\ \sum_i j_i = n}} (X(k) \otimes Y(i_1) \otimes \cdots \otimes Y(i_k)) \otimes Z(j_1) \otimes \cdots \otimes Z(j_{i_1+\dots+i_k}), \\
& (X \circ (Y \circ Z))(n) \\
&= \bigoplus_{\substack{k \geq 0 \\ i_1, \dots, i_k \geq 0 \\ \sum_i j_i = n}} X(k) \otimes (Y(i_1) \otimes Z(j_1) \otimes \cdots \otimes Z(j_{i_1})) \otimes \cdots \\
& \quad \cdots \otimes (Y(i_k) \otimes Z(j_{i_1+\dots+i_{k-1}+1}) \otimes \cdots \otimes Z(j_{i_1+\dots+i_k})).
\end{aligned}$$

It is simply given by rearranging tensor factors, but note that this involves signs since we are using the Koszul sign rule with respect to the total degree,

$$\begin{aligned}
& (x \otimes y_1 \otimes \cdots \otimes y_k) \otimes z_1 \otimes \cdots \otimes z_{i_1+\dots+i_k} \\
& \mapsto (-1)^\varepsilon x \otimes (y_1 \otimes z_1 \otimes \cdots \otimes z_{i_1}) \otimes \cdots \\
& \quad \cdots \otimes (y_k \otimes z_{i_1+\dots+i_{k-1}+1} \otimes \cdots \otimes z_{i_1+\dots+i_k}),
\end{aligned}$$

where

$$\varepsilon = \sum_{j=1}^{k-1} \left(\|y_{j+1}\| \sum_{l=1}^{i_1+\dots+i_j} \|z_l\| \right).$$

This monoidal category admits an extension where the elements of the sequence carry symmetric group actions, see also [16, 5.1].

Definition 1.3.2. A *collection* X is a sequence such that each $X(n)$, $n \geq 0$, is equipped with a right action of the symmetric group Σ_n , i.e. the group of automorphisms of the set $\{1, 2, \dots, n\}$.

A *morphism of collections* $f: X \rightarrow Y$ is an arity-wise equivariant morphism of sequences. We denote the resulting category by Coll_k .

The category Coll_k can be equipped with the following monoidal structure.

- The tensor product is the *symmetric circle product*

$$\circ_\Sigma: \text{Coll}_k \times \text{Coll}_k \rightarrow \text{Coll}_k,$$

where $(X \circ_\Sigma Y)(n)$ is defined by

$$\bigoplus_{k \geq 0} X(k) \otimes_{\Sigma_k} \left(\bigoplus_{i_1+\dots+i_k=n} (Y(i_1) \otimes \cdots \otimes Y(i_k)) \otimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} k[\Sigma_n] \right).$$

Here, given a group G , we use the tensor product \otimes_G of a left G -module and a right G -module, extended to graded complexes, and $k[G]$ denotes the group ring. The right Σ_k -module structure on $X(k)$ is the given action. The right $(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})$ -module structure on $Y(i_1) \otimes \cdots \otimes Y(i_k)$ is the tensor product of the right actions of Σ_{i_j} on $Y(i_j)$, $1 \leq j \leq k$. The left $(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})$ -module structure on $k[\Sigma_n]$ is given by the inclusion of the Young subgroup $\Sigma_{i_1} \times \cdots \times \Sigma_{i_k} \subset \Sigma_n$ associated to the following partition of $\{1, \dots, n\}$,

$$\{1, \dots, n\} = \coprod_{j=1}^k \{i_1 + \cdots + i_{j-1} + 1, \dots, i_1 + \cdots + i_j\}.$$

The left action of Σ_k on the big tensor factor on the right is given by

$$\tau \cdot (y_1 \otimes \cdots \otimes y_k \otimes \sigma) = y_{\tau^{-1}(1)} \otimes \cdots \otimes y_{\tau^{-1}(k)} \otimes \tau_{i_1, \dots, i_k} \sigma.$$

Here $\tau \in \Sigma_k$, and $\tau_{i_1, \dots, i_k} \in \Sigma_n$ is the block permutation which permutes the k blocks of the previous partition of $\{1, \dots, n\}$ according to τ . It is straightforward to check that this action is well defined, using the associativity and the symmetry of the tensor product of graded complexes. Finally, the right action of Σ_n on $(X \circ_{\Sigma} Y)(n)$ is simply given by

$$(x \otimes y_1 \otimes \cdots \otimes y_k \otimes \sigma) \cdot \nu = x \otimes y_1 \otimes \cdots \otimes y_k \otimes \sigma \nu,$$

for $\nu \in \Sigma_n$.

- The tensor unit is $I = (0, k, 0, \dots)$ and the associated structure isomorphisms r and l are

$$\begin{aligned} (X \circ_{\Sigma} I)(n) &= X(n) \otimes_{\Sigma_n} ((k \otimes \cdots \otimes k) \otimes_{\Sigma_1 \times \cdots \times \Sigma_1} k[\Sigma_n]) \\ &\cong X(n) \otimes_{\Sigma_n} k[\Sigma_n] \\ &\cong X(n), \end{aligned}$$

and

$$\begin{aligned} (I \circ_{\Sigma} X)(n) &= k \otimes_{\Sigma_1} (X(n) \otimes_{\Sigma_n} k[\Sigma_n]) \\ &\cong k \otimes X(n) \\ &\cong X(n). \end{aligned}$$

Moreover, the associator is defined essentially as in the nonsymmetric case, see Definition 1.3.1.

Remark 1.3.3. Let G be a group, X a right G -module, and Y a left G -module. We can endow Y with the right G -module action given by $y \cdot g = g^{-1} \cdot y$. The tensor product $X \otimes_G Y$ obviously coincides with the module of coinvariants $(X \otimes Y)_G$ of the diagonal right action on the tensor product over the ground commutative ring k .

Using this elementary fact, we can express $(X \circ_\Sigma Y)(n)$ as the direct sum of coinvariants

$$\bigoplus_{k \geq 0} \left(X(k) \otimes \left(\bigoplus_{i_1 + \dots + i_k = n} (Y(i_1) \otimes \dots \otimes Y(i_k)) \otimes_{\Sigma_{i_1} \times \dots \times \Sigma_{i_k}} k[\Sigma_n] \right) \right)_{\Sigma_k}$$

of the following action $\tau \in \Sigma_k$,

$$(x \otimes y_1 \otimes \dots \otimes y_k \otimes \sigma) \cdot \tau = x \cdot \tau \otimes y_{\tau(1)} \otimes \dots \otimes y_{\tau(k)} \otimes \tau_{i_1, \dots, i_k}^{-1} \sigma.$$

We will use this in order to compare the previous monoidal structure on collections to the second one below.

Remark 1.3.4. Given two collections X and Y , there is a natural transformation

$$X \circ Y \rightarrow X \circ_\Sigma Y$$

induced by the unit of the group rings $k[\Sigma_n]$ and the projection onto the coinvariants. This, together with the identity map in I , endows the forgetful functor from collections to sequences

$$(\text{Coll}_k, \circ_\Sigma, I) \rightarrow (\text{Seq}_k, \circ, I)$$

with a lax monoidal structure.

The forgetful functor has a left adjoint, called *symmetrization*:

$$\begin{aligned} - \otimes k[\Sigma]: \text{Seq}_k &\rightarrow \text{Coll}_k \\ X &\mapsto X \otimes k[\Sigma], \end{aligned}$$

where

$$(X \otimes k[\Sigma])(n) := X(n) \otimes k[\Sigma_n]$$

has the obvious action of Σ_n . Symmetrization becomes a strong monoidal functor

$$(\text{Seq}_k, \circ, I) \rightarrow (\text{Coll}_k, \circ_\Sigma, I)$$

when endowed with the identity map $\Psi: I = I \otimes k[\Sigma]$ and with the natural isomorphism

$$\phi = \phi_{X,Y}: (X \otimes k[\Sigma]) \circ_\Sigma (Y \otimes k[\Sigma]) \rightarrow (X \circ Y) \otimes k[\Sigma],$$

which in arity n is defined in the obvious way, using the symmetry isomorphism in the category of graded complexes, the isomorphism

$$k[\Sigma_{i_1}] \otimes \cdots \otimes k[\Sigma_{i_k}] \otimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} k[\Sigma_n] \cong k[\Sigma_n],$$

and $k[\Sigma_k] \otimes_{\Sigma_k} M \cong M$, where M is an arbitrary left Σ_k -module.

Remark 1.3.5. Given a group G and a right G -module M , the invariants are contained in the module $M^G \subset M$ and the module projects onto the coinvariants $M \rightarrow M_G$, so we obtain a natural map $M^G \rightarrow M_G$. If G is finite, we have the *norm map* in the opposite direction,

$$\begin{aligned} M_G &\rightarrow M^G, \\ [x] &\mapsto \sum_{g \in G} x \cdot g, \end{aligned} \tag{1.3.6}$$

going from coinvariants to invariants. The two composites

$$M^G \rightarrow M_G \rightarrow M^G, \quad M_G \rightarrow M^G \rightarrow M_G,$$

are given by multiplication by $|G|$, the number of elements of the group. In particular the two maps are isomorphisms if $|G|$ is invertible in k . The norm map is also an isomorphism in other cases, e.g. if $M = N \otimes k[G]$ where N is a G -module and M carries the diagonal action.

Unlike in the non-symmetric case, we need a different monoidal structure on collections in order to deal with symmetric cooperads. See also [16, 5.1.15].

Definition 1.3.7. We define a second monoidal structure on Coll_k as follows:

- The tensor product is $\bar{\circ}_\Sigma: \text{Coll}_k \times \text{Coll}_k \rightarrow \text{Coll}_k$, where $(X \bar{\circ}_\Sigma Y)(n)$ is defined as

$$\bigoplus_{k \geq 0} \left(X(k) \otimes \left(\bigoplus_{i_1 + \cdots + i_k = n} Y(i_1) \otimes \cdots \otimes Y(i_k) \right) \otimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} k[\Sigma_n] \right)^{\Sigma_k},$$

the module of invariants with respect to the right action of Σ_k defined in Remark 1.3.3. The right action of Σ_n on $(X \bar{\circ}_\Sigma Y)(n)$ is given by

$$(x \otimes y_1 \otimes \cdots \otimes y_k \otimes \sigma) \cdot \tau = x \otimes y_1 \otimes \cdots \otimes y_k \otimes \sigma \tau,$$

for $\tau \in \Sigma_n$. It is well-defined since the right actions of Σ_k and Σ_n commute.

- The tensor unit is $I = (0, k, 0, \dots)$, as above, and the structure isomorphisms l are r are given by $I \bar{\circ}_\Sigma X = I \circ_\Sigma X = I \circ X \cong X$ and

$$\begin{aligned}
(X \bar{\circ}_\Sigma I)(n) &= (X(n) \otimes ((k \otimes \cdots \otimes k) \otimes_{\Sigma_1 \times \cdots \times \Sigma_1} k[\Sigma_n]))^{\Sigma_n} \\
&\cong (X(n) \otimes k[\Sigma_n])^{\Sigma_n} \\
&\cong (X(n) \otimes k[\Sigma_n])_{\Sigma_n} \\
&= X(n) \otimes_{\Sigma_n} k[\Sigma_n] \\
&\cong X(n).
\end{aligned}$$

Here, in the third line, we use the inverse of the norm map in Remark 1.3.5, which is an isomorphism in this case. Moreover, the associator is again essentially as in the nonsymmetric case, compare Definition 1.3.1.

Remark 1.3.8. The two monoidal structures on the category of collections, $(\text{Coll}_k, \circ_\Sigma, I)$ and $(\text{Coll}_k, \bar{\circ}_\Sigma, I)$, can be compared in the following ways. Given two collections X and Y , there are two natural transformations,

$$X \bar{\circ}_\Sigma Y \rightarrow X \circ_\Sigma Y, \quad X \circ_\Sigma Y \rightarrow X \bar{\circ}_\Sigma Y,$$

given by the maps between invariants and coinvariants in Remark 1.3.5.

The second one together with the identity in I enhances the identity functor in Coll_k to two monoidal functors,

$$(\text{Coll}_k, \circ_\Sigma, I) \rightarrow (\text{Coll}_k, \bar{\circ}_\Sigma, I), \quad (\text{Coll}_k, \bar{\circ}_\Sigma, I) \rightarrow (\text{Coll}_k, \circ_\Sigma, I),$$

which are colax and lax, respectively. Moreover, they are both strong if $|G| \in k$ is invertible. Actually, $\text{Id}_{\text{Coll}_k} : (\text{Coll}_k, \circ_\Sigma, I) \rightarrow (\text{Coll}_k, \bar{\circ}_\Sigma, I)$ is close to being strong even if $|G| \in k$ is not invertible. More precisely, if Y is reduced then $X \circ_\Sigma Y \rightarrow X \bar{\circ}_\Sigma Y$ is an isomorphism (compare e.g. [24, 9.1, 9.6, 9.7, and 9.10]).

1.4 Operads and cooperads

We now introduce (non)symmetric (co)operads as monoids in the nonsymmetric monoidal categories introduced in the previous section.

Definition 1.4.1. A *nonsymmetric operad* $\mathcal{O} = (\mathcal{O}, \mu_{\mathcal{O}}, \eta_{\mathcal{O}})$, abbreviated *ns-operad*, is a monoid in the monoidal category (Seq_k, \circ, I) of Definition 1.3.1. A *morphism of ns-operads* $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of monoids. We denote by $\text{id}_{\mathcal{O}}$ the image of $1 \in k$ in arity 1 by $\eta_{\mathcal{O}}$ and call it the *identity*

operation. An *augmented ns-operad* $\mathcal{O} = (\mathcal{O}, \mu_{\mathcal{O}}, \eta_{\mathcal{O}}, \epsilon_{\mathcal{O}})$ is an augmented monoid in the sense of Definition 1.2.3. Morphisms between them are augmented monoid morphisms. We denote the resulting categories by **ns-Op** and **ns-AugOp** respectively.

If \mathcal{O} is augmented, denote $\bar{\mathcal{O}} = \text{Ker}(\epsilon_{\mathcal{O}})$, where Ker is the kernel in the category of sequences (of graded complexes). Since $\epsilon_{\mathcal{O}}\eta_{\mathcal{O}} = \mathbf{1}_I$, it follows that when an ns-operad \mathcal{O} is augmented it has a canonical decomposition as a direct sum

$$\mathcal{O}(n) = (I \oplus \bar{\mathcal{O}})(n) = \begin{cases} k \oplus \bar{\mathcal{O}}(1), & n = 1; \\ \bar{\mathcal{O}}(n), & n \neq 1. \end{cases} \quad (1.4.2)$$

Remark 1.4.3. The multiplication $\mu_{\mathcal{O}}: \mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$ consists of a series of multiplication morphisms, $1 \leq j \leq k$, $i_j \geq 0$,

$$\mu_{k; i_1, \dots, i_k}: \mathcal{O}(k) \otimes \mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_k) \longrightarrow \mathcal{O}(i_1 + \cdots + i_k)$$

satisfying associativity and unitality rules. We can equivalently define operads in terms of *infinitesimal composition* laws, $1 \leq i \leq p$, $q \geq 0$,

$$\circ_i: \mathcal{O}(p) \otimes \mathcal{O}(q) \longrightarrow \mathcal{O}(p+q-1): x \otimes y \mapsto x \circ_i y,$$

defined as

$$x \circ_i y = \mu_{p, 1, i-1, 1, q, 1, p-i, 1}(x \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}} \otimes y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}).$$

Such composition operations together with an identity cycle in arity 1 and bidegree $(0, 0)$ form an operad if and only if the following equations hold:

1. $x \circ_i (y \circ_j z) = (x \circ_i y) \circ_{i+j-1} z$.
2. $(x \circ_i y) \circ_j z = (x \circ_j z) \circ_{j-1+\text{arity of } z} y$, $j < i$.
3. $x \circ_i \text{id}_{\mathcal{O}} = x$.
4. $\text{id}_{\mathcal{O}} \circ_1 x = x$.

The circle product must satisfy the *operadic Leibniz rule* in all cases,

$$d(x \circ_i y) = d(x) \circ_i y + (-1)^{\|x\|} x \circ_i d(y).$$

Definition 1.4.4. An *operad* (in graded complexes) $\mathcal{O} = (\mathcal{O}, \mu_{\mathcal{O}}, \eta_{\mathcal{O}})$, is a monoid in the monoidal category $(\text{Coll}_k, \circ_{\Sigma}, I)$ of Definition 1.3.2. A *morphism of operads* $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of monoids. An *augmented operad* $\mathcal{O} = (\mathcal{O}, \mu_{\mathcal{O}}, \eta_{\mathcal{O}}, \epsilon_{\mathcal{O}})$ is an augmented monoid and maps between them are augmented monoid morphisms. We denote the resulting categories by **Op** and **AugOp** respectively.

Since $\epsilon_{\mathcal{O}}\eta_{\mathcal{O}} = \mathbb{1}_I$, it follows that augmented operads have the same direct sum decomposition as augmented ns-operads.

It follows from Proposition 1.2.2 and Remark 1.3.4 that the sequence underlying an operad is a nonsymmetric operad, and the symmetrization of a nonsymmetric operad is an operad. Similarly in the augmented case.

Operads can also be defined both in terms of multiplication morphisms and composition laws, as in Remark 1.4.3, satisfying obvious equivariance conditions with respect to the symmetric group actions.

Definition 1.4.5. A *nonsymmetric cooperad* $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}})$, abbreviated *ns-cooperad*, is a comonoid in (Seq_k, \circ, I) . A *morphism of ns-cooperads* $f: \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of comonoids. A *coaugmented ns-cooperad* $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}}, \eta_{\mathcal{C}})$ is a coaugmented comonoid in (Seq_k, \circ, I) and maps between them are coaugmented comonoid morphisms. We denote by $\text{id}_{\mathcal{C}}$ the image of $1 \in k$ by $\eta_{\mathcal{C}}$ and call it the *identity cooperation*. We denote the resulting categories by **ns-Coop** and **ns-CoaugCoop** respectively.

If \mathcal{C} is coaugmented, denote $\bar{\mathcal{C}} = \text{Coker}(\eta_{\mathcal{C}})$, where Coker is the cokernel in the category of sequences. Since $\epsilon_{\mathcal{C}}\eta_{\mathcal{C}} = \mathbb{1}_I$, it follows that when a ns-cooperad \mathcal{C} is coaugmented it has a decomposition as a direct sum

$$\mathcal{C}(n) = (I \oplus \bar{\mathcal{C}})(n) = \begin{cases} k \oplus \bar{\mathcal{C}}(1), & n = 1; \\ \bar{\mathcal{C}}(n), & n \neq 1. \end{cases}$$

Remark 1.4.6. In case $\mathcal{C}(0) = 0$, we adopt the following abuse of notation for the comultiplication $\Delta_{\mathcal{C}}$ acting on a element $x \in \mathcal{C}$.

$$\Delta_{\mathcal{C}}(x) = \sum_{\substack{k \geq 0, \\ i_1 + \dots + i_k = \text{arity of } x}} x_k \otimes x_{i_1} \otimes \dots \otimes x_{i_k}, \quad (1.4.7)$$

where $x_k \in \mathcal{C}(k)$, $k \geq 0$, and $x_{i_j} \in \mathcal{C}(i_j)$, $\sum i_j = \text{arity of } x$. Notice that the sum in this notation makes sense, because we assumed $\mathcal{C}(0) = 0$.

We will mostly work with coaugmented ns-cooperads satisfying a conilpotency condition, whose definition needs some preliminaries.

The compatibility of $\eta_{\mathcal{C}}$ with the comultiplication amounts to

$$\Delta_{\mathcal{C}}\eta_{\mathcal{C}} = \eta_{\mathcal{C}} \circ \eta_{\mathcal{C}}: I \cong I \circ I \rightarrow \mathcal{C} \circ \mathcal{C}.$$

Here, and below, we abuse a little of notation, dropping the natural isomorphism l or r from notation.

We decompose the coproduct $\Delta_{\mathcal{C}}$ as

$$\Delta_{\mathcal{C}} = \eta_{\mathcal{C}} \circ \mathbb{1}_{\mathcal{C}} + \mathbb{1}_{\mathcal{C}} \circ \eta_{\mathcal{C}} + \bar{\Delta}_{\mathcal{C}}, \quad (1.4.8)$$

where

$$\eta_{\mathcal{C}} \circ \mathbb{1}_{\mathcal{C}}: \mathcal{C} \cong I \circ \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}, \quad \mathbb{1}_{\mathcal{C}} \circ \eta_{\mathcal{C}}: \mathcal{C} \cong \mathcal{C} \circ I \rightarrow \mathcal{C} \circ \mathcal{C}.$$

In particular,

$$\bar{\Delta}_{\mathcal{C}} \eta_{\mathcal{C}} = -\eta_{\mathcal{C}} \circ \eta_{\mathcal{C}}.$$

Define

$$\begin{aligned} \hat{\Delta}_{\mathcal{C}}^0 &= \mathbb{1}_{\mathcal{C}}, \\ \hat{\Delta}_{\mathcal{C}}^1 &= \hat{\Delta}_{\mathcal{C}} = \bar{\Delta}_{\mathcal{C}} + \Delta_{\mathcal{C}} \eta_{\mathcal{C}} \epsilon_{\mathcal{C}}, \\ \hat{\Delta}_{\mathcal{C}}^n &= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \hat{\Delta}_{\mathcal{C}}^{n-1}, \quad n \geq 2. \end{aligned} \tag{1.4.9}$$

Then define

$$\begin{aligned} F_0 \mathcal{C} &:= I, \\ F_n \mathcal{C} &:= \ker \hat{\Delta}_{\mathcal{C}}^n, \quad n \geq 1. \end{aligned}$$

Clearly, $F_n \mathcal{C} \subset F_{n+1} \mathcal{C}$ for all n .

Definition 1.4.10. The filtration

$$F_0 \mathcal{C} \subset F_1 \mathcal{C} \subset F_2 \mathcal{C} \subset \cdots \subset F_n \mathcal{C} \subset F_{n+1} \mathcal{C} \subset \cdots$$

is called the *coradical filtration* of a coaugmented ns-cooperad \mathcal{C} . A coaugmented ns-cooperad is called *conilpotent* if its coradical filtration is exhaustive, i.e. $\text{colim}_n F_n \mathcal{C} = \cup_n F_n \mathcal{C} = \mathcal{C}$. We denote the resulting full subcategory of ns-CoaugCoop by ns-ConilCoop.

Apparently, our definition of the maps $\hat{\Delta}_{\mathcal{C}}^n$, and hence of the coradical filtration, does not coincide with the definition in [16, 5.8.5]. In fact, with the definition for $\hat{\Delta}_{\mathcal{C}}^n$ in [16, 5.8.5] it is not clear that the kernels $F_n \mathcal{C}$ define an increasing filtration. Both definitions are nevertheless equivalent, as we will now prove.

In [16, 5.8.5], the following maps are defined,

$$\begin{aligned} \hat{\Delta}_{\mathcal{C}}^0 &= \mathbb{1}_{\mathcal{C}}, \\ \hat{\Delta}_{\mathcal{C}}^n &= \tilde{\Delta}_{\mathcal{C}}^n - (\mathbb{1}_{\mathcal{C}}^{\circ n} \circ \eta_{\mathcal{C}}) \tilde{\Delta}_{\mathcal{C}}^{n-1}, \quad n \geq 1, \end{aligned} \tag{1.4.11}$$

where

$$\begin{aligned} \tilde{\Delta}_{\mathcal{C}}^0 &= \mathbb{1}_{\mathcal{C}}, \\ \tilde{\Delta}_{\mathcal{C}}^1 &= \tilde{\Delta}_{\mathcal{C}} = \bar{\Delta}_{\mathcal{C}} + \mathbb{1}_{\mathcal{C}} \circ \eta_{\mathcal{C}} + \Delta_{\mathcal{C}} \eta_{\mathcal{C}} \epsilon_{\mathcal{C}}, \\ \tilde{\Delta}_{\mathcal{C}}^n &= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \tilde{\Delta}_{\mathcal{C}}) \tilde{\Delta}_{\mathcal{C}}^{n-1}, \quad n \geq 2. \end{aligned}$$

Assume now that $\hat{\Delta}_{\mathcal{C}}^n$ is defined as in (1.4.11).

Lemma 1.4.12. *We have $\tilde{\Delta}_c^n = \sum_{k=0}^n \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k)}$.*

Proof. For $n = 0$ it is obvious. Now assume that

$$\tilde{\Delta}_c^{n-1} = \sum_{k=0}^{n-1} \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k-1)}$$

for some $n \geq 1$. Then

$$\begin{aligned} \hat{\Delta}_c^n &= \tilde{\Delta}_c^n - (\mathbb{1}_c^{\circ n} \circ \eta_c) \tilde{\Delta}_c^{n-1} \\ &= \tilde{\Delta}_c^n - \tilde{\Delta}_c^{n-1} \circ \eta_c \\ &= \tilde{\Delta}_c^n - \left(\sum_{k=0}^{n-1} \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k-1)} \right) \circ \eta_c \\ &= \tilde{\Delta}_c^n - \sum_{k=0}^{n-1} \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k)}. \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{\Delta}_c^n &= \hat{\Delta}_c^n + \sum_{k=0}^{n-1} \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k)} \\ &= \sum_{k=0}^n \hat{\Delta}_c^k \circ \eta_c^{\circ(n-k)}. \end{aligned}$$

□

We now check that both definitions (1.4.11) and (1.4.9) for $\hat{\Delta}_c^n$ coincide. This is established in the following lemma, where we assume again that $\hat{\Delta}_c^n$ is in principle defined as in (1.4.11).

Lemma 1.4.13. *We have*

$$\begin{aligned} \hat{\Delta}_c^0 &= \mathbb{1}_c, \\ \hat{\Delta}_c^1 &= \hat{\Delta}_c = \bar{\Delta}_c + \Delta_c \eta_c \epsilon_c, \\ \hat{\Delta}_c^n &= (\mathbb{1}_c^{\circ(n-1)} \circ \hat{\Delta}_c) \hat{\Delta}_c^{n-1}, n \geq 2. \end{aligned}$$

Proof. There is nothing to check for $n = 0$. For $n = 1$, we have

$$\begin{aligned} \hat{\Delta}_c^1 &= \tilde{\Delta}_c - (\mathbb{1}_c \circ \eta_c) \mathbb{1}_c \\ &= \bar{\Delta}_c + \mathbb{1}_c \circ \eta_c + \Delta_c \eta_c \epsilon_c - (\mathbb{1}_c \circ \eta_c) \mathbb{1}_c \\ &= \bar{\Delta}_c + \Delta_c \eta_c \epsilon_c = \hat{\Delta}_c. \end{aligned}$$

For $n \geq 2$,

$$\begin{aligned}
\hat{\Delta}_{\mathcal{C}}^n &= \tilde{\Delta}_{\mathcal{C}}^n - (\mathbb{1}_{\mathcal{C}}^{\circ n} \circ \eta_{\mathcal{C}}) \tilde{\Delta}_{\mathcal{C}}^{n-1} \\
&= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \tilde{\Delta}_{\mathcal{C}} - \mathbb{1}_{\mathcal{C}}^{\circ n} \circ \eta_{\mathcal{C}}) \tilde{\Delta}_{\mathcal{C}}^{n-1} \\
&= \sum_{k=0}^{n-1} (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ (\tilde{\Delta}_{\mathcal{C}} - \mathbb{1}_{\mathcal{C}} \circ \eta_{\mathcal{C}})) \left(\hat{\Delta}_{\mathcal{C}}^k \circ \eta_{\mathcal{C}}^{\circ(n-k-1)} \right) \\
&= \sum_{k=0}^{n-1} (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \left(\hat{\Delta}_{\mathcal{C}}^k \circ \eta_{\mathcal{C}}^{\circ(n-k-1)} \right) \\
&= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \hat{\Delta}_{\mathcal{C}}^{n-1} + \sum_{k=0}^{n-2} (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \left(\hat{\Delta}_{\mathcal{C}}^k \circ \eta_{\mathcal{C}}^{\circ(n-2-k)} \circ \eta_{\mathcal{C}} \right) \\
&= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \hat{\Delta}_{\mathcal{C}}^{n-1} + \sum_{k=0}^{n-2} \hat{\Delta}_{\mathcal{C}}^k \circ \eta_{\mathcal{C}}^{\circ(n-2-k)} \circ (\hat{\Delta}_{\mathcal{C}} \eta_{\mathcal{C}}) \\
&= (\mathbb{1}_{\mathcal{C}}^{\circ(n-1)} \circ \hat{\Delta}_{\mathcal{C}}) \hat{\Delta}_{\mathcal{C}}^{n-1},
\end{aligned}$$

where in the last identification it is used that

$$\begin{aligned}
\hat{\Delta}_{\mathcal{C}} \eta_{\mathcal{C}} &= \bar{\Delta}_{\mathcal{C}} \eta_{\mathcal{C}} + \Delta_{\mathcal{C}} \eta_{\mathcal{C}} \epsilon_{\mathcal{C}} \eta_{\mathcal{C}} \\
&= \bar{\Delta}_{\mathcal{C}} \eta_{\mathcal{C}} + \Delta_{\mathcal{C}} \eta_{\mathcal{C}} \\
&= -\eta_{\mathcal{C}} \circ \eta_{\mathcal{C}} + \eta_{\mathcal{C}} \circ \eta_{\mathcal{C}} \\
&= 0.
\end{aligned}$$

□

We now define a cooperadic analogue of the infinitesimal composition laws in Remark 1.4.3.

Definition 1.4.14. Given two sequences X and Y , we denote by $X \circ_{(1)} Y$ the sequence defined as

$$(X \circ_{(1)} Y)(n) = \bigoplus_{\substack{1 \leq j \leq p \\ p+q=n+1}} X(p) \otimes Y(q).$$

The *infinitesimal decomposition* $\Delta_{(1)}$ of an ns-cooperad \mathcal{C} is given by the following composite

$$\Delta_{(1)}: \mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \circ \mathcal{C} \xrightarrow{\text{Pr}} \mathcal{C} \circ_{(1)} \mathcal{C} \tag{1.4.15}$$

where Pr is given in arity n by the maps

$$\bigoplus_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n}} \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \dots \otimes \mathcal{C}(i_k) \longrightarrow \bigoplus_{\substack{1 \leq j \leq p \\ p+q=n+1}} \mathcal{C}(p) \otimes \mathcal{C}(q).$$

whose component, for fixed values of the indices, is

$$\mathbb{1}_{\mathcal{C}(k)} \otimes \epsilon_{\mathcal{C}(i_1)} \otimes \dots \otimes \epsilon_{\mathcal{C}(i_{j-1})} \otimes \mathbb{1}_{\mathcal{C}(i_j)} \otimes \epsilon_{\mathcal{C}(i_{j+1})} \otimes \dots \otimes \epsilon_{\mathcal{C}(i_k)}$$

if $p = k$, $q = i_j$, and $i_l = 1$ for all $l \neq j$, and zero otherwise.

Remark 1.4.16. We remark here, for later use, that given two sequences X and Y , there is a natural (split) inclusion

$$X \circ_{(1)} Y \rightarrow X \circ (I \oplus Y)$$

which, in arity n , and on the direct summand indexed by certain $1 \leq j \leq p$ and q with $n + 1 = p + q$, is defined as

$$x \otimes y \mapsto x \otimes \text{id} \otimes \dots \otimes \text{id} \otimes y \otimes \text{id} \otimes \dots \otimes \text{id}.$$

Here id is the generator of $I(1) = k$, which is also the identity operation of I regarded as the initial ns-operad.

Definition 1.4.17. A cooperad $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}})$ is a comonoid in the monoidal category $(\text{Coll}_k, \bar{\circ}_{\Sigma}, I)$ of Definition 1.3.7. A morphism of cooperads $f: \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of comonoids in the monoidal category $(\text{Coll}_k, \bar{\circ}_{\Sigma}, I)$. A coaugmented cooperad $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}}, \eta_{\mathcal{C}})$ is a coaugmented comonoid and maps between them are coaugmented comonoid maps. We denote by $\text{id}_{\mathcal{C}}$ the image of $1 \in k$ by $\eta_{\mathcal{C}}$ and call it the *identity cooperation*. We denote the resulting categories by Coop and CoaugCoop respectively.

Remark 1.4.18. It follows from the dual of Proposition 1.2.2 and Remarks 1.3.4 and 1.3.8 that the symmetrization of a nonsymmetric cooperad is a cooperad. Similarly in the coaugmented case.

Since $\eta_{\mathcal{C}}$ is a morphism of cooperads, we have $\epsilon_{\mathcal{C}} \eta_{\mathcal{C}} = \mathbb{1}_I$. It follows that coaugmented cooperads have the same direct sum decomposition as coaugmented ns-cooperads.

The previous discussion on the coradical filtration extends to the symmetric case, replacing \circ with $\bar{\circ}_{\Sigma}$. The full subcategory of CoaugCoop spanned by conilpotent coaugmented cooperads is denoted by ConilCoop .

We warn the reader familiar with [16] that the reduced decomposition map of [16, 5.8.1] only coincides with our $\bar{\Delta}_{\mathcal{C}}$ in the nonsymmetric case. In the symmetric case, our decomposition (1.4.8) is element-wise of the form

$$\Delta_{\mathcal{C}}(x) = \text{id}_{\mathcal{C}} \otimes x + \sum_{\sigma \in \Sigma_n} (x \cdot \sigma) \otimes \text{id}_{\mathcal{C}}^{\otimes n} \otimes \sigma^{-1} + \bar{\Delta}_{\mathcal{C}}(x).$$

Here $x \in \mathcal{C}(n)$.

We finally define the symmetric infinitesimal decomposition.

Definition 1.4.19. Given two collection X and Y , we denote by $X \circ_{\Sigma, (1)} Y$ the collection defined as

$$(X \circ_{\Sigma, (1)} Y)(n) = \bigoplus_{p+q=n+1} X(p) \otimes_{\Sigma_p} \left(\bigoplus_{1 \leq j \leq p} Y(q) \otimes_{\Sigma_q} k[\Sigma_n] \right).$$

Here, the left Σ_q -module structure on $k[\Sigma_n]$ is given by the inclusion of the Young subgroup

$$\Sigma_1 \times \overset{j-1}{\dots} \times \Sigma_1 \times \Sigma_q \times \Sigma_1 \times \overset{p-j}{\dots} \times \Sigma_1 \subset \Sigma_n.$$

Moreover, the left action of $\tau \in \Sigma_p$ on the module between brackets sends $y \otimes \sigma$ in the direct summand indexed by j to

$$y \otimes \tau_{1, j-1, 1, q, 1, p-j, 1} \sigma$$

in the direct summand indexed by $\tau(j)$. See Definition 1.3.2 for the definition of $\tau_{1, j-1, 1, q, 1, p-j, 1} \in \Sigma_n$.

The *infinitesimal decomposition* $\Delta_{(1)}$ of a reduced cooperad \mathcal{C} is given by the following composite

$$\Delta_{(1)}: \mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \circ_{\Sigma} \mathcal{C} \longrightarrow \mathcal{C} \circ_{\Sigma, (1)} \mathcal{C}$$

where the second arrow is the map of collections defined as in Definition 1.4.14 above.

Remark 1.4.20. As in Remark 1.4.16, given two collections X and Y there is a natural (split) inclusion

$$X \circ_{\Sigma, (1)} Y \rightarrow X \circ_{\Sigma} (I \oplus Y)$$

defined essentially by the same formula as therein.

1.5 Algebras over operads

In this section algebras over (ns-)operads are defined. To this end, we need to introduce the operad of endomorphisms.

Definition 1.5.1. The *operad of endomorphisms* $\mathcal{E}(X)$ of a graded complex X is defined in arity n by the following inner Hom,

$$\mathcal{E}(X)(n) = [X^{\otimes n}, X].$$

The action of Σ_n is given by the permutation action on $X^{\otimes n}$ in the closed symmetric monoidal category of graded complexes. More precisely, if we regard $f \in [X^{\otimes n}, X]$ as a homogeneous multilinear map $f: X^{\otimes n} \rightarrow X$, then for $\sigma \in \Sigma_n$, $1 \leq i \leq n$,

$$(f \cdot \sigma)(x_1 \otimes \cdots \otimes x_n) = (-1)^\epsilon f(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}),$$

where

$$\epsilon = \sum_{\substack{i < j \\ \sigma(i) > \sigma(j)}} \|x_i\| \|x_j\|.$$

The multiplication morphisms

$$\mu_{n;p_1, \dots, p_n}: [X^{\otimes n}, X] \otimes [X^{\otimes p_1}, X] \otimes \cdots \otimes [X^{\otimes p_n}, X] \longrightarrow [X^{\otimes(p_1 + \cdots + p_n)}, X]$$

are given by composition of homogeneous multilinear maps,

$$\mu_{n;p_1, \dots, p_n}(f_0 \otimes \cdots \otimes f_n) = f_0(f_1 \otimes \cdots \otimes f_n).$$

The identity operation is the identity map $\mathbb{1}_X \in [X, X]$.

Definition 1.5.2. An *algebra* over an (ns-)operad \mathcal{O} , or \mathcal{O} -*algebra* for short, is a graded complex X equipped with a morphism of (ns-)operads

$$\mathcal{O} \rightarrow \mathcal{E}(X).$$

Remark 1.5.3. Using the characterization of internal Homs as right adjoints to the tensor product, we see that an \mathcal{O} -algebra X can be alternatively defined in terms of structure morphisms of graded complexes, $n \geq 0$,

$$\mathcal{O}(n) \otimes X \otimes \cdots \otimes X \longrightarrow X,$$

satisfying certain conditions. Therefore, we can heuristically regard $\mathcal{O}(n)$ as the ‘space’ of arity n operations in \mathcal{O} -algebras. Morphisms in the category of \mathcal{O} -algebras are obviously the maps between the underlying graded complexes which are compatible with the structure morphisms.

Remark 1.5.4. The forgetful functor from \mathcal{O} -algebras to graded complexes has a well-known left adjoint given by

$$X \mapsto \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes X^{\otimes n}$$

in the nonsymmetric setting and

$$X \mapsto \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

in the symmetric setting, compare [16, Proposition 5.2.1]. Here, the right action of Σ_n on $\mathcal{O}(n)$ is part of the structure of a collection and the left action on $X^{\otimes n}$ is given by permutation of variables, as above. We can alternatively use coinvariants in the symmetric case,

$$\bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n} = \bigoplus_{n \geq 0} (\mathcal{O}(n) \otimes X^{\otimes n})_{\Sigma_n},$$

see Remark 1.3.3. These \mathcal{O} -algebras are called *free* and will be denoted $\mathcal{O}(X)$. In both cases, the structure \mathcal{O} -algebra maps, in the sense of Remark 1.5.3, are induced in the obvious way by the operadic multiplication morphisms in Remark 1.4.3.

Remark 1.5.5. By turning to the opposite monoidal category, we can consider not only cooperads, but also *coalgebras* over them, *cofree coalgebras*, etc.

Coproducts in the opposite category are products in the original category, and coinvariants become invariants. Hence, if \mathcal{C} is an (ns-)cooperad, the cofree \mathcal{C} -coalgebra on a graded complex X is

$$\prod_{n \geq 0} \mathcal{C}(n) \otimes X^{\otimes n}$$

in the nonsymmetric setting and

$$\prod_{n \geq 0} (\mathcal{C}(n) \otimes X^{\otimes n})^{\Sigma_n}$$

in the symmetric setting.

If \mathcal{C} is coaugmented and conilpotent, it is easy to notice that the direct sum

$$\begin{aligned} \bigoplus_{n \geq 0} \mathcal{C}(n) \otimes X^{\otimes n} &\subset \prod_{n \geq 0} \mathcal{C}(n) \otimes X^{\otimes n}, \\ \bigoplus_{n \geq 0} (\mathcal{C}(n) \otimes X^{\otimes n})^{\Sigma_n} &\subset \prod_{n \geq 0} (\mathcal{C}(n) \otimes X^{\otimes n})^{\Sigma_n}, \end{aligned}$$

forms a sub- \mathcal{C} -coalgebra in each case, that we denote $\mathcal{C}(X)$ by analogy with the previous remark.

Chapter 2

Koszul duality for operads

Koszul duality theory is a fabulous tool to construct small resolutions in homological algebra. It was initiated by Priddy [20] in the associative algebra setting and extended by Ginzburg and Kapranov to the realm of operads [11]. The authoritative monograph [16] by Loday and Vallette is an excellent introduction to the topic, that we mostly follow. We also crucially use Fresse's [8], which extends parts of the classical theory to operads over a commutative ground ring which is not necessarily a field, under suitable projectivity assumptions. Actually, we will be mainly interested in this more general case.

2.1 Free operads

Koszul duality theory relies heavily on the notion of free operad. Free operads are somewhat involved gadgets whose underlying combinatorics is that of trees. We thoroughly describe them in this section, both in the symmetric and in the nonsymmetric settings.

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category.

Definition 2.1.1. A monoid $(\mathcal{F}(X), \mu_{\mathcal{F}(X)}, \eta_{\mathcal{F}(X)})$ is a *free monoid generated by an object X* in \mathcal{C} if it is equipped with a map $X \rightarrow \mathcal{F}(X)$ in \mathcal{C} such that, if (A, μ_A, η_A) is another monoid and $X \rightarrow A$ is another map in \mathcal{C} , then there exists a unique monoid morphism $(\mathcal{F}(X), \mu_{\mathcal{F}(X)}, \eta_{\mathcal{F}(X)}) \rightarrow (A, \mu_A, \eta_A)$ such that the underlying map in \mathcal{C} fits in the following commutative triangle

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{F}(X) \\ & \searrow & \swarrow \\ & & A \end{array}$$

See [17, page 51].

The free monoid generated by an object, if it exists, is unique up to a canonical isomorphism defined by the previous universal property. If the free monoid happens to exist for any object X , then they assemble to a *free monoid functor* $\mathcal{F}: \mathcal{C} \rightarrow \mathbf{Mon}(\mathcal{C})$, which is left adjoint to the forgetful functor $U: \mathbf{Mon}(\mathcal{C}) \rightarrow \mathcal{C}$,

$$U((A, \mu_A, \eta_A)) = A.$$

Moreover, the map $X \rightarrow \mathcal{F}(X)$ is the unit of the adjunction.

The free ns-operad on a sequence and the free operad on a collection are defined according to Definition 2.1.1, since (ns-)operads are monoids in certain monoidal categories, see Definitions 1.4.1 and 1.4.4. They happen to exist always, so we have left adjoints \mathcal{F} to the forgetful functors. We now explicitly describe what free operads look like. We will start with the nonsymmetric version, following [19].

The combinatorics of operads is that of trees with additional structure. We now recall some necessary facts about trees from [19, section 3].

Definition 2.1.2. A *planted tree with leaves* is a contractible finite 1-dimensional (abstract) simplicial complex T with a set of vertices $V(T)$, a non-empty set of edges $E(T)$, a distinguished vertex $r(T) \in V(T)$ called *root*, and a set of distinguished vertices $L(T) \subset V(T) \setminus \{r(T)\}$ called *leaves*. The root and the leaves must have degree 1. Recall that the *degree* of $v \in V(T)$ is the number of edges containing v . The *arity* of v , denoted by \tilde{v} , is the degree minus one,

$$\tilde{v} = \text{degree}(v) - 1.$$

The *level* of a vertex $v \in V(T)$ is the number of edges in the shortest path to the root. The *height* of T is the maximum level of its vertices.

Definition 2.1.3. A *planted planar tree with leaves* is a planted tree with leaves T together with a total order \preceq in $V(T)$, called *planar order*, such that:

- If $\text{level}(v) < \text{level}(w)$ then $v \prec w$.
- If $\{v_1, v_2\}, \{w_1, w_2\} \in E(T)$ are edges with

$$\text{level}(v_1) = \text{level}(w_1) = \text{level}(v_2) - 1 = \text{level}(w_2) - 1,$$

and $v_1 \prec w_1$, then $v_2 \prec w_2$.

There is another total order \leq on $V(T)$, called *path order*, satisfying the following conditions. Given two vertices $v, w \in V(T)$:

- If v lies on the (shortest) path from $r(T)$ to w then $v < w$.
- Otherwise, assume that the path from $r(T)$ to v coincides with the path from $r(T)$ to w up to level n , and let v' and w' be the level $n + 1$ vertices on these paths. If $v' \prec w'$ then $v < w$.

We will order vertices according to the path order, unless specified otherwise. The set $E(T)$ is ordered according first to the bottom vertex of each edge, and second to the top vertex (in case they share the bottom vertex). In the set formed by the union of all vertices and edges, we identify an edge $\{u, v\}$ with the word uv and follow the lexicographic order (vertices are obviously words with a single letter).

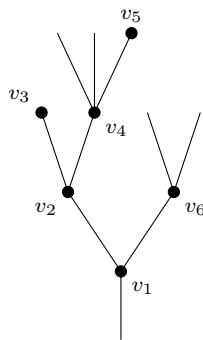
Given $e = \{v, w\} \in E(T)$ with $v < w$ we say that e is an *incoming edge* of v and the *outgoing edge* of w . Note that the arity of a vertex equals the number of its incoming edges.

An *inner vertex* is a vertex which is neither a leaf nor the root. The set of inner vertices will be denoted by $I(T)$,

$$V(T) = \{r(T)\} \sqcup I(T) \sqcup L(T).$$

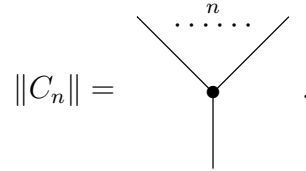
Abusing terminology, we say that an edge is the *root* or a *leaf* if it contains the root or a leaf vertex, respectively. The rest of edges are called *inner edges*.

The *geometric realization* of a planted planar tree with leaves $\|T\|$ is obtained from a standard geometric realization in the plain by removing the root and the leaves, e.g.



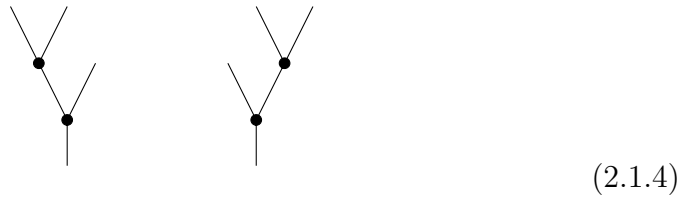
The root is depicted at the bottom, and the rest of edges are drawn so that the planar order can be read from bottom to top and from left to right. The path order on vertices is here indicated by the subscript.

For $n \geq 0$, the *corolla* with n leaves is the planted planar tree C_n with one inner vertex and n leaves,



An *isomorphism* of planted planar trees with leaves is a simplicial isomorphism preserving the root and the leaves. A *planar isomorphism* is an isomorphism which in addition preserves the planar order (and hence the path order).

In the following picture we see two isomorphic planted planar trees with leaves which are not planarly isomorphic,



Planted planar trees with leaves have no non-trivial planar automorphism. We choose a representative of each planar isomorphism class and denote the resulting set of planted planar trees with leaves by **PPTL**.

Definition 2.1.5. Given a sequence of graded complexes X , we define the *tree module* of a planted planar tree with leaves T to be the following graded complex,

$$X(T) = \bigotimes_{v \in I(T)} X(\tilde{v}).$$

In particular, $X(\bullet) = k$. Since the symmetry isomorphism in the closed symmetric monoidal category of graded complexes involves signs (we are using the Koszul sign rule), it is important to fix an order for the factors of this tensor product. We use the path order.

The underlying sequence of the free ns-operad on the sequence X is

$$\mathcal{F}(X)(n) = \bigoplus_{\substack{T \in \mathbf{PPTL} \\ \text{with } n \text{ leaves}}} X(T). \quad (2.1.6)$$

In this context, we call operations $x \in X(n)$ *generating operations* of $\mathcal{F}(X)$.

Below, we intensively use the following notation for these pondering signs. Given an ns-labeled planted planar tree with leaves m_T ,

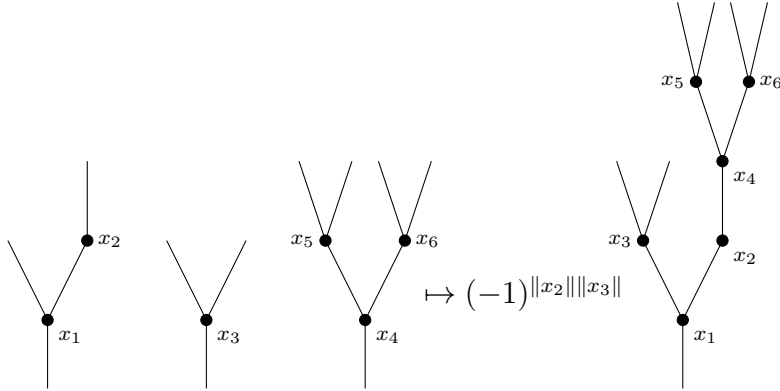
$$(-1)^{\|m_T\|_{<v}}$$

is the sum of the total degrees of the labels in m_T of the inner vertices of T that come (strictly) before v , with respect to the path order (always). Hence, $d_{\mathcal{F}(X)}(m_T)$ is a pondered sum indexed by the inner vertices of T of ns-labeled planted planar trees with leaves with underlying tree T . The summand corresponding to a certain $v \in I(T)$ has the same labels as m_T except at v , where it has $d_X(x_v)$ if m_T had x_v . The pondering sign of this summand is precisely $(-1)^{\|m_T\|_{<v}}$.

The multiplication maps $\mu_{\mathcal{F}(X)}$ of the free operad is essentially given, on ns-labeled planted planar trees with leaves, by grafting. More precisely, given ns-labeled planted planar trees with leaves $m_{T_0}, m_{T_1}, \dots, m_{T_n}$, where T_0 has n leaves,

$$\mu_{\mathcal{F}(X)}(m_{T_0} \otimes m_{T_1} \otimes \dots \otimes m_{T_n}) = \pm m_{T_0(T_1, \dots, T_n)},$$

where $m_{T_0(T_1, \dots, T_n)}$ is the ns-labeled planted planar tree with leaves obtained by grafting each m_{T_i} , $1 \leq i \leq n$, on the i^{th} leaf of m_{T_0} , counting from left to right, i.e. according to the path order. The sign is that of the symmetry isomorphism needed to obtain $m_{T_0(T_1, \dots, T_n)}$ from $m_{T_0} \otimes m_{T_1} \otimes \dots \otimes m_{T_n}$ permuting the tensor factors,



The identity operation $\text{id}_{\mathcal{F}(X)}$ is $|$ regarded as a labeled tree (with no labels, as this tree has no inner vertices). Finally, the map $X \rightarrow \mathcal{F}(X)$ is given by the inclusion of the direct summands indexed by corollas.

The free ns-operad is augmented. The augmentation $\epsilon_{\mathcal{F}(X)} : \mathcal{F}(X) \rightarrow I$ is determined by the fact that it vanishes on all tree modules except for $X(|)$.

We introduce a weight grading on the free ns-operad, see e.g. also [16, section 5.5.3].

Definition 2.1.8. The *weight* of a ns-labeled planted planar tree with leaves is the number of inner vertices of the underlying tree. We denote by $\mathcal{F}(X)^{(r)}$, $r \geq 0$, the sequence of homogeneous operations of weight r , i.e. the subsequence obtained by restricting the direct sum (2.1.6) to planted planar trees with leaves with precisely r inner vertices. In particular, $\mathcal{F}(X)^{(0)} = k$ concentrated in arity 1 generated by $\text{id}_{\mathcal{F}(X)}$, $\mathcal{F}(X)^{(1)} = X$, and $\mathcal{F}(X)^{(2)} = X \circ_{(1)} X$ in the sense of Definition 1.4.14.

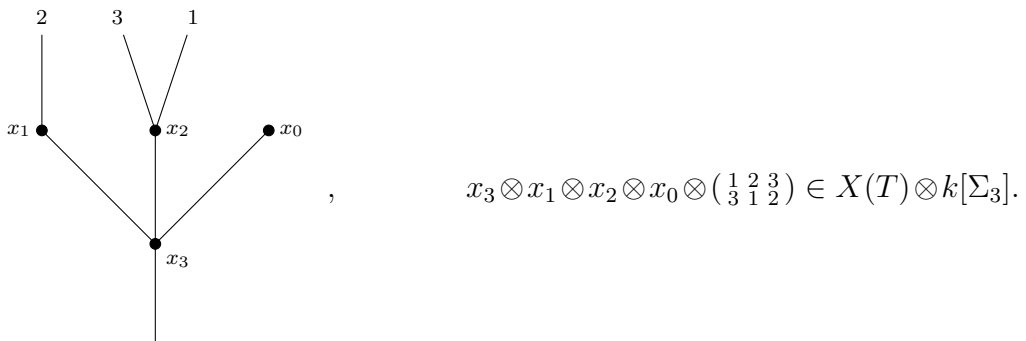
The ns-operadic multiplication is homogeneous with respect to the weight grading.

We now turn to the symmetric case. We describe the free operad on a collection by simplifying [4, 5.8], which works with general closed symmetric monoidal categories, while we restrict to the more familiar setting of graded complexes.

Let X be a collection. We will denote the free operad generated by X by $\mathcal{F}_{\Sigma}(X)$, in order to distinguish it from the free ns-operad $\mathcal{F}(X)$ on the underlying sequence of X . For its description, we need the following new kind of labeled trees.

Definition 2.1.9. A *labeled planted planar tree with leaves* m_T is a planted planar tree with leaves T where each inner vertex $v \in I(T)$ is labeled by an element of the corresponding arity $x \in X(\tilde{v})$, as in Definition 2.1.7 and each leaf is labeled with a different number in $\{1, \dots, n\}$, where n is the number of leaves.

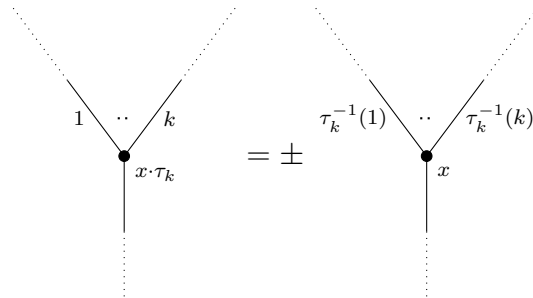
Remark 2.1.10. A labeled planted planar tree with leaves m_T can be regarded as an element of the *symmetrized tree module* $X(T) \otimes k[\Sigma_n]$, where n is the number of leaves of T . The tensor factors in X are as in the nonsymmetric case, and the permutation $\sigma \in \Sigma_n$ is defined by the fact that the i^{th} leaf, with respect to the path order, is labeled by $\sigma^{-1}(i)$, $1 \leq i \leq n$,



The free operad $\mathcal{F}_\Sigma(X)(n)$ generated by the collection X is, in arity n , a quotient of the graded complex

$$\bigoplus_{\substack{T \in \mathbf{PPTL} \\ \text{with } n \text{ leaves}}} X(T) \otimes k[\Sigma_n] = \mathcal{F}(X)(n) \otimes k[\Sigma_n]$$

by the following relations between labeled planted planar trees with leaves,



Here, the numbering of the incoming edges of the vertices is not to be regarded as part of the labeling. It just indicates how they are permuted in order to obtain the labeled planted planar tree with leaves on the right from the one on the left. The rest of labels do not change. The sign \pm comes from

the application of the Koszul sign rule. See the following explicit examples,

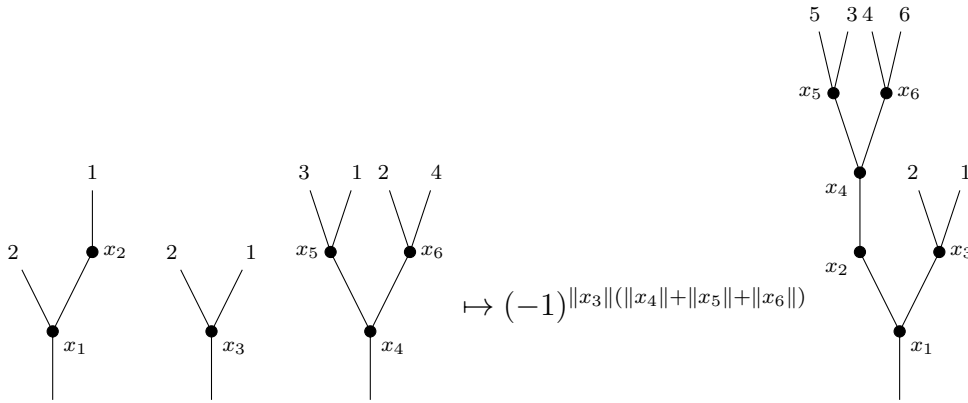
$$\begin{array}{ccc}
 \begin{array}{c} 3 \quad 1 \quad 4 \\ | \quad | \quad | \\ n_3 \bullet \quad n_0 \bullet \quad n_1 \bullet \\ \diagdown \quad \diagup \\ \bullet \\ m_3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{array} & = & \begin{array}{c} 2 \quad 3 \quad 1 \quad 4 \\ | \quad | \quad | \quad | \\ n_1 \bullet \quad n_3 \bullet \quad n_0 \bullet \\ \diagdown \quad \diagup \\ \bullet \\ m_3 \end{array} \\
 \\
 \begin{array}{c} (m_3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}) \otimes n_3 \otimes n_0 \otimes n_1 \otimes \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \\ = (-1)^{(\|n_3\| + \|n_0\|)\|n_1\|} m_3 \otimes n_1 \otimes n_3 \otimes n_0 \otimes \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{c} 3 \quad 1 \quad 4 \\ | \quad | \quad | \\ n_3 \bullet \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ \diagdown \quad \diagup \\ \bullet \\ m_3 \end{array} & = & \begin{array}{c} 1 \quad 4 \quad 3 \quad 2 \\ | \quad | \quad | \quad | \\ n_3 \bullet \quad n_0 \bullet \quad n_1 \bullet \\ \diagdown \quad \diagup \\ \bullet \\ m_3 \end{array} \\
 \\
 \begin{array}{c} m_3 \otimes (n_3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}) \otimes n_0 \otimes n_1 \otimes \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \\ = m_3 \otimes n_3 \otimes n_0 \otimes n_1 \otimes \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \end{array}
 \end{array}
 \end{array}$$

The right action of $\tau \in \Sigma_n$ on $\mathcal{F}_\Sigma(X)(n)$ is given by applying the permutation τ^{-1} to the labels of the leaves,

$$\begin{array}{ccc}
 \begin{array}{c} 2 \quad 3 \quad 1 \\ | \quad | \quad | \\ x_1 \bullet \quad x_2 \bullet \quad x_0 \bullet \\ \diagdown \quad \diagup \\ \bullet \\ x_3 \end{array} & \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & = & \begin{array}{c} 3 \quad 1 \quad 2 \\ | \quad | \quad | \\ x_1 \bullet \quad x_2 \bullet \quad x_0 \bullet \\ \diagdown \quad \diagup \\ \bullet \\ x_3 \end{array}
 \end{array}$$

There are no signs involved in this action.

We have defined the collection $\mathcal{F}_\Sigma(X)$ as a quotient of the symmetrization $\mathcal{F}(X) \otimes k[\Sigma]$ of the free nonsymmetric operad $\mathcal{F}(X)$ on the underlying sequence of X . The operad structure on $\mathcal{F}_\Sigma(X)$ is the unique one such that the natural projection $\mathcal{F}(X) \otimes k[\Sigma] \rightarrow \mathcal{F}_\Sigma(X)$ is a map of operads. This means that the multiplication of labeled planted planar trees with leaves is given by grafting, this time according to the order indicated by the labels, up to a sign determined by the Koszul rule, and shifting the numbering of the leaves of the trees grafted on top of the first one in the obvious way in order to avoid repetitions, e.g.



The identity operation is represented by $|$ regarded as a tree with no labels, as in the nonsymmetric case. The map $X \rightarrow \mathcal{F}_\Sigma(X)$ is again given by inclusion of corollas.

It is worth to notice that the relation defining the free operad only identifies labeled planted planar trees with leaves with (non-planarly) isomorphic underlying planted planar trees with leaves. Based on this observation, we now give a more sophisticated and useful description of $\mathcal{F}_\Sigma(X)$, which is actually closer to that in [4]. For this, we need to extend some facts on modules over groups to the wider context of groupoids.

Definition 2.1.11. A *groupoid* G is a category where all morphisms are isomorphisms. A *left G -module* is a functor $M: G \rightarrow \text{Mod}_k$.

The *module of invariants* of a left G -module M is simply the limit of the functor M , $M^G = \lim M$. It is the submodule of $\prod_{x \in \text{Ob } G} M(x)$ consisting of the elements $(m_x)_{x \in \text{Ob } G}$ such that for any $g: x \rightarrow y$ in $\text{Mor } G$, $M(g)(m_x) = m_y$.

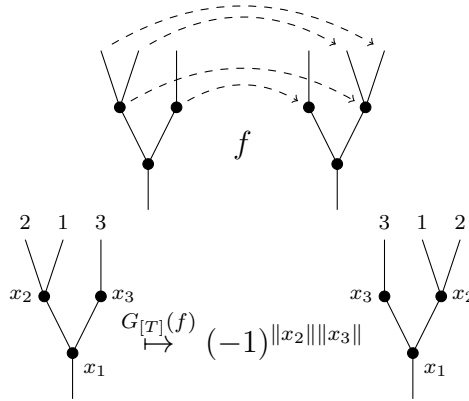
Similarly, the *module of coinvariants* is the colimit of the functor M , $M_G = \text{colim } M$. It is the quotient of $\bigoplus_{x \in \text{Ob } G} M(x)$ by all the elements of the form $m_x - M(g)(m_x)$, where $m_x \in M(x)$ for some $x \in \text{Ob } G$ and $g: x \rightarrow y$ is a map in G .

Definition 2.1.12. Given a planted planar tree with leaves T , denote by $[T] \subset \mathbf{PPTL}$ the subset formed by those planted planar trees with leaves which are (non-planarly) isomorphic to T , e.g. (2.1.4) is one of these subsets. This is obviously an equivalence class under the relation \sim of being (non-planarly) isomorphic in the set \mathbf{PPTL} . We consider the *tree groupoid* $G_{[T]}$ whose object set is $[T]$ and whose morphisms are (non-planar) isomorphisms in the sense of Definition 2.1.2, e.g. in case $[T]$ is the set in (2.1.4), the groupoid $G_{[T]}$ consists of two isomorphic objects each of which has two automorphisms, the identity and the automorphism which flips the two top leaves.

We consider the left $G_{[T]}$ -module $X_{[T]}$, called *symmetric tree module*, which sends a tree $T' \in [T]$ to the symmetrized tree module $X(T') \otimes k[\Sigma_n]$,

$$X_{[T]}(T') = X(T') \otimes k[\Sigma_n]. \quad (2.1.13)$$

Here n is the number of leaves of T (and of any other tree in $[T]$). If $f: T' \rightarrow T''$ is a map in $G_{[T]}$, the induced isomorphism $X_{[T]}(f): X(T') \rightarrow X(T'')$ sends a labeling of T' to the unique labeling of T'' such that labels of vertices and leaves match under f , up to a sign determined by the Koszul rule, e.g.



The previous definition of $\mathcal{F}_\Sigma(X)$ coincides with the following direct sum of coinvariants

$$\mathcal{F}_\Sigma(X)(n) = \bigoplus_{\substack{[T] \in \mathbf{PPTL} / \sim \\ n \text{ leaves}}} (X_{[T]})_{G_{[T]}}. \quad (2.1.14)$$

The action of Σ_n on $\mathcal{F}_\Sigma(X)$ is well defined on each of these direct factors, since the action of $G_{[T]}$ on $X_{[T]}$ commutes with the action of Σ_n . One of the advantages of this definition over the previous elementary one is that, here, we decompose $\mathcal{F}_\Sigma(X)(n)$ as a big direct sum of small factors, each of which can be dealt with independently, as in the nonsymmetric case.

The free operad is augmented. The augmentation is characterized by the fact that it vanishes on all direct factors different from $[[]]$.

The weight grading of [16, section 5.5.3] in the symmetric setting is defined as follows.

Definition 2.1.15. The *weight* of a labeled planted planar tree with leaves is the number of inner vertices of the underlying tree. We denote by $\mathcal{F}_\Sigma(X)^{(r)}$, $r \geq 0$, the collection of homogeneous operations of weight r , i.e. the subcollection obtained by restricting the direct sum (2.1.14) to the classes $[T]$ where T (and any other tree in the class) has precisely r inner vertices. In particular, $\mathcal{F}(X)^{(0)} = k$ concentrated in arity 1 generated by $\text{id}_{\mathcal{F}(X)}$, $\mathcal{F}(X)^{(1)} = X$, and $\mathcal{F}(X)^{(2)} = X \circ_{\Sigma, (1)} X$ in the sense of Definition 1.4.19.

The operadic multiplication is homogeneous with respect to the weight grading.

2.2 Cofree cooperads

Cofree (ns-)cooperads could be defined by using Definition 2.1.1. However, even if they existed, they would not be very interesting for us. For Koszul duality theory, we rather need something fitting in the following more restrictive situation.

Definition 2.2.1. Let $(\mathcal{C}, \otimes, I)$ be a closed symmetric monoidal category. Let $\mathcal{D} \subset \text{CoaugComon}(\mathcal{C})$ be a full subcategory of the category of coaugmented comonoids. A coaugmented comonoid $\langle \mathcal{F}^c(X), \Delta_{\mathcal{F}^c(X)}, \epsilon_{\mathcal{F}^c(X)}, \eta_{\mathcal{F}^c(X)} \rangle$ in \mathcal{D} is a *cofree coaugmented comonoid generated by X in \mathcal{D}* if it is equipped with a map $\mathcal{F}^c(X) \rightarrow X$ such that for any object $\langle C, \Delta_C, \epsilon_C, \eta_C \rangle$ in \mathcal{D} and any map $C \rightarrow X$ in \mathcal{C} there exists a unique coaugmented comonoid map $\langle C, \Delta_C, \epsilon_C, \eta_C \rangle \rightarrow \langle \mathcal{F}^c(X), \Delta_{\mathcal{F}^c(X)}, \epsilon_{\mathcal{F}^c(X)}, \eta_{\mathcal{F}^c(X)} \rangle$ such that the underlying map in \mathcal{C} fits in the following commutative triangle

$$\begin{array}{ccc} X & \longleftarrow & \mathcal{F}^c(X) \\ & \swarrow & \searrow \\ & C & \end{array}$$

Cofree coaugmented comonoids are unique up to canonical isomorphism, provided they exist.

Definition 2.2.1 is suitable to define conilpotent coaugmented (ns-)cooperads, taking \mathcal{D} to be the full subcategory spanned by those coaugmented (ns-)cooperads which are conilpotent. They exist for any reduced sequence

or collection. We will now describe them explicitly following [16, Theorem 5.8.3], starting with the non-symmetric case.

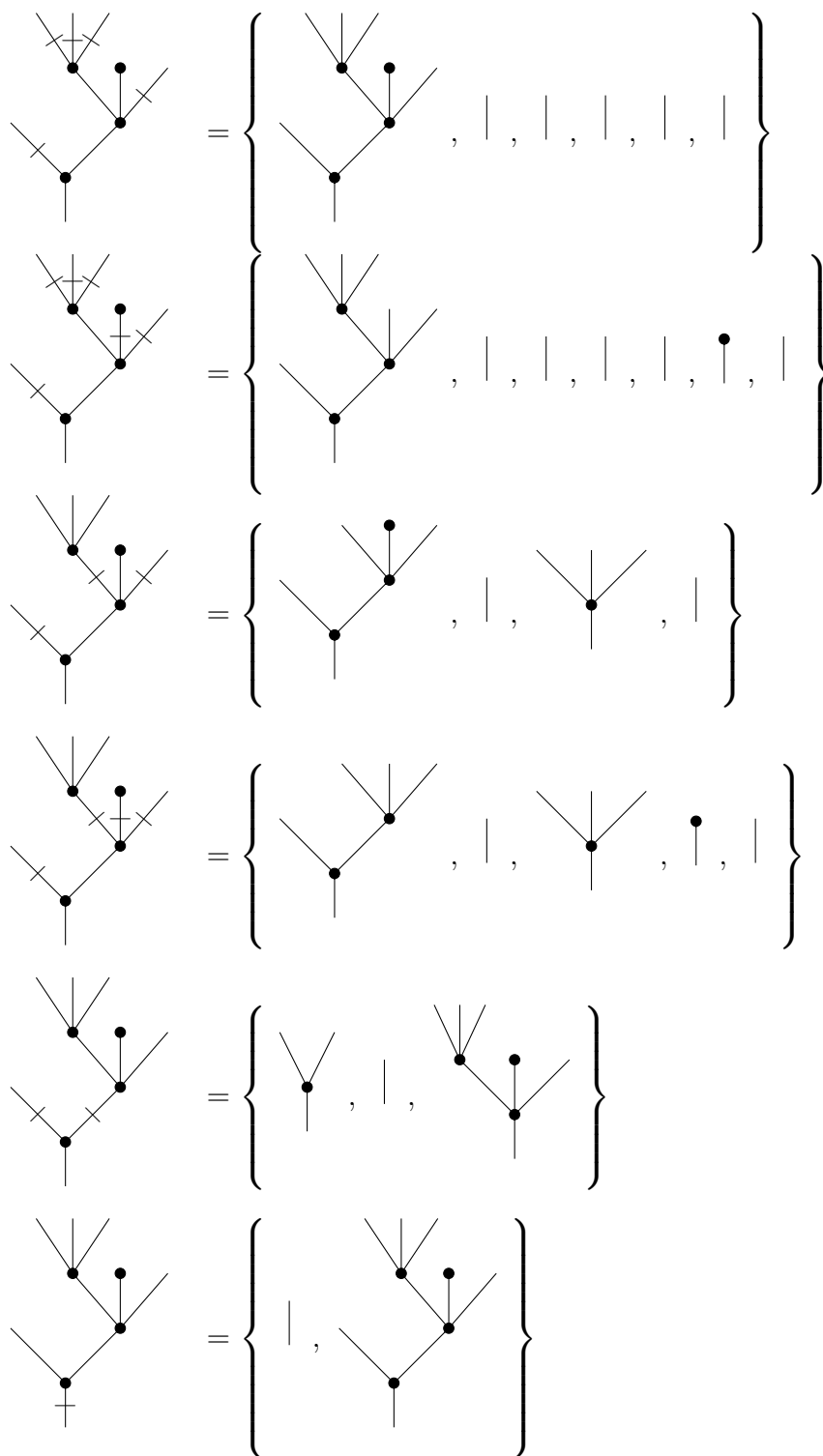
Given a reduced sequence X , the underlying sequence of $\mathcal{F}^c(X)$ is the same as that of $\mathcal{F}(X)$, the free operad on the same sequence described in the previous section in terms of planted planar trees with leaves. The counit of $\mathcal{F}^c(X)$ is the augmentation of $\mathcal{F}(X)$ and, similarly, the unit cooperation coincides with the unit operation (this defines the coaugmentation). Moreover, the map $\mathcal{F}^c(X) \rightarrow X$ is the projection onto the direct summands of (2.1.6) indexed by corollas.

It is only left to define the diagonal (comultiplication) of $\mathcal{F}^c(X)$. For this, we need the following tree notion.

Definition 2.2.2. Given planted planar trees with leaves T, S_0, S_1, \dots, S_n , we say that $\{S_0, S_1, \dots, S_n\}$ is a *cutting* of T if S_0 has n leaves and grafting each S_i , $1 \leq i \leq n$ onto the i^{th} leaf of S_0 (counting from left to right, i.e. with respect to the path order) we obtain T . We define the *degrafting* of T to be the set of all cuttings of T .

The following picture illustrates the degrafting of a planted planar tree with leaves. We see here that each cutting can be simply indicated by marking the edges where we have to cut. The key rule is that, in each path from the root to a leaf, there must be one and only one mark, and in each path from the root to an inner vertex of arity zero there may be one mark or no

mark at all.



Let $\{S_0, S_1, \dots, S_k\}$ be a cutting of a planted planar tree with leaves T . Let m_T be an ns-labeled planted planar tree with leaves with underlying tree T . For each $0 \leq i \leq n$, we define m_{S_i} as the ns-labeled planted planar tree with leaves with underlying tree S_i where the labels of its inner vertices are the same labels as these inner vertices have in T (recall that S_i is a subtree of T). The diagonal of $\mathcal{F}^c(X)$,

$$\Delta_{\mathcal{F}^c(X)}: \mathcal{F}^c(X) \rightarrow \mathcal{F}^c(X) \circ \mathcal{F}^c(X),$$

is defined as

$$\Delta_{\mathcal{F}^c(X)}(m_T) = \sum_{\text{cuttings of } T} \pm m_{S_0} \otimes m_{S_1} \otimes \cdots \otimes m_{S_n}.$$

Here, each pondering sign is that of the symmetry isomorphism needed to

obtain $m_{S_0} \otimes m_{S_1} \otimes \cdots \otimes m_{S_n}$ from m_T permuting the tensor factors, e.g.

$$\begin{aligned}
& \Delta_{\mathcal{F}^c(X)} \left(\begin{array}{c} \diagup \quad \diagdown \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \right) = \\
& \begin{array}{c} \diagup \quad \diagdown \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)}^{\otimes 5} + \begin{array}{c} \diagup \quad \diagdown \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)}^{\otimes 3} \otimes \begin{array}{c} \bullet \quad x_3 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \\
& + (-1)^{\|x_2\| \cdot \|x_3\|} \begin{array}{c} \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \otimes \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \\
& + \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \otimes \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ | \\ \bullet \end{array} \otimes \begin{array}{c} \bullet \quad x_3 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \\
& + + \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} \otimes \text{id}_{\mathcal{F}^c(X)} \otimes \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} + \text{id}_{\mathcal{F}^c(X)} \otimes \begin{array}{c} \diagdown \quad \diagup \\ \quad \bullet \quad x_3 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_2 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_1 \\ \diagdown \quad \diagup \\ \quad \bullet \quad x_0 \\ | \\ \bullet \end{array} .
\end{aligned}$$

Observe that the diagonal $\Delta_{\mathcal{F}^c(X)}$ is homogeneous with respect to the weight grading in Definition 2.1.8.

We now turn to the symmetric case. Let X be a reduced collection. The cofree conilpotent coaugmented cooperad on X will be denoted by $\mathcal{F}_\Sigma^c(X)$, in order to distinguish it from the cofree conilpotent coaugmented ns-cooperad $\mathcal{F}^c(X)$ generated by the underlying sequence of X .

The underlying collection of $\mathcal{F}_\Sigma^c(X)$ is $\mathcal{F}_\Sigma(X)$. This collection is reduced,

since X is. Hence, we can define the diagonal using the symmetric circle product, see Remark 1.3.8.

In the previous section, we have constructed the free operad $\mathcal{F}_\Sigma(X)$ by means of a natural projection $\mathcal{F}(X) \otimes k[\Sigma] \twoheadrightarrow \mathcal{F}_\Sigma(X)$ from the symmetrization of the free ns-operad on the underlying sequence. Hence, we have a natural projection

$$\mathcal{F}^c(X) \otimes k[\Sigma] \twoheadrightarrow \mathcal{F}_\Sigma^c(X). \quad (2.2.3)$$

Here $\mathcal{F}^c(X) \otimes k[\Sigma]$ is a coaugmented cooperad and $\mathcal{F}_\Sigma^c(X)$ is endowed with the unique cooperad structure which makes this natural projection a coaugmented cooperad morphism. The map $\mathcal{F}_\Sigma^c(X) \rightarrow X$ is the projection onto the direct factors of (2.1.14) indexed by corollas. The diagonal $\Delta_{\mathcal{F}_\Sigma^c(X)}$ is homogeneous with respect to the weight grading in Definition 2.1.15.

2.3 The bar and cobar constructions

In this section we describe in detail the bar and cobar constructions for (ns-)operads in graded complexes. We put some emphasis in checking that the differential squares to zero, as we feel that this fact, despite being elementary, is not adequately treated in the literature. We follow [16, Section 6.5].

Denote by $\mathbf{ns}\text{-AugOp}_r$, $\mathbf{ns}\text{-ConilCoop}_r$, and $\mathbf{ns}\text{-CoaugCoop}_r$ the full subcategories of $\mathbf{ns}\text{-AugOp}$, $\mathbf{ns}\text{-ConilCoop}$, and $\mathbf{ns}\text{-CoaugCoop}$, respectively, consisting of reduced objects. We are going to define a functor

$$\mathbf{B} : \mathbf{ns}\text{-AugOp}_r \rightarrow \mathbf{ns}\text{-ConilCoop}_r \subset \mathbf{ns}\text{-CoaugCoop}_r,$$

called the *nonsymmetric bar construction*.

For X a graded complex, its *suspension* sX is a graded complex equipped with a natural isomorphism $s : X \cong sX$ of bidegree $(0, 1)$. In particular, we can identify

$$\begin{aligned} (sX)_{i,j} &= X_{i,j-1}, \\ d^{sX}(sx) &= -s(d^X x). \end{aligned}$$

This construction extends arity-wise to sequences and collections.

Let \mathcal{O} be a reduced augmented ns-operad. The underlying sequence of bigraded k -modules of $\mathbf{B}(\mathcal{O})$ is that of $\mathcal{F}^c(s\bar{\mathcal{O}})$. Also, the coproduct $\Delta_{\mathbf{B}(\mathcal{O})} = \Delta_{\mathcal{F}^c(s\bar{\mathcal{O}})}$, the counit $\epsilon_{\mathbf{B}(\mathcal{O})} = \epsilon_{\mathcal{F}^c(s\bar{\mathcal{O}})}$, and the coaugmentation $\eta_{\mathbf{B}(\mathcal{O})} = \eta_{\mathcal{F}^c(s\bar{\mathcal{O}})}$ are those of $\mathcal{F}^c(s\bar{\mathcal{O}})$.

The ns-bar construction $\mathbf{B}(\mathcal{O})$ differs from $\mathcal{F}^c(s\bar{\mathcal{O}})$ only in its differential $d_{\mathbf{B}(\mathcal{O})}$, which is a perturbation of $d_{\mathcal{F}^c(s\bar{\mathcal{O}})}$,

$$d_{\mathbf{B}(\mathcal{O})} := d_{\mathcal{F}^c(s\bar{\mathcal{O}})} + d_2.$$

Let us define the second piece.

Define the following morphisms, $1 \leq i \leq p, q \geq 0$,

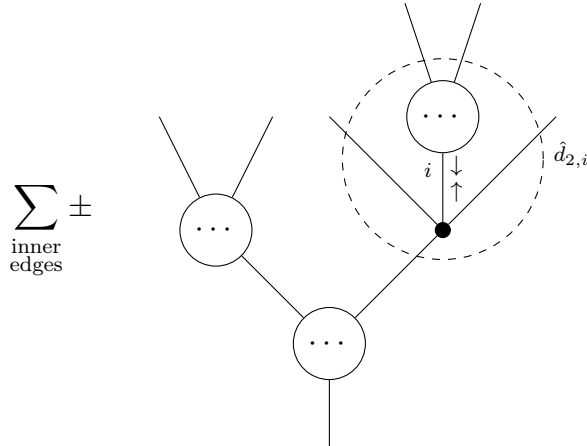
$$\begin{aligned} \hat{d}_{2,i}: s\bar{\mathcal{O}}(p) \otimes s\bar{\mathcal{O}}(q) &\longrightarrow s\bar{\mathcal{O}}(p+q-1), \\ \hat{d}_{2,i}(sx \otimes sy) &= (-1)^{\|x\|} s(x \circ_i y), \end{aligned} \quad (2.3.1)$$

which are well-defined, since $\mu_{\mathcal{O}}$ restricted to $\bar{\mathcal{O}} \circ \bar{\mathcal{O}}$ corestricts to $\bar{\mathcal{O}}$.

Define $d_2: \mathcal{F}^c(s\bar{\mathcal{O}}) \rightarrow \mathcal{F}^c(s\bar{\mathcal{O}})$ as follows. Let m_T be an ns-labeled planted planar tree with leaves. Then $d_2(m_T)$ is a sum of ns-labeled planted planar tree with leaves indexed by the inner edges of T , $\{v, w\}$, $v < w$, pondered by certain signs. The underlying planted planar tree with leaves of the summand indexed by $\{v, w\}$ is obtained by contracting this edge of T . All inner vertices, except for v and w , keep their old labels from m_T . If v and w are labeled in m_T with sx_v and sx_w , respectively, and w is in the i^{th} incoming edge of v , then the inner vertex obtained by contraction is labeled with $\hat{d}_{2,i}(sx_v \otimes sx_w)$. The pondering sign is

$$(-1)^{\|m_T\|_{<v}} (-1)^{\|sx_w\| \|m_T\|_{>v}^v}. \quad (2.3.2)$$

Here $\|m_T\|_{<v}$ is the sum of the total degrees of the labels in m_T of the inner vertices of T that come (strictly) before v , always with respect to the path order. If we write $\|m_T\|_{<v}$, we add the total degree of v . Similarly, $\|m_T\|_{<w}^v$ denotes the sum of the total degrees of the labels in m_T of the inner vertices of T that come before w and after v . An intuitive picture of this summation is



We also provide an explicit example,

$$\begin{aligned}
 d_2 \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right) &= \\
 & (-1)^{\|x_1\|} \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \end{array} + (-1)^{\|sx_1\| + \|x_2\|} \begin{array}{c} \text{Diagram 6} \\ \text{Diagram 7} \end{array} \\
 & + (-1)^{\|sx_1\| + \|sx_3\| + \|sx_4\| + \|x_2\|} \begin{array}{c} \text{Diagram 8} \\ \text{Diagram 9} \end{array}
 \end{aligned}$$

Here, some signs come from the pondering and some other signs come from the definition of $\hat{d}_{2,i}$.

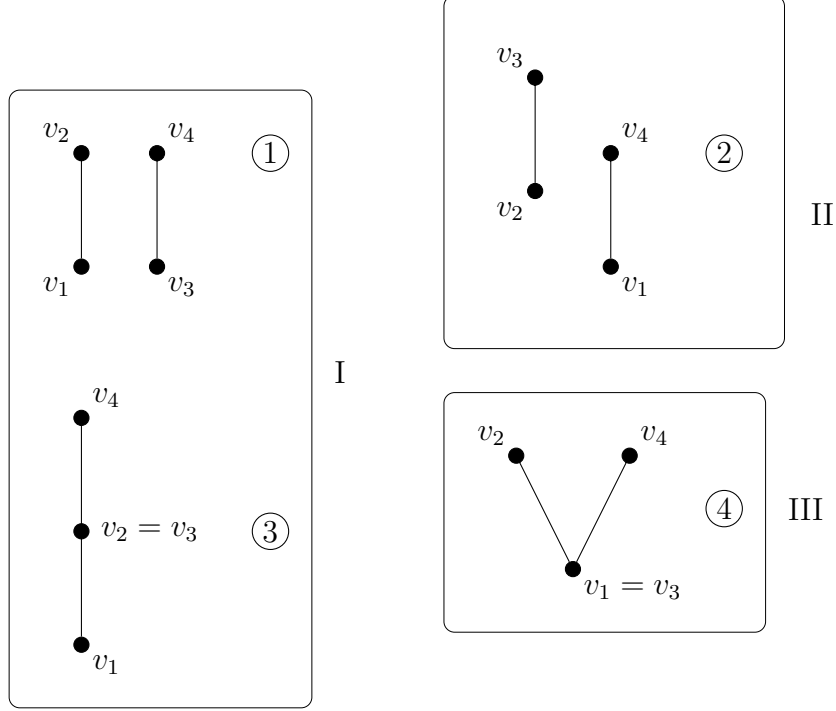
Let us check that d_2 is a differential.

Lemma 2.3.3. $(d_2)^2 = 0$.

Proof. Notice that, when d_2 is applied twice to a ns-labeled planted planar tree with leaves m_T , we end up with a summation containing two summands for each pair of different inner edges of T . Each of the two summands corresponds to one of the two different orders of contracting the two edges. The underlying tree is the same in both cases, as well as the labels up to sign. We now study the signs and check that the two summands cancel each other out in all cases.

The four possible configurations of two different inner edges in T are

depicted below (subscripts indicate the path order),



We give three different arguments for the cancellation of the two summands, I, II and III, adapted to the cases indicated in that figure. We will denote the label of the vertex v_i by sx_i .

Situation I. If we first contract $\{v_1, v_2\}$, we get an extra minus sign compared to first contracting $\{v_3, v_4\}$. This is caused by encountering the label $s(x_1 \circ_i x_2)$, of total degree $\|x_1\| + \|x_2\| + 1$, compared to the separate labels sx_1 and sx_2 , of total degree $\|x_1\| + 1$ and $\|x_2\| + 1$, respectively.

Situation II. First contracting $\{v_1, v_4\}$ and then $\{v_2, v_3\}$ results in a sign

$$(-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_4\|(\|m_T\|_{<v_2}^{>v_1} + \|sx_2\| + \|m_T\|_{<v_3}^{>v_2} + \|sx_3\| + \|m_T\|_{<v_4}^{>v_3})} (-1)^{\|x_1\|} \\ (-1)^{\|m_T\|_{<v_1} + \|x_1\| + \|x_4\| + 1 + \|m_T\|_{<v_2}^{>v_1}} (-1)^{\|sx_3\| \|m_T\|_{<v_3}^{>v_2}} (-1)^{\|x_2\|}.$$

Here, and below, we consider jointly the pondering sign and the sign in the definition of $\hat{d}_{2,i}$. If we first contract $\{v_2, v_3\}$ and then $\{v_1, v_4\}$, the sign is

$$(-1)^{\|m_T\|_{<v_1} + \|sx_1\| + \|m_T\|_{<v_2}^{>v_1}} (-1)^{\|sx_3\| \|m_T\|_{<v_3}^{>v_2}} (-1)^{\|x_2\|} \\ (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_4\|(\|m_T\|_{<v_2}^{>v_1} + \|m_T\|_{<v_3}^{>v_2} + \|m_T\|_{<v_4}^{>v_3} + \|x_2\| + \|x_3\| + 1)} (-1)^{\|x_1\|}.$$

Checking the exponents it is straightforward to see that the two previous signs are opposite.

Situation III. On the one hand, first contracting $\{v_1, v_2\}$ and then $\{v_1, v_4\}$ results in a sign

$$\begin{aligned} & (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_2\| \|m_T\|_{<v_2}^{>v_1}} (-1)^{\|x_1\|} \\ & (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_4\| (\|m_T\|_{<v_2}^{>v_1} + \|m_T\|_{<v_4}^{>v_2})} (-1)^{\|x_1\| + \|x_2\|}. \end{aligned}$$

On the other hand, first contracting $\{v_1, v_4\}$, then $\{v_1, v_2\}$, results in a sign

$$\begin{aligned} & (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_4\| (\|m_T\|_{<v_2}^{>v_1} + \|sx_2\| + \|m_T\|_{<v_4}^{>v_2})} (-1)^{\|x_1\|} \\ & (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_2\| \|m_T\|_{<v_2}^{>v_1}} (-1)^{\|x_1\| + \|x_4\|} (-1)^{\|x_2\| \|x_4\|}. \end{aligned}$$

Here, the last $(-1)^{\|x_2\| \|x_4\|}$ comes from the fact that, if v_2 and v_4 are in the i^{th} and j^{th} incoming edges of $v_1 = v_3$, $i < j$, respectively, then

$$(x_1 \circ_i x_2) \circ_{j+\tilde{v}_2-1} x_4 = (-1)^{\|x_2\| \|x_4\|} (x_1 \circ_j x_4) \circ_i x_2.$$

This equation relates the labels of the vertex obtained by contraction in each of the two previous orders. Again, the two global signs are opposite. \square

Now we check that the differential of the whole bar construction squares to zero.

Proposition 2.3.4. $(d_{\mathcal{B}(\mathcal{O})})^2 = 0$.

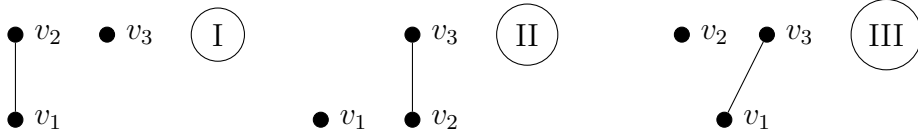
Proof. Since both $d_{\mathcal{F}^c(s\bar{\mathcal{O}})}$ and d_2 are differentials, this boils down to

$$d_{\mathcal{F}^c(s\bar{\mathcal{O}})} d_2 + d_2 d_{\mathcal{F}^c(s\bar{\mathcal{O}})} = 0.$$

On the one hand, $d_{\mathcal{F}^c(s\bar{\mathcal{O}})} d_2$ applied to a ns-labeled planted planar tree with leaves m_T is a double summation of such labeled trees indexed by first the inner edges $\{v, w\}$ of T , $v < w$, and then the inner vertices of T with $\{v, w\}$ contracted. On the other hand, $d_2 d_{\mathcal{F}^c(s\bar{\mathcal{O}})}(m_T)$ is a double summation indexed by first the inner vertices of T and then the inner edges of T . The inner vertices of T with $\{v, w\}$ contracted, away from the vertex created by contraction, are in bijection with the inner vertices of T different from v and w . We will see that the two corresponding factors cancel each other out. The underlying tree is the same in both cases, as well as the labels up to sign. Moreover, we will see that the three factors corresponding on the one hand to $\{v, w\}$ and to the inner vertex obtained by contraction, and on the other hand to the vertices v and w and to the edge $\{v, w\}$, also cancel. In this case the underlying tree is the same for the three summands, and labels out of the vertex obtained by contraction are the same, so we just have to check

that the three labels of the troublesome vertex with the right signs add up to zero.

We first consider the cancellation of the two factors associated to an inner edge and an external inner vertex. Below we depict the possible relative positions (subscripts reflect the path order). We give a different argument in each case. We denote the label of the vertex v_i by sx_i .



Situation I. When first contracting $\{v_1, v_2\}$ and then applying $d_{s\bar{O}}$ to the label of v_3 , we get an extra minus than if we proceed in the opposite order. This is caused by encountering before v_3 the label $s(x_1 \circ_i x_2)$, of total degree $\|x_1\| + \|x_2\| + 1$, compared to the separate labels sx_1 and sx_2 of degrees $\|x_1\| + 1$ and $\|x_2\| + 1$, respectively. Here v_2 is in the i^{th} incoming edge of v_1 .

Situation II. When first contracting $\{v_2, v_3\}$ and then applying $d_{s\bar{O}}$ to the label v_1 , we also get an extra minus sign than if we proceed in the other way. This comes from the pondering sign in the definition of d_2 , which includes -1 up to the sum of the total degrees of the labels before v_2 . The only different label is that of v_1 , which is $s(d_{s\bar{O}}(sx_1))$ in the first case, of total degree $\|x_1\|$, and sx_1 in the second case, of total degree $\|x_1\| + 1$.

Situation III. First contracting $\{v_1, v_3\}$ and then applying $d_{\mathcal{F}^c(s\bar{O})}$ to v_2 results in a sign

$$(-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} (-1)^{\|m_T\|_{<v_3}^{\geq v_1}} (-1)^{\|x_1\|} (-1)^{\|m_T\|_{<v_1}} (-1)^{\|x_1\| + \|x_3\| + 1} (-1)^{\|m_T\|_{<v_2}^{\geq v_1}}.$$

Here and below, we jointly consider the pondering signs of $d_{\mathcal{F}^c(s\bar{O})}$ and d_2 and the sign in the definition of $\hat{d}_{2,i}$.

On the other hand, first applying $d_{s\bar{O}}$ to v_2 and then contracting $\{v_1, v_3\}$ results in a sign

$$(-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_1\|} (-1)^{\|m_T\|_{<v_2}^{\geq v_1}} (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} (\|m_T\|_{<v_3}^{\geq v_1} - 1) (-1)^{\|x_1\|},$$

which is opposite to the previous one.

We finally tackle the troublesome situation, which involves an inner edge $\{v_1, v_2\}$, $v_1 < v_2$; three related vertices, v_1 , v_2 , and the vertex created by contraction; and the corresponding three summands. We have remarked above that these summands have the same underlying tree and only differ in the labels of the vertex created by contraction. In the analysis below, we

incorporate signs to this label. Assume that v_2 is in the i^{th} incoming edge of v_1 .

If we first contract $\{v_1, v_2\}$ and then apply $d_{s\bar{\mathcal{O}}}$ to the vertex obtained by contraction, the label of that vertex will be

$$-(-1)^{\|m_T\|_{<v_1}}(-1)^{\|sx_2\|\|m_T\|_{<v_2}^{>v_1}}(-1)^{\|x_1\|}(-1)^{\|m_T\|_{<v_1}}sd_{\bar{\mathcal{O}}}(x_1 \circ_i x_2).$$

If we first apply $d_{s\bar{\mathcal{O}}}$ to the label of v_1 and then contract $\{v_1, v_2\}$, the label of the vertex created by contraction is

$$-(-1)^{\|m_T\|_{<v_1}}(-1)^{\|m_T\|_{<v_1}}(-1)^{\|sx_2\|\|m_T\|_{<v_2}^{>v_1}}(-1)^{\|x_1\|-1}s(d_{\bar{\mathcal{O}}}(x_1) \circ_i x_2).$$

If we first apply $d_{s\bar{\mathcal{O}}}$ to the label of v_2 and then contract $\{v_1, v_2\}$, the label is

$$\begin{aligned} & -(-1)^{\|m_T\|_{<v_1} + \|sx_1\| + \|m_T\|_{<v_2}^{>v_1}} \\ & (-1)^{\|m_T\|_{<v_1}}(-1)^{(\|sx_2\|-1)\|m_T\|_{<v_2}^{>v_1}}(-1)^{\|x_1\|}s(x_1 \circ_i d_{\bar{\mathcal{O}}}(x_2)). \end{aligned}$$

The sum of the three previous labels vanishes by the operadic Leibniz rule, which says that

$$d_{\bar{\mathcal{O}}}(x_1 \circ_i x_2) = d_{\bar{\mathcal{O}}}(x_1) \circ_i x_2 + (-1)^{\|x_1\|}x_1 \circ_i d_{\bar{\mathcal{O}}}(x_2).$$

□

Denote by AugOp_r , ConilCoop_r , and CoaugCoop_r the full subcategories of AugOp , ConilCoop , and CoaugCoop , respectively, consisting of reduced objects. In full analogy with the ns-bar construction, we define a functor

$$\text{B}_\Sigma : \text{AugOp}_r \rightarrow \text{ConilCoop}_r \subset \text{CoaugCoop}_r,$$

called the *bar construction*. The underlying collection of bigraded k -modules of $\text{B}_\Sigma(\mathcal{O})$ is that of $\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})$. Also, the coproduct $\Delta_{\text{B}_\Sigma(\mathcal{O})} = \Delta_{\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})}$, the counit $\epsilon_{\text{B}_\Sigma(\mathcal{O})} = \epsilon_{\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})}$, and the coaugmentation $\eta_{\text{B}_\Sigma(\mathcal{O})} = \eta_{\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})}$. The differential, however, is perturbed

$$d_{\text{B}_\Sigma(\mathcal{O})} := d_{\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})} + d_2$$

by d_2 , which is defined as above in terms of the morphisms $\hat{d}_{2,i}$. It is just a matter of checking that the definition of d_2 does not depend on the equivalence relation defining $\mathcal{F}_\Sigma^c(s\bar{\mathcal{O}})$ as a quotient. The previous proofs that d_2 and $d_{\text{B}_\Sigma(\mathcal{O})}$ square to zero also work in the symmetric case. Moreover, we can also regard \mathcal{O} as an ns-operad, forgetting the symmetric group actions. Then, the natural projection (2.2.3) induces a coaugmented cooperad map

$$\text{B}(\mathcal{O}) \otimes k[\Sigma] \twoheadrightarrow \text{B}_\Sigma(\mathcal{O}), \quad (2.3.5)$$

which is also surjective.

The nonsymmetric bar construction has a left adjoint

$$\Omega : \text{ns-CoaugCoop}_r \rightarrow \text{ns-AugOp}_r,$$

called the *nonsymmetric cobar construction*.

The *desuspension* s^{-1} in the category of graded complexes is the inverse of the suspension functor above. More precisely, given a graded complex X , its desuspension $s^{-1}X$ is a graded complex equipped with a natural isomorphism $s^{-1}: X \cong s^{-1}X$ of bidegree $(0, -1)$. In particular,

$$\begin{aligned} (s^{-1}X)_{i,j} &= X_{i,j+1}, \\ d^{s^{-1}X}(s^{-1}x) &= -s^{-1}(d^X x). \end{aligned}$$

Given a reduced nonsymmetric coaugmented cooperad \mathcal{C} , the underlying sequence of bigraded k -modules of $\Omega(\mathcal{C})$ is that of $\mathcal{F}(s^{-1}\bar{\mathcal{C}})$. Also, the product $\mu_{\Omega(\mathcal{C})} = \mu_{\mathcal{F}(s^{-1}\bar{\mathcal{C}})}$, the unit $\eta_{\Omega(\mathcal{C})} = \eta_{\mathcal{F}(s^{-1}\bar{\mathcal{C}})}$, and the augmentation $\epsilon_{\Omega(\mathcal{C})} = \epsilon_{\mathcal{F}(s^{-1}\bar{\mathcal{C}})}$. The ns-cobar construction $\Omega(\mathcal{C})$ differs from $\mathcal{F}(s^{-1}\bar{\mathcal{C}})$ only in its differential $d_{\Omega(\mathcal{C})}$, which is

$$d_{\Omega(\mathcal{C})} := d_{\mathcal{F}(s^{-1}\bar{\mathcal{C}})} + d_2.$$

Let us define the second summand.

Since $\Omega(\mathcal{C})$ is free as an operad of bigraded modules, $\Omega(\mathcal{C}) = \mathcal{F}(s^{-1}\bar{\mathcal{C}})$, there exists a unique degree -1 map $d_2: \Omega(\mathcal{C}) \rightarrow \Omega(\mathcal{C})$ of sequences of bigraded modules satisfying the operadic Leibniz rule with a fixed restriction to the generating sequence $s^{-1}\bar{\mathcal{C}}$. The restriction of d_2 to the generating sequence will be a certain map

$$\hat{d}_2: s^{-1}\bar{\mathcal{C}} \longrightarrow (s^{-1}\bar{\mathcal{C}}) \circ_{(1)} (s^{-1}\bar{\mathcal{C}}) = \mathcal{F}(s^{-1}\bar{\mathcal{C}})^{(2)} \subset \Omega(\mathcal{C})$$

landing in the weight 2 part of the free operad $\mathcal{F}(s^{-1}\bar{\mathcal{C}})$, see Definition 2.1.8. In order to define \hat{d}_2 , we consider the infinitesimal decomposition $\Delta_{(1)}: \mathcal{C} \rightarrow \mathcal{C} \circ_{(1)} \mathcal{C}$ in Definition 1.4.14. Given $x \in \bar{\mathcal{C}}(n)$, if

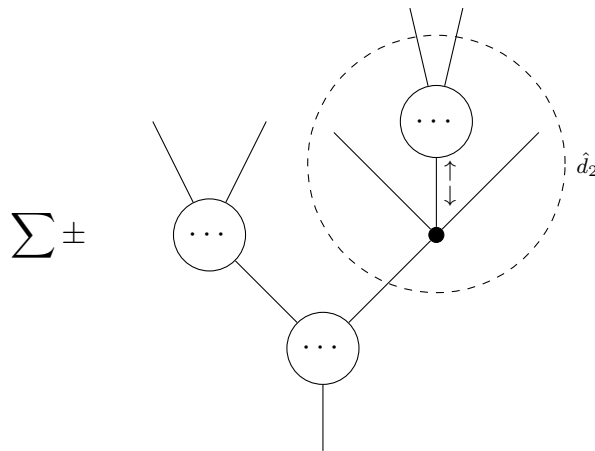
$$\Delta_{(1)}(x) - \text{id}_{\mathcal{C}} \otimes x - x \otimes \text{id}_{\mathcal{C}}^{\otimes n} = \sum_l x_{1,l} \otimes x_{2,l} \quad (2.3.6)$$

then we set

$$\hat{d}_2(s^{-1}x) = - \sum_l (-1)^{\|x_{1,l}\|} s^{-1}x_{1,l} \otimes s^{-1}x_{2,l}. \quad (2.3.7)$$

In [16, 6.5.2], the authors seem to have forgotten the subtraction of $\text{id}_{\mathcal{O}} \otimes x$ and $x \otimes \text{id}_{\mathcal{O}}^{\otimes n}$ from $\Delta_{(1)}(x)$. This is necessary since otherwise $\Delta_{(1)}(x)$ lands in $\mathcal{C} \circ_{(1)} \mathcal{C}$, not in the smaller direct summand $\bar{\mathcal{C}} \circ_{(1)} \bar{\mathcal{C}}$.

Graphically, when we apply d_2 to an ns-labeled tree m_T with leaves we obtain a summation of such labeled trees whose underlying shapes are obtained from T by blowing up inner vertices, creating new inner edges. The number of summands is impossible to quantify in general, since it depends on the decomposition of $\Delta_{(1)}(x)$ as a sum of tensors. Nevertheless, the following picture is illustrative of how d_2 works.



Proving that both d_2 and $d_{\Omega(\mathcal{C})}$ square to zero is much easier than for the bar construction, since it is enough to check it on free operad generators. This follows almost immediately from the coassociativity of Δ , hence we omit the details.

The same procedure defines the *cobar construction* in the symmetric context,

$$\Omega_{\Sigma} : \text{CoaugCoop}_r \rightarrow \text{AugOp}_r,$$

which is left adjoint to the bar construction, i.e. $\Omega_{\Sigma}(\mathcal{C})$ is $\mathcal{F}_{\Sigma}(s^{-1}\bar{\mathcal{C}})$ with differential perturbed by a certain d_2 defined in terms of the infinitesimal decomposition in Definition 1.4.19 and the weight filtration in Definition 2.1.15.

2.4 Quadratic operads and cooperads

Quadratic operads appear in nature quite frequently as models for algebras of different kind. Quadratic cooperads arise in Koszul duality theory as Koszul duals of quadratic operads.

Definition 2.4.1. An *ns-operadic quadratic data* (E, R) is a reduced sequence E with trivial differential $d_E = 0$, together with a subsequence $R \subset E \circ_{(1)} E$. The elements of E are called *generating operations* and the elements of R are called *relators*.

An *ns-operadic ideal* of an ns-operad \mathcal{O} is a subsequence $\mathcal{J} \subset \mathcal{O}$ such that $x \circ_i y \in \mathcal{J}$ if either $x \in \mathcal{J}$ or $y \in \mathcal{J}$.

Equivalently, an ns-operadic ideal is a subsequence $\mathcal{J} \subset \mathcal{O}$ such that the ns-operad structure of \mathcal{O} passes to the (arity-wise) quotient \mathcal{O}/\mathcal{J} . We refer to \mathcal{O}/\mathcal{J} as the *quotient ns-operad*.

Given ns-operadic quadratic data (E, R) , we define the associated *quadratic ns-operad* $(E|R) = \mathcal{F}(E)/(R)$ as the quotient of the free ns-operad on the generating operations by the ns-operadic ideal $(R) \subset \mathcal{F}(E)$ generated by $R \subset E \circ_{(1)} E = \mathcal{F}(E)^{(2)} \subset \mathcal{F}(E)$ in weight 2, see Definition 2.1.8. We say that (E, R) is a *presentation* of $(E|R)$. Notice that $(E|R)$ is reduced since E is reduced.

The ns-operad $(E|R)$ is universal (initial) among operads \mathcal{O} fitting into a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & \mathcal{F}(E) & \longrightarrow & \mathcal{O} \\ & & \searrow & \nearrow & \\ & & & & 0 \end{array}$$

Since (R) is a weight-homogeneous ideal (generated in weight 2), the weight grading passes to $(E|R)$. Moreover, $(E|R)$ coincides in weights 0 and 1 with $\mathcal{F}(E)$. In particular, $(E|R)$ is augmented. The augmentation is the projection onto the weight 0 part. The weight ≤ 2 part is

$$\begin{aligned} (E|R)^{(0)} &= I, \\ (E|R)^{(1)} &= E, \\ (E|R)^{(2)} &= (E \circ_{(1)} E)/R. \end{aligned}$$

The formulas in higher weights are slightly more complicated, but still friendly quotients.

The previous definitions carry over to the symmetric setting with minor modifications.

Definition 2.4.2. An *operadic quadratic data* (E, R) is a reduced collection E with trivial differential $d_E = 0$, together with a subcollection $R \subset E \circ_{\Sigma, (1)} E$. The elements of E are called *generating operations* and the elements of R are called *relators*.

An *operadic ideal* of an operad \mathcal{O} is an ns-operadic ideal such that $\mathcal{J} \subset \mathcal{O}$ is a subcollection, i.e. closed under the action of the permutation groups.

Equivalently, a subcollection $\mathcal{J} \subset \mathcal{O}$ is an operadic ideal if the operad structure of \mathcal{O} passes to the *quotient operad* \mathcal{O}/\mathcal{J} , see e.g. [16, 5.2.14].

Given operadic quadratic data (E, R) , we define the *quadratic operad* $(E|R) = \mathcal{F}_\Sigma(E)/(R)$ as the quotient of the free operad on the generating operations by the operadic ideal $(R) \subset \mathcal{F}_\Sigma(E)$ generated in weight 2 by $R \subset E \circ_{\Sigma, (1)} E = \mathcal{F}_\Sigma(E)^{(2)} \subset \mathcal{F}_\Sigma(E)$, see Definition 2.1.15. We say that (E, R) is a *presentation* of $(E|R)$. We see that $(E|R)$ is reduced since E is reduced.

The quadratic operad $(E|R)$ is universal in the same way as its nonsymmetric counterpart. Moreover, it is weight graded since R is weight homogeneous, it is augmented by the projection onto the weight 0 part, and admits the same description in weights ≤ 2 .

We now go for the cooperadic analogues.

Definition 2.4.3. A *sub-ns-cooperad* \mathcal{C} of an ns-cooperad \mathcal{D} is a cooperad which is also a subsequence $\mathcal{C} \subset \mathcal{D}$ in such a way that this inclusion is a morphism of ns-cooperads.

Given ns-operadic quadratic data (E, R) , the associated *quadratic ns-cooperad* $(E|R)_c$ is the sub-ns-cooperad of the cofree conilpotent coaugmented ns-cooperad $\mathcal{F}^c(E)$ which is universal (final) among the ns-cooperads \mathcal{C} fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{F}^c(E) \longrightarrow (E \circ_1 E)/R. \\ & \searrow & \nearrow \\ & & 0 \end{array}$$

Here the first horizontal map must be a cooperad map, and \twoheadrightarrow is the composition of the projection onto the weight 2 part $\mathcal{F}^c(E)^{(2)} = E \circ_{(1)} E$ with the natural projection onto the quotient $(E \circ_{(1)} E)/R$. We say that (E, R) is a *copresentation* of $(E|R)_c$. In this framework, we refer to the elements of E as the *generating cooperations* and to the elements of R as the *corelators*. Notice that $(E|R)_c$ is reduced since it is a sub-ns-cooperad of $\mathcal{F}^c(E)$, the latter being reduced since E is.

The quadratic ns-cooperad $(E|R)_c$ is a sub-ns-cooperad of $\mathcal{F}^c(E)$ and inherits its weight grading. In lower weights it looks like

$$\begin{aligned} (E|R)_c^{(0)} &= I, \\ (E|R)_c^{(1)} &= E, \\ (E|R)_c^{(2)} &= R. \end{aligned}$$

It is coaugmented by the inclusion of the weight 0 piece.

Let us go symmetric.

Definition 2.4.4. Given a cooperad \mathcal{D} a *subcooperad* $\mathcal{C} \subset \mathcal{D}$ is a subcollection and a cooperad such that the inclusion is a morphism of cooperads.

Given operadic quadratic data (E, R) , the associated *quadratic cooperad* $(E|R)_c$ is the sub-cooperad of the cofree conilpotent coaugmented cooperad $\mathcal{F}_\Sigma^c(E)$ with the same universal property as its nonsymmetric analogue. Again, we say that (E, R) is a *copresentation* of $(E|R)_c$. Notice that $(E|R)_c$ is reduced since it is a sub-cooperad of $\mathcal{F}_\Sigma^c(E)$, the latter being reduced since E is.

The quadratic cooperad $(E|R)_c$ inherits the weight grading from the cofree conilpotent coaugmented cooperad, see [16, 7.1.4], it is given by the same formulas as in the nonsymmetric case in weights 0, 1 and 2, and it is coaugmented by the inclusion of the weight 0 part.

2.5 The Koszul dual cooperad

The Koszul duality theory of quadratic operads (in the category of plain chain complexes) allows to construct quasi-isomorphic operads with underlying free graded operads which are minimal in a precise sense. Algebras over these new operads are called homotopy algebras. We here review the relevant notions and methods of computation, in the category of graded complexes, where quasi-isomorphisms are defined in the obvious way. Everything extends naturally from complexes to graded complexes since, except for the bigrading, all constructions depend only on the underlying total complex in the sense of Example 1.1.12, and the total complex functor is strong monoidal.

Definition 2.5.1. The *Koszul dual ns-cooperad* of the quadratic ns-operad $(E|R)$ is the quadratic ns-cooperad

$$\mathcal{O}^i := (sE|s^2R)_c.$$

This makes sense, i.e. (sE, s^2R) are ns-operadic quadratic data since $R \subset E \circ_{(1)} E$ and

$$(sE) \circ_{(1)} (sE) \cong s^2(E \circ_{(1)} E): sx \otimes sy \mapsto (-1)^{\|x\|} s^2(x \otimes y).$$

We now explain how to compute this quadratic ns-cooperad from the nonsymmetric bar construction of $\mathcal{O} = (E|R)$.

Remember that both quadratic ns-operads and cofree ns-cooperads are weight graded, see Definition 2.1.8. There is an induced weight degree on $B(E|R) = \mathcal{F}^c(s\bar{\mathcal{O}})$. The weight of an ns-labeled planted planar tree with

leaves with labels sx_1, \dots, sx_k is the sum of the weights of the labels after removing s ,

$$w(x_1) + \dots + w(x_k).$$

Notice that $\bar{\mathcal{O}}$ is concentrated in weight degrees ≥ 1 , hence the weight 0 part of $B(E|R)$ is just I . In order to remedy this situation, we consider the following related grading. The *syzygy degree* of an ns -labeled planted planar tree with leaves m_T in $B(E|R)$ is its weight degree minus the number of inner vertices of T , i.e. if the labels of m_T are sx_1, \dots, sx_k , its syzygy degree is

$$w(x_1) + \dots + w(x_k) - k.$$

The component of $B(E|R)$ of syzygy degree d is denoted by $B^d(E|R)$, whereas the component of homological bidegree (r, s) is denoted by $(B(E|R))_{r,s}$. Note that

$$B^0(E|R) = \mathcal{F}^c(E)$$

since $(E|R)^{(1)} = E$.

Since E carries a trivial differential, then so does $(E|R)$, and the differential on $B(E|R)$ reduces to d_2 . Clearly, the differential d_2 , defined in Section 2.3, preserves the weight and raises the syzygy degree by $+1$. It follows that $B(E|R)$ is a cochain complex with respect to the syzygy degree, which splits with respect to the weight grading. Hence the associated cohomology groups will be bigraded by the syzygy degree and by the weight degree.

By definition

$$\mathcal{O}^i \subset \mathcal{F}^c(sE) = B^0(E|R) \subset B(E|R).$$

Proposition 2.5.2. *Assume that $\mathcal{O} = (E|R)$ is aritywise projective. Then the previous ns -cooperad inclusion $i: \mathcal{O}^i \hookrightarrow B(E|R)$ induces an isomorphism of ns -cooperads with trivial differential:*

$$i: \mathcal{O}^i \xrightarrow{\cong} H^0(B^\bullet \mathcal{O}).$$

The proof is similar in nature to the proof of [16, Proposition 7.3.1]. However, in [16, Proposition 7.3.1] the authors are working over a ground field, whereas we work over an arbitrary commutative ring. In [8, 5.2.5] it is proved that, over a commutative ring, it is enough to assume that $\mathcal{O} = (E|R)$ is aritywise projective (which is a standing assumption in that paper [8, 0.1]) in order to assure that the previous proposition still holds. Moreover, $\mathcal{O} = (E|R)$ is connected in the sense of [8, 5.1.3] since E , and hence \mathcal{O} , is reduced and we have seen in the previous section that \mathcal{O} reduces to I in weight 0.

As we have remarked in the first paragraph of this section, working in the slightly more general setting of graded complexes does not compromise the validity of the quoted results, which are in principle for chain complexes, since the previous constructions are defined in terms of tensor products, and the totalization functor in Example 1.1.12 is strong monoidal.

Definition 2.5.3. A quadratic ns-operad $\mathcal{O} = (E|R)$ is *Koszul* if the inclusion $i: \mathcal{O}^i \hookrightarrow B(E|R)$ is a quasi-isomorphism of ns-cooperads, i.e. if $H^n(B^\bullet \mathcal{O})$ is trivial in syzygy degree $n \geq 1$.

Since $(\bar{\mathcal{O}}^i)^{(1)} = sE$, this gives a natural projection $s^{-1}\bar{\mathcal{O}}^i \twoheadrightarrow E$, the projection onto the weight 1 part. This induces a surjective ns-operad map $p: \Omega\mathcal{O}^i \twoheadrightarrow \mathcal{O}$ given by

$$\mathcal{F}(s^{-1}\bar{\mathcal{O}}^i) \xrightarrow{\mathcal{F}(s^{-1}\bar{\mathcal{O}}^i \rightarrow E)} \mathcal{F}(E) \twoheadrightarrow (E|R).$$

Theorem 2.5.4. *Assume that $\mathcal{O} = (E|R)$ and \mathcal{O}^i are aritywise projective. Then the quadratic ns-operad $(E|R)$ being Koszul is equivalent to $p: \Omega\mathcal{O}^i \twoheadrightarrow \mathcal{O}$ being a quasi-isomorphism of ns-operads.*

The proof is given in [8, Proposition 5.2.12] when working over complexes of modules over a commutative ring. The conditions for this Proposition are satisfied in our case, because of remarks [8, 5.1.2 and 5.1.3]. As before, this theorem stays valid for graded complexes. Note that the projectivity hypothesis for \mathcal{O}^i holds automatically if the ground ring is a principal ideal domain, or more generally a Dedekind ring. Indeed, by Proposition 2.5.2, if \mathcal{O} is projective then \mathcal{O}^i is a submodule of a projective module. The projectivity hypothesis on \mathcal{O}^i also holds if the inclusion $i: \mathcal{O}^i \hookrightarrow B(E|R)$ is not only a quasi-isomorphism, but the inclusion of a strong deformation retract, as it often is. In this last case, k can be any commutative ground ring.

Definition 2.5.5. If \mathcal{O} is an aritywise projective Koszul quadratic ns-operad with \mathcal{O}^i also aritywise projective, we call $\mathcal{O}_\infty = \Omega\mathcal{O}^i$ the *minimal model* of \mathcal{O} .

The word *minimal* indicates that $\Omega\mathcal{O}^i$ is as small as possible within a class of operad determined by \mathcal{O} satisfying particularly good properties. We prefer not to dive into the subtleties of minimality, for they are not relevant in what follows, and refer the reader to [16, 6.3.4].

Everything above makes sense in the symmetric settings. Moreover, the references provided above actually work in this more general setting. Hence, below, we simply provide succinct symmetric statements without further comments.

Definition 2.5.6. The *Koszul dual cooperad* of a quadratic operad $(E|R)$ is the quadratic cooperad

$$\mathcal{O}^i := (sE|s^2R)_c.$$

Proposition 2.5.7. *Assume that $\mathcal{O} = (E|R)$ is aritywise projective. Then natural cooperad inclusion $i: \mathcal{O}^i \hookrightarrow B_\Sigma(E|R)$ induces an isomorphism of cooperads with trivial differential:*

$$i: \mathcal{O}^i \xrightarrow{\cong} H^0(B_\Sigma^\bullet \mathcal{O}).$$

Definition 2.5.8. A quadratic operad $\mathcal{O} = (E|R)$ is *Koszul* if the inclusion $i: \mathcal{O}^i \hookrightarrow B_\Sigma(E|R)$ is a quasi-isomorphism of cooperads, i.e. if $H^n(B_\Sigma^\bullet \mathcal{O})$ is trivial in syzygy degree $n \geq 1$.

Theorem 2.5.9. *Assume that $\mathcal{O} = (E|R)$ and \mathcal{O}^i are aritywise projective over the ground ring. Then the quadratic operad $(E|R)$ being Koszul is equivalent to $p: \Omega_\Sigma \mathcal{O}^i \rightarrow \mathcal{O}$ being a quasi-isomorphism of operads.*

As above, the projectivity hypothesis for \mathcal{O}^i holds automatically if the ground ring is a principal ideal domain or a Dedekind ring, or if $i: \mathcal{O}^i \hookrightarrow B_\Sigma(E|R)$ is the inclusion of a strong deformation retract.

Definition 2.5.10. If \mathcal{O} is an aritywise projective Koszul quadratic operad with \mathcal{O}^i also aritywise projective over the ground ring, we call $\mathcal{O}_\infty = \Omega_\Sigma \mathcal{O}^i$ the *minimal model* of \mathcal{O} .

Chapter 3

Derived operads and their algebras

- In the process of its internal development and prompted by its inner logic, mathematics, too, creates virtual worlds of great complexity and internal beauty which defy any attempt to describe them in natural language but challenge the imagination of a handful of professionals in many successive generations. -

Yuri I. Manin in “Mathematics as metaphor”

In our way of approaching the construction of minimal models for operadic algebras over a general commutative ground ring, we need to appropriately extend classical algebra notions to bicomplexes. This is done in this chapter in a way that will permit the later use of powerful tools from Koszul duality theory for operads.

3.1 Derived operads

Recall that the *ring of dual numbers* is the quotient of a polynomial ring on one variable Δ by its square,

$$\frac{k[\Delta]}{(\Delta^2)} \cong k \cdot 1 \oplus k \cdot \Delta.$$

As a module, it is free of rank two, with basis formed by the unit and the variable.

Definition 3.1.1. The *(ns-)operad of dual numbers* \mathcal{D} is given in arity 1 by

the ring of dual numbers

$$\mathcal{D}(1) = \frac{k[\Delta]}{(\Delta^2)},$$

with Δ of bidegree $(-1, 0)$, and zero elsewhere $\mathcal{D}(n) = 0$, $n \neq 1$. The identity operation is clearly $\text{id}_{\mathcal{D}} = 1$.

Note that both operads and ns-operads concentrated in arity 1 are the same as unital algebras.

Lemma 3.1.2. *The (ns-)operad of dual numbers is quadratic associated to the (ns-)operadic quadratic data $(k \cdot \Delta, k \cdot (\Delta \otimes \Delta))$, where $k \cdot \Delta$ and $k \cdot (\Delta \otimes \Delta)$ are free modules of rank 1 regarded as collections (resp. sequences) concentrated in arity 1,*

$$\mathcal{D} = (k \cdot \Delta | k \cdot (\Delta \otimes \Delta)).$$

This lemma is a mere observation. Note that the second variable of the previous quadratic data is as big as possible, since Δ is in arity 1,

$$(k \cdot \Delta) \circ_{(1)} (k \cdot \Delta) = k \cdot (\Delta \otimes \Delta).$$

As any quadratic operad, the operad of dual numbers is augmented. Its augmentation $\epsilon_{\mathcal{D}}: \mathcal{D} \rightarrow I$ is defined by $\epsilon_{\mathcal{D}}(\Delta) = 0$.

Corollary 3.1.3. *A \mathcal{D} -algebra is the same as a graded complex (X, d^X) equipped with an extra (horizontal) differential of bidigree $(-1, 0)$, $d_h: X \rightarrow X$, such that, if $d_v = d^X$, $d_h d_v + d_v d_h = 0$, i.e. a bicomplex (X, d_h, d_v) .*

Proof. A map $\psi: \mathcal{D} \rightarrow \mathcal{E}(X)$ providing a \mathcal{D} -algebra structure is determined by the choice of $\psi(\Delta) = d_h$ satisfying $d_h^2 = 0$. Moreover, since the dual numbers have trivial differential, the compatibility of ψ with differentials is equivalent to $0 = d^{[X, X]}(d_h) = d^X d_h + d_h d^X$. \square

Definition 3.1.4. Let \mathcal{O} be an ns-operad. We define the morphism of sequences $\varphi: \mathcal{D} \circ \mathcal{O} \rightarrow \mathcal{O} \circ \mathcal{D}$ by

$$\begin{aligned} \varphi(1 \otimes x) &= x \otimes 1 \otimes \cdots \otimes 1, \\ \varphi(\Delta \otimes x) &= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes 1 \otimes \cdots \otimes 1. \end{aligned}$$

Indeed, φ is a well-defined map of sequences since $\mathcal{D} \circ \mathcal{O} = (k[\Delta]/(\Delta^2)) \otimes \mathcal{O}$.

Lemma 3.1.5. *The map of sequences φ in Definition 3.1.4 is a distributive law.*

Proof. We must check that the diagrams in Definition 1.2.4 commute. The commutativity of 3 and 4 is equivalent to the following two formulas,

$$\begin{aligned}\varphi(\Delta \otimes \text{id}_{\mathcal{O}}) &= \text{id}_{\mathcal{O}} \otimes \Delta, \\ \varphi(1 \otimes x) &= x \otimes 1 \otimes \cdots \otimes 1,\end{aligned}$$

for $x \in \mathcal{O}$. In order to check the commutativity of 1, we must consider four kinds of elements in $\mathcal{D} \circ (\mathcal{D} \circ \mathcal{O})$:

1. $1 \otimes (1 \otimes x)$.
2. $1 \otimes (\Delta \otimes x)$.
3. $\Delta \otimes (1 \otimes x)$.
4. $\Delta \otimes (\Delta \otimes x)$.

Again $x \in \mathcal{O}$. We will deal with these four cases separately. In each of the cases, we must check that the result of two series of equations coincide.

Case 1. On the one hand,

$$\begin{aligned}(\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{D}} \circ \varphi)(1 \otimes (1 \otimes x)) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbf{1}_{\mathcal{D}})(1 \otimes (x \otimes 1 \otimes \cdots \otimes 1)) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbf{1}_{\mathcal{D}})((1 \otimes x) \otimes 1 \otimes \cdots \otimes 1) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})((x \otimes 1 \otimes \cdots \otimes 1) \otimes 1 \otimes \cdots \otimes 1) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)) \\ &= x \otimes 1 \otimes \cdots \otimes 1.\end{aligned}$$

Here, and below, we use the associators of \circ . On the other hand,

$$\begin{aligned}\varphi(\mu_{\mathcal{D}} \circ \mathbf{1}_{\mathcal{O}})(1 \otimes (1 \otimes x)) &= \varphi(\mu_{\mathcal{D}} \circ \mathbf{1}_{\mathcal{O}})((1 \otimes 1) \otimes x) \\ &= \varphi(1 \otimes x) \\ &= x \otimes 1 \otimes \cdots \otimes 1.\end{aligned}$$

Case 2. On the one hand,

$$\begin{aligned}
& (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbb{1}_{\mathcal{D}})(\mathbb{1}_{\mathcal{D}} \circ \varphi)(1 \otimes (\Delta \otimes x)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbb{1}_{\mathcal{D}})(1 \otimes (x \otimes 1 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes 1)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes (1 \otimes 1) \otimes \cdots \otimes \underbrace{(1 \otimes \Delta)}_{i\text{-th place}} \otimes \cdots \otimes (1 \otimes 1)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi(\mu_{\mathcal{D}} \circ \mathbb{1}_{\mathcal{O}})((1 \otimes \Delta) \otimes x) &= \varphi(\Delta \otimes x) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes 1.
\end{aligned}$$

Case 3. On the one hand,

$$\begin{aligned}
& (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbb{1}_{\mathcal{D}})(\mathbb{1}_{\mathcal{D}} \circ \varphi)(\Delta \otimes (1 \otimes x)) \\
&= (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbb{1}_{\mathcal{D}})(\Delta \otimes (x \otimes 1 \otimes \cdots \otimes 1)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes (1 \otimes 1) \otimes \cdots \otimes \underbrace{(\Delta \otimes 1)}_{i\text{-th place}} \otimes \cdots \otimes (1 \otimes 1)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes 1.
\end{aligned}$$

And on the other hand,

$$\begin{aligned}
\varphi(\mu_{\mathcal{D}} \circ \mathbb{1}_{\mathcal{O}})((\Delta \otimes 1) \otimes x) &= \varphi(\Delta \otimes x) \\
&= (-1)^{\|x\|} \sum x \otimes 1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1,
\end{aligned}$$

Case 4. In this case,

$$\begin{aligned}
& (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{D}} \circ \varphi)(\Delta \otimes (\Delta \otimes x)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(\varphi \circ \mathbf{1}_{\mathcal{D}})(\Delta \otimes (x \otimes 1 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes 1)) \\
&= (-1)^{\|x\|} \sum_{i,j=1}^{\text{arity of } x} (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})((x \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots) \otimes \cdots \otimes \underbrace{\Delta}_{j\text{-th place}} \otimes \cdots) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} \sum_{j=i+1}^{\text{arity of } x} (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes \cdots \otimes \underbrace{(\Delta \otimes 1)}_{i\text{-th place}} \otimes \cdots \\
&\quad \cdots \otimes \underbrace{(1 \otimes \Delta)}_{j\text{-th place}} \otimes \cdots) \\
&\quad + (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes \cdots \otimes \underbrace{(\Delta \otimes \Delta)}_{i\text{-th place}} \otimes \cdots) \\
&\quad - (-1)^{\|x\|} \sum_{j=1}^{\text{arity of } x} \sum_{i=j+1}^{\text{arity of } x} (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{D}})(x \otimes \cdots \otimes \underbrace{(1 \otimes \Delta)}_{j\text{-th place}} \otimes \cdots \\
&\quad \cdots \otimes \underbrace{(\Delta \otimes 1)}_{i\text{-th place}} \otimes \cdots) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} \sum_{j=i+1}^{\text{arity of } x} (x \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots \otimes \underbrace{\Delta}_{j\text{-th place}} \otimes \cdots) \\
&\quad - (-1)^{\|x\|} \sum_{j=1}^{\text{arity of } x} \sum_{i=j+1}^{\text{arity of } x} (x \otimes \cdots \otimes \underbrace{\Delta}_{j\text{-th place}} \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots) \\
&= 0.
\end{aligned}$$

Here we use that $\mu_{\mathcal{D}}(\Delta \otimes \Delta) = 0$ and the fact that Δ has odd total degree. For this very same reason,

$$\varphi(\mu_{\mathcal{D}} \circ \mathbf{1}_{\mathcal{O}})((\Delta \otimes \Delta) \otimes x) = 0$$

In order to check the commutativity of 2, it is enough to consider two kinds of elements in $(\mathcal{D} \circ \mathcal{O}) \circ \mathcal{O}$:

1. $(1 \otimes x_0) \otimes x_1 \otimes \cdots \otimes x_n$.
2. $(\Delta \otimes x_0) \otimes x_1 \otimes \cdots \otimes x_n$.

Here $x_i \in \mathcal{O}$ and x_0 has arity n . Again, we consider each case separately.

Case 1. On the one hand,

$$\begin{aligned}
& (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)(\varphi \circ \mathbf{1}_{\mathcal{O}})((1 \otimes x_0) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)((x_0 \otimes 1 \otimes \cdots \otimes 1) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)(x_0 \otimes (1 \otimes x_1) \otimes \cdots \otimes (1 \otimes x_n)) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(x_0 \otimes (x_1 \otimes 1 \otimes \cdots \otimes 1) \otimes \cdots \otimes (x_n \otimes 1 \otimes \cdots \otimes 1)) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})((x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes 1 \otimes \cdots \otimes 1) \\
&= \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes 1 \otimes \cdots \otimes 1.
\end{aligned}$$

Here, and below, we use the associators of \circ . On the other hand,

$$\begin{aligned}
& \varphi(\mathbf{1}_{\mathcal{D}} \circ \mu_{\mathcal{O}})(1 \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) \\
&= \varphi(1 \otimes \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)) \\
&= \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes 1 \otimes \cdots \otimes 1.
\end{aligned}$$

Case 2. On the one hand,

$$\begin{aligned}
& (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)(\varphi \circ \mathbf{1}_{\mathcal{O}})((\Delta \otimes x_0) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (-1)^{\|x_0\|} \sum_{i=1}^{\text{arity of } x_0} (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)((x_0 \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (-1)^{\sum_{j=0}^{i-1} \|x_j\|} \sum_{i=1}^{\text{arity of } x_0} (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(\mathbf{1}_{\mathcal{O}} \circ \varphi)(x_0 \otimes (1 \otimes x_1) \otimes \cdots \\
&\quad \cdots \otimes (\Delta \otimes x_i) \otimes \cdots \otimes (1 \otimes x_n)) \\
&= (-1)^{\sum_{j=0}^i \|x_j\|} \sum_{i=1}^{\text{arity of } x_0} \sum_{l=1}^{\text{arity of } x_i} (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})(x_0 \otimes (x_1 \otimes 1 \otimes \cdots \otimes 1) \otimes \cdots \\
&\quad \cdots \otimes (x_i \otimes \cdots \otimes \underbrace{\Delta}_{l\text{-th place}} \otimes \cdots) \otimes \cdots \otimes (x_n \otimes 1 \otimes \cdots \otimes 1)) \\
&= (-1)^{\sum_{j=0}^n \|x_j\|} \sum_{i=1}^{\sum_{l=1}^n \text{arity of } x_l} (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{D}})((x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots) \\
&= (-1)^{\|x_0 \otimes x_1 \otimes \cdots \otimes x_n\|} \sum_{i=1}^{\text{arity of } x_0 \otimes x_1 \otimes \cdots \otimes x_n} (\mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \varphi(\mathbf{1}_{\mathcal{D}} \circ \mu_{\mathcal{O}})(\Delta \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) \\
&= \varphi(\Delta \otimes \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)) \\
&= (-1)^{\|\mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)\|} \sum_{i=1}^{\text{arity of } \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)} \mu_{\mathcal{O}}(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots .
\end{aligned}$$

This computation concludes the proof. \square

Definition 3.1.6. The *derived ns-operad* of an ns-operad \mathcal{O} is

$$d\mathcal{O} := \mathcal{O} \circ_{\varphi} \mathcal{D}.$$

The operad structure is given by Proposition 1.2.5.

We now consider the symmetric version.

Definition 3.1.7. Let \mathcal{O} be an operad. We define the morphism of collections $\varphi: \mathcal{D} \circ_{\Sigma} \mathcal{O} \rightarrow \mathcal{O} \circ_{\Sigma} \mathcal{D}$ by

$$\begin{aligned}
\varphi(1 \otimes x) &= x \otimes 1 \otimes \cdots \otimes 1 \otimes \text{id}, \\
\varphi(\Delta \otimes x) &= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes 1 \otimes \cdots \otimes 1 \otimes \text{id}.
\end{aligned}$$

Here id is the identity permutation.

Lemma 3.1.8. *The map φ in Definition 3.1.7 is a distributive law.*

Proof. It is a map of sequences by the same reason as in the non-symmetric case. Indeed, $\mathcal{D} \circ_{\Sigma} \mathcal{O} = \mathcal{D} \circ \mathcal{O}$ since \mathcal{D} is concentrated in arity 1.

Moreover, φ is compatible with the symmetric group actions since, given $x \in \mathcal{O}(n)$ and $\tau \in \Sigma_n$, on the one hand,

$$\begin{aligned}
\varphi((1 \otimes x) \cdot \tau) &= \varphi(1 \otimes (x \cdot \tau)) \\
&= (x \cdot \tau) \otimes 1 \otimes \cdots \otimes 1 \otimes \text{id} \\
&= x \otimes 1 \otimes \cdots \otimes 1 \otimes \tau \\
&= \varphi(1 \otimes x) \cdot \tau,
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
& \varphi((\Delta \otimes x) \cdot \tau) \\
&= \varphi(\Delta \otimes (x \cdot \tau)) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} (x \cdot \tau) \otimes (1 \otimes \cdots \otimes 1 \otimes \Delta \otimes 1 \otimes \cdots \otimes 1 \otimes \text{id}_n) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes (1 \otimes \cdots \otimes 1 \otimes \Delta \otimes 1 \otimes \cdots \otimes 1 \otimes \tau) \\
&= \varphi(\Delta \otimes x) \cdot \tau.
\end{aligned}$$

Hence, φ is indeed a morphism of collections. Here we use the relations imposed by the tensor product over Σ_n in

$$(\mathcal{O} \circ_{\Sigma} \mathcal{D})(n) = \mathcal{O}(n) \otimes_{\Sigma_n} (\mathcal{D}(1) \otimes \cdots \otimes \mathcal{D}(1) \otimes k[\Sigma_n]).$$

Recall that Σ_n acts on the left Σ_n -module by permuting the first n copies of $\mathcal{D}(1)$ and in the standard way on $k[\Sigma_n]$.

The fact that φ satisfies the properties required to a distributive law follows as in Lemma 3.1.5. \square

Definition 3.1.9. The *derived operad* of an operad \mathcal{O} is

$$d\mathcal{O} := \mathcal{O} \circ_{\Sigma, \varphi} \mathcal{D}.$$

The operad structure is given by Proposition 1.2.5.

3.2 Derived algebras

In this section we characterize algebras over derived operads.

Definition 3.2.1. Given an (ns-)operad \mathcal{O} , a *derived \mathcal{O} -algebra* is a $d\mathcal{O}$ -algebra.

Proposition 3.2.2. *A derived \mathcal{O} -algebra is an \mathcal{O} -algebra with underlying graded complex (X, d^X) equipped with a bidegree $(-1, 0)$ differential d_h such that $d_h d_v + d_v d_h = 0$, $d_v = d^X$, so X becomes a bicomplex (X, d_h, d_v) , and such that the \mathcal{O} -algebra structure morphisms $\rho_n: \mathcal{O}(n) \otimes X \otimes \cdots \otimes X \rightarrow X$, $n \geq 0$, in the sense of Remark 1.5.3, are compatible with the horizontal differential on both sides, taking the trivial horizontal differential on \mathcal{O} , i.e.*

$$d_h \rho(y \otimes x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^n (-1)^{\|y\| + \sum_{j=1}^{i-1} \|x_j\|} \rho(y \otimes \cdots \otimes d_h(x_i) \otimes \cdots).$$

Proof. Let us consider the nonsymmetric case. Since $d\mathcal{O} = \mathcal{D} \circ_{\varphi} \mathcal{O}$, we use the universal property of the circle product of two operads in Theorem 1.2.6. As per that theorem, a derived \mathcal{O} -algebra X consists of a \mathcal{D} -algebra structure and an \mathcal{O} -algebra structure on the graded complex X , given by maps

$$\xi: \mathcal{O} \rightarrow \mathcal{E}(X), \quad \psi: \mathcal{D} \rightarrow \mathcal{E}(X),$$

for which the following pentagonal diagram commutes

$$\begin{array}{ccc}
 \mathcal{O} \circ \mathcal{D} & \xrightarrow{\xi \circ \psi} & \mathcal{E}(X) \circ \mathcal{E}(X) \\
 \uparrow \varphi & & \downarrow \mu_{\mathcal{E}(X)} \\
 \mathcal{D} \circ \mathcal{O} & & \mathcal{E}(X) \\
 \searrow \psi \circ \xi & & \nearrow \mu_{\mathcal{E}(X)} \\
 & \mathcal{E}(X) \circ \mathcal{E}(X) &
 \end{array}$$

By Corollary 3.1.3, the \mathcal{D} -algebra structure is equivalent to giving a horizontal differential $\psi(\Delta) = d_h$ on X which, together with the intrinsic (vertical) differential of the graded complex X , forms a bicomplex. Let us check that the pentagon condition is equivalent to the compatibility of d_h with the action of \mathcal{O} .

The pentagon condition is equivalent to, for any $x \in \mathcal{O}$,

$$\begin{aligned}
 \mu_{\mathcal{E}(X)}(\psi \circ \xi)(1 \otimes x) &= \mu_{\mathcal{E}(X)}(\xi \circ \psi)\varphi(1 \otimes x), \\
 \mu_{\mathcal{E}(X)}(\psi \circ \xi)(\Delta \otimes x) &= \mu_{\mathcal{E}(X)}(\xi \circ \psi)\varphi(\Delta \otimes x).
 \end{aligned}$$

The first equation is satisfied since,

$$\begin{aligned}
 \mu_{\mathcal{E}(X)}(\psi \circ \xi)(1 \otimes x) &= \mu_{\mathcal{E}(X)}(\mathbf{1}_X \otimes \xi(x)) \\
 &= \xi(x), \\
 \mu_{\mathcal{E}(X)}(\xi \circ \psi)\varphi(1 \otimes x) &= \mu_{\mathcal{E}(X)}(\xi \circ \psi)(x \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) \\
 &= \mu_{\mathcal{E}(X)}(\xi(x) \otimes \mathbf{1}_X \otimes \cdots \otimes \mathbf{1}_X) \\
 &= \xi(x).
 \end{aligned}$$

For the second equation to hold, the results of the two following series of

equations should coincide

$$\begin{aligned}
\mu_{\mathcal{E}(X)}(\psi \circ \xi)(\Delta \otimes x) &= \mu_{\mathcal{E}(X)}(d_h \otimes \xi(x)) \\
&= d_h \xi(x), \\
\mu_{\mathcal{E}(X)}(\xi \circ \psi)\varphi(\Delta \otimes x) &= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} \mu_{\mathcal{E}(X)}(\xi \circ \psi)(x \otimes \cdots \otimes \underbrace{\Delta}_{i\text{-th place}} \otimes \cdots) \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} \mu_{\mathcal{E}(X)}(\xi(x) \otimes \cdots \otimes \underbrace{d_h}_{i\text{-th place}} \otimes \cdots), \\
&= (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} \xi(x)(\dots, \underbrace{d_h}_{i\text{-th place}}, \dots).
\end{aligned}$$

Since \mathcal{O} has no horizontal differential, this is equivalent to say that the structure morphisms of the \mathcal{O} -algebra X are maps of bicomplexes.

The symmetric case is completely analogous. \square

3.3 A presentation for quadratic derived operads

Let $\mathcal{O} = (E|R)$ and $\mathcal{P} = (F|S)$ be two quadratic ns-operads and let $\varphi: \mathcal{P} \circ \mathcal{O} \rightarrow \mathcal{O} \circ \mathcal{P}$ be a distributive law. We are interested in obtaining a presentation of $\mathcal{O} \circ_{\varphi} \mathcal{P}$, which particularizes to a presentation of the derived operad of a quadratic operad. We will obtain it under certain hypotheses on φ , specified in Theorem 3.3.2 below.

Definition 3.3.1. An *ns-rewriting rule* is a morphism of sequences $\lambda: F \circ_{(1)} E \rightarrow E \circ_{(1)} F$. The graph of the ns-rewriting rule λ is the image of

$$\begin{pmatrix} \mathbf{1}_{F \circ_{(1)} E} \\ -\lambda \end{pmatrix}: F \circ_{(1)} E \rightarrow F \circ_{(1)} E \oplus E \circ_{(1)} F.$$

Now define $\mathcal{O} \vee_{\lambda} \mathcal{P}$ as the quadratic ns-operad

$$\mathcal{O} \vee_{\lambda} \mathcal{P} = (E \oplus F | R \oplus D_{\lambda} \oplus S).$$

The definition of $\mathcal{O} \vee_{\lambda} \mathcal{P}$ makes sense since

$$\begin{aligned}
R \oplus D_{\lambda} \oplus S &\subset E \circ_{(1)} E \oplus (F \circ_{(1)} E \oplus E \circ_{(1)} F) \oplus F \circ_{(1)} F \\
&= (E \oplus F) \circ_{(1)} (E \oplus F) \\
&= \mathcal{F}(E \oplus F)^{(2)}.
\end{aligned}$$

The universal property satisfied by the quadratic ns-operad \mathcal{O} provides a unique morphism of ns-operads $\alpha: \mathcal{O} \rightarrow \mathcal{O} \vee_\lambda \mathcal{P}$ making the following diagram commutative:

$$\begin{array}{ccc}
& & 0 \\
& \curvearrowright & \\
R & \xrightarrow{\quad} & \mathcal{F}(E) \xrightarrow{\quad} \frac{\mathcal{F}(E)}{(R)} =: (E|R) \\
& \searrow & \searrow \\
& & \mathcal{F}(E \oplus F) \\
& \searrow & \searrow \\
& & \frac{\mathcal{F}(E \oplus F)}{(R \oplus D_\lambda \oplus S)} =: \mathcal{O} \vee_\lambda \mathcal{P} \\
& \curvearrowleft & \\
& & 0
\end{array}$$

Similarly, the universal property satisfied by \mathcal{P} provides a morphism of ns-operads $\beta: \mathcal{P} \rightarrow \mathcal{O} \vee_\lambda \mathcal{P}$.

Theorem 3.3.2. *Suppose $\varphi: \mathcal{P} \circ \mathcal{O} \rightarrow \mathcal{O} \circ \mathcal{P}$ is a distributive law which (co)restricts to an ns-rewriting rule $\lambda: F \circ_{(1)} E \rightarrow E \circ_{(1)} F$, in the sense that the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{P} \circ \mathcal{O} & \xrightarrow{\quad \varphi \quad} & \mathcal{O} \circ \mathcal{P} \\
\uparrow (\mathcal{F}(F) \rightarrow \mathcal{P}) \circ (\mathcal{F}(E) \rightarrow \mathcal{O}) & & \uparrow (\mathcal{F}(E) \rightarrow \mathcal{O}) \circ (\mathcal{F}(F) \rightarrow \mathcal{P}) \\
\mathcal{F}(F) \circ \mathcal{F}(E) & & \mathcal{F}(E) \circ \mathcal{F}(F) \\
\uparrow (F \hookrightarrow \mathcal{F}(F)) \circ (\eta_{\mathcal{F}(E)}, E \hookrightarrow \mathcal{F}(E)) & & \uparrow (E \hookrightarrow \mathcal{F}(E)) \circ (\eta_{\mathcal{F}(F)}, F \hookrightarrow \mathcal{F}(F)) \\
F \circ (I \oplus E) & & E \circ (I \oplus F) \\
\uparrow \text{Remark 1.4.16} & & \uparrow \text{Remark 1.4.16} \\
F \circ_{(1)} E & \xrightarrow{\quad \lambda \quad} & E \circ_{(1)} F
\end{array}$$

Then the map $p = \mu_{\mathcal{O} \vee_\lambda \mathcal{P}}(\alpha \circ \beta): \mathcal{O} \circ_\varphi \mathcal{P} \rightarrow \mathcal{O} \vee_\lambda \mathcal{P}$ is an isomorphism of ns-operads. Moreover, $\varphi := p^{-1} \mu_{\mathcal{O} \vee_\lambda \mathcal{P}}(\beta \circ \alpha)$.

Proof. We are going to define an operad map

$$p^{-1}: \mathcal{O} \vee_\lambda \mathcal{P} \rightarrow \mathcal{O} \circ_\varphi \mathcal{P}$$

by using the universal property of the quadratic operad $\mathcal{O} \vee_\lambda \mathcal{P}$. We will later show that this map is the inverse of p , but for the moment p^{-1} will only be an independent symbol. To this end, define $p^{-1}|_E: E \rightarrow \mathcal{O} \circ_\varphi \mathcal{P}$ as the composition

$$E \hookrightarrow \mathcal{F}(E) \twoheadrightarrow \mathcal{O} \cong \mathcal{O} \circ I \xrightarrow{1_{\mathcal{O}} \circ \eta_{\mathcal{P}}} \mathcal{O} \circ_\varphi \mathcal{P}$$

and $p^{-1}|_F: F \rightarrow \mathcal{O} \circ_\varphi \mathcal{P}$ as

$$F \hookrightarrow \mathcal{F}(F) \twoheadrightarrow \mathcal{P} \cong I \circ \mathcal{P} \xrightarrow{\eta_{\mathcal{O}} \circ 1_{\mathcal{P}}} \mathcal{O} \circ_\varphi \mathcal{P}.$$

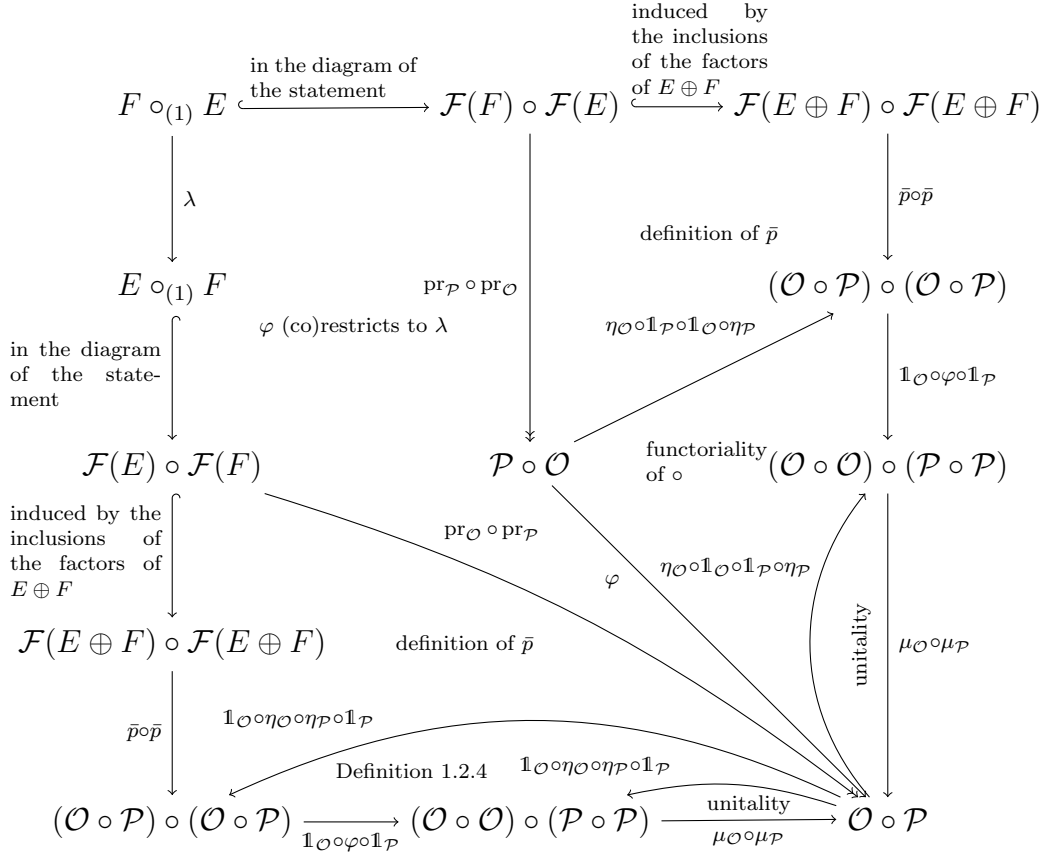
By the universal property of $\mathcal{F}(E \oplus F)$, the maps $p^{-1}|_E$ and $p^{-1}|_F$ induce a morphism of ns-operads $\bar{p}: \mathcal{F}(E \oplus F) \rightarrow \mathcal{O} \circ_\varphi \mathcal{P}$. In order to check that the map p^{-1} is well-defined, we need to prove that \bar{p} vanishes on $R \oplus D_\lambda \oplus S$. That \bar{p} vanishes on R is a consequence of the following commutative diagram,

$$\begin{array}{ccc}
 & 0 & \\
 R & \xrightarrow{\quad} & \mathcal{O} \\
 & \searrow & \uparrow \\
 & \mathcal{F}(E) & \\
 & \downarrow \mathcal{F}(E \hookrightarrow E \oplus F) & \\
 \mathcal{F}(E \oplus F) & \xrightarrow{\quad \bar{p} \quad} & \mathcal{O} \circ_\varphi \mathcal{P} \\
 & & \uparrow 1_{\mathcal{O}} \circ \eta_{\mathcal{P}} \\
 & & \mathcal{O}
 \end{array}$$

Similarly, \bar{p} vanishes on S since the following diagram commutes,

$$\begin{array}{ccc}
 & 0 & \\
 S & \xrightarrow{\quad} & \mathcal{P} \\
 & \searrow & \uparrow \\
 & \mathcal{F}(F) & \\
 & \downarrow \mathcal{F}(F \hookrightarrow E \oplus F) & \\
 \mathcal{F}(E \oplus F) & \xrightarrow{\quad \bar{p} \quad} & \mathcal{O} \circ_\varphi \mathcal{P} \\
 & & \uparrow \eta_{\mathcal{O}} \circ 1_{\mathcal{P}} \\
 & & \mathcal{P}
 \end{array}$$

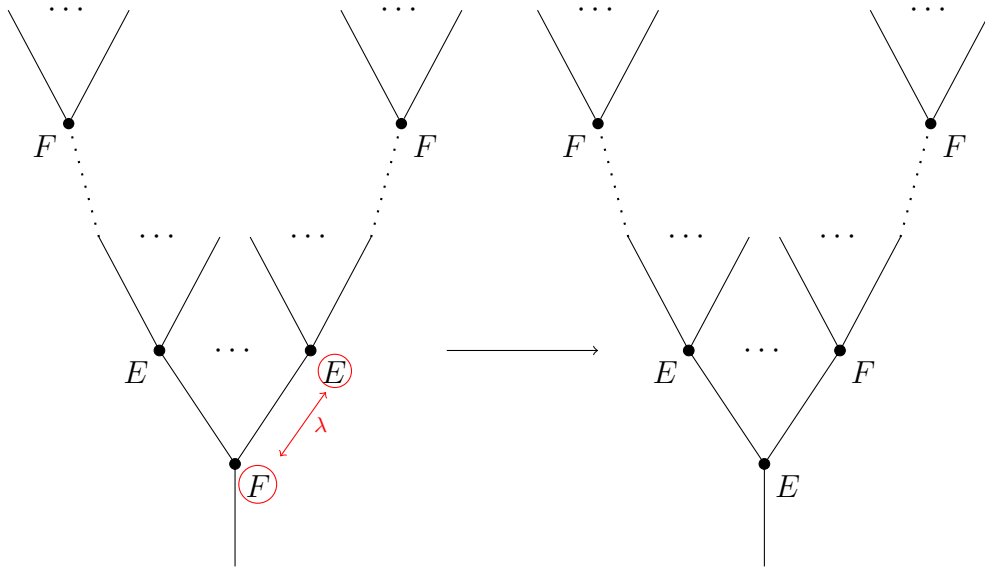
For the vanishing on D_λ , we consider the following diagram,



Here we denote by $\text{pr}_{\mathcal{O}}: \mathcal{F}(E) \twoheadrightarrow \mathcal{O}$ and $\text{pr}_{\mathcal{P}}: \mathcal{F}(F) \twoheadrightarrow \mathcal{P}$ the natural projections onto the quotient ns-operads. Each subdiagram commutes by the indicated reason. The two loops called ‘unitality’ around the bottom right corner compose to the identity in $\mathcal{O} \circ \mathcal{P}$. Moreover, the two composites $E \circ_{(1)} F \rightarrow \mathcal{O} \circ \mathcal{P}$ and $F \circ_{(1)} E \rightarrow \mathcal{O} \circ \mathcal{P}$ in the outer square going through the bottom left corner and the top right corner, respectively, coincide with the restriction of \bar{p} to $E \circ_{(1)} F$ and $F \circ_{(1)} E$. Therefore, \bar{p} vanishes on D_{λ} , since $\bar{p}(D_{\lambda})$ is the image of the difference of the two maps $E \circ_{(1)} F \rightarrow \mathcal{O} \circ \mathcal{P}$ in the outer square, see Definition 3.3.1.

Now, the universal property of the quadratic operad $\mathcal{O} \vee_{\lambda} \mathcal{P}$ shows that \bar{p} factors through a map that we call $p^{-1}: \mathcal{O} \vee_{\lambda} \mathcal{P} \rightarrow \mathcal{O} \circ_{\varphi} \mathcal{P}$. The fact that $p^{-1}p = \mathbb{1}_{\mathcal{O} \circ_{\varphi} \mathcal{P}}$ is a consequence of the commutativity of the following diagram,

generators of $\mathcal{O} \vee_{\lambda} \mathcal{P}$ are represented by ns-labeled planted planar trees with leaves, where each label is in E or in F . Moreover, generators of $\mathcal{O} \circ_{\varphi} \mathcal{P}$ are represented by an ns-labeled planted planar tree with leaves where the labels are in E where we have grafted on top ns-labeled planted planar trees with leaves with labels in F . Note that the latter are part of the former. The map p is induced by this inclusion. Actually, since the latter class is a small part of the former, it looks like if they would not be enough to generate $\mathcal{O} \vee_{\lambda} \mathcal{P}$. Nevertheless, they are. Indeed, the relations imposed by D_{λ} allow to use the ns-rewriting rule λ so that, each time we have an inner edge with bottom label in F and top label in E , we replace it with another one where the bottom label is in E and the top label in F instead. In this way, proceeding from bottom to top, we end up with an ns-labeled planted planar tree with leaves in the image of p , see the following picture as a way of illustration



This is actually why λ is called a rewriting rule.

Since p is bijective, it is an isomorphism. Moreover, since p^{-1} is left inverse to the isomorphism p , then p^{-1} is actually the inverse isomorphism.

The formula for φ at the end of the statement is a consequence of the following commutative diagram,

The previous theorem applies to the distributive law in Definition 3.1.4 in case \mathcal{O} is quadratic. Hence we obtain the following corollary.

Corollary 3.3.3. *Let $\mathcal{O} = (E|R)$ be a quadratic ns-operad. Consider the ns-rewriting rule*

$$\lambda: k \cdot \Delta \circ_{(1)} E \rightarrow E \circ_{(1)} k \cdot \Delta$$

$$\Delta \otimes x \mapsto (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta}_{i\text{-th position}} \otimes 1 \otimes \cdots \otimes 1.$$

Then the ns-operad $d\mathcal{O}$ admits the following quadratic presentation

$$d\mathcal{O} = (k \cdot \Delta \oplus E | k \cdot (\Delta \otimes \Delta) \oplus D_\lambda \oplus R).$$

We now state the corresponding definition and results in the symmetric setting, where everything works mutatis mutandis. Let $\mathcal{O} = (E|R)$ and $\mathcal{P} = (F|S)$ be now two quadratic operads.

Definition 3.3.4. A *rewriting rule* is a morphism of collections $\lambda: F \circ_{\Sigma, (1)} E \rightarrow E \circ_{\Sigma, (1)} F$. From such a rewriting rule, we can define D_λ and $\mathcal{O} \vee_{\Sigma, \lambda} \mathcal{P}$ as in Definition 3.3.1.

We can also define the maps $\alpha: \mathcal{O} \rightarrow \mathcal{O} \vee_{\Sigma, \lambda} \mathcal{P}$ and $\beta: \mathcal{P} \rightarrow \mathcal{O} \vee_{\Sigma, \lambda} \mathcal{P}$ as above.

Theorem 3.3.5. *For any distributive law $\varphi: \mathcal{P} \circ_{\Sigma} \mathcal{O} \rightarrow \mathcal{O} \circ_{\Sigma} \mathcal{P}$ which (co)restricts to a rewriting rule $\lambda: F \circ_{\Sigma, (1)} E \rightarrow E \circ_{\Sigma, (1)} F$ in the sense of Theorem 3.3.2, the map $p = \mu_{\mathcal{O} \vee_{\lambda} \mathcal{P}}(\alpha \circ_{\Sigma} \beta): \mathcal{O} \circ_{\varphi} \mathcal{P} \rightarrow \mathcal{O} \vee_{\lambda} \mathcal{P}$ is an isomorphism of operads. Moreover, $\varphi = p^{-1} \mu_{\mathcal{O} \vee_{\lambda} \mathcal{P}}(\beta \circ_{\Sigma} \alpha)$.*

In the statement of this theorem, the symmetric analogue of the commutative diagram in Theorem 3.3.2 should use Remark 1.4.20 instead of Remark 1.4.16.

Corollary 3.3.6. *Let $\mathcal{O} = (E|R)$ be a quadratic operad. Consider the rewriting rule*

$$\lambda: k \cdot \Delta \circ_{\Sigma, (1)} E \rightarrow E \circ_{\Sigma, (1)} k \cdot \Delta$$

$$\Delta \otimes x \mapsto (-1)^{\|x\|} \sum_{i=1}^{\text{arity of } x} x \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta}_{i\text{-th position}} \otimes 1 \otimes \cdots \otimes 1 \otimes \text{id}.$$

Here id denotes the identity permutation. Then the ns-operad $d\mathcal{O}$ admits the following quadratic presentation

$$d\mathcal{O} = (k \cdot \Delta \oplus E | k \cdot (\Delta \otimes \Delta) \oplus D_\lambda \oplus R).$$

Chapter 4

Koszul duality of derived operads

- He deals the cards to find the answer

The sacred geometry of chance

The hidden laws of a probable outcome

The numbers lead a dance -

Sting and Dominic Miller, “The Shape Of My Heart”

This is the main chapter of this thesis, insofar as the results here will allow in the next chapter the definition of a nice notion of derived homotopy algebra over an operad where minimal models of operadic algebras live. In fact, the applications and explicit computations in the next chapter are more or less formal consequences of strong results on the Koszul duality theory of derived operads proven here. We show that, under the standard projectivity assumptions, the derived operad of a quadratic Koszul operad is again Koszul, and we explicitly compute its Koszul dual cooperad. We start working in the nonsymmetric context and later deduce the symmetric versions under the extra assumption that the ground ring contains the rationals. This assumption, which seems to us unavoidable, is also standard in the homotopy theory of symmetric operads [12, 13].

The results of this chapter can be regarded as far reaching generalizations of what Livernet, Roitzheim, and Whitehouse did in [15] for the associative operad. Our techniques are however rather different, since we work over rather generic quadratic Koszul operads.

4.1 The trivial distributive law

Dealing directly with derived operads would be utterly complicated. This is why we first need to deal with similar operads constructed from an easier distributive law, that we now define. In this whole section we place ourselves in the nonsymmetric setting.

Definition 4.1.1. Let \mathcal{O} and \mathcal{P} be augmented ns-operads. Recall that they decompose as sequences, $\mathcal{O} = I \oplus \bar{\mathcal{O}}$, $\mathcal{P} = I \oplus \bar{\mathcal{P}}$, see (1.4.2). Actually, $\bar{\mathcal{O}}$ and $\bar{\mathcal{P}}$ are operadic ideals. We define the *trivial distributive law*

$$\varphi_0: \mathcal{P} \circ \mathcal{O} \rightarrow \mathcal{O} \circ \mathcal{P}$$

as

$$\varphi_0(y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) = \text{id}_{\mathcal{O}} \otimes y, \quad (4.1.2)$$

$$\varphi_0(\text{id}_{\mathcal{P}} \otimes x) = x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}, \quad (4.1.3)$$

$$\varphi_0(y \otimes x_1 \otimes \cdots \otimes x_n) = 0 \text{ if } y \in \bar{\mathcal{P}} \text{ and } x_i \in \bar{\mathcal{O}} \text{ for some } 1 \leq i \leq n.$$

Here, φ_0 is a well-defined map of sequences since

$$(\mathcal{P} \circ \mathcal{O})(n) = \bigoplus_{\substack{k \geq 0, \\ l_1 + \cdots + l_k = n}} (I \oplus \bar{\mathcal{P}})(k) \otimes (I \oplus \bar{\mathcal{O}})(l_1) \otimes \cdots \otimes (I \oplus \bar{\mathcal{O}})(l_k).$$

This sequence contains

$$\mathcal{P}(n) \otimes (k \cdot \text{id}_{\mathcal{O}}) \otimes \cdots \otimes (k \cdot \text{id}_{\mathcal{O}}) \cong \mathcal{P}(n)$$

and

$$(k \cdot \text{id}_{\mathcal{P}}) \otimes \mathcal{O}(n) \cong \mathcal{O}(n)$$

as direct summands. The formula for φ_0 maps these two summands isomorphically to the direct summands

$$(k \cdot \text{id}_{\mathcal{O}}) \otimes \mathcal{P}(n) \cong \mathcal{P}(n)$$

and

$$\mathcal{O}(n) \otimes (k \cdot \text{id}_{\mathcal{P}}) \otimes \cdots \otimes (k \cdot \text{id}_{\mathcal{P}}) \cong \mathcal{O}(n)$$

of $(\mathcal{O} \circ \mathcal{P})(n)$, respectively, and the rest of direct summands of $(\mathcal{P} \circ \mathcal{O})(n)$ to zero, $n \geq 0$.

The trivial distributive law has been considered in [16, 8.6.4] for the special case of quadratic operads. However, it was wrongly stated that φ_0 is identically trivial, which would violate some of the conditions that a distributive law must satisfy (Definition 1.2.4). Actually, checking in general that the trivial distributive law is indeed a distributive law requires some work. We now tackle this task.

Lemma 4.1.4. *The map of sequences φ_0 in Definition 4.1.1 is indeed a distributive law.*

Proof. We check the commutativity of the four diagrams in Definition 1.2.4. Diagrams 3 and 4 are equivalent to (4.1.2) and (4.1.3), respectively. This is why a distributive law cannot be identically zero.

The commutativity of 2 is equivalent to saying that, given $t = y_0 \otimes z_1 \otimes \cdots \otimes z_n \in \mathcal{P} \circ (\mathcal{P} \circ \mathcal{O})$, with $z_i = y_i \otimes x_{i,1} \otimes \cdots \otimes x_{i,l_i}$, $y_i \in \mathcal{P}$, $x_{i,j} \in \mathcal{O}$,

$$(\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{P}} \circ \varphi_0)(t) = \varphi_0(\mu_{\mathcal{P}} \circ \mathbf{1}_{\mathcal{O}})(t). \quad (4.1.5)$$

We will distinguish several cases.

Assume that, for some $1 \leq i \leq n$ and some $1 \leq j \leq l_i$, $y_i \in \bar{\mathcal{P}}$ and $x_{i,j} \in \bar{\mathcal{O}}$. Then $\varphi_0(z_i) = 0$, hence $(\mathbf{1}_{\mathcal{P}} \circ \varphi_0)(t) = 0$, in particular the left hand side of (4.1.5) vanishes. Concerning the right hand side,

$$\varphi_0(\mu_{\mathcal{P}} \circ \mathbf{1}_{\mathcal{O}})(t) = \pm \varphi_0(\mu_{\mathcal{P}}(y_0 \otimes y_1 \otimes \cdots \otimes y_n) \otimes x_{1,1} \otimes \cdots \otimes x_{n,l_n}) = 0.$$

Here we use that $\bar{\mathcal{P}}$ is an operadic ideal, so $\mu_{\mathcal{P}}(y_0 \otimes y_1 \otimes \cdots \otimes y_n) \in \bar{\mathcal{P}}$ since $y_i \in \bar{\mathcal{P}}$.

Otherwise, for each $1 \leq i \leq n$, we can suppose that either $y_i = \text{id}_{\mathcal{P}}$ or $x_{i,1} = \cdots = x_{i,l_i} = \text{id}_{\mathcal{O}}$. We now distinguish some subcases of this one.

Suppose that $x_{i,1} = \cdots = x_{i,l_i} = \text{id}_{\mathcal{O}}$ for all $1 \leq i \leq n$, so $\varphi_0(z_i) = \text{id}_{\mathcal{O}} \otimes y_i$. Then, on the one hand, the left hand side of (4.1.5) is

$$\begin{aligned} & (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{P}} \circ \varphi_0)(t) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbf{1}_{\mathcal{P}})(y_0 \otimes (\text{id}_{\mathcal{O}} \otimes y_1) \otimes \cdots \otimes (\text{id}_{\mathcal{O}} \otimes y_n)) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbf{1}_{\mathcal{P}})((y_0 \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \otimes y_1 \otimes \cdots \otimes y_n) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})((\text{id}_{\mathcal{O}} \otimes y_0) \otimes y_1 \otimes \cdots \otimes y_n) \\ &= (\mathbf{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\text{id}_{\mathcal{O}} \otimes (y_0 \otimes y_1 \otimes \cdots \otimes y_n)) \\ &= \text{id}_{\mathcal{O}} \otimes \mu_{\mathcal{P}}(y_0 \otimes y_1 \otimes \cdots \otimes y_n). \end{aligned}$$

Here and below we use the associativity isomorphism for the circle product. On the other hand, the right hand side of (4.1.5) is

$$\begin{aligned} & \varphi_0(\mu_{\mathcal{P}} \circ \mathbf{1}_{\mathcal{O}})(t) \\ &= \varphi_0(\mu_{\mathcal{P}} \circ \mathbf{1}_{\mathcal{O}})((y_0 \otimes y_1 \otimes \cdots \otimes y_n) \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \\ &= \varphi_0(\mu_{\mathcal{P}}(y_0 \otimes y_1 \otimes \cdots \otimes y_n) \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \\ &= \text{id}_{\mathcal{O}} \otimes \mu_{\mathcal{P}}(y_0 \otimes y_1 \otimes \cdots \otimes y_n). \end{aligned}$$

Otherwise, for some $1 \leq i \leq n$ and some $1 \leq j \leq l_i$, $y_i = \text{id}_{\mathcal{P}}$, so $l_i = 1 = j$, and $x_{i,j} \in \bar{\mathcal{O}}$, hence $\varphi_0(z_i) = x_{i,j} \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}$. If in addition $y_0 \in \bar{\mathcal{P}}$, the left hand side of (4.1.5) vanishes since

$$(\varphi_0 \circ \mathbb{1}_{\mathcal{P}})(\mathbb{1}_{\mathcal{P}} \circ \varphi_0)(t) = \varphi_0(y_0 \otimes \cdots \otimes x_{i,j} \otimes \cdots) \otimes \cdots = 0.$$

The right hand side is

$$\varphi_0(\mu_{\mathcal{P}} \circ \mathbb{1}_{\mathcal{O}})(t) = \pm \varphi_0(\mu_{\mathcal{P}}(y_0 \otimes \cdots) \otimes \cdots \otimes x_{i,j} \otimes \cdots) = 0.$$

Here we use that $\mu_{\mathcal{P}}(y_0 \otimes \cdots) \in \bar{\mathcal{P}}$ since $\bar{\mathcal{P}}$ is an operadic ideal.

Alternatively, $y_0 = \text{id}_{\mathcal{P}}$ and $t = \text{id}_{\mathcal{P}} \otimes (\text{id}_{\mathcal{P}} \otimes x_{i,j})$. In this case, calling $x = x_{i,j}$, the left hand side of (4.1.5) is

$$\begin{aligned} & (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbb{1}_{\mathcal{P}})(\mathbb{1}_{\mathcal{P}} \circ \varphi_0)(\text{id}_{\mathcal{P}} \otimes (\text{id}_{\mathcal{P}} \otimes x)) \\ &= (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbb{1}_{\mathcal{P}})(\text{id}_{\mathcal{P}} \otimes (x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}})) \\ &= (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(\varphi_0 \circ \mathbb{1}_{\mathcal{P}})((\text{id}_{\mathcal{P}} \otimes x) \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \\ &= (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})((x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \\ &= (\mathbb{1}_{\mathcal{O}} \circ \mu_{\mathcal{P}})(x \otimes (\text{id}_{\mathcal{P}} \otimes \text{id}_{\mathcal{P}}) \otimes \cdots \otimes (\text{id}_{\mathcal{P}} \otimes \text{id}_{\mathcal{P}})) \\ &= x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}. \end{aligned}$$

The right hand side of (4.1.5) is

$$\begin{aligned} \varphi_0(\mu_{\mathcal{P}} \circ \mathbb{1}_{\mathcal{O}})(\text{id}_{\mathcal{P}} \otimes (\text{id}_{\mathcal{P}} \otimes x)) &= \varphi_0(\mu_{\mathcal{P}} \circ \mathbb{1}_{\mathcal{O}})((\text{id}_{\mathcal{P}} \otimes \text{id}_{\mathcal{P}}) \otimes x) \\ &= \varphi_0(\text{id}_{\mathcal{P}} \otimes x) \\ &= x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}. \end{aligned}$$

The commutativity of 1 is equivalent to saying that, given $t = y \otimes z_1 \otimes \cdots \otimes z_n \in \mathcal{P} \circ (\mathcal{O} \circ \mathcal{O})$, with $z_i = x_i \otimes x_{i,1} \otimes \cdots \otimes x_{i,l_i}$, $y \in \mathcal{O}$, $x_i \in \mathcal{O}$, $x_{i,j} \in \mathcal{O}$,

$$(\mu_{\mathcal{O}} \circ \mathbb{1}_{\mathcal{P}})(\mathbb{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbb{1}_{\mathcal{O}})(t) = \varphi_0(\mathbb{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(t). \quad (4.1.6)$$

We will distinguish several cases again.

Suppose that $y \in \bar{\mathcal{P}}$. As a first subcase of this one, assume further that, for some $1 \leq i \leq n$, $x_i \in \bar{\mathcal{O}}$. Then

$$\begin{aligned} (\varphi_0 \circ \mathbb{1}_{\mathcal{O}})(t) &= \pm (\varphi_0 \circ \mathbb{1}_{\mathcal{O}})((y \otimes x_1 \otimes \cdots \otimes x_n) \otimes x_{1,1} \otimes \cdots \otimes x_{n,l_n}) \\ &= 0, \end{aligned}$$

hence the left hand side of (4.1.6) vanishes. Concerning the right hand side,

$$\begin{aligned} & \varphi_0(\mathbb{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(t) \\ &= \varphi_0(\mathbb{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(y \otimes (x_1 \otimes x_{1,1} \otimes \cdots \otimes x_{1,l_1}) \otimes \cdots \otimes (x_n \otimes x_{n,1} \otimes \cdots \otimes x_{n,l_n})) \\ &= \pm \varphi_0(y \otimes \mu_{\mathcal{O}}(x_1 \otimes x_{1,1} \otimes \cdots \otimes x_{1,l_1}) \otimes \cdots \otimes \mu_{\mathcal{O}}(x_n \otimes x_{n,1} \otimes \cdots \otimes x_{n,l_n})) \\ &= 0. \end{aligned}$$

Here we use the associativity isomorphism for the circle product and that $\bar{\mathcal{O}}$ is an operadic ideal, so $\mu_{\mathcal{O}}(x_i \otimes x_{i,1} \otimes \cdots \otimes x_{i,l_i}) \in \bar{\mathcal{O}}$ since $x_i \in \bar{\mathcal{O}}$.

As a second subcase, assume that $x_1 = \cdots = x_n = \text{id}_{\mathcal{O}}$, in particular $l_i = 1$ for all $1 \leq i \leq n$. We further split this subcase up in the case that for some $1 \leq i \leq n$, $x_{i,1} \in \bar{\mathcal{O}}$ and the case that $x_{1,1} = \cdots = x_{n,1} = \text{id}_{\mathcal{O}}$.

In the former situation, the right hand side of (4.1.6) is

$$\begin{aligned} & \varphi_0(\mathbf{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(t) \\ &= \varphi_0(\mathbf{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(y \otimes (\text{id}_{\mathcal{O}} \otimes x_{1,1}) \otimes \cdots \otimes (\text{id}_{\mathcal{O}} \otimes x_{n,1})) \\ &= \pm \varphi_0(y \otimes x_{1,1} \otimes \cdots \otimes x_{n,1}) \\ &= 0. \end{aligned}$$

On the other hand, the left hand side of (4.1.6) vanishes, since

$$(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbf{1}_{\mathcal{O}})(t) = \text{id}_{\mathcal{O}} \otimes \varphi_0(y \otimes \cdots \otimes x_{i,1} \otimes \cdots) = 0.$$

In the latter situation, the right hand side of (4.1.6) is

$$\begin{aligned} & \varphi_0(\mathbf{1}_{\mathcal{P}} \otimes \mu_{\mathcal{O}})(y \otimes (\text{id}_{\mathcal{O}} \otimes \text{id}_{\mathcal{O}}) \otimes \cdots \otimes (\text{id}_{\mathcal{O}} \otimes \text{id}_{\mathcal{O}})) \\ &= \varphi_0(y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \\ &= \text{id}_{\mathcal{O}} \otimes y. \end{aligned}$$

And the left hand side of (4.1.6) is

$$\begin{aligned} & (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbf{1}_{\mathcal{O}})(y \otimes (\text{id}_{\mathcal{O}} \otimes \text{id}_{\mathcal{O}}) \otimes \cdots \otimes (\text{id}_{\mathcal{O}} \otimes \text{id}_{\mathcal{O}})) \\ &= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbf{1}_{\mathcal{O}})((y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \\ &= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)((\text{id}_{\mathcal{O}} \otimes y) \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}}) \\ &= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\text{id}_{\mathcal{O}} \otimes (y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}})) \\ &= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\text{id}_{\mathcal{O}} \otimes (\text{id}_{\mathcal{O}} \otimes y)) \\ &= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})((\text{id}_{\mathcal{O}} \otimes \text{id}_{\mathcal{O}}) \otimes y) \\ &= \text{id}_{\mathcal{O}} \otimes y. \end{aligned}$$

Alternatively, $y = \text{id}_{\mathcal{P}}$ and $t = (\text{id}_{\mathcal{P}} \otimes x_1) \otimes x_{1,1} \otimes \cdots \otimes x_{1,l_1}$. In this case, calling $x = x_1$, $x_j = x_{1,j}$, and $l_1 = n$, the right hand side of (4.1.6) is

$$\begin{aligned} & \varphi_0(\mathbf{1}_{\mathcal{P}} \circ \mu_{\mathcal{O}})(\text{id}_{\mathcal{P}} \otimes (x \otimes x_1 \otimes \cdots \otimes x_n)) \\ &= \varphi_0(\text{id}_{\mathcal{P}} \otimes \mu_{\mathcal{O}}(x \otimes x_1 \otimes \cdots \otimes x_n)) \\ &= \mu_{\mathcal{O}}(x \otimes x_1 \otimes \cdots \otimes x_n) \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}. \end{aligned}$$

The left hand side of (4.1.6) is

$$\begin{aligned}
& (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbf{1}_{\mathcal{O}})(\text{id}_{\mathcal{P}} \otimes (x \otimes x_1 \otimes \cdots \otimes x_n)) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(\varphi_0 \circ \mathbf{1}_{\mathcal{O}})((\text{id}_{\mathcal{P}} \otimes x) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)((x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \otimes x_1 \otimes \cdots \otimes x_n) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(\mathbf{1}_{\mathcal{O}} \circ \varphi_0)(x \otimes (\text{id}_{\mathcal{P}} \otimes x_1) \otimes \cdots \otimes (\text{id}_{\mathcal{P}} \otimes x_n)) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})(x \otimes (x_1 \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \otimes \cdots \otimes (x_n \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}})) \\
&= (\mu_{\mathcal{O}} \circ \mathbf{1}_{\mathcal{P}})((x \otimes x_1 \otimes \cdots \otimes x_n) \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}) \\
&= \mu_{\mathcal{O}}(x \otimes x_1 \otimes \cdots \otimes x_n) \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}}.
\end{aligned}$$

This concludes the proof. \square

For quadratic operads, the circle product with respect to the trivial distributive law is again quadratic.

Proposition 4.1.7. *Given quadratic ns-operads $\mathcal{O} = (E|R)$ and $\mathcal{P} = (F|S)$. The ns-operad $\mathcal{O} \circ_{\varphi_0} \mathcal{P}$ is quadratic associated to the following ns-quadratic data,*

$$(E \oplus F, R \oplus F \circ_{(1)} E \oplus S).$$

Proof. This follows from Theorem 3.3.2. The hypothesis holds since φ_0 (co)restricts to the trivial rewriting rule $0: F \circ_{(1)} E \rightarrow E \circ_{(1)} F$. \square

The symmetric case works equally well.

Definition 4.1.8. Two augmented operads \mathcal{O} and \mathcal{P} decompose as collections, $\mathcal{O} = I \oplus \bar{\mathcal{O}}$, $\mathcal{P} = I \oplus \bar{\mathcal{P}}$, with $\bar{\mathcal{O}}$ and $\bar{\mathcal{P}}$ operadic ideals. We define the *trivial distributive law*

$$\varphi_{\Sigma,0}: \mathcal{P} \circ_{\Sigma} \mathcal{O} \rightarrow \mathcal{O} \circ_{\Sigma} \mathcal{P}$$

as

$$\begin{aligned}
\varphi_{\Sigma,0}(y \otimes \text{id}_{\mathcal{O}} \otimes \cdots \otimes \text{id}_{\mathcal{O}} \otimes \sigma) &= \text{id}_{\mathcal{O}} \otimes y \otimes \sigma, \\
\varphi_{\Sigma,0}(\text{id}_{\mathcal{P}} \otimes x \otimes \sigma) &= x \otimes \text{id}_{\mathcal{P}} \otimes \cdots \otimes \text{id}_{\mathcal{P}} \otimes \sigma, \\
\varphi_{\Sigma,0}(y \otimes x_1 \otimes \cdots \otimes x_n \otimes \sigma) &= 0 \text{ if } y \in \bar{\mathcal{P}} \text{ and } x_i \in \bar{\mathcal{O}} \text{ for some } 1 \leq i \leq n.
\end{aligned}$$

This is a well-defined map of collections since, using the definition of the symmetric circle product \circ_{Σ} as a big direct sum, we see that $\mathcal{P} \circ_{\Sigma} \mathcal{O}$ and $\mathcal{O} \circ_{\Sigma} \mathcal{P}$ contain

$$\begin{aligned}
\mathcal{P}(n) \otimes_{\Sigma_n} ((k \cdot \text{id}_{\mathcal{O}})^{\otimes n} \otimes k[\Sigma_n]) &\cong \mathcal{P}(n), \\
(k \cdot \text{id}_{\mathcal{P}}) \otimes \mathcal{O}(n) \otimes_{\Sigma_n} k[\Sigma_n] &\cong \mathcal{O}(n),
\end{aligned}$$

and

$$\begin{aligned} (k \cdot \text{id}_{\mathcal{O}}) \otimes \mathcal{P}(n) \otimes_{\Sigma_n} k[\Sigma_n] &\cong \mathcal{P}(n) \\ \mathcal{O}(n) \otimes_{\Sigma_n} ((k \cdot \text{id}_{\mathcal{O}})^{\otimes n} \otimes k[\Sigma_n]) &\cong \mathcal{O}(n), \end{aligned}$$

as direct summands, respectively. As a morphism of collections, $\varphi_{\Sigma,0}$ maps isomorphically these factors and vanishes on the rest of direct factors of the source.

The previous proofs work with minor modifications in order to show the two following results.

Lemma 4.1.9. *The map of sequences $\varphi_{\Sigma,0}$ in Definition 4.1.8 is indeed a distributive law.*

Proposition 4.1.10. *Given quadratic operads $\mathcal{O} = (E|R)$ and $\mathcal{P} = (F|S)$. The operad $\mathcal{O} \circ_{\varphi_{\Sigma,0}} \mathcal{P}$ is quadratic associated to the following quadratic data,*

$$(E \oplus F, R \oplus F \circ_{\Sigma, (1)} E \oplus S).$$

4.2 Mock derived ns-operads and their bar construction

In this section, again in the nonsymmetric setting, we consider the operads defined in the same way as derived operads but using the trivial distributive law instead. We compute a smaller model for their bar construction.

Definition 4.2.1. The *mock derived ns-operad* of an augmented ns-operad \mathcal{O} is defined as

$$d'\mathcal{O} = \mathcal{O} \circ_{\varphi_0} \mathcal{D}.$$

Here we use the trivial distributive law in Definition 4.1.1 instead of the distributive law defining $d\mathcal{O}$ in Definition 3.1.4.

The mock derived ns-operad of a quadratic ns-operad is quadratic by Proposition 4.1.7.

Corollary 4.2.2. *Given a quadratic ns-operad $\mathcal{O} = (E|R)$, its mock derived ns-operad $d'\mathcal{O}$ is quadratic associated to the following ns-quadratic data*

$$(E \oplus k \cdot \Delta, R \oplus (k \cdot \Delta) \otimes E \oplus k \cdot (\Delta \otimes \Delta)).$$

Here, for a fixed quadratic ns-operad $\mathcal{O} = (E|R)$, we construct a relatively small model of the bar construction of $d'\mathcal{O}$. It will be used in the next section for the computation of the Koszul dual ns-cooperad of $d\mathcal{O}$. This small model will be part of a strong deformation retraction, in the sense of the following definition.

Definition 4.2.3. A *strong deformation retraction*, or simply *SDR*, consists of two graded complexes X and Y and a diagram

$$X \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} Y \begin{array}{c} \curvearrowright h \\ \curvearrowleft \end{array}$$

where i and p are maps of graded complexes, h is a homotopy from ip to $\mathbb{1}_Y$, i.e a bidegree $(0, 1)$ map of bigraded modules satisfying the *chain homotopy equation*

$$ip - \mathbb{1}_Y = dh + hd, \quad (4.2.4)$$

and the following equations are satisfied,

$$pi = \mathbb{1}_X, \quad ph = 0, \quad hi = 0, \quad h^2 = 0.$$

Strong deformation retractions are usually defined for chain complexes, but they make equal sense for graded chain complexes, sequences, collections, etc. They are a very convenient kind of homotopy equivalence, in particular they preserve homology. They are often use to reduce homology computations for the big object Y to computations on the small one X .

We can now state the main result of this section.

Theorem 4.2.5. *There is an SDR of sequences*

$$B(\mathcal{D}) \circ B(\mathcal{O}) \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} B(d'\mathcal{O}) \begin{array}{c} \curvearrowright h \\ \curvearrowleft \end{array}$$

preserving the syzygy degree whose maps are defined below.

We prove this theorem at the end of this section. As a warm-up, we start by calculating the bar construction of \mathcal{D} an its Koszul dual cooperad \mathcal{D}^i , see also [16, 10.3.7].

Remark 4.2.6. Clearly,

$$\bar{\mathcal{D}} = k \cdot \Delta$$

concentrated in arity 1 and bidegree $(-1, 0)$. Consequently,

$$s\bar{\mathcal{D}} = k \cdot (s\Delta),$$

concentrated in arity 1 and bidegree $(-1, 1)$. It follows that the underlying sequence of $B(\mathcal{D})$ is

$$\mathcal{F}^c(s\bar{\mathcal{D}}) = \bigoplus_{n \geq 0} k \cdot (s\Delta)^{\otimes n}. \quad (4.2.7)$$

We sometimes simplify $(s\Delta)^{\otimes n}$ as $(s\Delta)^n$. The coproduct $\Delta_{B(\mathcal{D})}$, being de-grafting, takes the form

$$\begin{aligned} \Delta_{B(\mathcal{D})}: B(\mathcal{D}) &\rightarrow B(\mathcal{D}) \circ B(\mathcal{D}), \\ (s\Delta)^{\otimes n} &\mapsto \sum_{i+j=n} (s\Delta)^{\otimes i} \otimes (s\Delta)^{\otimes j}. \end{aligned} \quad (4.2.8)$$

The counit $\epsilon_{B(\mathcal{D})}: B(\mathcal{D}) \rightarrow I$ is defined by the equations $\epsilon_{B(\mathcal{D})}((s\Delta)^{\otimes 0}) = 1$ and $\epsilon_{B(\mathcal{D})}((s\Delta)^{\otimes n}) = 0$, $n > 0$, and the coaugmentation $\eta_{B(\mathcal{D})}: I \rightarrow B(\mathcal{D})$ by $\epsilon_{B(\mathcal{D})}(1) = (s\Delta)^{\otimes 0}$. Finally, the differential $d_{B(\mathcal{D})}$ is trivial since

$$\begin{aligned} \hat{d}_{2,1}: s\bar{\mathcal{D}}(1) \otimes s\bar{\mathcal{D}}(1) &\rightarrow s\bar{\mathcal{D}}(1) \\ \hat{d}_{2,1}(s\Delta \otimes s\Delta) &= -s(\Delta \circ_1 \Delta) \\ &= -s\mu_{\mathcal{D}}(\Delta \otimes \Delta) \\ &= 0. \end{aligned}$$

All this means that the cooperad $B(\mathcal{D})$ is the polynomial coalgebra on one variable of bidegree $(-1, 1)$ regarded as an ns-cooperad concentrated in arity 1.

The syzygy degree of $(s\Delta)^{\otimes n}$ is 0 for all $n \geq 0$, i.e. $B(\mathcal{D})$ is concentrated in syzygy degree 0. Hence \mathcal{D} is trivially Koszul and

$$\mathcal{D}^i = H^0(B^\bullet \mathcal{D}) = B(\mathcal{D}).$$

We can now obtain an interesting corollary of the previous theorem.

Corollary 4.2.9. *If the quadratic ns-operad \mathcal{O} is Koszul, then so is the mock derived operad $d'\mathcal{O}$.*

Proof. It is enough to check that the cohomology of $B(\mathcal{D}) \circ B(\mathcal{O})$ is concentrated in syzygy degree 0. Since $B(\mathcal{D})$ is concentrated in arity 1, $B(\mathcal{D}) \circ B(\mathcal{O}) = B(\mathcal{D}) \otimes B(\mathcal{O})$. Moreover, $B(\mathcal{D})$ is k -free and has trivial differential, therefore

$$H^*(B(\mathcal{D}) \otimes B(\mathcal{O})) = B(\mathcal{D}) \otimes H^*(B(\mathcal{O})).$$

The ns-operad \mathcal{O} being Koszul means that $H^*(B(\mathcal{O}))$ is concentrated in syzygy degree 0, hence we are done. \square

In order to define the components of the strong deformation retraction in Theorem 4.2.5 we must analyze the sequence underlying the bar construction of the mock derived operad.

As a sequence, $d'\mathcal{O}$ and its augmentation ideal $\overline{d'\mathcal{O}}$ are given by

$$\begin{aligned}
(d'\mathcal{O})(1) &= \mathcal{O}(1) \otimes \mathcal{D}(1) \\
&= (k \cdot \text{id}_{\mathcal{O}} \oplus \bar{\mathcal{O}}(1)) \otimes (k \cdot \text{id}_{\mathcal{D}} \oplus k \cdot \Delta) \\
&= (k \cdot \text{id}_{\mathcal{O}} \otimes k \cdot \text{id}_{\mathcal{D}}) \oplus (k \cdot \text{id}_{\mathcal{O}} \otimes k \cdot \Delta) \\
&\quad \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \text{id}_{\mathcal{D}}) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \Delta) \\
&= k \cdot \text{id}_{d'\mathcal{O}} \oplus (k \cdot \text{id}_{\mathcal{O}} \otimes k \cdot \Delta) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \text{id}_{\mathcal{D}}) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \Delta), \\
(\overline{d'\mathcal{O}})(1) &= ((k \cdot \text{id}_{\mathcal{O}} \otimes k \cdot \Delta) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \text{id}_{\mathcal{D}})) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \Delta),
\end{aligned}$$

for $n = 1$; and for $n \neq 1$,

$$\begin{aligned}
(\overline{d'\mathcal{O}})(n) &= (d'\mathcal{O})(n) \\
&= \mathcal{O}(n) \otimes \mathcal{D}(1)^{\otimes n} \\
&= \bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}} \oplus k \cdot \Delta)^{\otimes n} \\
&\cong \bigoplus_{0 \leq p \leq n} \bigoplus_{\binom{n}{p}} \bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes(n-p)} \otimes (k \cdot \Delta)^{\otimes p} \\
&= \bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes n} \\
&\quad \oplus \bigoplus_{1 \leq p \leq n} \bigoplus_{\binom{n}{p}} \bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes(n-p)} \otimes (k \cdot \Delta)^{\otimes p}.
\end{aligned}$$

Here the isomorphism is a plain symmetry constraint. In the rest of this chapter, we use the isomorphisms

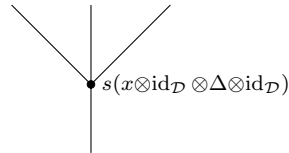
$$\begin{aligned}
k \cdot \text{id}_{\mathcal{O}} \otimes k \cdot \Delta &\cong k \cdot \Delta, \\
\bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes n} &\cong \bar{\mathcal{O}}(n), \quad n \geq 0,
\end{aligned}$$

as identifications. With this notation, for $n \neq 1$,

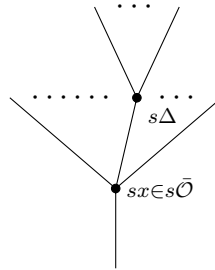
$$\begin{aligned}
(\overline{d'\mathcal{O}})(1) &= (k \cdot \Delta \oplus \bar{\mathcal{O}}(1)) \oplus (\bar{\mathcal{O}}(1) \otimes k \cdot \Delta), \tag{4.2.10} \\
(\overline{d'\mathcal{O}})(n) &= \bar{\mathcal{O}}(n) \oplus \left(\bigoplus_{1 \leq p \leq n} \bigoplus_{\binom{n}{p}} \bar{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes(n-p)} \otimes (k \cdot \Delta)^{\otimes p} \right).
\end{aligned}$$

In $B(d'\mathcal{O})$, which is $\mathcal{F}^c(s(\overline{d'\mathcal{O}}))$ as a coaugmented cooperad, and hence $\mathcal{F}(s(\overline{d'\mathcal{O}}))$ as a sequence, we will slightly change the notation of ns-labeled planted planar trees with leaves.

Definition 4.2.11. An *admissible labeled tree* in $\mathcal{F}(s(\overline{d'\mathcal{O}}))$ is a ns-labeled planted planar tree with leaves whose labels are either $s\Delta$ or sx with x in $\overline{\mathcal{O}}(n)$ or in one of the direct factors isomorphic to $\overline{\mathcal{O}}(n) \otimes (k \cdot \text{id}_{\mathcal{D}})^{\otimes(n-p)} \otimes (k \cdot \Delta)^{\otimes p}$ for some $1 \leq p \leq n$. This third kind of labels are called *mixed labels*, e.g.

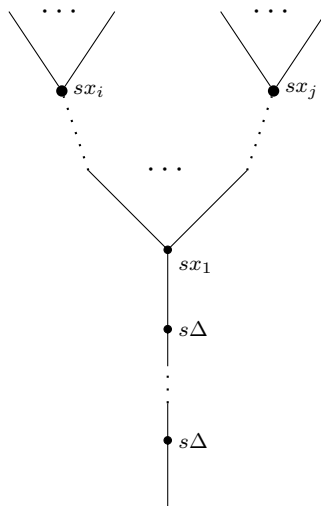


An admissible labeled tree is *good* if it contains neither a mixed label nor an inner edge with top label $s\Delta$ and bottom label in $s\overline{\mathcal{O}}$,



Those which are not good will be called *bad*.

Good admissible labeled trees look like



(4.2.12)

with the x 's in $\overline{\mathcal{O}}$.

We apologize for this choice of terminology. Admissible, good, and bad are overworked words, but they will only appear in technical parts so it is not worth to look for more elaborate names.

Remark 4.2.13. Admissible labeled trees are still k -linear generators of $B(d'\mathcal{O})$ by the decompositions in (4.2.10).

Moreover, the tree modules (Definition 2.1.5) appearing in the direct sum decomposition of the underlying sequence of $\mathcal{F}(s(\overline{d'\mathcal{O}}))$ (2.1.6), also decompose as direct sums according to (4.2.10) and to the additivity of the tensor product. Each of these new direct summands is k -linearly spanned by either good or bad labeled trees, hence $\mathcal{F}(s(\overline{d'\mathcal{O}}))$ is the direct sum of the subsequences generated by either of the two kinds of admissible labeled trees.

Furthermore, good admissible labeled trees can also be regarded as elements of $\mathcal{F}(k \cdot s\Delta) \circ \mathcal{F}(s\overline{\mathcal{O}})$, the underlying sequence of $B(\mathcal{D}) \circ B(\mathcal{O})$, since they are obtained by grafting arbitrary an n s-labeled planted planar tree with leaves in $\mathcal{F}(s\overline{\mathcal{O}})$ on the unique leaf of an element in $\mathcal{F}(k \cdot s\Delta)$. Therefore, the direct summand of $\mathcal{F}(s(\overline{d'\mathcal{O}}))$ which is k -linearly spanned by the good admissible labeled trees is clearly isomorphic to $\mathcal{F}(k \cdot s\Delta) \circ \mathcal{F}(s\overline{\mathcal{O}})$.

This defines the inclusion

$$i: B(\mathcal{D}) \circ B(\mathcal{O}) \rightarrow B(d'\mathcal{O}) \quad (4.2.14)$$

in the statement of Theorem 4.2.5. Moreover, the map

$$p: B(d'\mathcal{O}) \rightarrow B(\mathcal{D}) \circ B(\mathcal{O}) \quad (4.2.15)$$

is the obvious projection, killing the bad admissible labeled trees and preserving the good ones. In particular ,

$$pi = \mathbb{1}_{B(\mathcal{D}) \circ B(\mathcal{O})}.$$

Note that both i and p preserve the syzygy grading.

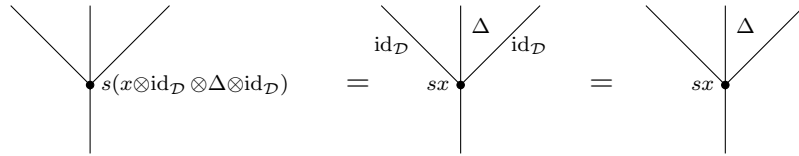
In principle, i and p are just maps of sequences of bigraded modules. They clearly preserve the syzygy grading. We now prove compatibility with the bar constructions differentials.

Lemma 4.2.16. *The map i in (4.2.14) is a morphism of sequences of graded complexes.*

Proof. The differential on $B(\mathcal{D}) \circ B(\mathcal{O}) = B(\mathcal{D}) \otimes B(\mathcal{O})$ is $\mathbb{1}_{B(\mathcal{D})} \otimes d_{B(\mathcal{O})}$, which is given by contracting in (4.2.12) inner edges with both labels in $s\overline{\mathcal{O}}$. The differential on $B(d'\mathcal{O})$ would be given by contracting all kinds of inner

edges in (4.2.12). The contraction of an inner edge with both labels $s\Delta$ yields zero since $\Delta^2 = 0$ in the dual numbers. Moreover, the contraction of the only possible inner edge with $s\Delta$ at the bottom and top label in $s\bar{\mathcal{O}}$ also vanishes since the definition of $d'\mathcal{O}$ uses the trivial distributive law. Hence the differential of $B(d'\mathcal{O})$ restricted to $B(\mathcal{D}) \circ B(\mathcal{O})$ is also $\mathbb{1}_{B(\mathcal{D})} \otimes d_{B(\mathcal{O})}$ (signs obviously match). \square

Remark 4.2.17. We will use the following convenient new notation for mixed labels. Any mixed label of arity n is of the form $s(x_0 \otimes x_1 \otimes \cdots \otimes x_n)$ where $x_0 \in \bar{\mathcal{O}}(n)$ and $x_i = \text{id}_{\mathcal{D}}$ or $x_i = \Delta$ (there must be at least one Δ). With our new notation, we will keep sx_0 labeling the inner vertex and move the x_i , $1 \leq i \leq n$, to the incoming edges. We may even drop the identity edge labels, e.g.

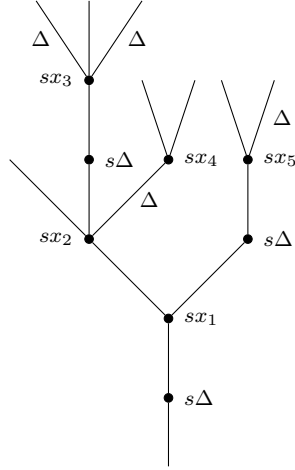


Accordingly, inner vertices with labels in $s\bar{\mathcal{O}}$ can also be enhanced by labeling the incoming edges with $\text{id}_{\mathcal{D}}$ (these are not mixed), e.g.



With this new notation, an admissible labeled tree m_T is a planted planar tree with leaves where inner vertices are labeled with $s\Delta$ or an element in $s\bar{\mathcal{O}}$, and some inner edges whose bottom label is in $s\bar{\mathcal{O}}$ are labeled with Δ ,

e.g.



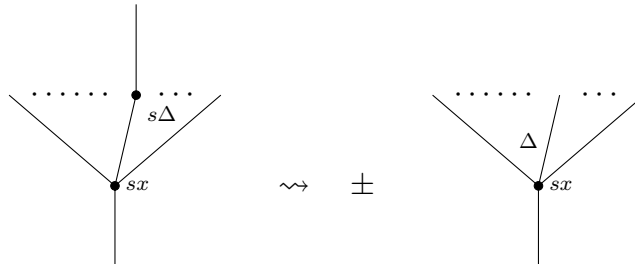
(4.2.18)

Moreover, in the computation of $d(m_T)$ by contraction of inner edges, where $d = d_2$ is the differential of the bar construction $B(d'\mathcal{O})$, contracting an inner edge labeled Δ or with bottom vertex labeled $s\Delta$ is always zero. Indeed, if the top vertex is labeled with $s\Delta$ it follows from the fact that $\Delta^2 = 0$ in the dual numbers, and if the top vertex's label is in $s\bar{\mathcal{O}}$ then it is a consequence of the use of the trivial distributive law in the definition of $d'\mathcal{O}$. Furthermore, contracting an unlabeled (or labeled with $\text{id}_{\mathcal{D}}$) inner edge with top vertex labeled $s\Delta$ and bottom vertex's label in $s\bar{\mathcal{O}}$ amounts to removing the top inner vertex, turning its label $s\Delta$ into a Δ labeling the new inner edge created by the deletion.

Lemma 4.2.19. *The map p in (4.2.15) is a morphism of sequences of graded complexes.*

Proof. We must check that the differential of $B(d'\mathcal{O})$ restricts to the direct summand linearly spanned by bad admissible labeled trees (which is the kernel of p). Let m_T be a bad admissible labeled tree in $B(d'\mathcal{O})$.

Suppose that m_T has an inner edge where the top label is $s\Delta$ and the bottom label is in $s\bar{\mathcal{O}}$. If we contract this inner edge, we obtain a mixed label, and hence a bad summand, e.g.



If we contract other inner edges, the problematic inner edge will stay there, therefore we will also obtain bad summands.

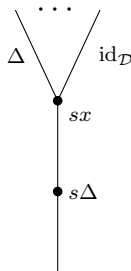
If m_T contains a mixed label in a given inner vertex, then contracting non-adjacent inner edges we still get the mixed label. All these summands are bad. If we contract an inner edge containing the mixed label, the result is either zero or another mixed label, hence a bad summand. Indeed, let us review the possible configurations.

If mixed label is at the bottom and the inner edge is labeled with Δ , e.g.



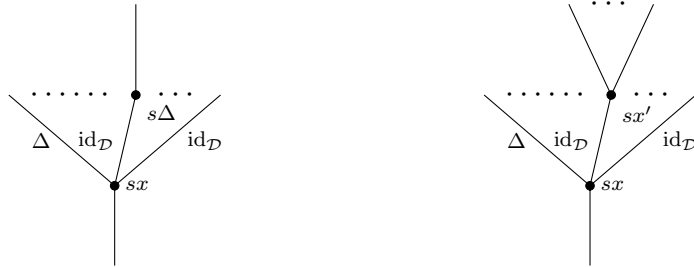
with $x, x' \in \bar{\mathcal{O}}$, then the contraction yields zero, either because $\Delta^2 = 0$ in the dual numbers or because the definition of $d'\mathcal{O}$ uses the trivial distributive law, respectively. Note that in the second case the top label could be mixed.

If the mixed label is on top and the bottom label is $s\Delta$, we also obtain zero by contraction because we are using the trivial distributive law,



Suppose that the mixed label is at the bottom and that inner edge is labeled with the identity operation. The top label may be $s\Delta$, something in

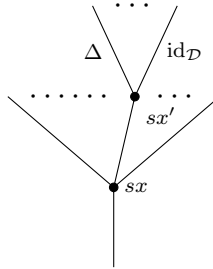
$s\bar{\mathcal{O}}$ or mixed, e.g.



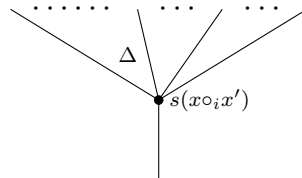
The picture on the right covers the two last possible cases. In all cases, we obtain a mixed label by contraction,



Similarly, if the mixed label is on top and the bottom label is in $s\bar{\mathcal{O}}$, e.g.



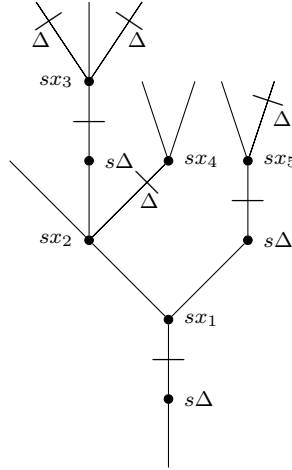
we obtain a mixed vertex by contraction,



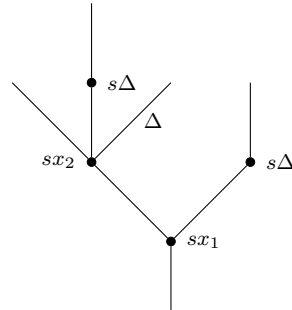
□

We finally define h , the remaining ingredient in the statement of Theorem 4.2.5.

Definition 4.2.20. Given an admissible labeled tree m_T , as in Remark 4.2.17, we cut after any symbol Δ , either in an aedge or $s\Delta$ on an inner vertex. The bottommost piece will possibly be an element in $\mathcal{F}(k \cdot \Delta)$, i.e. a linear tree with all labels $s\Delta$. If this is the case, we define the *essential block* of m_T to be the next piece, otherwise the essential is the bottommost piece, e.g. if m_T is (4.2.18)



and the essential block is

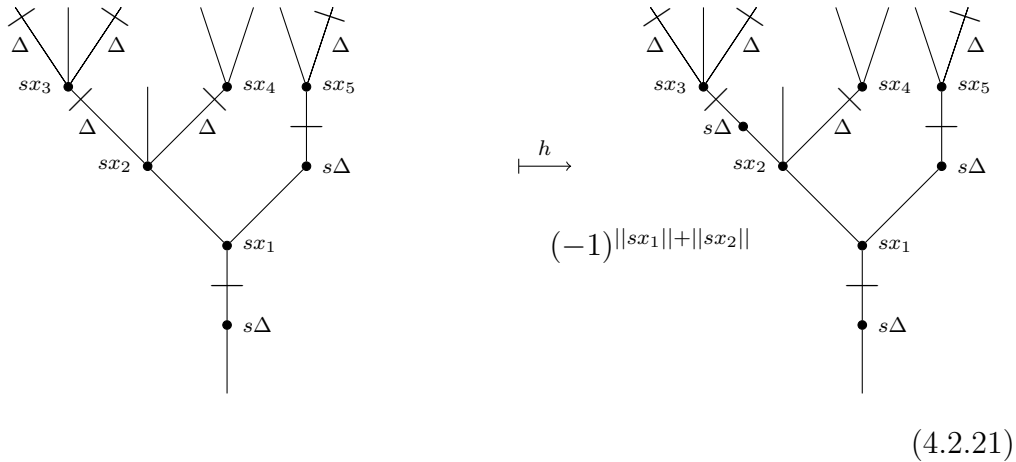


We define $h(m_T)$ as follows, according to the leftmost label of the essential block containing Δ . Indeed, by the previous way of cutting, labels containing Δ in the essential block are linearly ordered from left to right with respect to the path order. Moreover, if the essential block does not contain any Δ then m_T is good, and conversely.

1. If it is an $s\Delta$ attached to an inner vertex w (e.g. in the previous picture, where w is the vertex adjacent to sx_2 in the middle incoming edge) or if there is no Δ in the essential block, then $h(m_T) = 0$.
2. If it is a plain Δ attached to an edge with bottom vertex v , we subdivide that edge drawing a new inner vertex in the middle labeled $s\Delta$ (this

replaces the old Δ labeling the egde) and multiply by $(-1)^{\|sx\|+\|m_T\|_{<v}}$, where $sx \in s\bar{\mathcal{O}}$ is the label of v in the new labeling style. This (sign included) gives rise to a new admissible labeled tree $h(m_T)$. Recall from Section 2.3 that $\|m_T\|_{<v}$ is the sum of the total degrees of the labels of the vertices preceding v (strictly, in the path order).

Let us give an example of (2),



Here v is the vertex labeled sx_2 .

Note that h preserves the arity and the horizontal (\mathbb{N} -)grading, and increases the syzygy degree and the vertical (\mathbb{Z} -)grading by $+1$.

Remark 4.2.22. The cuttings carried out in this definition do not have anything to do with the mainstream cuttings used in the definition of the diagonal of a cofree conilpotent cooperad (Definition 2.2.2). They are just used for the purpose of defining h . Note that the definition of h depends strongly on the planar structure (as we have to look for the leftmost Δ). This is why we later have to take a different approach in the symmetric case.

The leftmost label of the essential block containing Δ can be located without any reference to the essential block. It is the first label containing Δ which comes after a label in $s\bar{\mathcal{O}}$ in the path order. While the essential block could be easily bypassed in the nonsymmetric setting, it helps in some arguments, notably in the closed formula defining the symmetric version of h .

For the computation of $\|m_T\|_{<v}$ in (2), we should use the old labeling style, i.e. the labels Δ on edges should go back to their bottom mixed vertex, see Remark 4.2.17. Nevertheless this is irrelevant in this case since, by the previous paragraph, there is no mixed vertex before v . Note however that $\|sx\| + \|m_T\|_{<v}$ in (2) is not the same as $\|m_T\|_{\leq v}$ since the latter would also take into account the number of edges with bottom vertex v labeled Δ .

We are now ready to prove the main theorem of this section.

Proof of Theorem 4.2.5. It is only left to check the equations in Definition 4.2.3. Let m_T be an admissible labeled tree.

Equation $h^2 = 0$ follows from the fact that, by definition, either $h(m_T) = 0$ or the leftmost Δ in $h(m_T)$ is an $s\Delta$, labeling the new inner vertex created by subdivision.

Equation $hi = 0$ follows from the fact that good admissible labeled trees do not have mixed vertices, in particular they contain no edge labeled Δ , hence Definition 4.2.20(1) applies.

Equation $ph = 0$ is a consequence of the fact that, for any admissible labeled tree m_T , if $h(m_T)$ is non-trivial then it is bad, since it must then contain an inner edge with top label $s\Delta$ and bottom label in $s\bar{\mathcal{O}}$ (at least the one whose top inner vertex has just been created by subdivision).

Equation $pi = \mathbb{1}_{B(\mathcal{D}) \circ B(\mathcal{O})}$ follows from the very definition of i and p in Remark 4.2.13.

We now turn to (4.2.4), which is the most difficult part. We must check it for m_T good or bad.

If m_T is good then it is in the image of i , and (4.2.4) follows from the other equations.

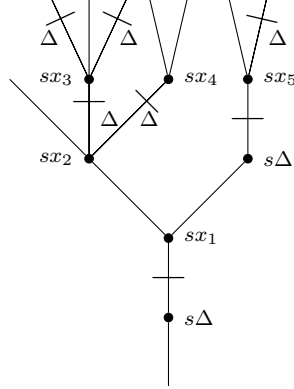
Now let m_T be bad. Then $p(m_T) = 0$, so we must check that

$$-m_T = dh(m_T) + hd(m_T), \quad (4.2.23)$$

where $d = d_2$ is the bar construction differential. We distinguish two cases.

If Definition 4.2.20(1) applies to m_T , then $h(m_T) = 0$. Moreover, since m_T is bad, its essential block must contain at least one Δ . Furthermore, $d(m_T)$ is a sum of admissible labeled trees obtained by contracting inner edges. The essential blocks of these summands (the non-trivial ones) either coincide with the essential block n_T of m_T or they are one of the summands appearing in $d(n_T)$. Therefore, all these summands also fit in Definition 4.2.20(1), except for the one obtained by contracting the inner edge with top vertex w . In the example of the previous definition, which is (4.2.18), this

summand (modulo signs) is



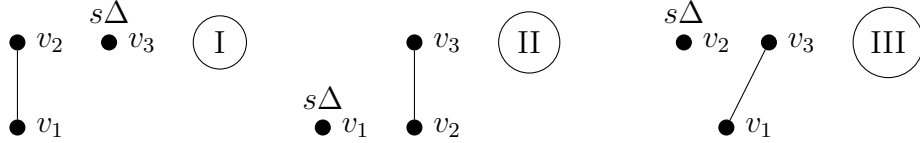
We have incorporated the cuttings in Definition 4.2.20 in order to make the observation that, now, applying h to this admissible labeled tree, we recover the original one (4.2.18) up to sign. We must now check that the product of the two signs we have ignored yield a minus sign. The first sign, coming from the bar construction differential d , is $(-1)^{\|x\|}(-1)^{\|m_T\|_{<v}}$, where sx is the label of w . The second factor in the product (2.3.2) is $+1$ since $\|s\Delta\| = 0$. The second sign, from h , is $(-1)^{\|sx\|+\|m_T\|_{<v}} = -(-1)^{\|x\|}(-1)^{\|m_T\|_{<v}}$. Hence we are done with this case.

Assume now that Definition 4.2.20(2) applies to m_T . Then $-m_T$ appears as a summand in $dh(m_T)$. More precisely, it is the summand obtained by contracting the inner edge in $h(m_T)$ whose top vertex is the new inner vertex created by subdivision. It is clear that this summand is m_T modulo signs. Let us check that the sign is -1 . The sign coming from h is $(-1)^{\|sx\|+\|m_T\|_{<v}}$, where sx is the label of v . The sign coming from d is $(-1)^{\|x\|}(-1)^{\|m_T\|_{<v}}$. Again, the second factor of the product (2.3.2) is $+1$ since $\|s\Delta\| = 0$. The product of both signs is -1 , as in the previous paragraph.

Suppose that the leftmost Δ in the essential part of m_T was labeling an inner edge, e.g. in (4.2.21). Then the summand of $d(m_T)$ obtained by contracting this inner edge vanishes. Indeed, since $d'\mathcal{O}$ uses the trivial distributive law and $\Delta^2 = 0$ in the dual numbers, contracting an inner edge labeled with Δ always yields zero. For the same reason, contracting an inner edge with bottom label $s\Delta$ is always zero. Hence, in $dh(m_T)$, contracting the incoming edge of the newly created inner vertex also vanishes.

We now check that the remaining summands in $dh(m_T)$ cancel pairwise with the remaining summands of $hd(m_T)$. Both collections of remaining summands are indexed by the inner edges of the tree underlying m_T different from the one containing the leftmost Δ in the essential part. We have pointed

out before that the essential part of any summand in $d(m_T)$ coincides with the essential part n_T of m_T or with a summand in $d(n_T)$. For this reason, these remaining summands in both $dh(m_T)$ and $hd(m_T)$ pairwise coincide modulo signs. We just have to check that the signs are opposite. We distinguish three cases, according to the relative position of the inner edge and the inner vertex created by subdivision, illustrated in the following picture,



Here we depict the indexing inner edge and the inner vertex created by subdivision, which is labeled $s\Delta$. The labels v_i only indicate the positions of the vertices in the path order. Let sx_i be the label of v_i .

Case I. The only difference in the summands indexed by the depicted inner edge is that, on the one hand, in $hd(m_T)$, when computing h , we find the total degree of $s(x_1 \circ_i x_2)$, which is $\|x_1\| + \|x_2\| + 1$. This label is obtained by contracting the inner edge, which becomes an inner vertex preceding v_3 . On the other hand, in $dh(m_T)$ we separately find the total degrees of sx_1 and sx_2 , which add up to $\|x_1\| + \|x_2\| + 2$.

Case II. In this case, the different in signs is explained by the fact that in $dh(m_T)$, when contracting the inner edge in the formula for d , we find $s\Delta$ before v_2 , which has total degree 0, with in $hd(m_T)$ we simply find a Δ , of total degree -1 .

Case III. Finally, in this case, the sign of the factor corresponding to the depicted inner edge is, in $hd(m_T)$,

$$\begin{aligned} (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} \|m_T\|_{<v_3}^{>v_1} (-1)^{\|x_1\|} (-1)^{\|sx\| + \|m_T\|_{<v} + \|x_3\|} = \\ (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} \|m_T\|_{<v_3}^{>v_1} (-1)^{\|x_1\|} \\ (-1)^{\|m_T\|_{<v_1}} (-1)^{\|x_1\| + \|x_3\| + 1} (-1)^{\|sx\| + \|m_T\|_{<v}^{>v_1}}. \end{aligned}$$

Here v , without subscripts, is the vertex in Definition 4.2.20(2), and sx is its label. This vertex is bigger or equal than v_1 in the path order, but v is always strictly before v_2 . The sign in $dh(m_T)$ is

$$\begin{aligned} (-1)^{\|sx\| + \|m_T\|_{<v}} (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} (\|m_T\|_{<v_2}^{>v_1} + 1) (-1)^{\|x_1\|} = \\ (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_1\|} (-1)^{\|m_T\|_{<v}^{>v_1}} (-1)^{\|sx\|} \\ (-1)^{\|m_T\|_{<v_1}} (-1)^{\|sx_3\|} (\|m_T\|_{<v_3}^{>v_1} + 1) (-1)^{\|x_1\|}. \end{aligned}$$

The two computed signs are clearly opposite. This concludes the proof of equation (4.2.4) and hence of the theorem. \square

4.3 The Koszul dual ns-cooperad of a derived ns-operad

Still in the nonsymmetric setting, we perturb the previous section's small model for the bar construction of mock derived operads in order to obtain a similar small model for the bar construction of an honest derived operad. In this way we obtain a Koszulity result under minimal assumptions and compute its Koszul dual cooperad.

Let $\mathcal{O} = (E|R)$ be a quadratic ns-operad which is aritywise projective.

Definition 4.3.1. Define the map of sequences $\varphi^i: \mathcal{D}^i \circ \mathcal{O}^i \rightarrow \mathcal{O}^i \circ \mathcal{D}^i$ by

$$(s\Delta)^i \otimes x \mapsto \sum_{j_1 + \dots + j_n = i} x \otimes (s\Delta)^{j_1} \otimes \dots \otimes (s\Delta)^{j_n}$$

where n is the arity of x .

The main result of this section, which is the following theorem, is on the Koszul duality of the derived operad $d\mathcal{O}$.

Theorem 4.3.2. *The map φ^i in Definition 4.3.1 is a coaugmented codistributive law, i.e. an augmented distributive law in the opposite monoidal category of the category of sequences endowed with the circle product, and there is a coaugmented ns-cooperad isomorphism*

$$(d\mathcal{O})^i \cong \mathcal{D}^i \circ_{\varphi^i} \mathcal{O}^i.$$

Moreover, if \mathcal{O} is Koszul then so is $d\mathcal{O}$.

Computations here depend heavily on the previous section's calculations.

Remark 4.3.3. We have seen in Definition 3.3.1, Theorem 3.3.2, and Corollary 3.3.3 that the ns-operad $d\mathcal{O}$ is of the form $\mathcal{O} \vee_{\lambda} \mathcal{D}$ for a certain rewriting rule λ . Therefore, experts might wonder why we do not invoke [16, Theorem 8.6.4] (or an extension to our wider context) in order to prove the Koszulity of $d\mathcal{O}$. The proof of that theorem, unfortunately, contains a flaw. Namely, the increasing filtration F_p by number of inversions in the bar construction is not compatible with the differential. We now illustrate this with a simple example.

Consider two algebras of dual numbers, $k[x]/(x^2)$ and $k[y]/(y^2)$, that we can regard as operads of chain complexes concentrated in arity 1 and degree 0 (bidegree $(0, 0)$ if we insisted to work with graded complexes). We consider the rewriting rule defined by $\lambda(y \otimes x) = x \otimes y$. Hence

$$k[x]/(x^2) \vee_\lambda k[y]/(y^2) = k[x, y]/(x^2, y^2),$$

a commutative finite-dimensional k -algebra linearly spanned by the unit and the monomials x , y , and xy . The bar construction $B(k[x, y]/(x^2, y^2))$ is the tensor algebra on the free module of rank 3 generated by the previous monomials. Generating tensors can be regarded as liner trees with inner vertices labeled with these three monomials. For the sake of simplicity, we here use the tensor notation. An *inversion* in such a tree is an inner edge linking a y to an x . In terms of tensors, they look like one of the following examples

$$\begin{aligned} & \cdots \otimes y \otimes x \otimes \cdots, \\ & \cdots \otimes xy \otimes x \otimes \cdots, \\ & \cdots \otimes y \otimes xy \otimes \cdots, \\ & \cdots \otimes xy \otimes xy \otimes \cdots. \end{aligned}$$

There are no more kinds of inversions, and a given tensor may contain several. For counting inversions, we must fix the notation xy for the monomial otherwise equal to yx . The filtration F_p is defined by letting $F_p B(k[x, y]/(x^2, y^2))$ be spanned by tensors with $\leq p$ inversions, and it is claimed in [16, Theorem 8.6.4] that the bar construction differential preserves this filtration. The tensor

$$y \otimes y \otimes x \otimes x$$

contains exactly one inversion (in the middle). Nevertheless, the differential of this tensor is

$$y \otimes xy \otimes x,$$

which contains two. This disproves the claim.

The reader could complain that the previous example is not of the form $d\mathcal{O}$. Nevertheless, the same argument works to show that the same happens with $d\mathcal{O} = \mathcal{O} \vee_\lambda \mathcal{D}$ for $\mathcal{O} = \mathcal{D}$. Hence not only [16, Theorem 8.6.4] fails in general, but also in our case of interest.

The spectral sequence of this faulty filtration is also used in [16, Proposition 8.6.6], which would have provided some partial information on $(d\mathcal{O})^i$, at least as a sequence, and over a ground field. These are the reasons why we have had to develop a completely new approach for the computation of the Koszul dual cooperad $(d\mathcal{O})^i$.

We would also like to stress that, since our counterexamples are concentrated in arity 1, they work in both the symmetric and the nonsymmetric contexts.

Theorem 4.3.2 will be derived from the following one.

Definition 4.3.4. Define the map of sequences $\bar{\varphi}: B(\mathcal{D}) \circ B(\mathcal{O}) \rightarrow B(\mathcal{O}) \circ B(\mathcal{D})$ by the same formula as in Definition 4.3.1.

Theorem 4.3.5. *The previous map $\bar{\varphi}$ is a coaugmented codistributive law and there is an SDR of sequences*

$$B(\mathcal{D}) \circ_{\bar{\varphi}} B(\mathcal{O}) \xleftarrow[p]{i'} B(d\mathcal{O}) \rightrightarrows h'$$

preserving the syzygy degree, where p is defined as in Theorem 4.2.5, and i' is a coaugmented ns-cooperad morphism.

Proof of Theorem 4.3.2. Since $B(\mathcal{D})$ is k -free and has trivial differential, the syzygy-graded cohomology is

$$H^*(B(d\mathcal{O})) \cong B(\mathcal{D}) \circ H^*(B(\mathcal{O})).$$

In particular, we get the desired coaugmented ns-cooperad isomorphism in syzygy degree zero, since $\bar{\varphi}$ restricts to φ^i in H^0 via i' , and we derive the Koszulity statement. \square

Theorem 4.3.5 will be obtained from Theorem 4.2.5 by applying techniques from homological perturbation theory. We start by recalling the following Lemma and Remark, whose versions underneath can be found in [19, Lemma 1.17 and Remark 1.18].

Lemma 4.3.6 (Basic Perturbation Lemma [6]). *Given an SDR of graded complexes*

$$(X, d^X) \xleftarrow[p]{i} (Y, d^Y) \rightrightarrows h$$

and a bidegree $(0, -1)$ map of modules $\partial: Y \rightarrow Y$, called perturbation, such that $\partial^2 + d^Y \partial + \partial d^Y = 0$ and the infinite sum $\Sigma_\infty = \sum_{n \geq 0} (\partial h)^n \partial$ is well defined, i.e. almost all summands vanish when evaluated at a given $y \in Y$, then there is a new SDR

$$(X, d_X + p\Sigma_\infty i) \xleftarrow[p+p\Sigma_\infty h]{i+h\Sigma_\infty i} (Y, d_Y + \partial) \rightrightarrows h+h\Sigma_\infty h.$$

Remark 4.3.7. The vanishing condition is fulfilled if Y is equipped with an exhaustive increasing filtration

$$0 = F_{-1}Y \subset F_0Y \subset \cdots \subset F_nY \subset F_{n+1}Y \subset \cdots \subset Y, \quad Y = \bigcup_{n \geq 0} F_nY,$$

such that, for $n \geq 0$,

$$\partial(F_nY) \subset F_{n-1}Y, \quad h(F_nY) \subset F_nY.$$

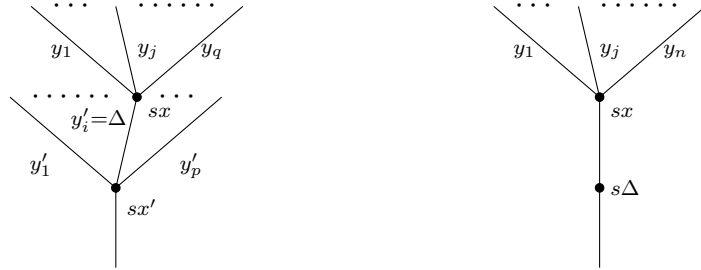
This implies that, if $y \in F_nY$, then $\Sigma_\infty(y) = \sum_{j=0}^{n-1} (\partial h)^j \partial(y)$. The maps i and p often preserve the filtration, like h .

Note that the bar constructions of the derived operad and the mock derived operad of \mathcal{O} , $B(d\mathcal{O})$ and $B(d'\mathcal{O})$, have the same underlying sequence of bigraded modules.

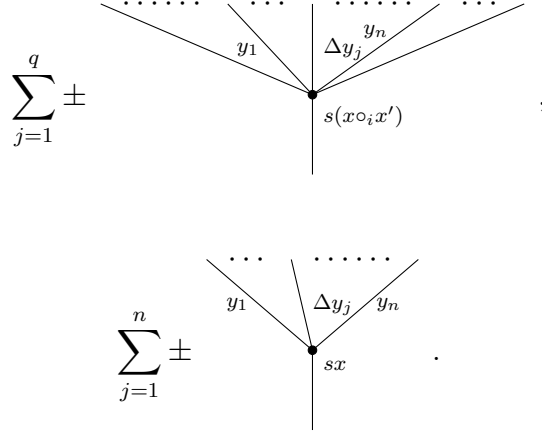
Lemma 4.3.8. *The map $\partial = d_{B(d\mathcal{O})} - d_{B(d'\mathcal{O})}: B(d'\mathcal{O}) \rightarrow B(d'\mathcal{O})$ is a perturbation for the SDR in Theorem 4.2.5 in the sense of Lemma 4.3.6.*

Remark 4.3.9. Consider an admissible labeled tree m_T , with the notation in Remark 4.2.17. Let us look at $\partial(m_T) = (d_{B(d\mathcal{O})} - d_{B(d'\mathcal{O})})(m_T)$. Both $d_{B(d\mathcal{O})}(m_T)$ and $d_{B(d'\mathcal{O})}(m_T)$ produce sums, indexed by the inner edges $\{v, w\}$, $v < w$. The summand indexed by $\{v, w\}$ is an admissible labeled tree with underlying tree $T/\{v, w\}$, the tree obtained from T by contracting $\{v, w\}$, or maybe a sum of such admissible labeled trees. The label of the new inner vertex u created by contraction and its incoming edges is defined by the operadic composition in $d\mathcal{O}$ and $d'\mathcal{O}$, and that is the only difference. These operads are very similar, they share the underlying sequence $\mathcal{O} \circ \mathcal{D}$, but they are defined by different distributive laws (Definitions 3.1.4 and 4.1.1). This is why the new label may sometimes be the same in both cases. Indeed, the behavior of these two distributive laws is only different on $\Delta \otimes x$, $x \in \bar{\mathcal{O}}$.

Suppose m_T has an inner edge labeled Δ (hence the bottom vertex' label is in $s\bar{\mathcal{O}}$) and with the top vertex's label in $s\bar{\mathcal{O}}$ (and call it type I for later use) or an unlabeled inner edge with bottom vertex labeled $s\Delta$ and top vertex's label in $s\bar{\mathcal{O}}$ (type II), e.g.



Here, each y_j and y'_j is either Δ or the empty label (except for the indicated y'_i , labeling $\{v, w\}$, which is Δ). Then the summand in $d_{B(d'\mathcal{O})}(m_T)$ indexed by such an inner edge is trivial, whereas the summand in $d_{B(d\mathcal{O})}(m_T)$ is not. The latter is itself a sum where each summand is obtained by labeling u with $s(x' \circ_i x)$ in case I and sx in case II, and multiplying by Δ one of the old incoming edges of w at a time, with signs coming from the Koszul convention and from the definition of the differential d_2 of the bar construction, e.g.



Note that the j^{th} summand is zero if $y_j = \Delta$. Summands in $d_{B(d'\mathcal{O})}(m_T)$ and $d_{B(d\mathcal{O})}(m_T)$ indexed by inner edges which are not of type I or II are equal, so they cancel in $\partial(m_T)$. Hence, $\partial(m_T)$ is a sum indexed by the inner edges of T of type I and II, and each summand is itself a summation as above.

Proof. We need to check the two conditions satisfied by a perturbation according to 4.3.6. The first one is

$$\begin{aligned}
 0 &= d_{B(d\mathcal{O})}^2 \\
 &= (d_{B(d'\mathcal{O})} + \partial)^2 \\
 &= d_{B(d'\mathcal{O})}^2 + \partial^2 + d_{B(d'\mathcal{O})}\partial + \partial d_{B(d'\mathcal{O})} \\
 &= \partial^2 + d_{B(d'\mathcal{O})}\partial + \partial d_{B(d'\mathcal{O})}.
 \end{aligned}$$

In order to see that Σ_∞ is well defined, we will use Remark 4.3.7. To this end, we should equip the graded complex $B(d'\mathcal{O})(n)$ with an appropriate filtration.

Consider an admissible labeled tree m_T . For each inner vertex v of T not labeled by $s\Delta$, count the number $N_{m_T, v}$ of labels containing a Δ in the shortest path from the root $r(T)$ to v . Let N_{m_T} be the sum of all the $N_{m_T, v}$. Let $F_s(B(d'\mathcal{O}))$ be the subsequence of bigraded modules spanned

by the admissible labeled trees m_T with $N_{m_T} \leq s$. This obviously defines an exhaustive increasing filtration. Even more, $d_{\mathbb{B}(d'\mathcal{O})} = d_2$ preserves this filtration in the sense that

$$d_{\mathbb{B}(d'\mathcal{O})}(F_s(\mathbb{B}(d'\mathcal{O})(n))) \subset F_s(\mathbb{B}(d'\mathcal{O})(n)). \quad (4.3.10)$$

Let us argue for (4.3.10). Recall that $d_2(m_T)$ is a sum, indexed by inner edges $\{v, w\}$, $v < w$, of T , of admissible labeled trees $m_{T/\{v,w\}}$ in which the inner edge $\{v, w\}$ of T is contracted. We have to check that for each $m_{T/\{v,w\}}$ we have $N_{m_{T/\{v,w\}}} \leq N_{m_T}$. We distinguish between different cases. They are associated with different types of edges $\{v, w\}$ of m_T .

Case 1: Let $\{v, w\}$ be labeled by Δ or having bottom vertex labeled by $s\Delta$. In Remark 4.2.17 it is explained that $m_{T/\{v,w\}} = 0$ in this case, hence obviously $N_{m_{T/\{v,w\}}} \leq N_{m_T}$.

Case 2: Otherwise, $\{v, w\}$ carries no label and the bottom vertex' label is in $s\bar{\mathcal{O}}$. The tree $T/\{v, w\}$ underlying $m_{T/\{v,w\}}$ has one inner vertex less than T underlying m_T , caused by the contraction of $\{v, w\}$. We denote the new vertex in $T/\{v, w\}$ created by the contraction of $\{v, w\}$ by u .

In case the top vertex w is labeled by $s\Delta$, it will not be summed over in the calculation of N_{m_T} . As noted in Remark 4.2.17, $T/\{v, w\}$ is up to sign obtained by removing the inner vertex w and replacing its label $s\Delta$ with a Δ labeling the edge. Therefore, $N_{m_{T/\{v,w\}},v} = N_{m_T,u}$, and for the other inner vertices $t \neq v, w$ not labeled by $s\Delta$, we have $N_{m_T,t} = N_{m_{T/\{v,w\}},t}$. Here we use that the inner vertices of $T/\{v, w\}$ different from u are the inner vertices of T different from v and w . It follows that $N_{m_{T/\{v,w\}}} = N_{m_T}$.

In case the top vertex' label is in $s\bar{\mathcal{O}}$, it will only contribute extra to the number N_{m_T} . For the rest of inner vertices, the situation is very similar to the previous paragraph. Hence $N_{m_{T/\{v,w\}}} \leq N_{m_T}$.

It remains to check, for $s \geq 0$, that

$$\partial(F_s(\mathbb{B}(d'\mathcal{O})(n))) \subset F_{s-1}(\mathbb{B}(d'\mathcal{O})(n)), \quad (4.3.11)$$

and that

$$h(F_s(\mathbb{B}(d'\mathcal{O})(n))) \subset F_s(\mathbb{B}(d'\mathcal{O})(n)). \quad (4.3.12)$$

Equation (4.3.11) is satisfied. Indeed, looking at the formula for $\partial(m_T)$ in Remark 4.3.9 we see that, in each summand $m_{T/\{v,w\}}$ of $\partial(m_T)$, there is a Δ or $s\Delta$ in T which has jumped upwards over a label in $s\bar{\mathcal{O}}$, hence we get a strict inequality $N_{m_{T/\{v,w\}}} < N_{m_T}$.

In order to check (4.3.12) we distinguish two cases.

Case 1: When h is defined via Definition 4.2.20(1), then $h(m_T) = 0$, so there is nothing to check.

Case 2: Now let h be defined via Definition 4.2.20(2), as exemplified by (4.2.21). The newly created inner vertex v labeled $s\Delta$ is not summed over in the calculation of $N_{h(m_T)}$. Furthermore, for inner vertices $t \neq v$ in the planted planar tree with leaves underlying $h(m_T)$, we have $N_{h(m_T),t} = N_{m_T,t}$, where the inner vertices t of T are the ones corresponding in the obvious way to the inner vertices $t \neq v$ of the planted planar tree with leaves underlying $h(m_T)$. It follows that $N_{h(m_T)} = N_{m_T}$. The proof is now complete. \square

Lemma 4.3.13. *The compositions $p\Sigma_\infty i$ and $p\Sigma_\infty h$ equal zero.*

Proof. When applying ∂ to an admissible labeled tree m_T , $\partial(m_T)$ is a sum of admissible labeled trees with at least one mixed vertex, see Remark 4.3.9. Hence $p\partial = 0$.

Therefore,

$$\begin{aligned} p\Sigma_\infty i &= p(\partial + (\partial h)\partial + (\partial h)(\partial h)\partial + \cdots)i \\ &= (p\partial)(\mathbf{1}_{\mathcal{F}^c(s\overline{\mathcal{O}}\circ\mathcal{D})} + h\partial + h(\partial h)\partial + \cdots)i \\ &= (p\partial)(\mathbf{1}_{\mathcal{F}^c(s\overline{\mathcal{O}}\circ\mathcal{D})} + h\Sigma_\infty)i \\ &= 0, \end{aligned}$$

and, similarly,

$$\begin{aligned} p\Sigma_\infty h &= (p\partial)(\mathbf{1}_{\mathcal{F}^c(s\overline{\mathcal{O}}\circ\mathcal{D})} + h\Sigma_\infty)h \\ &= 0. \end{aligned}$$

\square

Proof of Theorem 4.3.5. That there is an SDR of sequences

$$\mathbf{B}(\mathcal{D}) \circ_{\bar{\varphi}} \mathbf{B}(\mathcal{O}) \xleftarrow[p]{i'} \mathbf{B}(d\mathcal{O}) \xrightarrow{h'}$$

preserving the syzygy degree is a direct consequence of Theorem 4.2.5 and Lemmas 4.3.6, 4.3.8, and 4.3.13. Here $i' = i + h\Sigma_\infty i$ and $h' = h + h\Sigma_\infty h$.

In order to check that $\bar{\varphi}$ is a coaugmented codistributive law, we must check that the dual versions of the five diagrams in Definition 1.2.4 commute. In order to check commutativity of the dual of 3, we must consider two types of elements in $\mathbf{B}(\mathcal{D}) \circ \mathbf{B}(\mathcal{O})$:

1. $(s\Delta)^i \otimes \text{id}_{\mathbf{B}(\mathcal{O})}$.
2. $(s\Delta)^i \otimes x$.

Here $x \in \overline{B(\mathcal{O})}$. In the second case, $\epsilon_{B(\mathcal{O})}(x) = 0$ assures that both paths from source to sink in the dual of 3 give 0. In the first case commutativity of the dual of 3 is equivalent to the following formula, which holds by definition,

$$\bar{\varphi}((s\Delta)^i \otimes \text{id}_{B(\mathcal{O})}) = \text{id}_{B(\mathcal{O})} \otimes (s\Delta)^i.$$

To check commutativity of the dual of 4, we must again consider two types of elements in $B(\mathcal{D}) \circ B(\mathcal{O})$:

1. $(s\Delta)^0 \otimes x$.
2. $(s\Delta)^i \otimes x, i > 0$.

Here $x \in B(\mathcal{O})$. In the second case $\epsilon_{B(\mathcal{D})}((s\Delta)^i) = 0$ for $i > 0$ assures that both paths from source to sink in the dual of 4 give 0. In the first case commutativity of the dual of 4 is equivalent to the following formula, which also holds by definition,

$$\bar{\varphi}((s\Delta)^0 \otimes x) = x \otimes (s\Delta)^0 \otimes \cdots \otimes (s\Delta)^0.$$

Checking the commutativity of the dual of diagram 2 is equivalent to checking the following equation.

$$\begin{aligned} (\bar{\varphi} \circ \mathbf{1}_{B(\mathcal{D})})(\mathbf{1}_{B(\mathcal{D})} \circ \bar{\varphi})(\Delta_{B(\mathcal{D})} \circ \mathbf{1}_{B(\mathcal{O})})((s\Delta)^i \otimes x) & \quad (4.3.14) \\ = (\mathbf{1}_{B(\mathcal{O})} \circ \Delta_{B(\mathcal{D})})\bar{\varphi}((s\Delta)^i \otimes x), & \end{aligned}$$

for $x \in B(\mathcal{O})(n)$.

On the one hand, the left hand side of equation (4.3.14) equals

$$\begin{aligned}
& (\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})(\mathbf{1}_{\mathcal{B}(\mathcal{D})} \circ \bar{\varphi})(\Delta_{\mathcal{B}(\mathcal{D})} \circ \mathbf{1}_{\mathcal{B}(\mathcal{O})})((s\Delta)^i \otimes x) \\
&= \sum_{j+q=i} (\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})(\mathbf{1}_{\mathcal{B}(\mathcal{D})} \circ \bar{\varphi})(((s\Delta)^j \otimes (s\Delta)^q) \otimes x) \\
&= \sum_{j+q=i} (\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})(\mathbf{1}_{\mathcal{B}(\mathcal{D})} \circ \bar{\varphi})((s\Delta)^j \otimes ((s\Delta)^q \otimes x)) \\
&= \sum_{\substack{q_1+\dots+q_n=q \\ j+q=i}} (\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})((s\Delta)^j \otimes (x \otimes (s\Delta)^{q_1} \otimes \dots \otimes (s\Delta)^{q_n})) \\
&= \sum_{\substack{q_1+\dots+q_n=q \\ j+q=i}} (\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})(((s\Delta)^j \otimes x) \otimes (s\Delta)^{q_1} \otimes \dots \otimes (s\Delta)^{q_n}) \\
&= \sum_{\substack{q_1+\dots+q_n=q \\ j_1+\dots+j_n=j \\ j+q=i}} (x \otimes (s\Delta)^{j_1} \otimes \dots \otimes (s\Delta)^{j_n}) \otimes (s\Delta)^{q_1} \otimes \dots \otimes (s\Delta)^{q_n} \\
&= \sum_{\substack{q_1+\dots+q_n=q \\ j_1+\dots+j_n=j \\ j+q=i}} x \otimes ((s\Delta)^{j_1} \otimes (s\Delta)^{q_1}) \otimes \dots \otimes ((s\Delta)^{j_n} \otimes (s\Delta)^{q_n}) \\
&= \sum_{\substack{i_1+\dots+i_n=i \\ j_1+q_1=i_1 \\ \vdots \\ j_n+q_n=i_n}} x \otimes ((s\Delta)^{j_1} \otimes (s\Delta)^{q_1}) \otimes \dots \otimes ((s\Delta)^{j_n} \otimes (s\Delta)^{q_n}),
\end{aligned}$$

using the associator of \circ .

On the other hand, the right hand side of equation (4.3.14) equals

$$\begin{aligned}
& (\mathbf{1}_{\mathcal{B}(\mathcal{O})} \circ \Delta_{\mathcal{B}(\mathcal{D})})\bar{\varphi}((s\Delta)^i \otimes x) \\
&= \sum_{i_1+\dots+i_n=i} (\mathbf{1}_{\mathcal{B}(\mathcal{O})} \circ \Delta_{\mathcal{B}(\mathcal{D})})(x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n}) \\
&= \sum_{\substack{i_1+\dots+i_n=i \\ j_1+q_1=i_1 \\ \vdots \\ j_n+q_n=i_n}} x \otimes ((s\Delta)^{j_1} \otimes (s\Delta)^{q_1}) \otimes \dots \otimes ((s\Delta)^{j_n} \otimes (s\Delta)^{q_n}).
\end{aligned}$$

Checking the commutativity of the dual of diagram 1 is equivalent to checking the following equation.

$$\begin{aligned}
& (\mathbf{1}_{\mathcal{B}(\mathcal{O})} \circ \bar{\varphi})(\bar{\varphi} \circ \mathbf{1}_{\mathcal{B}(\mathcal{O})})(\mathbf{1}_{\mathcal{B}(\mathcal{D})} \circ \Delta_{\mathcal{B}(\mathcal{O})})((s\Delta)^i \otimes x) \\
&= (\Delta_{\mathcal{B}(\mathcal{O})} \circ \mathbf{1}_{\mathcal{B}(\mathcal{D})})\bar{\varphi}((s\Delta)^i \otimes x),
\end{aligned} \tag{4.3.15}$$

for $x \in B(\mathcal{O})(n)$.

On the one hand, the left hand side of equation (4.3.15) is the following element, where we use the notation introduced in (1.4.7) for $\Delta_{B(\mathcal{O})}(x)$,

$$\begin{aligned}
& (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})(\bar{\varphi} \circ \mathbb{1}_{B(\mathcal{O})})(\mathbb{1}_{B(\mathcal{D})} \circ \Delta_{B(\mathcal{O})})((s\Delta)^i \otimes x) \\
&= (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})(\bar{\varphi} \circ \mathbb{1}_{B(\mathcal{O})})((s\Delta)^i \otimes \Delta_{B(\mathcal{O})}(x)) \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n}} (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})(\bar{\varphi} \circ \mathbb{1}_{B(\mathcal{O})})((s\Delta)^i \otimes (x_k \otimes x_{i_1} \otimes \dots \otimes x_{i_k})) \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n}} (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})(\bar{\varphi} \circ \mathbb{1}_{B(\mathcal{O})})(((s\Delta)^i \otimes x_k) \otimes x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n \\ j_1 + \dots + j_k = i}} (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})((x_k \otimes (s\Delta)^{j_1} \otimes \dots \otimes (s\Delta)^{j_k}) \otimes x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n \\ j_1 + \dots + j_k = i}} (\mathbb{1}_{B(\mathcal{O})} \circ \bar{\varphi})(x_k \otimes ((s\Delta)^{j_1} \otimes x_{i_1}) \otimes \dots \otimes ((s\Delta)^{j_k} \otimes x_{i_k})) \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n \\ q_{l,1} + \dots + q_{l,i_l} = j_l \\ 1 \leq l \leq k}} x_k \otimes \dots \otimes (x_{i_l} \otimes (s\Delta)^{q_{l,1}} \otimes \dots \otimes (s\Delta)^{q_{l,i_l}}) \otimes \dots \\
&= \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n \\ q_{1,1} + \dots + q_{k,i_k} = i}} (x_k \otimes x_{i_1} \otimes \dots \otimes x_{i_k}) \otimes (s\Delta)^{q_{1,1}} \otimes \dots \otimes (s\Delta)^{q_{k,i_k}} \\
&= \sum_{q_{1,1} + \dots + q_{k,i_k} = i} \Delta_{B(\mathcal{O})}(x) \otimes (s\Delta)^{q_{1,1}} \otimes \dots \otimes (s\Delta)^{q_{k,i_k}}.
\end{aligned}$$

Here we use again the associator of \circ .

On the other hand, the right hand side of equation (4.3.15) equals

$$\begin{aligned}
& (\Delta_{B(\mathcal{O})} \circ \mathbb{1}_{B(\mathcal{D})})\bar{\varphi}((s\Delta)^i \otimes x) \\
&= \sum_{i_1 + \dots + i_n = i} (\Delta_{B(\mathcal{O})} \circ \mathbb{1}_{B(\mathcal{D})})(x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n}) \\
&= \sum_{i_1 + \dots + i_n = i} \Delta_{B(\mathcal{O})}(x) \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n}.
\end{aligned}$$

The commutativity of the dual of diagram 5 amounts to $\bar{\varphi}((s\Delta)^0 \otimes \text{id}_{\mathcal{O}}) = \text{id}_{\mathcal{O}} \otimes (s\Delta)^0$, which holds by definition.

It is left to check that i' is an ns-cooperad morphism. To this end, we first spell out the map i' . Define $\phi_n: \mathbf{B}(\mathcal{D}) \circ \mathbf{B}(\mathcal{O}) \rightarrow \mathbf{B}(d\mathcal{O})$ as $\phi_n := (h\partial)^n i$ for $n \geq 0$, so $\phi_{n+1} = (h\partial)\phi_n$. Then

$$i' = i + h\Sigma_\infty i = \sum_{n \geq 0} \phi_n.$$

The image of i is linearly spanned by good admissible labeled trees, which do not contain mixed labels. We are going to describe $(h\partial)(m_T)$ for m_T an admissible labeled tree without mixed labels. We will prove that $(h\partial)(m_T)$ is a sum of admissible labeled trees without mixed labels and deduce a formula for i' .

First, by Remark 4.3.9, $\partial(m_T)$ is a sum indexed by the type II inner edges. Each summand is again a sum of admissible labeled trees obtained by removing the bottom vertex, labeled with $s\Delta$, and labeling with Δ one incoming edge of the top vertex at a time, with certain signs,

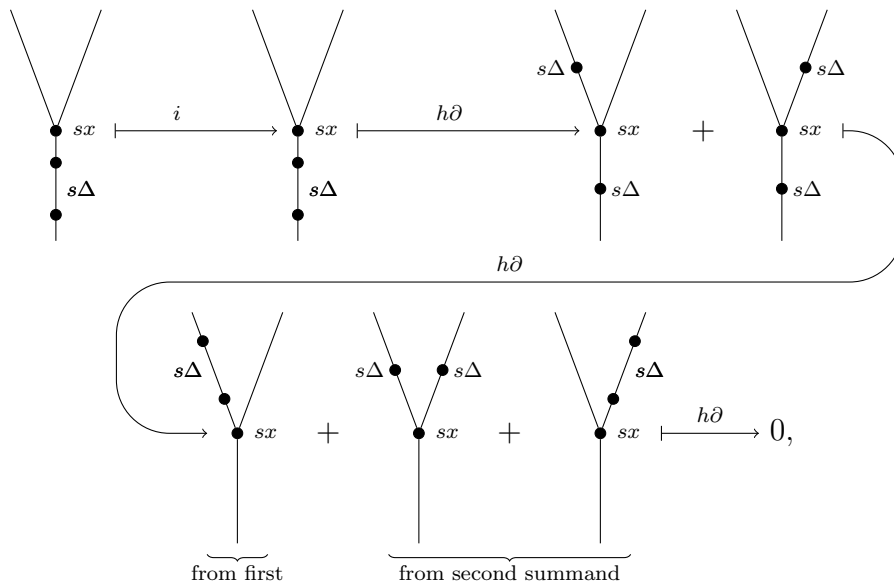
(4.3.16)

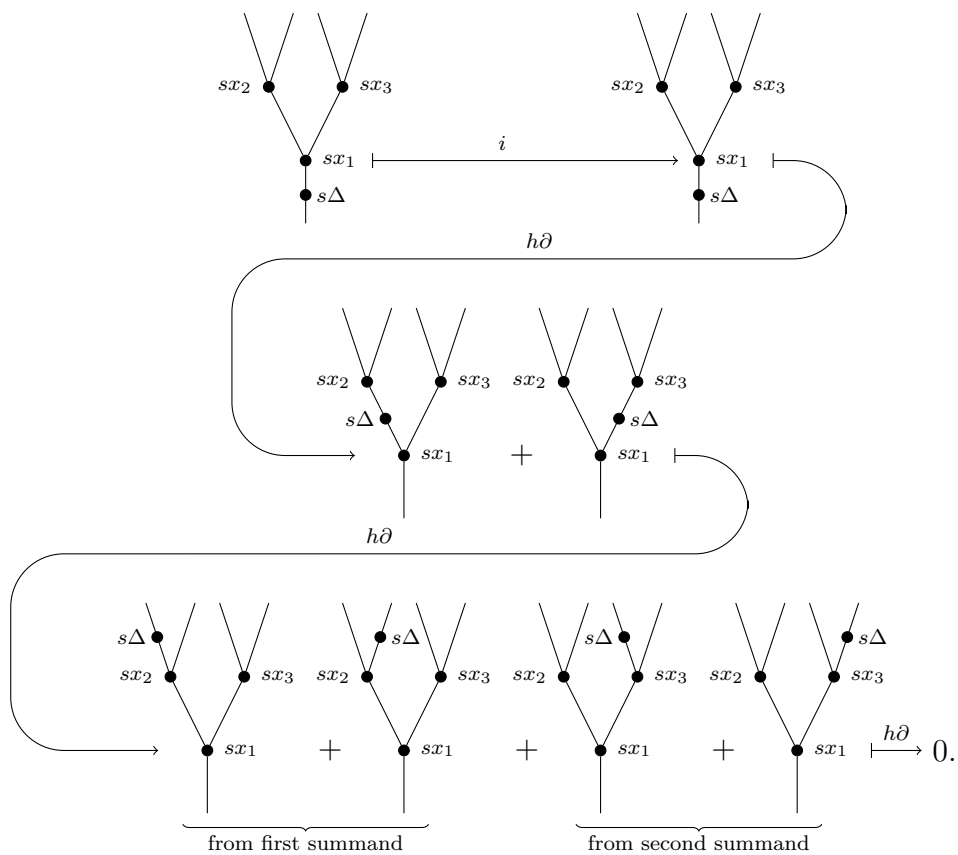
Only those summands where Δ is leftmost in the essential block survive in $h\partial(m_T)$, and they get subdivided as in Definition 4.2.20(2), e.g.

(4.3.17)

In particular, there are no mixed labels. The signs coming from ∂ and from h cancel each other, so we get a formula without signs.

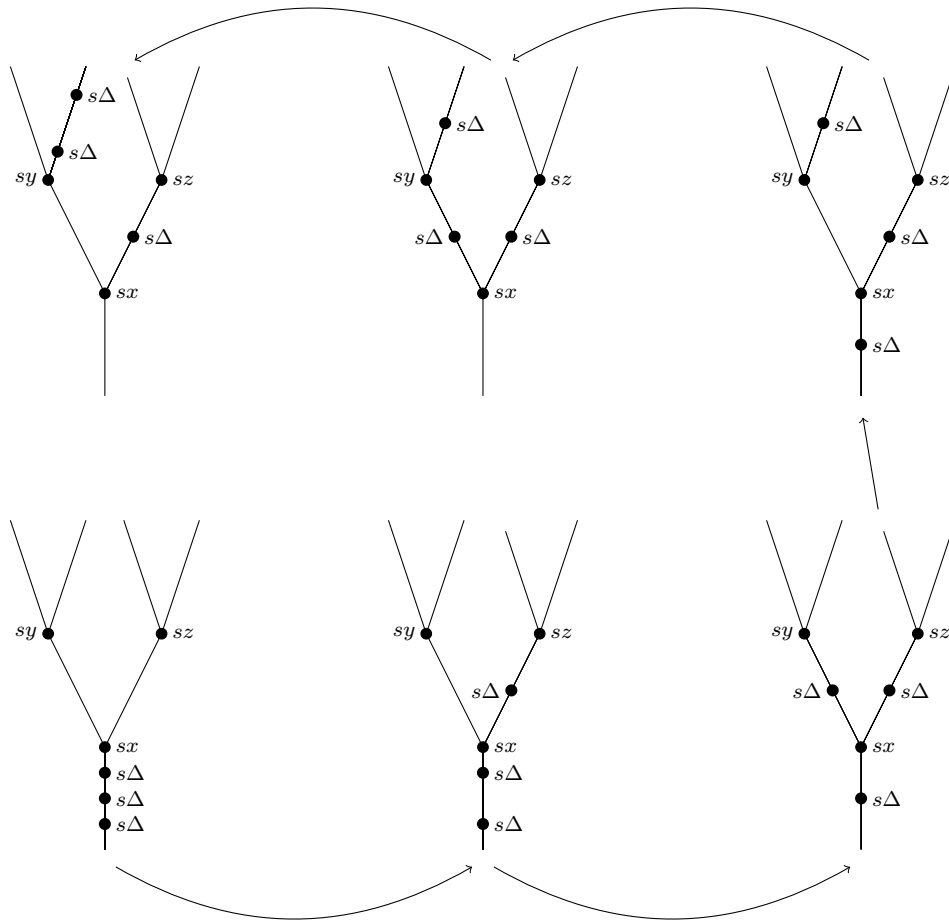
If we start with a good admissible labeled tree like (4.2.12), which is of the form $m_T = (s\Delta)^i \otimes m_{T'}$, where $m_{T'}$ has all labels in $s\bar{\mathcal{O}}$, the previous computation shows that, if T' has n edges, $i'(m_T)$ is a sum indexed by the set of non-negative integers $j_1, \dots, j_n \geq 0$ with $j_1 + \dots + j_n = i$. Each factor, denoted $(m_{T'})_{j_1, \dots, j_n}$, is obtained by subdividing the l^{th} edge of T' , adding j_l new inner vertices labeled with $s\Delta$, $1 \leq l \leq n$, e.g.





Each summand in i' of some good admissible labeled tree can be uniquely traced back through the iterations of $h\partial$. In the following picture, we start

with a certain one, and arrows represent that the target shows up in $h\partial$:



The procedure at each step is taking the essential block and ‘de-jump’ the leftmost $s\Delta$ therein. This is another way of checking our formula for i' .

With the previous description of i' , we now check that it is a morphism of coaugmented ns-cooperads. Compatibility with the coaugmentation is obvious. Proving that i' is a map of ns-cooperads amounts to checking that

the following diagram commutes

$$\begin{array}{ccc}
\mathrm{B}(\mathcal{D}) \circ \mathrm{B}(\mathcal{O}) & \xrightarrow{\Delta_{\mathrm{B}(\mathcal{D})} \circ \Delta_{\mathrm{B}(\mathcal{O})}} & (\mathrm{B}(\mathcal{D}) \circ \mathrm{B}(\mathcal{D})) \circ (\mathrm{B}(\mathcal{O}) \circ \mathrm{B}(\mathcal{O})) \\
\downarrow i' & & \downarrow \mathbb{1}_{\mathrm{B}(\mathcal{D})} \circ \bar{\varphi} \circ \mathbb{1}_{\mathrm{B}(\mathcal{O})} \\
& & (\mathrm{B}(\mathcal{D}) \circ \mathrm{B}(\mathcal{O})) \circ (\mathrm{B}(\mathcal{D}) \circ \mathrm{B}(\mathcal{O})) \\
& & \uparrow p \circ p \\
\mathrm{B}(d\mathcal{O}) & \xrightarrow{\Delta_{\mathrm{B}(d\mathcal{O})}} & \mathrm{B}(d\mathcal{O}) \circ \mathrm{B}(d\mathcal{O}).
\end{array}$$

is commutative.

If we apply $(p \circ p)\Delta_{\mathrm{B}(d\mathcal{O})}i'$ to a good admissible labeled tree $m_T = (s\Delta)^i \otimes m_{T'}$, we first obtain a summation $i'(m_T)$ as above whose summands $(m_{T'})_{j_1, \dots, j_n}$ are then cut, in the sense of Definition 2.2.2, to obtain $\Delta_{\mathrm{B}(d\mathcal{O})}i'(m_T)$. In $(p \circ p)\Delta_{\mathrm{B}(d\mathcal{O})}i'(m_T)$, only those cuttings whose pieces are good admissible labeled trees survive. The result is the same as cutting separately $(s\Delta)^i$ and $m_{T'}$, giving $\Delta_{\mathrm{B}(\mathcal{D})}((s\Delta)^i)$ and $\Delta_{\mathrm{B}(\mathcal{O})}(m_{T'})$, and then applying $\mathbb{1}_{\mathrm{B}(\mathcal{D})} \circ \bar{\varphi} \circ \mathbb{1}_{\mathrm{B}(\mathcal{O})}$ to $\Delta_{\mathrm{B}(\mathcal{D})}((s\Delta)^i) \otimes \Delta_{\mathrm{B}(\mathcal{O})}(m_{T'})$. Signs match since they are just given by the Koszul sign rule. The proof is now finished. \square

4.4 An alternative chain homotopy

Assume throughout this section that our ground commutative ring contains the rationals, $\mathbb{Q} \subset k$.

In this section we construct a new chain homotopy

$$\bar{h}: \mathrm{B}(d'\mathcal{O}) \rightarrow \mathrm{B}(d'\mathcal{O})$$

fitting in Theorem 4.2.5, for the same i and p , whose perturbation

$$\bar{h}': \mathrm{B}(d\mathcal{O}) \rightarrow \mathrm{B}(d\mathcal{O})$$

in the sense of Lemma 4.3.6 fits in Theorem 4.3.5, for the same i' and p . This will be necessary to obtain symmetric versions of the previous results in this chapter.

We use the same notation and terminology as in the construction of h in Definition 4.2.20.

Definition 4.4.1. Let m_T be an admissible labeled tree. In the essential block, we consider the set $\{e_\Delta^i\}_{i=1}^{n_\Delta}$ of edges labeled Δ , $e_\Delta^i = \{v_i, w_i\}$, $v_i < w_i$,

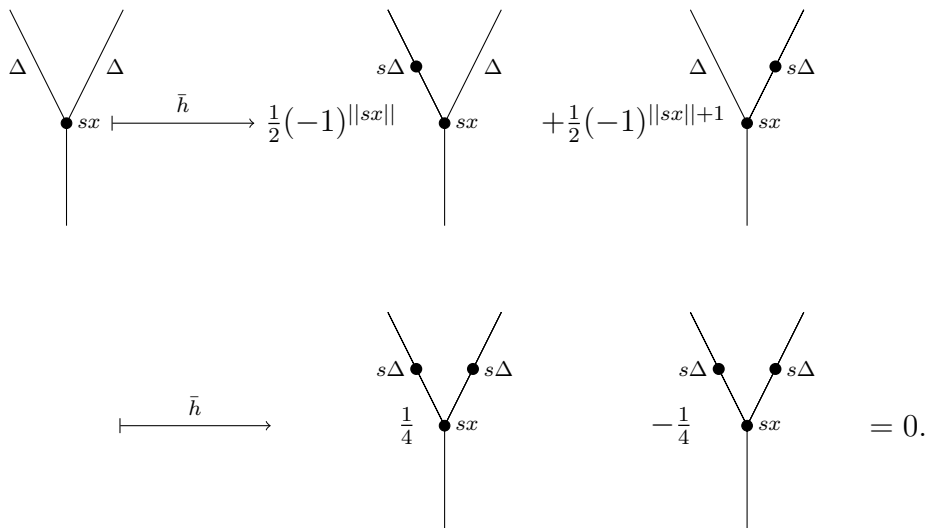
and the set $\{u_{s\Delta}^j\}_{j=1}^{n_{s\Delta}}$ of inner vertices labeled $s\Delta$. We denote by $sx_i \in s\bar{\mathcal{O}}$ the label of v_i in the new labeling style introduced in Remark 4.2.17. Let n_i be the number of edges labeled Δ with bottom vertex v_i which come before e_Δ^i . Moreover, let m_{T,e_Δ^i} be the admissible labeled tree obtained from m_T by subdividing the edge e_Δ^i , drawing a new inner vertex in the middle labeled $s\Delta$, replacing the previous label Δ of e_Δ^i . We define

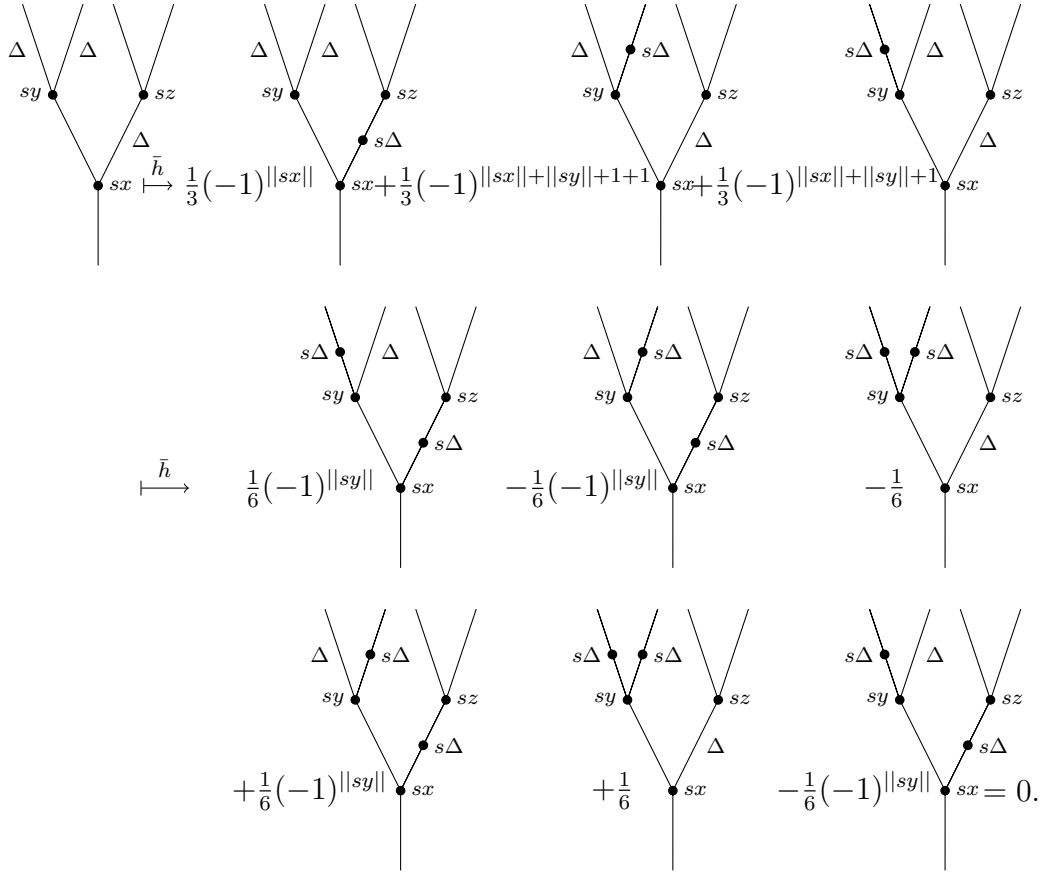
$$\bar{h}(m_T) = \frac{1}{n_\Delta + n_{s\Delta}} \sum_{i=1}^{n_\Delta} (-1)^{\|sx_i\| + \|m_T\|_{<v_i} + n_i} m_{T,e_\Delta^i}.$$

Here $\|m_T\|_{<v_i}$, defined in Section 2.3, is the sum of the total degrees of the labels (old labeling style, see Remark 4.2.22) of the vertices preceding v_i (strictly, in the path order). With respect to the new labeling style, $\|m_T\|_{<v_i}$ is the sum of the total degrees of the labels of the vertices *and edges* preceding v_i , and moreover $\|sx_i\| + \|m_T\|_{<v_i} + n_i$ is the sum of the total degrees of all labels (vertices and edges) preceding e_Δ^i . We will write $n_\Delta = n_{\Delta,m_T}$, $n_{s\Delta} = n_{s\Delta,m_T}$, and $n_i = n_{i,m_T}$ when confusions are possible.

The formula for $\bar{h}(m_T)$ explains why we require $\mathbb{Q} \subset k$. Had we started with this homotopy, the range of applicability of our nonsymmetric results would have been tighter. In the symmetric case, the condition $\mathbb{Q} \subset k$ is very common since many aspects of the theory of operads need all representations of symmetric groups over the ground ring to be projective, e.g. in order to have a model structure which allows from a well-behaved homotopy theory.

The following two pictures illustrate the formula for \bar{h} and show in two special cases that $\bar{h}^2 = 0$.





The equation $\bar{h}^2 = 0$ is checked as part of the following theorem.

Theorem 4.4.2. *There is an SDR of sequences*

$$\mathbb{B}(\mathcal{D}) \circ \mathbb{B}(\mathcal{O}) \xleftarrow[p]{i} \mathbb{B}(d'\mathcal{O}) \xrightarrow{\bar{h}}$$

preserving the syzygy degree where i and p are the same maps as in Theorem 4.2.5 and \bar{h} is the map in Definition 4.4.1.

Proof. We have to check the equations in Definition 4.2.3. The equation $pi = \mathbb{1}_{\mathbb{B}(\mathcal{D}) \circ \mathbb{B}(\mathcal{O})}$ was checked in the proof of Theorem 4.2.5, and $\bar{h}i = 0$ and $p\bar{h} = 0$ also follow from the same reasons as in that proof.

Let us now prove that $\bar{h}^2 = 0$. It is obviously true on good admissible labeled trees, since \bar{h} vanishes on them. Let m_T be a bad admissible labeled tree. Note that, if n_T denotes the essential block of m_T , then the essential block of each summand of $\bar{h}(m_T)$ is a summand of $\bar{h}(n_T)$, up to signs. In particular, if $m_{T'}$ is a summand of $\bar{h}(m_T)$, then, if $n_{\Delta, m_T} > 0$, $n_{\Delta, m_{T'}} = n_{\Delta, m_T} - 1$ and $n_{s\Delta, m_{T'}} = n_{s\Delta, m_T} + 1$, so $n_{\Delta, m_T} + n_{s\Delta, m_T} = n_{\Delta, m_{T'}} + n_{s\Delta, m_{T'}}$.

Hence, by definition, $\bar{h}^2(m_T)$ is a summation indexed by the ordered pairs (e_Δ^i, e_Δ^j) with distinct coordinates $i \neq j$. The summands indexed by (e_Δ^i, e_Δ^j) and (e_Δ^j, e_Δ^i) are scalar multiples of the same admissible labeled tree $m_{T, e_\Delta^i, e_\Delta^j}$, obtained from m_T by subdividing the edges e_Δ^i and e_Δ^j in the usual way. The absolute values of the scalar coefficients of these summands are clearly the same, namely $1/(n_\Delta + n_{s\Delta})^2$. Therefore, in order to check that $\bar{h}^2(m_T) = 0$ it suffices to prove that these two summands have opposite sign. This is indeed easy. Suppose without loss of generality that e_Δ^i comes before e_Δ^j in the path order. Then, in the summand indexed by (e_Δ^i, e_Δ^j) , we first subdivide e_Δ^j and then e_Δ^i , and the total degree of the label Δ of e_Δ^i contributes to the sign, which is -1 . In the summand indexed by (e_Δ^j, e_Δ^i) we first subdivide e_Δ^i and then e_Δ^j , and we find the total degree of the label $s\Delta$ of the new inner vertex subdividing e_Δ^i , which is 0 . The rest of total degrees contributing to signs are the same, hence the final signs are opposite.

Let us finally check the homotopy equation (4.2.4). For admissible labeled trees m_T with a fixed number $l = n_\Delta + n_{s\Delta}$ of labels containing Δ in its essential block, we could have defined $h_i(m_T)$, $1 \leq i \leq l$, as in Definition 4.2.20, replacing the role of the leftmost label in the essential block containing Δ by the i^{th} label starting from the left in the essential block containing Δ . Note that this number l defines a filtration of $B(d'\mathcal{O})$ compatible with the bar complex differential. The same arguments as in the proof of Theorem 4.2.5 for $h_1 = h$ show that, for all $1 \leq i \leq l$,

$$ip(m_T) - m_T = dh_i(m_T) + h_id(m_T).$$

Observe that

$$\bar{h}(m_T) = \frac{1}{l} \sum_{i=1}^l h_i(m_T).$$

Indeed, the $n_{s\Delta}$ summands corresponding to labels $s\Delta$ vanish, and the remaining ones are those in the formula for $\bar{h}(m_T)$ in Definition 4.4.1. Then

$$\begin{aligned} d\bar{h}(m_T) + \bar{h}d(m_T) &= \frac{1}{l} \sum_{i=1}^l (dh_i(m_T) + h_id(m_T)) \\ &= \frac{1}{l} \sum_{i=1}^l (ip(m_T) - m_T) \\ &= ip(m_T) - m_T. \end{aligned}$$

This completes the proof. \square

Lemma 4.4.3. *The map $\partial = d_{B(d\mathcal{O})} - d_{B(d'\mathcal{O})}: B(d'\mathcal{O}) \rightarrow B(d'\mathcal{O})$ is a perturbation for the SDR in Theorem 4.4.2 in the sense of Lemma 4.3.6.*

Proof. It suffices to prove that \bar{h} is compatible with the filtration of $B(d'\mathcal{O})$ in the proof of Lemma 4.3.8. The argument given in the final paragraph of the proof of that lemma (Case 2) works to show that all summands in $\bar{h}(m_T)$ have filtration degree \leq than the filtration degree of m_T itself. \square

Lemma 4.4.4. *The composition $p\Sigma_\infty\bar{h}$ equals zero.*

Proof. The proof of the analogous fact for the chain homotopy h in Lemma 4.3.13 only uses that $p\partial = 0$. Hence it also works for \bar{h} . \square

Theorem 4.4.5. *Using the coaugmented codistributive law $\bar{\varphi}$ in Definition 4.3.1, there is an SDR of sequences*

$$B(\mathcal{D}) \circ_{\bar{\varphi}} B(\mathcal{O}) \xleftarrow[p]{i'} B(d\mathcal{O}) \xrightarrow{\bar{h}'}$$

preserving the syzygy degree obtained by applying Lemma 4.3.6 to the SDR in Theorem 4.4.2 and to the perturbation ∂ in Lemma 4.4.3. The maps p and i' are the same as in Theorem 4.3.5.

Proof. It is enough to check that i' in Theorem 4.3.5 satisfies

$$i' = \sum_{n \geq 0} (\bar{h}\partial)^n i. \quad (4.4.6)$$

The right hand side of this equation can be computed as in the proof of Theorem 4.3.5, setting $\bar{\phi}_n = (\bar{h}\partial)^n i$ and noting that these maps are inductively defined from $\bar{\phi}_0 = i$ by $\bar{\phi}_{n+1} = (\bar{h}\partial)\bar{\phi}_n$. The map ∂ is the same as in the previous section, so the description of $\partial(m_T)$ for m_T an admissible labeled tree without mixed labels in the proof of Theorem 4.3.5 remains valid, see (4.3.16). Each summand of $\partial(m_T)$ contains a single edge e_Δ labeled Δ . The only difference here is that all summands such that e_Δ is in the essential block survive $\bar{h}\partial(m_T)$.

More precisely, if $m_{T'}$ is an admissible labeled tree with a single edge e_Δ labeled Δ in the essential block, then

$$\bar{h}(m_{T'}) = \pm \frac{1}{1 + n_{n_{s\Delta}, m_{T'}}} m_{T', e_\Delta}.$$

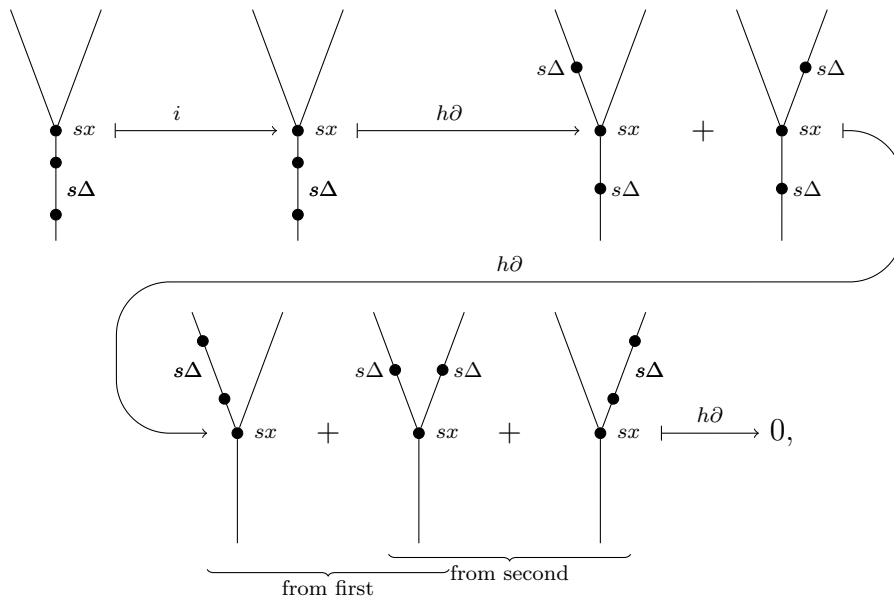
Hence, $\bar{h}\partial(m_T)$ is a sum where each summand contains an admissible labeled tree without mixed labels $m_{T''}$ obtained from m_T by taking an inner vertex labeled $s\Delta$ with top adjacent label in $s\mathcal{O}$ and making it jump over the top adjacent vertex, see (4.3.17), in such a way that the new vertex lays in the essential block. These summands have certain positive coefficients (the signs

coming from ∂ and \bar{h} cancel each other, as in the proof of Theorem 4.3.5). We claim that all coefficients of a given $m_{T''}$ add up to 1. Indeed, if we look at the inner vertices labeled $s\Delta$ in the essential block of $m_{T''}$, $\{u_{s\Delta}^j\}_{j=1}^{n_{s\Delta, m_{T''}}}$, then $m_{T''}$ has arisen in the inductive process by applying \bar{h} to the admissible labeled trees $m_{T'', u_{s\Delta}^j}$ obtained by desubdividing each of these $u_{s\Delta}^j$, up to signs. There are $n_{s\Delta, m_{T''}}$ of these, and the coefficient of $\bar{h}(m_{T'', u_{s\Delta}^j})$ is

$$\frac{1}{1 + n_{s\Delta, m_{T''}, u_{s\Delta}^j}} = \frac{1}{n_{s\Delta, m_{T''}}},$$

as indicated above. Hence the claim follows.

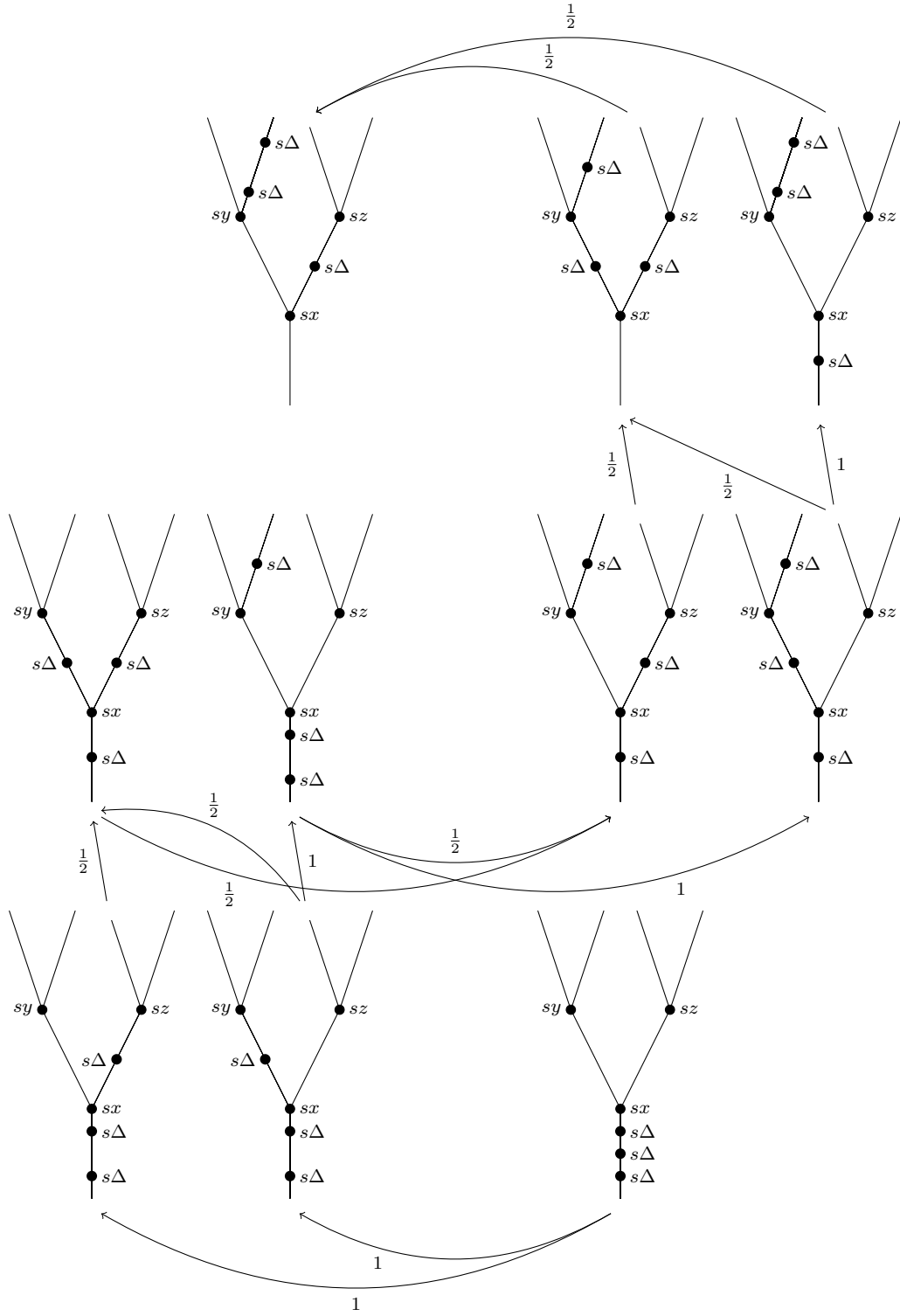
The following picture illustrates why i' is the same as in Theorem 4.3.5, but for different reasons. The starting point is identical as in the first example of the proof of that theorem:



Here, in step 4, the middle term arises from the two summands in step 3. Each of them contribute to it by a factor of $\frac{1}{2}$, adding up to 1.

Also as in the proof of Theorem 4.3.5, we wish to illustrate how each summand in i' of some good admissible labeled tree can be traced back. In the following picture, arrows represent that the target shows up in $\bar{h}\partial$ of the

source with coefficient indicated in the label:



The procedure at each step is taking the essential block and ‘de-jump’ each of the $s\Delta$ ’s found therein. Note that here we have the same starting point as in the proof of Theorem 4.3.5, but the trace-back procedure is different. \square

4.5 The Koszul dual cooperad of a derived operad

In this final section, relying on the previous one, we extend our previous results for nonsymmetric operads to the symmetric settings. Hence, we finally achieve the main technical results of this thesis.

Let $\mathcal{O} = (E|R)$ be a quadratic operad which is aritywise projective and assume $\mathbb{Q} \subset k$.

Definition 4.5.1. Define the morphism of collections $\varphi_{\Sigma}^i: \mathcal{D}^{i \circ_{\Sigma} \mathcal{O}^i} \rightarrow \mathcal{O}^{i \circ_{\Sigma} \mathcal{D}^i}$ by

$$\varphi_{\Sigma}^i((s\Delta)^i \otimes x) = \sum_{i_1 + \dots + i_n = i} x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n} \otimes \text{id}_n,$$

where n is the arity of x and $\text{id}_n \in \Sigma_n$ is the identity permutation.

It is clear that φ_{Σ}^i is a map of sequences. Let us prove that it is equivariant, and hence a map of collections. Let $\tau \in \Sigma_n$. On the one hand,

$$\begin{aligned} \varphi_{\Sigma}^i(((s\Delta)^i \otimes x) \cdot \tau) &= \varphi_{\Sigma}^i((s\Delta)^i \otimes (x \cdot \tau)) \\ &= \sum_{i_1 + \dots + i_n = i} (x \cdot \tau) \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n} \otimes \text{id}_n \\ &= \sum_{i_1 + \dots + i_n = i} x \otimes \tau \cdot ((s\Delta)^{i_{\tau^{-1}(1)}} \otimes \dots \otimes (s\Delta)^{i_{\tau^{-1}(n)}} \otimes \text{id}_n) \\ &= \sum_{i_1 + \dots + i_n = i} x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n} \otimes \tau. \end{aligned}$$

Here, in the last step we simply do a change of variables. On the other hand,

$$\begin{aligned} \varphi_{\Sigma}^i((s\Delta)^i \otimes x) \cdot \tau &= \sum_{i_1 + \dots + i_n = i} (x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n} \otimes \text{id}_n) \cdot \tau \\ &= \sum_{i_1 + \dots + i_n = i} x \otimes (s\Delta)^{i_1} \otimes \dots \otimes (s\Delta)^{i_n} \otimes \tau. \end{aligned}$$

Furthermore, that φ_{Σ}^i is a codistributive law follows from the same proof as φ^i being a codistributive law in the nonsymmetric case with only some minor modifications.

In this section we use the results of the previous section to compute the coaugmented cooperad $d\mathcal{O}^i$ under our standing assumptions.

Theorem 4.5.2. *The previous map φ^i is a coaugmented codistributive law, i.e. an augmented distributive law in the opposite monoidal category of the category of collections endowed with the symmetric circle product, and there is a coaugmented cooperad isomorphism*

$$(d\mathcal{O})^i \cong \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{O}^i.$$

Moreover, if \mathcal{O} is Koszul then so is $d\mathcal{O}$.

This theorem can be derived from Theorem 4.5.4 below in exactly the same way as Theorem 4.3.2.

Definition 4.5.3. Define the map of collections $\bar{\varphi}: B(\mathcal{D}) \circ_{\Sigma} B_{\Sigma} \mathcal{O} \rightarrow B_{\Sigma} \mathcal{O} \circ_{\Sigma} B(\mathcal{D})$ by the same formula as in Definition 4.5.1.

Note that $B(\mathcal{D}) = B_{\Sigma}(\mathcal{D})$ since \mathcal{D} is concentrated in arity 1.

Theorem 4.5.4. *The previous map $\bar{\varphi}$ is a coaugmented codistributive law and there is an SDR of collections*

$$B(\mathcal{D}) \circ_{\Sigma, \bar{\varphi}} B_{\Sigma} \mathcal{O} \xrightleftharpoons[p_{\Sigma}]^{i'_{\Sigma}} B_{\Sigma}(d\mathcal{O}) \curvearrowright h'_{\Sigma}$$

preserving the syzygy degree, and i'_{Σ} is a cooperad morphism.

As in the nonsymmetric case, Theorem 4.5.4 follows by perturbing a more elementary SDR in Theorem 4.5.11 below, via the basic perturbation lemma (Lemma 4.3.6), compare the proof of Theorem 4.4.5. The fact that $\bar{\varphi}$ in Definition 4.5.3 is a coaugmented codistributive law follows since this $\bar{\varphi}$ is plainly a symmetric version of that in Definition 4.3.4, which was checked to be a coaugmented codistributive law within the proof of Theorem 4.3.5.

Definition 4.5.5. The *mock derived operad* of an augmented operad \mathcal{O} is defined as

$$d'\mathcal{O} = \mathcal{O} \circ_{\Sigma, \varphi_0} \mathcal{D}.$$

Here we use the trivial distributive law in Definition 4.1.8 instead of the distributive law defining $d\mathcal{O}$ in Definition 3.1.7.

The mock derived operad of a quadratic operad is quadratic by Proposition 4.1.10.

Corollary 4.5.6. *Given a quadratic operad $\mathcal{O} = (E|R)$, its mock derived operad $d'\mathcal{O}$ is quadratic associated to the following quadratic data*

$$(E \oplus k \cdot \Delta, R \oplus (k \cdot \Delta) \otimes E \oplus k \cdot (\Delta \otimes \Delta)).$$

Definition 4.5.7. Consider the symmetrization of the nonsymmetric i in Theorem 4.2.5. This symmetrization is the top arrow in the diagram below. It is straightforward to check that it induces a map i_Σ below. This is how we define i in Theorem 4.5.11.

$$\begin{array}{ccc} (\mathcal{B}\mathcal{D} \circ \mathcal{B}\mathcal{O}) \otimes k[\Sigma] & \xrightarrow{i \otimes k[\Sigma]} & \mathcal{B}(d'\mathcal{O}) \otimes k[\Sigma] \\ \parallel & & \downarrow \\ \mathcal{B}\mathcal{D} \circ_\Sigma ((\mathcal{B}\mathcal{O}) \otimes k[\Sigma]) & & \\ \downarrow & & \downarrow \\ \mathcal{B}\mathcal{D} \circ_\Sigma \mathcal{B}_\Sigma \mathcal{O} & \xrightarrow{i_\Sigma} & \mathcal{B}_\Sigma(d'\mathcal{O}) \end{array}$$

Here, the vertical projections are given by (2.3.5), hence they are induced by the relations defining the free symmetric operad as a quotient of the free ns-operad, see also (2.2.3).

The map p_Σ in Theorem 4.5.11 is similarly defined from the symmetrization of p in Theorem 4.2.5 by using the following diagram.

$$\begin{array}{ccc} \mathcal{B}(d'\mathcal{O}) \otimes k[\Sigma] & \xrightarrow{p \otimes k[\Sigma]} & (\mathcal{B}\mathcal{D} \circ \mathcal{B}\mathcal{O}) \otimes k[\Sigma] & (4.5.8) \\ \downarrow & & \parallel & \\ & & \mathcal{B}\mathcal{D} \circ_\Sigma ((\mathcal{B}\mathcal{O}) \otimes k[\Sigma]) & \\ \downarrow & & \downarrow & \\ \mathcal{B}_\Sigma(d'\mathcal{O}) & \xrightarrow{p_\Sigma} & \mathcal{B}\mathcal{D} \circ_\Sigma \mathcal{B}_\Sigma \mathcal{O} & \end{array}$$

Also the map \bar{h}_Σ in Theorem 4.5.11 is defined from the symmetrization of \bar{h} in Theorem 4.4.2 by using the following diagram.

$$\begin{array}{ccc} \mathcal{B}(d'\mathcal{O}) \otimes k[\Sigma] & \xrightarrow{\bar{h} \otimes k[\Sigma]} & \mathcal{B}(d'\mathcal{O}) \otimes k[\Sigma] & (4.5.9) \\ \downarrow & & \downarrow & \\ \mathcal{B}_\Sigma(d'\mathcal{O}) & \xrightarrow{\bar{h}_\Sigma} & \mathcal{B}_\Sigma(d'\mathcal{O}) & \end{array}$$

We cannot do the same with the chain homotopy h in Theorem 4.2.5, because the following diagram cannot be filled

$$\begin{array}{ccc} \mathbb{B}(d'\mathcal{O}) \otimes k[\Sigma] & \xrightarrow{h \otimes k[\Sigma]} & \mathbb{B}(d'\mathcal{O}) \otimes k[\Sigma] \\ \downarrow & & \downarrow \\ \mathbb{B}_\Sigma(d'\mathcal{O}) & \dashrightarrow & \mathbb{B}_\Sigma(d'\mathcal{O}) \end{array}$$

The filler does not exist since, in Definition 4.2.20, we have to search for a certain leftmost label containing Δ , which is not compatible with the relations defining symmetric free operads. Tree isomorphisms may change the relative position of this label.

The existence of i_Σ and p_Σ filling the diagrams above is clear, since the definitions of i and p are very easy. The existence of the filler \bar{h}_Σ is also convincing since the definition of \bar{h} , unlike that of h , does not depend on orderings. However, there may be suspicions about signs. We argue in the following lemma that the signs in the definition of \bar{h} are correctly chosen, in the sense that they are compatible with the tree groupoid action on the symmetrization.

Lemma 4.5.10. *The filler h_Σ in diagram (4.5.9) indeed exists.*

Proof. Let $m_T \in X(T)$, $X = s(\overline{d'\mathcal{O}})$, be an admissible labeled tree and $f: T \rightarrow T'$ a non-planar isomorphism. Assume that T has n leaves and let $\sigma \in \Sigma_n$. We must check that

$$\bar{h}(m_T) \otimes \sigma = (\bar{h} \otimes k[\Sigma])X_{[T]}(f)(m_T \otimes \sigma)$$

in $\mathbb{B}_\Sigma(d'\mathcal{O})$.

Let $l = n_\Delta + n_{s\Delta}$ be the number of labels containing Δ in the essential block of m_T . We use the linear operators h_i in the proof of Theorem 4.4.2, which satisfy

$$\bar{h}(m_T) = \frac{1}{l} \sum_{i=1}^l h_i(m_T).$$

Let i , $1 \leq i \leq l$, be such that the i^{th} label in the essential block of m_T containing Δ is attached to an edge $e = \{v, w\}$, $v < w$, so $h_i(m_T)$ is possibly non-trivial. Similarly, let j be the position of the label Δ of the edge $f(e)$ among the labels containing Δ in the essential block of $X_{[T]}(f)(m_T \otimes \sigma)$ (in order to define the essential block of this, remove the labels from leaves so as to get an honest admissible labeled tree). Let T_e be the tree obtained from T by subdividing the edge e , similarly $T'_{f(e)}$, and let $f_e: T_e \rightarrow T'_{f(e)}$ be

the obvious isomorphism with the same underlying homeomorphism as f . It suffices to prove that

$$h_i(m_T) \otimes \sigma = (h_j \otimes k[\Sigma])X_{[T]}(f)(m_T \otimes \sigma)$$

in $B_\Sigma(d'\mathcal{O})$. Here,

$$h_i(m_T) \otimes \sigma = X_{[T_e]}(f_e)(h_i(m_T) \otimes \sigma)$$

by the relations defining $B_\Sigma(d'\mathcal{O})$ as a quotient of $B(d'\mathcal{O}) \otimes k[\Sigma]$. Moreover, by the construction of f_e , the equality

$$X_{[T_e]}(f_e)(h_i(m_T) \otimes \sigma) = (h_j \otimes k[\Sigma])X_{[T]}(f)(m_T \otimes \sigma)$$

holds already in $B(d'\mathcal{O}) \otimes k[\Sigma]$ up to sign. It is therefore enough to notice that the signs match.

There are signs coming from the tree groupoid actions and from the definition of h_i and h_j . The (total) degrees of the labels of m_T and $h_i(m_T)$ coincide, except for the labels of v , whose degrees differ by -1 (we have removed a Δ by application of h_i), and for the extra inner vertex of $h_i(m_T)$, whose label is $s\Delta$, of total degree 0, so it does not contribute to signs. Hence the difference between signs coming from the tree groupoid actions on both sides is -1 up to the sum of the degrees of the labels of the vertices exchanging their order with v by f plus the sum of the edges labeled Δ adjacent to v which jump over e . This is exactly the difference between the signs coming from h_i and h_j , hence we are done. \square

As a consequence of Definition 4.5.7 and Theorem 4.4.2, we have the following SDR in the symmetric setting.

Theorem 4.5.11. *There is an SDR of collections*

$$B(\mathcal{D}) \circ_\Sigma B_\Sigma \mathcal{O} \xrightleftharpoons[p_\Sigma]{i_\Sigma} B_\Sigma(d'\mathcal{O}) \curvearrowright \bar{h}_\Sigma$$

preserving the syzygy degree.

This SDR can be perturbed.

Lemma 4.5.12. *The map $\partial = d_{B_\Sigma(d\mathcal{O})} - d_{B_\Sigma(d'\mathcal{O})}: B_\Sigma(d'\mathcal{O}) \rightarrow B_\Sigma(d'\mathcal{O})$ is a perturbation for the SDR in Theorem 4.5.11 in the sense of Lemma 4.3.6.*

This theorem can be derived from Lemma 4.4.3. The filtration of $B(d'\mathcal{O})$ constructed in the proof of that lemma extends to $B(d'\mathcal{O}) \otimes k[\Sigma]$, the symmetrization, and passes to its quotient $B_\Sigma(d'\mathcal{O})$.

Lemma 4.5.13. *The compositions $p_{\Sigma_\infty} i_\Sigma$ and $p_{\Sigma_\infty} \bar{h}_\Sigma$ equal zero.*

This follows immediately from Definition 4.5.7 and Lemmas 4.3.13 and 4.4.4. We have finally established the conditions to apply the basic perturbation lemma which proves Theorem 4.5.4, and hence Theorem 4.5.2. The explicit description of i' in the proof of Theorem 4.4.5 also works for i'_Σ , incorporating labels to the leaves reflecting the symmetric situation.

Chapter 5

Derived homotopy algebras, applications and examples

We finally come to the definition of derived homotopy algebras, the main topic of this thesis. Their definition and properties, as well as the explicit computations, depend heavily on the strongest results of the previous chapter. We could even say that they are almost formal consequences of them. This includes the construction of minimal models for operadic algebras, extending Sagave's theory [21] beyond the associative case. We explicitly compute the generating operations and their relations in the associative, commutative, and Lie cases, as a way of illustrating our powerful tools.

5.1 Derived homotopy algebras

We are now in the position to define the notion on which we based the title of this thesis.

Definition 5.1.1. Let $\mathcal{O} = (E|R)$ be a Koszul operad (symmetric or not) which is aritywise projective over the ground ring. We also assume that \mathcal{O}^i is aritywise projective over the ground ring. Denote by $d\mathcal{O}_\infty = \Omega_\Sigma(d\mathcal{O})^i$ the minimal model, in the sense of Definition 2.5.5 or 2.5.10, of the derived operad $d\mathcal{O}$ from Definition 3.1.6 or 3.1.9. A *derived homotopy \mathcal{O} -algebra* is an algebra over $d\mathcal{O}_\infty$.

Recall that the base symmetric monoidal category we work over is that of graded complexes, as spelled out in Example 1.1.11, where the symmetry isomorphism uses the Koszul sign convention with respect to the total degree. Derived (ns-)operads $d\mathcal{O}$ were defined in section 3.1 and the algebras over them, called derived \mathcal{O} -algebras, were characterized in section 3.2.

We recall here further that we performed the calculation of $(d\mathcal{O})^i$ for ns-operads in Theorem 4.3.2, and for operads in Theorem 4.3.2. Notice that in the symmetric setting we had to further impose that the ground ring contained the rationals, $\mathbb{Q} \subset k$, whereas in the nonsymmetric setting this restriction was not needed. We henceforth assume $\mathbb{Q} \subset k$ when dealing with (symmetric) operads.

Proposition 5.1.2. *Derived \mathcal{O} -algebras are particular instances of derived homotopy \mathcal{O} -algebras.*

Proof. Theorem 2.5.9 tells us that for any Koszul operad \mathcal{P} that is arity-wise projective, and such that \mathcal{P}^i is aritywise projective, the surjective map $p: \Omega_\Sigma \mathcal{P}^i \rightarrow \mathcal{P}$ defined earlier in section 2.5 is a quasi-isomorphism of operads. Since an algebra over \mathcal{P} is a graded complex X equipped with a morphism of operads $\mathcal{P} \rightarrow \mathcal{E}(X)$, see Definition 1.5.2, it then follows by precomposition that an algebra over \mathcal{P} is also an algebra over $\Omega_\Sigma \mathcal{P}^i$,

$$\Omega_\Sigma \mathcal{P}^i \twoheadrightarrow \mathcal{P} \rightarrow \mathcal{E}(X).$$

Hence it suffices to check that derived operads $d\mathcal{O}$ are Koszul, aritywise projective, and that $(d\mathcal{O})^i$ is aritywise projective. That $d\mathcal{O}$ is Koszul is already proved in Theorems 4.3.2 and 4.5.2, since \mathcal{O} is Koszul. That $d\mathcal{O}$ is aritywise projective follows easily from the decomposition $d\mathcal{O}(n) = \mathcal{O}(n) \otimes \mathcal{D}(1)^{\otimes n}$, $n \geq 0$. Indeed, the factor $\mathcal{O}(n)$ is by assumption projective and the factors $\mathcal{D}(1)$ are free, hence projective. As a consequence the tensor product of these factors is projective as well. That $(d\mathcal{O})^i$ is aritywise projective is a direct consequence of \mathcal{O}^i being aritywise projective, since $(d\mathcal{O})^i(n) = \mathcal{D}^i(1) \otimes \mathcal{O}^i(n)$ and $\mathcal{D}^i(1)$ is free. \square

Derived homotopy \mathcal{O} -algebras can be made into a category with a class of morphisms, called ∞ -morphisms, bigger than plain $d\mathcal{O}_\infty$ -morphisms, see [16, section 10.2.2]. We define them below.

A derived homotopy \mathcal{O} -algebra structure on X , $\theta = \theta_X: d\mathcal{O}_\infty \rightarrow \mathcal{E}(X)$, can be alternatively described as an extra differential d_θ^r of bidegree $(0, -1)$ on the cofree $(d\mathcal{O})^i$ -coalgebra $(d\mathcal{O})^i(X)$ described in Remark 1.5.5 such that $d_\theta = d_{(d\mathcal{O})^i(X)} + d_\theta^r$ is a $(d\mathcal{O})^i$ -coalgebra differential on $(d\mathcal{O})^i(X)$, see [16, Proposition 10.1.11] and compare [15, Definition 4.1].

Definition 5.1.3. An ∞ -morphism of derived homotopy algebras $X \rightsquigarrow Y$ is a $(d\mathcal{O})^i$ -coalgebra morphism $((d\mathcal{O})^i(X), d_{\theta_X}) \rightarrow ((d\mathcal{O})^i(Y), d_{\theta_Y})$.

Remark 5.1.4. Any ∞ -morphism $X \rightsquigarrow Y$ has an underlying graded complexes map $X \rightarrow Y$ which is obtained as the following composition,

$$X \subset (d\mathcal{O})^i(1) \otimes X \subset (d\mathcal{O})^i(X) \rightarrow (d\mathcal{O})^i(Y) \twoheadrightarrow (d\mathcal{O})^i(1) \otimes Y \twoheadrightarrow Y.$$

Here we use the counit and the coaugmentation of $(d\mathcal{O})^i$ and the direct sum decomposition of $(d\mathcal{O})^i(X)$, see Remark 1.5.5.

An honest $d\mathcal{O}_\infty$ -morphism $f: X \rightarrow Y$ gives rise to an ∞ -morphism $(d\mathcal{O})^i(f)$. This defines a faithful but not full inclusion of $d\mathcal{O}_\infty$ -morphisms into ∞ -morphisms of derived homotopy \mathcal{O} -algebras, see the diagram at the end of [16, section 10.2.5].

General ∞ -morphisms will only play a role below, in the definition of minimal for derived homotopy models for \mathcal{O} -algebras. This is why we do not pursue them further.

In the remaining sections of this chapter, instances of derived homotopy \mathcal{O} -algebras are discussed for \mathcal{O} the associative operad, the commutative operad, and the Lie operad, respectively. In particular, we provide calculations of $\Omega_\Sigma(d\mathcal{O})^i$ (or of $\Omega(d\mathcal{O})^i$ for the associative operad). We proceed in each case as follows. We first describe the derived \mathcal{O} -algebras and afterwards the derived homotopy \mathcal{O} -algebras. Then we define the corresponding operads $d\mathcal{O}_\infty$ and finally show that they are the result of the calculation of $\Omega_\Sigma(d\mathcal{O})^i$.

We will not include the formulas for ∞ -morphisms, so as not to get into further technicalities. In the associative case, the computation yields the same result as the ad-hoc definition of Sagave in [21], up to different but equivalent sign conventions. This was checked in [15, Theorem 4.4]. The similarity between derived homotopy algebras over the associative and commutative operads is similarly satisfied at the level of ∞ -morphisms.

5.2 Twisted complexes

The most easy examples of derived homotopy algebras are twisted complexes, which are actually dI_∞ -algebras, where I is the initial operad, the tensor unit for the circle product, see sections 1.2 and 1.3. The operad dI is simply \mathcal{D} . This operad is obviously aritywise projective (actually free). In Remark 4.2.6 we checked following [16, 10.3.7] that \mathcal{D} is Koszul and computed \mathcal{D}^i , which is also aritywise projective (free). Recall that by Corollary 3.1.3 a \mathcal{D} -algebra is a bicomplex (X, d_h, d_v) , i.e. a bigraded module X equipped with differentials d_v, d_h , of bidegrees $(0, -1), (-1, 0)$, respectively, such that $d_h d_v + d_v d_h = 0$.

Definition 5.2.1. A *twisted complex* $(X, \{d_i\}_{i \geq 0})$ is a bigraded module X together with maps

$$d_i: X \rightarrow X$$

of bidegree $(-i, i - 1)$, $i \geq 0$, satisfying the following equation for any fixed

$i \geq 0$,

$$\sum_{p+l=i} d_p d_l = 0.$$

Remark 5.2.2. It can be easily checked that a bicomplex is a twisted complex with $d_i = 0$ for $i \geq 2$. The surviving maps d_0 and d_1 then play the role of the vertical differential d_v and the horizontal differential d_h , respectively, of the bicomplex.

Equivalently, one can regard a twisted complex as a graded complex (X, d_0) together with additional maps d_i , $i \geq 1$, satisfying

$$d_{\text{End}(X)}(d_i) = d_0 d_i - (-1)^1 d_i d_0 = - \sum_{\substack{p+l=i \\ p,l \geq 1}} d_p d_l.$$

With the latter formulation of a twisted complex it is immediate that the operad dI_∞ , also denoted \mathcal{D}_∞ , determining a twisted complex is given in the following definition.

Definition 5.2.3. Define the ns-operad

$$dI_\infty = \mathcal{D}_\infty = \mathcal{F} \left(\left\{ 0, \bigoplus_{i \geq 1} k \cdot d_i, 0, \dots \right\} \right), \quad (5.2.4)$$

i.e. freely generated by the homogeneous operations d_i , $i \geq 1$, of arity 1 and bidegree $(-i, i-1)$, equipped with differential

$$d_{dI_\infty}(d_i) = - \sum_{p+l=i} d_p \circ_1 d_l. \quad (5.2.5)$$

Note that $dI_\infty = \mathcal{D}_\infty$ is concentrated in arity 1, like $dI = \mathcal{D}$.

The following computation was also carried out in [16, 10.3.7].

Proposition 5.2.6. *The minimal model of $dI = \mathcal{D}$ equals $dI_\infty = \mathcal{D}_\infty$, as defined in Definition 5.2.3.*

Proof. We have computed the coaugmented cooperad \mathcal{D}^i in Remark 4.2.6. We have in particular that it is concentrated in arity 1 and $\bar{\mathcal{D}}^i(1) = \bigoplus_{i \geq 1} k \cdot$

$(s\Delta)^i$. It follows that, calling $d_i = s^{-1}(s\Delta)^i$, the underlying sequence of the cobar construction $\Omega(\mathcal{D}^i) = \mathcal{F}(s^{-1}\bar{\mathcal{D}}^i)$ on \mathcal{D}^i is given by (5.2.4).

On generators, the differential d_2 coincides with \hat{d}_2 , which is defined from $\Delta_{(1)}$, see section 2.3. In the case of \mathcal{D}^i , the infinitesimal composition $\Delta_{(1)}$,

see Definition 1.4.15, clearly coincides with $\Delta_{\mathcal{D}^i}$ in (4.2.8). Hence it follows by (2.3.6) and (2.3.7) that

$$\hat{d}_2(s^{-1}((s\Delta)^{\otimes i})) = - \sum_{\substack{p+l=i \\ p,l \geq 1}} (s^{-1}(s\Delta)^{\otimes p}) \circ_1 (s^{-1}(s\Delta)^{\otimes l}),$$

as in (5.2.5). □

Remark 5.2.7. Notice that since I is the initial operad there exists a unique map $I \rightarrow \mathcal{O}$ for any operad \mathcal{O} . It is precisely the unit $\eta_{\mathcal{O}}$, see section 1.2. This operad map induces another one $dI_{\infty} \rightarrow d\mathcal{O}_{\infty}$, so an algebra over $d\mathcal{O}_{\infty}$, i.e. a derived homotopy \mathcal{O} -algebra, has an underlying algebra structure over dI_{∞} obtained by precomposition,

$$dI_{\infty} \rightarrow d\mathcal{O}_{\infty} \rightarrow \mathcal{E}(X),$$

i.e. an underlying twisted complex.

5.3 Minimal derived homotopy models

Let \mathcal{O} be a quadratic Koszul operad such that both \mathcal{O} and \mathcal{O}^i are aritywise projective. We assume in addition that \mathcal{O} (and hence \mathcal{O}^i) is concentrated in horizontal degree 0, i.e. it is an operad in plain graded (rather than bigraded) modules (which can always be regarded as bigraded modules concentrated in the vertical axis).

Recall that the underlying (vertically) graded complex of a twisted complex $(X, \{d_i\}_{i \geq 0})$ is (X, d_0) . We denote by $H_{ij}^v(X)$ the homology bigraded complex of (X, d_0) , where v stands for *vertical*.

Definition 5.3.1. A morphism of twisted complexes $f: X \rightarrow Y$ is an E^1 -equivalence if it is a quasi-isomorphism at the level of underlying (vertically) graded complexes, i.e. if it induces isomorphisms $H_{ij}^v(f): H_{ij}^v(X) \rightarrow H_{ij}^v(Y)$ in all bidegrees $i, j \in \mathbb{Z}$. A morphism of derived homotopy \mathcal{O} -algebras is an E^1 -equivalence if the underlying morphism of twisted complexes is. More generally, an ∞ -morphism $X \rightsquigarrow Y$ of derived homotopy \mathcal{O} -algebras is an E^1 -equivalence if the morphism of graded complexes $f: X \rightarrow Y$ of $X \rightsquigarrow Y$, in the sense of Remark 5.1.4, is an E^1 -equivalence.

This terminology comes from the fact that the horizontal homology is the E^1 -term of the spectral sequence of a filtered complex obtained by totalization, see [21].

In a twisted complex the equation $d_0d_1 + d_1d_0 = 0$ also holds, therefore d_1 induces a *horizontal* differential on $H_{ij}^v(X)$, i.e. if bidegree $(-1, 0)$, whose bigraded homology we can denote in either of the following two ways

$$H_{ij}^h H_{**}^v(X) = E_{ij}^2(X).$$

The second notation comes from the fact that this is the E^2 -term of the aforementioned spectral sequence.

Definition 5.3.2. A morphism of twisted complexes $f: X \rightarrow Y$ is an E^2 -equivalence if it induces isomorphisms $E_{ij}^2(f): E_{ij}^2(X) \rightarrow E_{ij}^2(Y)$ in all bidegrees $i, j \in \mathbb{Z}$. A morphism of derived homotopy \mathcal{O} -algebras is an E^2 -equivalence if the underlying morphism of twisted complexes is. More generally, an ∞ -morphism $X \rightsquigarrow Y$ of derived homotopy \mathcal{O} -algebras is an E^2 -equivalence if the maps $E_{ij}^2(f): E_{ij}^2(X) \rightarrow E_{ij}^2(Y)$ induced by the underlying morphism $X \rightarrow Y$ of $X \rightsquigarrow Y$, in the sense of Remark 5.1.4, are isomorphisms.

Remark 5.3.3. Obviously E^1 -equivalences are E^2 -equivalences. The fact that the underlying morphism $X \rightarrow Y$ of an ∞ -morphism induces a map, not only on vertical homology, but also on E_{ij}^2 , is straightforward. However it depends on a slightly deeper analysis of the structure of ∞ -morphisms that we have decided not to pursue in order to avoid further technicalities.

Definition 5.3.4. A twisted complex $(X, \{d_i\}_{i \geq 0})$ is *minimal* if $d_0 = 0$. A derived homotopy \mathcal{O} -algebra is *minimal* if its underlying twisted complex, in the sense of Remark 5.2.7, is minimal.

Remark 5.3.5. In a minimal twisted complex, the equation $d_0d_2 + d_1d_1 + d_2d_0 = 0$ reduces to $d_1^2 = 0$. Hence (X, d_1) is a graded complex with horizontal (rather than the usual vertical) differentials, called the *underlying horizontal graded complex* of $(X, \{d_i\}_{i \geq 0})$.

Proposition 5.3.6. *If Y is a derived homotopy \mathcal{O} -algebra such that $H_{ij}^v(Y)$ is projective for all $i, j \in \mathbb{Z}$, then there is a minimal derived homotopy \mathcal{O} -algebra X with underlying bigraded module $H_{ij}^v(Y)$ and an E^1 -equivalence $X \rightsquigarrow Y$.*

Proof. Since the vertical homology of Y is projective, the bigraded module $X = H_{ij}^v(Y)$, regarded as a graded complex with trivial differential, is a strong deformation retract of Y , in the sense of Definition 4.2.3. The standard homotopy transfer theorem [16, Theorem 10.3.1], which is a formal consequence of Koszul duality theory, endows X with the an appropriate

derived homotopy \mathcal{O} -algebra structure, which is an E^1 -equivalence by construction, and enhances the inclusion $i: X \hookrightarrow Y$, which is part of the SDR, to an ∞ -morphism. \square

The derived homotopy \mathcal{O} -algebra X can be explicitly computed in terms of well-known formulas involving trees.

Proposition 5.3.7. *Given an \mathcal{O} -algebra A in the category of chain complexes, there is a derived homotopy \mathcal{O} -algebra Y concentrated in non-negative horizontal degrees such that $H_{ij}^v(Y)$ is a projective bigraded module and a morphism of $d\mathcal{O}_\infty$ -algebras $Y \rightarrow A$ which is an E^2 -equivalence.*

Proof. This proof uses some strong and well-known results from homotopy theory, albeit at a user level. Sagave proved in [21, section 3] that the underlying chain complex of A has a simplicial resolution Y' , cofibrant in Bousfield's resolution model structure, whose associated bicomplex $Y = C(Y')$, with horizontal differential defined as usual (the alternating sum of simplicial face operators), satisfies the required hypotheses. Now, it suffices extend the bicomplex structure Y to a derived homotopy \mathcal{O} -algebra structure compatible with the resolution map $Y \rightarrow A$. We can regard $\mathcal{O}_\infty = \Omega_\Sigma \mathcal{O}^i$ (just $\Omega \mathcal{O}^i$ in the nonsymmetric case) as a constant simplicial chain operad. It is cofibrant in the model structure of [4] (resp. [19]). Hence we can transfer the \mathcal{O}_∞ -algebra structure of A (coming from the \mathcal{O} -algebra structure via the quasi-isomorphism $\mathcal{O}_\infty \rightarrow \mathcal{O}$) to Y' . Since the functor $C(-)$ is lax symmetric monoidal (via the Eilenberg–Zilber map), this induces an \mathcal{O}_∞ -algebra structure on the bicomplex $C(Y)$. By Proposition 3.2.2, this is the same as a $d(\mathcal{O}_\infty)$ -algebra structure on the underlying vertical graded complex of $C(Y)$. This induces a $(d\mathcal{O})_\infty$ -algebra structure by using the obvious surjective map $(d\mathcal{O})_\infty \rightarrow d(\mathcal{O}_\infty)$ whose kernel is the operadic ideal generated by the image of the operations $d_i \in dI_\infty$, $i \geq 2$, under the map $dI_\infty \rightarrow (d\mathcal{O})_\infty$ in Remark 5.2.7. \square

Combining the two previous propositions we obtain the following result, which ensures the existence of minimal models.

Corollary 5.3.8. *Any \mathcal{O} -algebra A in the category of chain complexes is E^2 -equivalent to a minimal derived homotopy \mathcal{O} -algebra X with underlying projective bigraded module concentrated in non-negative horizontal degrees.*

The underlying horizontal graded complex of X , in the sense of Remark 5.3.5, is therefore a projective resolution of $H_*(A)$, the homology of A .

Remark 5.3.9. Sagave computes in [21] derived homotopy associative minimal models for two differential graded algebras, see [21, Examples 5.1 and 5.3]. In the first example, the differential graded algebra is actually commutative and the minimal model is derived homotopy commutative.

5.4 Derived homotopy associative algebras

Let \mathcal{A} denote the (non-unital) associative operad, as defined in [16, 9.1.2]. This is a nonsymmetric operad. It is arity-wise projective since $\mathcal{A}(0) = 0$ and $\mathcal{A}(n) = k$, the ground ring concentrated in degree 0, for $n \geq 1$. The ns-operad \mathcal{A} is Koszul, see [16, Theorem 9.1.5]. Moreover, \mathcal{A}^i is also aritywise projective, as recalled in the proof of Theorem 5.4.7.

Definition 5.4.1. A *derived associative algebra*, *derived \mathcal{A} -algebra*, or *$d\mathcal{A}$ -algebra*, is a bicomplex (X, d_h, d_v) , i.e. an bigraded module X equipped with differentials d_v, d_h , of bidegrees $(0, -1), (-1, 0)$, respectively, such that $d_h d_v + d_v d_h = 0$, together with a morphism of bicomplexes $m: X \otimes X \rightarrow X$ of bidegree $(0, 0)$ which is associative, called *multiplication*, i.e. if we denote $m(x, y) = xy$ then $x(yz) = (xy)z$,

$$\begin{aligned} d_v(xy) &= d_v(x)y + (-1)^{\|x\|} x d_v(y), \\ d_h(xy) &= d_h(x)y + (-1)^{\|x\|} x d_h(y), \end{aligned}$$

where $\|x\|$ is the total degree of x .

Notice that Definition 5.4.1 is a particular instance of Proposition 3.2.2 for the associative operad.

Definition 5.4.2. A *derived homotopy associative algebra*, *derived homotopy \mathcal{A} -algebra*, or *$d\mathcal{A}_\infty$ -algebra*, is an bigraded module X together with maps

$$m_{in}: X^{\otimes n} \rightarrow X$$

of bidegree $(-i, i + n - 2)$, $i \geq 0$, $n \geq 1$, satisfying the following equation for any fixed $i \geq 0$ and $n \geq 1$,

$$\sum_{\substack{p+l=i, \\ k+q=n+1, \\ r+1+t=k}} (-1)^{q(k-r-1)-r} m_{pk}(\mathbf{1}^{\otimes r} \otimes m_{lq} \otimes \mathbf{1}^{\otimes t}) = 0.$$

Our derived homotopy associative algebras coincide with Sagave's derived A -infinity algebras, introduced in [21], up to signs. The different signs are

a consequence of our sign convention in the symmetry isomorphism for the tensor product of bigraded modules, see Example 1.1.10. Our convention uses the Koszul sign rule with respect to the total degree, whereas in [21] and [15] the Koszul rule is used with respect to horizontal and vertical degrees separately. Nevertheless, the resulting symmetric monoidal categories with both choices for the symmetry isomorphism turn out to be isomorphic.

Remark 5.4.3. It can be easily checked that a derived associative algebra a derived homotopy \mathcal{A} -algebra with $m_{in} = 0$ for $i + n \geq 3$ (in [21] and [15] also referred to as a bidga). The surviving maps m_{01} , m_{11} , and m_{02} will then play the role of the vertical differential d_v , the horizontal differential d_h , and the multiplication m , respectively, of the derived associative algebra.

Homotopy \mathcal{A} -algebras, or \mathcal{A}_∞ -algebras, as first introduced in [23], form another class of examples of derived homotopy \mathcal{A} -algebras when concentrated in the vertical axis, so $m_{in} = 0$ for $i > 0$. Obviously, the maps m_{0n} then play the role of the maps m_n defining the \mathcal{A}_∞ -algebra structure.

An easy check shows that the underlying twisted complex, see Remark 5.2.7, of a derived homotopy associative algebra is $(X, \{m_{i1}\}_{i \geq 0})$.

One can equivalently regard a derived homotopy \mathcal{A} -algebra as a graded complex (X, m_{01}) together with additional maps m_{in} satisfying

$$\begin{aligned} d_{\text{End}(X)}(m_{in}) &= m_{01}m_{in} - \sum_{r+t+1=n} (-1)^{n-2} m_{in}(\mathbf{1}^{\otimes r} \otimes m_{01} \otimes \mathbf{1}^{\otimes t}) \\ &= - \sum_{\substack{p+l=i, \\ k+q=n+1, \\ r+1+t=k \\ (p,k),(l,q) \neq (0,1)}} (-1)^{q(k-r-1)-r} m_{pk}(\mathbf{1}^{\otimes r} \otimes m_{lq} \otimes \mathbf{1}^{\otimes t}). \end{aligned}$$

With this last formulation, it can be easily checked that the ns-operad $d\mathcal{A}_\infty$ determining $d\mathcal{A}_\infty$ -algebras is given in the following definition.

Definition 5.4.4. Define the ns-operad

$$d\mathcal{A}_\infty = \mathcal{F} \left(\left\{ 0, \bigoplus_{i \geq 1} k \cdot \mu_{i1}, \dots, \bigoplus_{i \geq 0} k \cdot \mu_{in} \ (n \geq 2), \dots \right\} \right), \quad (5.4.5)$$

i.e. freely generated by the homogeneous operations μ_{in} , $i \geq 0$, $n \geq 1$, $(i, n) \neq (0, 1)$, of arity n and bidegree $(-i, i+n-2)$, equipped with differential

$$d_{d\mathcal{A}_\infty}(\mu_{in}) = - \sum_{\substack{p+l=i \\ q+k=n+1 \\ r+1+t=k}} (-1)^{q(k-r-1)-r} \mu_{pk} \circ_{r+1} \mu_{lq}. \quad (5.4.6)$$

As discussed in the introduction, Livernet, Roitzheim, and Whitehouse studied in [15] the minimal model of $d\mathcal{A}$, being $d\mathcal{A}_\infty$ up to signs. In the remainder of this section we show that our approach for calculating the minimal model of $d\mathcal{A}$ yields the same outcome up to sign convention, i.e. $\Omega(d\mathcal{A})^i$ calculated using Theorem 4.3.2 equals $d\mathcal{A}_\infty$ in Definition 5.4.4.

Theorem 5.4.7. *The minimal model of $d\mathcal{A}$ equals $d\mathcal{A}_\infty$, as defined in Definition 5.4.4.*

Proof. The coaugmented ns-cooperad structure on \mathcal{A}^i is provided in [16, section 9.1.5]. We spell out this structure here, since we need it in our computation.

The underlying sequence of \mathcal{A}^i is given by

$$\begin{aligned}\mathcal{A}^i(0) &= 0, \\ \mathcal{A}^i(n) &= k \cdot \mu_n^c, \quad n \geq 1,\end{aligned}\tag{5.4.8}$$

see also [16, Proposition 9.1.3], where μ_n^c , $n \geq 1$, are homogeneous elements of bidegree $(0, n - 1)$. For the sake of simplicity, we here use the notation $\mu_1^c = \text{id}_{\mathcal{A}^i}$ for the identity cooperation.

The comultiplication $\Delta_{\mathcal{A}^i}: \mathcal{A}^i \rightarrow \mathcal{A}^i \circ \mathcal{A}^i$ is given by

$$\Delta_{\mathcal{A}^i}(\mu_n^c) = \sum_{i_1 + \dots + i_k = n} (-1)^{\sum_{j=1}^k (i_j + 1)(k - j)} \mu_k^c \otimes \mu_{i_1}^c \otimes \dots \otimes \mu_{i_k}^c,$$

see also [16, Lemma 9.1.2].

The counit $\epsilon_{\mathcal{A}^i}: \mathcal{A}^i \rightarrow I$ vanishes on all μ_n^c except for μ_1^c , which is sent to $1 \in k = I(1)$, and the coaugmentation $\eta_{\mathcal{A}^i}: I \rightarrow \mathcal{A}^i$ is determined by being a splitting of the counit, sending therefore $1 \in k = I(1)$ to μ_1^c .

We are now in the position to determine the coaugmented ns-cooperad structure on $(d\mathcal{A})^i$. First of all, by Theorem 4.3.2, the underlying sequence of $(d\mathcal{A})^i = \mathcal{D}^i \circ_{\varphi^i} \mathcal{A}^i$ equals

$$\begin{aligned}d\mathcal{A}^i(0) &= 0, \\ d\mathcal{A}^i(n) &= \mathcal{D}^i(1) \otimes \mathcal{A}^i(n) \\ &= \bigoplus_{i \geq 0} k \cdot (s\Delta)^i \otimes k \cdot \mu_n^c \\ &= \bigoplus_{i \geq 0} k \cdot ((s\Delta)^i \otimes \mu_n^c), \quad n \geq 1.\end{aligned}$$

For $n \geq 1$, in the second equality we use (5.4.8) and (4.2.7) for the underlying sequence of $\mathcal{D}^i = \text{B}(\mathcal{D})$.

The comultiplication $\Delta_{(d\mathcal{A})^i}: (d\mathcal{A})^i \rightarrow (d\mathcal{A})^i \circ (d\mathcal{A})^i$ is $\Delta_{(d\mathcal{A})^i} = (\mathbb{1} \circ \varphi^i \circ \mathbb{1})(\Delta_{\mathcal{D}^i} \circ \Delta_{\mathcal{A}^i})$, compare Proposition 1.2.5. Here the codistributive law φ^i is given in Definition 4.3.1, $\Delta_{\mathcal{D}^i}$ is given in (4.2.8), and $\Delta_{\mathcal{A}^i}$ is above. Hence, on generators we get the following formula, where $a_k = \sum_{j=1}^k (i_j + 1)(k - j)$,

$$\begin{aligned}
& \Delta_{(d\mathcal{A})^i}((s\Delta)^i \otimes \mu_n^c) = \\
& (\mathbb{1} \circ \varphi^i \circ \mathbb{1})(\Delta_{\mathcal{D}^i} \circ \Delta_{\mathcal{A}^i})((s\Delta)^i \otimes \mu_n^c) = \\
& \sum_{\substack{p+l=i \\ i_1+\dots+i_k=n}} (-1)^{a_k} (\mathbb{1} \circ \varphi^i \circ \mathbb{1})(((s\Delta)^p \otimes (s\Delta)^l) \otimes (\mu_k^c \otimes \mu_{i_1}^c \otimes \dots \otimes \mu_{i_k}^c)) = \\
& \sum_{\substack{p+l=i \\ i_1+\dots+i_k=n}} (-1)^{a_k} (\mathbb{1} \circ \varphi^i \circ \mathbb{1})((s\Delta)^p \otimes ((s\Delta)^l \otimes \mu_k^c) \otimes \mu_{i_1}^c \otimes \dots \otimes \mu_{i_k}^c) = \\
& \sum_{\substack{p+l=i \\ i_1+\dots+i_k=n \\ p_1+\dots+p_k=l}} (-1)^{a_k} (s\Delta)^p \otimes (\mu_k^c \otimes (s\Delta)^{p_1} \otimes \dots \otimes (s\Delta)^{p_k}) \otimes \mu_{i_1}^c \otimes \dots \otimes \mu_{i_k}^c = \\
& \sum_{\substack{p+l=i \\ i_1+\dots+i_k=n \\ p_1+\dots+p_k=l}} (-1)^{a_k} ((s\Delta)^p \otimes \mu_k^c) \otimes ((s\Delta)^{p_1} \otimes \mu_{i_1}^c) \otimes \dots \otimes ((s\Delta)^{p_k} \otimes \mu_{i_k}^c).
\end{aligned}$$

In the third and last equality we use the associator for \circ , as defined in section 1.3.

As a next step, we have to determine the infinitesimal decomposition $\Delta_{(1)}$, as defined in (1.4.15). This gives, on generators,

$$\begin{aligned}
& \Delta_{(1)}((s\Delta)^i \otimes \mu_n^c) = \text{Pr } \Delta_{(d\mathcal{A})^i}((s\Delta)^i \otimes \mu_n^c) = \\
& \sum_{\substack{p+l=i \\ q+k=n+1 \\ r+1+t=k}} (-1)^{(q+1)(k-r-1)} ((s\Delta)^p \otimes \mu_k^c) \otimes \text{id}_{d\mathcal{A}^i}^{\otimes r} \otimes ((s\Delta)^l \otimes \mu_q^c) \otimes \text{id}_{d\mathcal{A}^i}^{\otimes t}.
\end{aligned}$$

We finally come to the calculation of $\Omega(d\mathcal{A})^i$. To this end, denote $\mu_{in} = s^{-1}((s\Delta)^i \otimes \mu_n^c)$. These are homogeneous elements of bidegree

$$i(-1, 1) + (0, n - 1) + (0, -1) = (-i, i + n - 2).$$

Actually, they are the free generators of the bigraded modules $s^{-1}\overline{(d\mathcal{A})^i}(n)$ for $(i, n) \neq (0, 1)$. So we see that the underlying sequence of bigraded modules of $\Omega((d\mathcal{A})^i) = \mathcal{F}(s^{-1}\overline{(d\mathcal{A})^i})$ equals the right hand side of (5.4.5).

The differential $d_{\Omega(d\mathcal{A})^i}$ reduces to d_2 , since $d\mathcal{A}$ has trivial differential. It is enough to compute it on the bigraded module generators of $s^{-1}\overline{(d\mathcal{A})^i}$. It then extends to the cobar construction as an operadic derivation. On generators,

d_2 coincides with \hat{d}_2 , which is defined from $\Delta_{(1)}$, see section 2.3. By (2.3.6) and (2.3.7) it now follows that

$$\begin{aligned} \hat{d}_2(\mu_{in}) &= - \sum_{\substack{p+l=i \\ q+k=n+1 \\ r+1+t=k \\ (p,k),(l,q) \neq (0,1)}} (-1)^{q(k-r-1)-r} \mu_{pk} \otimes \text{id}_{d_{Ai}}^{\otimes r} \otimes \mu_{lq} \otimes \text{id}_{d_{Ai}}^{\otimes t} \\ &= - \sum_{\substack{p+l=i \\ q+k=n+1 \\ r+1+t=k \\ (p,k),(l,q) \neq (0,1)}} (-1)^{q(k-r-1)-r} \mu_{pk} \circ_{r+1} \mu_{lq}, \end{aligned}$$

which equals (5.4.6). □

5.5 Derived homotopy commutative algebras

Let \mathcal{C} denote the (non-unital) commutative operad, as defined in [16, 13.1.3]. This is a symmetric operad, hence we assume that the ground ring contains the rationals, $\mathbb{Q} \subset k$. It is arity-wise projective over the ground ring k since $\mathcal{C}(0) = 0$ and $\mathcal{C}(n) = k$, for $n \geq 1$. The operad \mathcal{C} is Koszul, see [16, Proposition 13.1.2]. The cooperad $\mathcal{C}^!$ is also aritywise projective, compare the proof of Theorem 5.5.5.

Definition 5.5.1. A *derived commutative algebra*, *derived \mathcal{C} -algebra*, or *d \mathcal{C} -algebra*, is a bicomplex (X, d_h, d_v) , i.e. a bigraded module X equipped with differentials d_v, d_h of bidegrees $(0, -1), (-1, 0)$, respectively, such that $d_h d_v + d_v d_h = 0$, together with a morphism of bicomplexes $m: X \otimes X \rightarrow X$ of bidegree $(0, 0)$ which is both associative and commutative, called *multiplication*, i.e. if we denote $m(x, y) = xy$ then $x(yz) = (xy)z$, $xy = (-1)^{\|x\|\|y\|}yx$,

$$\begin{aligned} d_v(xy) &= d_v(x)y + (-1)^{\|x\|} x d_v(y), \\ d_h(xy) &= d_h(x)y + (-1)^{\|x\|} x d_h(y). \end{aligned}$$

Notice that Definition 5.5.1 is a particular instance of Proposition 3.2.2 for the commutative operad.

In order to define derived homotopy algebras for the commutative operad, we need the shuffle product in the tensor algebra $T(X) = \bigoplus_{n \geq 0} X^n$ of a bigraded module X . A (p, q) -*shuffle* is a permutation $\sigma \in \Sigma_{p+q}$ such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(q)$. The *shuffle product* is defined as

$$(x_1 \otimes \dots \otimes x_p) * (x_{p+1} \otimes \dots \otimes x_{p+q}) = \sum_{\sigma \in \{(p, q)\text{-shuffles}\}} \text{sign}(\sigma) \sigma \cdot (x_1 \otimes \dots \otimes x_{p+q}).$$

Here $\sigma \cdot (x_1 \otimes \cdots \otimes x_{p+q})$ is the effect of the left action of Σ_{p+q} on $X^{\otimes(p+q)}$ given by the symmetry isomorphism of the symmetric monoidal category of bigraded modules. This action involves not only permutation of tensor factors but also signs coming from the Koszul rule, compare Definition 1.5.1. This is why we do not explicitly write the outcome. We denote $\bar{T}(X) = \bigoplus_{n \geq 1} X^n \subset T(X)$.

Definition 5.5.2. A *derived homotopy commutative algebra*, *derived homotopy \mathcal{C} -algebra*, or *$d\mathcal{C}_\infty$ -algebra*, is a derived homotopy associative algebra in the sense of Definition 5.4.2 such that the structure maps $m_{in}: X^{\otimes n} \rightarrow X$ vanish on $\bar{T}(X) * \bar{T}(X)$.

Remark 5.5.3. Like in the derived associative algebra case, a derived commutative algebra is a derived homotopy commutative algebra with $m_{in} = 0$ for $i + n \geq 3$. The surviving maps m_{01} , m_{11} , and m_{02} will again play the role of the vertical differential d_v , the horizontal differential d_h , and the multiplication m , respectively, of the derived commutative algebra. The extra commutativity condition will arise as a consequence of the vanishing of m_{02} on $x \otimes y - (-1)^{\|x\|\|y\|} y \otimes x$.

Furthermore, another class of examples of derived homotopy commutative algebras are the homotopy commutative algebras, as defined in [16, Proposition 13.1.6]. They are \mathcal{A}_∞ -algebras satisfying the same vanishing condition as in Definition 5.5.2. They coincide with derived homotopy commutative algebras concentrated in the vertical axis, so $m_{in} = 0$ for $i > 0$. In this case, the homotopy commutative algebra structure is given by the maps $m_n = m_{0n}$.

Clearly a derived homotopy commutative algebra has underlying twisted complex $(X, \{m_{i1}\}_{i \geq 0})$, see Remarks 5.2.7 and 5.4.3.

In the definition of the operad $d\mathcal{C}_\infty$, we will need the right sub- Σ_n -modules $Sh_n \subset k[\Sigma_n]$ such that

$$Sh_n \otimes_{\Sigma_n} (X^{\otimes n}) = (\bar{T}(X) * \bar{T}(X)) \cap (X^{\otimes n}).$$

The action of Σ_n on $X^{\otimes n}$ is the obvious one, given by the symmetry isomorphism for the tensor product of bigraded modules. See also [16, 1.3.3].

Definition 5.5.4. Define the operad

$$d\mathcal{C}_\infty = \mathcal{F}_\Sigma \left(\left\{ 0, \bigoplus_{i \geq 1} k \cdot \mu_{i1}, \dots, \left(\bigoplus_{i \geq 0} k \cdot \mu_{in} \right) \otimes \frac{k[\Sigma_n]}{Sh_n} (n \geq 2), \dots \right\} \right),$$

where the μ_{in} are as in Definition 5.4.5. When setting $\mu_{in} = \mu_{in} \otimes \text{id}_n$, the differential $d_{d\mathcal{C}_\infty}$ is given by (5.4.6).

Theorem 5.5.5. *The minimal model of $d\mathcal{C}$ equals $d\mathcal{C}_\infty$, as defined in Definition 5.5.4.*

Remark 5.5.6. Throughout the proof we make without further mention use of the fact that taking the Koszul dual cooperad \mathcal{O}^i of a quadratic operad \mathcal{O} and applying the cobar construction $\Omega_\Sigma \mathcal{C}$ to a coaugmented cooperad \mathcal{C} are compatible with symmetrization $-\otimes k[\Sigma]$. That is, $(\mathcal{O} \otimes k[\Sigma])^i = \mathcal{O}^i \otimes k[\Sigma]$ and $\Omega_\Sigma(\mathcal{C} \otimes k[\Sigma]) = \Omega \mathcal{C} \otimes k[\Sigma]$. This follows from the monoidal properties of symmetrization reviewed in the initial chapters.

Proof of Theorem 5.5.5. We first describe the coaugmented cooperad structure on \mathcal{C}^i in terms of a known one. It follows from the proof of [16, Proposition 13.1.6] that there is a surjective coaugmented cooperad map $f: \mathcal{A}^i \otimes k[\Sigma] \twoheadrightarrow \mathcal{C}^i$ such that in arity n it factors as

$$\mathcal{A}^i(n) \otimes k[\Sigma_n] \twoheadrightarrow \mathcal{A}^i(n) \otimes \frac{k[\Sigma_n]}{Sh_n} \cong \mathcal{C}^i(n).$$

Hence the coaugmented cooperad structure on \mathcal{C}^i can be obtained from the coaugmented cooperad structure on $\mathcal{A}^i \otimes k[\Sigma]$, which in turn can be deduced from the structure of \mathcal{A}^i , described in the proof of Theorem 5.4.7, see Remark 1.4.18. Note that \mathcal{C}^i is aritywise projective since $\frac{k[\Sigma_n]}{Sh_n}$ is a projective k -module. Indeed, the definition of Sh_n is purely combinatorial, hence $\frac{k[\Sigma_n]}{Sh_n} = \frac{\mathbb{Q}[\Sigma_n]}{Sh_n} \otimes k$, which is actually free.

The coaugmented cooperad structure on $d\mathcal{C}^i$ can now be argued for as follows. Let \mathcal{O} and \mathcal{P} be quadratic operads which are aritywise projective. Since applying $\mathcal{D}^i \circ_\Sigma -$ and $-\circ_\Sigma \mathcal{D}^i$ to \mathcal{O}^i are functorial operations, and $\varphi^i: \mathcal{D}^i \circ_\Sigma - \rightarrow -\circ_\Sigma \mathcal{D}^i$ as defined in Definition 4.3.1 is natural, it follows that a coaugmented cooperad map $g: \mathcal{O}^i \rightarrow \mathcal{P}^i$ induces a coaugmented cooperad map

$$(d\mathcal{O})^i \cong \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{O}^i \xrightarrow{\mathcal{D}^i \circ_\Sigma g} \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{P}^i \cong (d\mathcal{P})^i.$$

Here we have used Theorem 4.5.2 as well. In particular, our surjection $f: \mathcal{A}^i \otimes k[\Sigma] \twoheadrightarrow \mathcal{C}^i$ induces a surjection of coaugmented cooperads

$$(d\mathcal{A})^i \otimes k[\Sigma] = (\mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{A}^i) \otimes k[\Sigma] \cong \mathcal{D}^i \circ_{\Sigma, \varphi^i} (\mathcal{A}^i \otimes k[\Sigma]) \xrightarrow{\mathcal{D}^i \circ_\Sigma f} \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{C}^i = (d\mathcal{C})^i$$

which factors arity-wise as

$$(d\mathcal{A})^i(n) \otimes k[\Sigma_n] \twoheadrightarrow (d\mathcal{A})^i(n) \otimes \frac{k[\Sigma_n]}{Sh_n} \cong (d\mathcal{C})^i(n).$$

Hence we can obtain the coaugmented cooperad structure on $(d\mathcal{C})^i$ from the coaugmented cooperad structure on $(d\mathcal{A})^i \otimes k[\Sigma]$, which we know from the proof of Theorem 5.4.7.

Now this theorem follows from the computations in the proof of Theorem 5.4.7. \square

5.6 Derived homotopy Lie algebras

Let \mathcal{L} denote the Lie operad, as defined in [16, 13.2.3]. This is a symmetric operad, so we assume $\mathbb{Q} \subset k$. This operad is arity-wise projective since free Lie algebras split out of free associative algebras, compare [16, Proposition 13.2.1 and Proposition 1.3.3]. Note that [16, Proposition 1.3.3], which goes back to Quillen and Wigner, assumes that the ground ring is a characteristic zero field, but it is enough to divide by integers, not by arbitrary non-zero elements in the ground ring, since the proof just relies on the combinatorics of Lie algebras, hence our hypothesis $\mathbb{Q} \subset k$ suffices. The operad \mathcal{L} is Koszul, see [16, section 8.3]. Moreover, \mathcal{L}^i is also aritywise projective, see the proof of Theorem 5.6.6.

Definition 5.6.1. A *derived Lie algebra*, *derived \mathcal{L} -algebra*, or *$d\mathcal{L}$ -algebra*, is a bicomplex (X, d_h, d_v) , i.e. a bigraded module X equipped with differentials d_v, d_h of bidegrees $(0, -1), (-1, 0)$, respectively, such that $d_h d_v + d_v d_h = 0$, together with a morphism of bicomplexes $[-, -]: X \otimes X \rightarrow X$ of bidegree $(0, 0)$, called the *Lie bracket*, which is anti-commutative and satisfies the Jacobi identity, i.e.

$$\begin{aligned} [x, y] &= -(-1)^{\|x\|\|y\|} [y, x], \\ (-1)^{\|x\|\|z\|} [x, [y, z]] + (-1)^{\|y\|\|x\|} [y, [z, x]] + (-1)^{\|z\|\|y\|} [z, [x, y]] &= 0, \\ d_v([x, y]) &= [d_v(x), y] + (-1)^{\|x\|} [x, d_v(y)], \\ d_h([x, y]) &= [d_h(x), y] + (-1)^{\|x\|} [x, d_h(y)]. \end{aligned}$$

Notice that Definition 5.6.1 is a particular instance of Proposition 3.2.2 for the Lie operad.

We say that a homogeneous multilinear map $f: X^{\otimes n} \rightarrow X$ of graded complexes in $\mathcal{E}(X)(n)$ is *skew-symmetric* if for $\sigma \in \Sigma_n$,

$$f \cdot \sigma = \text{sign}(\sigma) f.$$

Recall Definition 1.5.1 for the action of Σ_n on the operad of endomorphisms $\mathcal{E}(X)(n)$ in arity n .

Definition 5.6.2. A *derived homotopy \mathcal{L} -algebra*, *derived \mathcal{L}_∞ -algebra*, or *$d\mathcal{L}_\infty$ -algebra*, is a bigraded module X together with skew symmetric maps

$\lambda_{in}: X^{\otimes n} \rightarrow X$ of bidegree $(-i, i+n-2)$, $i \geq 0$, $n \geq 1$, satisfying the following equation for any fixed $i \geq 0$ and $n \geq 1$,

$$\sum_{\substack{p+l=i \\ k+q=n+1 \\ \sigma \in \{(q, n-q)\text{-shuffles}\}}} \text{sign}(\sigma)(-1)^{(k+1)q} \lambda_{pk}(\lambda_{lq} \otimes \mathbf{1}^{k-1}) \cdot \sigma^{-1} = 0.$$

Remark 5.6.3. One can check again that a derived Lie algebra is a derived homotopy \mathcal{L} -algebra with $\lambda_{in} = 0$ for $i+n \geq 3$. The surviving maps $-\lambda_{01}$, $-\lambda_{11}$, and λ_{02} will then play the role of the vertical differential d_v , the horizontal differential d_h , and the Lie bracket $[-, -]$, respectively, of the derived Lie algebra.

Also in this last example, homotopy \mathcal{L} -algebras, or \mathcal{L}_∞ -algebras, as defined in [16, Proposition 10.1.7], form a group of examples of derived homotopy \mathcal{L} -algebras when concentrated in the vertical axis, so $\lambda_{in} = 0$ for $i > 0$.

A derived homotopy Lie algebra has underlying twisted complex given by $(X, \{-\lambda_{i1}\}_{i \geq 0})$, see Remark 5.2.7.

For the Lie case one can also equivalently regard a derived homotopy \mathcal{L} -algebra as a graded complex $(X, -\lambda_{01})$ together with additional maps λ_{in} satisfying

$$\begin{aligned} d_{\text{End}(X)}(\lambda_{in}) &= (-\lambda_{01})\lambda_{in} - \sum_{r+t+1=n} (-1)^{n-2} \lambda_{in}(\mathbf{1}^{\otimes r} \otimes (-\lambda_{01}) \otimes \mathbf{1}^{\otimes t}) \\ &= \sum_{\substack{p+l=i \\ k+q=n+1 \\ \sigma \in \{(q, n-q)\text{-shuffles}\} \\ (p,k), (l,q) \neq (0,1)}} \text{sign}(\sigma)(-1)^{(k+1)q} \lambda_{pk}(\lambda_{lq} \otimes \mathbf{1}^{k-1}) \cdot \sigma^{-1}. \end{aligned}$$

With the latter formulation, it can be easily checked that the operad $d\mathcal{L}_\infty$ determining $d\mathcal{L}_\infty$ -algebras is given in the following definition.

Definition 5.6.4. Define the operad

$$d\mathcal{L}_\infty = \mathcal{F}_\Sigma \left(\left\{ 0, \bigoplus_{i \geq 1} k \cdot \lambda_{i1}, \dots, \bigoplus_{i \geq 0} k \cdot \lambda_{in} \ (n \geq 2), \dots \right\} \right). \quad (5.6.5)$$

Here λ_{in} has bidegree $(-i, i+n-2)$ and the action of Σ_n on the arity n part of the generating collection is simply given by

$$\lambda_{in} \cdot \sigma = \text{sign}(\sigma) \lambda_{in}.$$

The differential is defined by

$$d_{d\mathcal{L}_\infty}(\lambda_{in}) = \sum_{\substack{p+l=i \\ k+q=n+1 \\ \sigma \in \{(q, n-q)\text{-shuffles}\}}} \text{sign}(\sigma)(-1)^{(k+1)q}(\lambda_{pk} \circ_1 \lambda_{lq}) \cdot \sigma^{-1}.$$

Theorem 5.6.6. *The minimal model of $d\mathcal{L}$ equals $d\mathcal{L}_\infty$, as defined in Definition 5.6.4.*

Proof. During this proof we will make use of the statements in Remark 5.5.6. We know from the proof of [16, Proposition 10.1.7] that $\mathcal{L}^i(0) = 0$ and $\mathcal{L}^i(n) = k \cdot \lambda_n^c$ concentrated in bidegree $(0, n-1)$ with right Σ_n -action given by the sign of permutations,

$$\lambda_n^c \cdot \sigma = \text{sign}(\sigma)\lambda_n^c.$$

Moreover, an injective map of coaugmented cooperads is defined therein

$$f: \mathcal{L}^i \hookrightarrow \mathcal{A}^i \otimes k[\Sigma].$$

This map is explicitly defined by

$$f(\lambda_n^c) = \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma)\mu_n^c \otimes \sigma.$$

By the same functoriality argument as in the proof of Theorem 5.5.5,

$$\mathcal{D}^i \circ_\Sigma f: d\mathcal{L}^i = \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{L}^i \hookrightarrow \mathcal{D}^i \circ_{\Sigma, \varphi^i} \mathcal{A}^i \otimes k[\Sigma] = d\mathcal{A}^i \otimes k[\Sigma]$$

is an injective map of coaugmented cooperads, see Theorem 4.5.2. Taking the cobar construction, this map induces an injective operad morphism

$$d\mathcal{L}_\infty = \Omega_\Sigma(d\mathcal{L})^i \hookrightarrow \Omega(d\mathcal{A})^i \otimes k[\Sigma] = d\mathcal{A}_\infty \otimes k[\Sigma].$$

As a collection,

$$\begin{aligned} d\mathcal{L}^i(0) &= 0, \\ d\mathcal{L}^i(n) &= \mathcal{D}^i(1) \otimes \mathcal{L}^i(n) \\ &= \bigoplus_{i \geq 0} k \cdot (s\Delta)^i \otimes k \cdot \lambda_n^c \\ &= \bigoplus_{i \geq 0} k \cdot ((s\Delta)^i \otimes \lambda_n^c), \quad n \geq 1. \end{aligned}$$

The action of Σ_n is given by the product with the sign, as above. Moreover, the subcollection $(\overline{d\mathcal{L}})^i$ is simply given by removing the direct summand $i = 0$ for $n = 1$.

Let us denote

$$\lambda_{in} = s^{-1}((s\Delta)^i \otimes \lambda_n^c) \in d\mathcal{L}_\infty = \Omega_\Sigma(d\mathcal{L})^i.$$

With this notation, the above argument shows that the generating collection of the cobar construction $d\mathcal{L}_\infty = \Omega_\Sigma(d\mathcal{L})^i$ is as indicated in Definition 5.6.4.

Furthermore, the injective map $d\mathcal{L}_\infty \hookrightarrow d\mathcal{A}_\infty \otimes k[\Sigma]$ is given by

$$\lambda_{in} \mapsto \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) \mu_{in} \otimes \sigma.$$

Now, a tedious but straightforward computation shows that the only possible formula for the differential of $d\mathcal{L}_\infty$ compatible with this inclusion is the formula in Definition 5.6.4. \square

Remark 5.6.7. The map $d\mathcal{L}_\infty \rightarrow d\mathcal{A}_\infty \otimes k[\Sigma]$, explicitly computed in the previous proof, induced by the classical operad map $\mathcal{L} \rightarrow \mathcal{A} \otimes k[\Sigma]$, gives rise to an underlying derived homotopy Lie algebra structure $(X, \{\lambda_{in}\}_{i \geq 0, n \geq 1})$ on any derived homotopy associative algebra $(X, \{\mu_{in}\}_{i \geq 0, n \geq 1})$, explicitly defined by

$$\lambda_{in} = \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) (\mu_{in} \cdot \sigma).$$

Remark 5.6.8. The formula for the differential of the operad \mathcal{L}_∞ in [16, 13.2.9] has a slight mistake. The permutation σ cannot be a (p, q) -unshuffle since then σ would belong to Σ_{p+q} . But σ is acting on an operation of arity $n = p + q - 1$, so σ should belong to Σ_{p+q-1} . Actually, σ , or rather σ^{-1} with the notation therein, should run over the set of $(q, n - q)$ -shuffles. This fits with the usual conventions for \mathcal{L}_∞ -algebras in the literature.

Future directions

- *The road ahead never gives away a promise*
The road ahead is a highway or a dead-end street
The road ahead never answers any questions
And nothing is sure along the way
Not even tomorrow
With miles of the unknown ahead of you. -
City To City, “The Road Ahead”

As noted in the last paragraph of [15], current results in the literature allow to place E^1 -equivalences of derived homotopy algebras in a modern homotopical framework. Nevertheless, minimal models require also E^2 -equivalences. Therefore, defining model categories of derived homotopy algebras with E^2 -equivalences as weak equivalences is an important problem whose achievement should boost the development of this theory. This problem is widely open, even for the associative operad.

Triangulated categories are nowadays widely used in algebra, geometry, and mathematical physics. Differential graded categories (a categorified version of differential graded algebras [22]) provide algebraic models for them [26]. Moreover, A_∞ -categories (which in turn categorify A_∞ -algebras) provide *minimal* models for triangulated categories defined over a field (or with Hom sets which are projective over the ground ring) [5]. Unfortunately, to this day we lack of minimal models for triangulated categories defined over general ground rings. It would be very interesting to see whether categorified derived A_∞ -algebras can be used for this. This seems like an important and difficult project. Our contributions open the possibility of categorifying other derived homotopy algebra structures which could serve as models for triangulated categories with extra structure (e.g. symmetric monoidal).

Sagave’s seminal paper [21] contains more results than those that we generalize here to arbitrary quadratic Koszul operads. For instance, he has a strictification result [21, Theorem 1.2], turning derived homotopy algebras

into E^2 -equivalent derived algebras. We think that the operadic analogue could be derived from the development of a homotopy theory as indicated in the first paragraph. Sagave's construction does not generalize, as it relies on endomorphism algebras, which are always just associative.

Another interesting part of Sagave's paper is the cohomological theory of derived A_∞ -algebras. He extends the first obstruction to the formality of a differential graded algebra over a field, considered by Benson, Krause, and Schwede [3], to an arbitrary commutative ring, by using what he calls derived Hochschild cohomology. This class is defined in terms of a minimal derived A_∞ -model. The class of Benson, Krause, and Schwede has been generalized in a different direction to algebras over operads, over a ground field. This was done by Dimitrova in [7]. It would be interesting to explore the cohomological theory of derived homotopy \mathcal{O} -algebras, for general \mathcal{O} , in order to simultaneously extend the results of Benson, Krause, and Schwede and Dimitrova to algebras over a quadratic Koszul operad \mathcal{O} defined over an arbitrary ground ring.

We should also mention that Sagave's derived Hochschild cohomology, despite its name, is not known to coincide with the honest derived functor of Hochschild cohomology, in the sense of homotopical algebra. Honest derived Hochschild cohomology is also known as Shukla cohomology and has been considered in the literature, see [2] for instance. Hence, establishing a connection between both notions of derived Hochschild cohomology would be very interesting. The same question can of course be asked about the cohomology theories arising from the Koszul duality theory of general derived operads.

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Index

- admissible labeled tree, 103
- algebra over an operad, 43
 - free, 44
- arity, 30, 46
- associator, 15
- augmentation, 24

- bar construction, 67
 - nonsymmetric, 61
- bidegree, 20
- bigraded module, 20

- chain homotopy equation, 100
- circle product, 30
 - symmetric, 31
- co-notions, 16
- coalgebra over a cooperad, 44
 - cofree, 44
- cobar construction, 69
 - nonsymmetric, 68
- coinvariants, 54
- collection, 31
- complex, 19
- cooperad, 41
 - coaugmented, 41
 - conilpotent, 41
 - nonsymmetric, 37
 - coaugmented, 37
 - quadratic, 71
 - quadratic, 72
- cooperation
 - generating, 71
 - identity, 37
- copresentation, 71

- coradical filtration, 38
- corelators, 71
- corolla, 48
- cutting, 57

- degree, 46
 - horizontal , 21
 - total, 21
 - vertical, 21
- derived algebra, 83
 - commutative, 152
 - associative, 148
 - Lie, 155
- derived homotopy algebra, 141
 - associative, 148
 - commutative, 153
 - Lie, 155
 - minimal, 146
- derived operad, 83
 - mock, 136
 - nonsymmetric, 82
 - mock, 99
- desuspension, 68
- differential, 19
- distributive law, 25
 - trivial, 94, 98
- dual numbers, 76

- edge
 - incoming, 47
 - outgoing, 47
- equivalence
 - E^1 -, 145
 - E^2 -, 146

- essential block, 109
- generating operations, 48
- geometric realization, 47
- graded complex, 22
- graded map, 18
- graded module, 18
- groupoid, 54
- height, 46
- homogeneous element, 18
- infinitesimal
 - composition, 36
 - decomposition, 40, 42
- inner
 - edge, 47
 - vertex, 47
- inner Hom, 17
- internal morphism, 17
- invariants, 54
- Koszul dual
 - nonsymmetric, 72
- Koszul operad
 - nonsymmetric, 74
- Koszul sign convention, 19
- leaf
 - edge, 47
 - vertex, 46
- level, 46
- minimal model, 74, 75
- mixed label, 103
- monoid, 24
 - augmented, 24
 - free, 45
- monoidal category, 15
 - symmetric, 16
 - closed, 16
- monoidal functor
 - lax, 16
 - strict, 16
 - strong, 16
- morphism, ∞ -, 142
- norm map, 34
- operad, 36
 - augmented, 36
 - nonsymmetric, 35
 - augmented, 36
 - quadratic, 70
 - quotient, 70
 - quadratic, 71
 - presentation, 71
 - quotient, 71
- operadic ideal
 - nonsymmetric, 70
- operadic Leibniz rule, 36
- operadic quadratic data, 70
 - nonsymmetric, 70
- operation, 30
 - generating, 70
 - identity, 36
- opposite, 16
- order
 - path, 47
 - planar, 46
- planar isomorphism, 48
- planted tree with leaves, 46
 - planar, 46
 - labeled, 51
- relators, 70
- rewriting rule, 92
 - nonsymmetric, 85
- root
 - edge, 47
 - vertex, 46
- SDR, 100
- sequence, 30

- reduced, 30
- shuffle, 152
- shuffle product, 152
- skew-symmetric, 155
- strong deformation retraction, 100
- suspension, 61
- symmetrization, 33
- symmetry isomorphism, 16
- syzygy degree, 73

- tensor unit, 15
- total complex, 23
- tree groupoid, 55
- tree module, 48
 - symmetric, 55
 - symmetrized, 51
- twisted complex, 143
 - minimal, 146

- vertical differential, 22

- weight, 51

List of Notations

$\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$	(closed) (symmetric) monoidal category
$[-, -]$	inner Hom
	sometimes it also stands for Lie brackets
k	ground commutative ring with unit $1 \in k$
$\otimes = \otimes_k$	tensor product of k -modules over ground ring k
$\mathbb{1}_X$	identity map on an object X
\mathbb{N}	the set $\{0, 1, \dots\}$
$ x $	(bi)degree of an element x in a (bi)graded module
$\ x\ $	total degree of an element x in a bigraded module
GrCh_k	closed symm. monoidal category of graded complexes
Seq_k	monoidal category of sequences in GrCh_k
\circ	tensor product of sequences
I	tensor unit in sequences/collections
Σ_n	group of automorphisms of the set $\{1, 2, \dots, n\}$
id_n	identity on $\{1, 2, \dots, n\}$
Coll_k	monoidal category of collections in GrCh_k
$k[G]$	group algebra of G
\otimes_G	tensor product over $k[G]$ of left-right modules over G
X_G	space of coinvariants of a right G -module X
\circ_{Σ}	tensor product of collections in coinvariant setting
X^G	space of invariants of right G -module X
$\bar{\circ}_{\Sigma}$	tensor product of collections in invariant setting
$A = \langle A, \mu_A, \eta_A \rangle$	monoid
$C = \langle C, \Delta_C, \epsilon_C \rangle$	comonoid
$\epsilon_{\mathcal{O}}$	augmentation of an (ns-)operad \mathcal{O}
$\text{id}_{\mathcal{O}}$	identity operation in an (ns-)operad \mathcal{O}
$\bar{\mathcal{O}} = \text{Ker}(\epsilon_{\mathcal{O}})$	kernel of $\epsilon_{\mathcal{O}}$ in the category of sequences/collections
\circ_i	infinitesimal composition as defined in Remark 1.4.3
$\eta_{\mathcal{C}}$	identity cooperation in an (ns-)cooperad \mathcal{C}
$\text{id}_{\mathcal{C}}$	image of $1 \in k$ by $\eta_{\mathcal{C}}$
$\bar{\mathcal{C}} = \text{Coker}(\eta_{\mathcal{C}})$	cokernel of $\eta_{\mathcal{C}}$ in the category of sequences/collections

$\bar{\Delta}_C$	decomposition factor of Δ_C (1.4.8)
$\Delta_{(1)}$	infinitesimal decomposition (1.4.15)
$\mathcal{E}(X)$	(ns-)operad of endomorphisms
$V(T)$	set of vertices of a planted planar tree T with leaves
$E(T)$	set of edges of a planted planar tree T with leaves
\leq	path order in $V(T)$
U	unit for grafting
	sometimes it also designates a forgetful functor
$X(T)$	tree module as defined in Definition 2.1.5
$X_{[T]}$	symmetric tree module
$\mathcal{F}(X)$	free ns-operad
$\mathcal{F}_\Sigma(X)$	free operad
$\mathcal{F}^c(X)$	cofree coaugmented conilpotent ns-cooperad
$\mathcal{F}_\Sigma^c(X)$	cofree coaugmented conilpotent cooperad
B	nonsymmetric bar construction
$\hat{d}_{2,i}$	infinitesimal d_2 on (ns-)bar construction (2.3.1)
B_Σ	bar construction
Ω	nonsymmetric cobar construction
\hat{d}_2	infinitesimal d_2 on (ns-)cobar construction (2.3.6) (2.3.7)
Ω_Σ	cobar construction
$(E R)$	(ns-)operadic quadratic data
\mathcal{O}^i	Koszul dual (ns-)cooperad of an (ns-)operad \mathcal{O}
φ	a relevant distributive law
φ^i	a relevant codistributive law
$\bar{\varphi}$	a related relevant codistributive law
\mathcal{D}	(ns-)operad of dual numbers
	sometimes it also designates a generic (ns-)cooperad
$d\mathcal{O}$	derived (ns-)operad as defined in section 3.1
$d'\mathcal{O}$	mock derived (ns-)operad as defined in sections 4.2 and 4.5
λ	(ns-)rewriting rule as defined in Definitions 3.3.1 and 3.3.4
∂	perturbation as defined in Lemma 4.3.6
\mathcal{A}	(non-unital) associative ns-operad
\mathcal{C}	(non-unital) commutative operad
\mathcal{L}	Lie operad
X_*	X in homological (bi)degree $*$
$X^{(w)}$	X in weight degree (w)
X^\bullet	X in syzygy degree \bullet
$X(n)$	X in arity n