The Gröbner fan of an A_n-module

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Abstract

Let I be a non-zero left ideal of the Weyl algebra A_n of order n over a field $\mathbf{k}$ and let $L : \mathbf{R}^{2n} \longrightarrow \mathbf{R}$ be a linear form defined by $L(\alpha, \beta) = \sum_{i=1}^{n} e_i\alpha_i + \sum_{i=1}^{n} f_i\beta_i$. If $e_i + f_i \geq 0$, then L defines a filtration $F_{\bullet}^{L}$ on A_n. Let $\mathrm{gr}^L(I)$ be the graded ideal associated to the filtration induced by $F_{\bullet}^{L}$ on I. Let finally U denote the set of all linear forms L for which $e_i + f_i \geq 0$ for all $1 \leq i \leq n$. The aim of this paper is to study, by using the theory of Gröbner bases, the stability of $\mathrm{gr}^L(I)$ when L varies in U. In a previous paper, we obtained finiteness results for some particular linear forms (used in order to study the regularity of a $\mathcal{D}$-module along a smooth hypersurface). Here we generalize these results by adapting the theory of Gröbner fan of Mora-Robbiano to the $\mathcal{D}$-module case. Our main tool is the homogenization technique initiated in our previous paper, and recently clarified in a work of F. Castro-Jiménez and L. Narváez-Macarro.

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1 Introduction

Let $A_n(\mathbf{k})$ denote the Weyl algebra of order n over a field $\mathbf{k}$: $A_n(\mathbf{k})$ (A_n for short) is the central $\mathbf{k}$-algebra generated by $x_i, D_i, i = 1, \ldots, n$ with relations $[x_i, x_j] = [D_i, D_j] = 0$ and $[D_i, x_j] = \delta_{ij}$. Let $P = \sum_{\alpha,\beta} p_{\alpha,\beta} \underline{x}^{\alpha} \underline{D}^{\beta}$ be a non-zero element of A_n and denote by $\mathcal{N}(P)$ the Newton diagram of P, namely

$$\mathcal{N}(P) = \{(\alpha, \beta) \in \mathbf{N}^{2n};\, p_{\alpha,\beta} \neq 0\}.$$

If $L : \mathbf{R}^{2n} \longrightarrow \mathbf{R}$ is the linear form defined by $L(\alpha, \beta) = \sum_{i=1}^{n} e_i\alpha_i + \sum_{i=1}^{n} f_i\beta_i$, then the L-order $\mathrm{ord}_L(P)$ of P is defined to be the maximal element in the set of $L(\alpha, \beta), (\alpha, \beta) \in \mathcal{N}(P)$. If furthermore $e_i + f_i \geq 0$ for all $1 \leq i \leq n$, then $\mathrm{ord}_L(P.Q) = \mathrm{ord}_L(P) + \mathrm{ord}_L(Q)$ for all non-zero elements $P, Q \in A_n$, in particular L defines a filtration on A_n (where for all $k \in \mathbf{R}, F_k^L(A_n) = \{P \in A_n; \mathrm{ord}_L(P) \leq k\}$). If $e_i + f_i > 0$ for all $i = 1, \ldots, l$ (resp. $e_i + f_i = 0$ for all $i = l + 1, \ldots, n$), then the associated graded algebra is:

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$$\mathrm{gr}^L(A_n) \simeq \mathbf{k}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_l][D_{l+1}, \ldots, D_n]$$

with relations:

$$x_i x_j = x_j x_i,\ x_i \xi_m = \xi_m x_i,\ x_i D_p = D_p x_i - \delta_{ip},\ \xi_m D_p = D_p \xi_m$$

for all $1 \leq i, j \leq n$, $1 \leq m \leq l$ and $l + 1 \leq p \leq n$. The principal symbol of P is the element of $\mathrm{gr}^L(A_n)$,

$$\sigma^L(P) = \sum_{L(\alpha,\beta)=\mathrm{ord}_L(P)} p_{\alpha\beta}\underline{x}^\alpha \xi_1^{\beta_1} \cdots \xi_l^{\beta_l} D_{l+1}^{\beta_{l+1}} \cdots D_n^{\beta_n}$$

Let us point out that in the commutative case, there is no condition of the type $e_i + f_i \geq 0$. Here, since $-x_i D_i + D_i x_i = 1$, we must require $\mathrm{ord}_L(1) \leq \mathrm{ord}_L(x_i) + \mathrm{ord}_L(D_i)$.

Let I be a non-zero left ideal of A_n and let $\mathrm{gr}^L(I)$ be the graded ideal associated with the filtration induced by $F_\bullet^L$ on I. Let finally U denote the set of all linear form L for which $e_i + f_i \geq 0$ for all $1 \leq i \leq n$. The aim of this paper is to study, by using the theory of standard and Gröbner bases, the stability of $\mathrm{gr}^L(I)$ when L varies in U.

Let Y be the hypersurface of $\mathbf{k}^n$ defined by $x_1 = 0$. Given two non negative reals p, q, we define the linear form $L_{p,q}$ on $\mathbf{R}^{2n}$ by $L_{p,q}(\alpha,\beta) = p.(\sum_{i=1}^n \beta_i) + q.(\beta_1 - \alpha_1)$: this is an interpolation between the filtration F by the order of operators ($q = 0$) and the V-filtration of Malgrange-Kashiwara ($p = 0$). In [10], Y. Laurent proved, using 2-microdifferential operators, that the radical ideal $\sqrt{\mathrm{gr}^{L_{p,q}}(I)}$ is not a (F, V)- homogeneous ideal for only a finite set of rational numbers $r = p/q$ (eventually empty), when p, q vary in $\mathbf{R}_+^2$ (in [11] and [13], an analytic interpretation of these numbers is given). C. Sabbah and F. Castro proved in [15] the same result by using a local flattener. In [2] we obtained, using the theory of standard bases, a constructive proof of this result. This allowed us to give an algorithm for the calculation of these numbers. So, it was natural to think about general finiteness results when L varies in U. Recall that the theory of Gröbner bases (cf [5]) works very well in the Weyl algebra A_n (cf [6],[7] and [8]). However, when the coefficients of the linear form $L \in U$ are negative, the division process in A_n can be infinite. In [2], in order to avoid this difficulty, we worked in $A_n[t]$ by homogenizing with respect to the total order in a way inspired by [12]. However, the non commutativity of $A_n[t]$ causes some difficulty, since divisions by homogeneous elements do not produce necessarily homogeneous remainders. Although our algorithm (which consists in rehomogenizing the remainders and iterating division) allows us the calculation of standard bases with respect to any $L \in U$, it does not seem to be adapted to the question we are working on. In [9], this difficulty is avoided in the following natural way: consider the graded $\mathbf{k}$-algebra B, generated by $x_i, D_i, i = 1, \ldots, n$ and t with homogeneous relations:

$$[t, x_i] = [t, D_i] = [x_i, x_j] = [D_i, D_j] = 0, [D_i, x_j] = \delta_{ij} t^2.$$

This $\mathbf{k}$-algebra coincides with the Rees algebra associated with the Bernstein filtration on A_n. The homogenization process between A_n and B verify the same properties as in the commutative case, in particular the notion of the homogenized ideal $h(I)$ of I is well defined. On the other

hand, we can get the different graded ideals $\mathrm{gr}^L(I)$ from calculations of the graded ideals of $h(I)$. Since the notion of reduced standard bases exists for ideals in B, then the natural way in order to study our question is in adapting to the $\mathcal{D}$–module case the theory of Gröbner fan developed by T. Mora and L. Robbiano in [14].
Let us summarize the structure of the paper: in Section 2 we recall some of the results of [9] related to the homogenization problem. We also prove that a standard basis w.r.t. the Bernstein filtration of an ideal I in A_n gives us a generating system of $h(I)$ in $B = A_n[t]$. In section 3, paragraph 3.2., we obtain finiteness results for the set of graded ideals $\mathrm{gr}^L(J)$ where J is an homogeneous ideal of $A_n[t]$. The main tool we use here is the Hilbert function of J. This notion has also been used by one of us [1] in order to prove similar results in the commutative case. The results of paragraph 3.2. are then applied in order to prove that the set of graded ideals $\mathrm{gr}^L(I), L \in U$ is finite (paragraph 3.3.). Finally, in section 4, we study the repartition of the graded ideals $\mathrm{gr}^L(h(I))$, where L varies in U. We define first the notion of privileged exponent (or stairs) $\mathrm{Exp}_{\prec_L}(h(I))$ of an ideal associated with a fixed well ordering on $\mathbf{N}^{1+2n}$ (see 3.3.). Our main result is then the following, which generalizes the results in the commutative case as found in [1], [14] and [17]:

Theorem 1.1 *There exists a partition $\mathcal{E}$ of U into convex rational polyhedral cones, such that for all element $\sigma \in \mathcal{E}$, $\mathrm{gr}^L(h(I))$ and $\mathrm{Exp}_{\prec_L}(h(I))$ do not depend on $L \in \sigma$ (and the same is true for $\mathrm{gr}^L(I)$).*

Some results of this article has been used in [16].

2 Homogenization

We shall use here the results of [9]. Let $A_n[t]$ denote the algebra

$$A_n[t] = \mathbf{k}[t, \underline{x}][\underline{D}] = \mathbf{k}[t, x_1, \ldots, x_n][D_1, \ldots, D_n]$$

with relations:

$$[t, x_i] = [t, D_i] = [x_i, x_j] = [D_i, D_j] = 0, [D_i, x_j] = \delta_{ij} t^2.$$

The algebra $A_n[t]$ is a graded algebra, the degree of the monomial $t^k \underline{x}^\alpha \underline{D}^\beta$ being $k + |\alpha| + |\beta|$. In fact, the $\mathbf{k}$–algebra $A_n[t]$ is isomorphic to the Rees algebra associated with the Bernstein filtration on A_n. The algebra $\mathbf{k}[t]$ is central in $A_n[t]$, and the quotient algebra $A_n[t]/\langle t-1\rangle$ is isomorphic to A_n.

Let $P = \sum_{\alpha,\beta} p_{\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta$ be a non-zero operator of A_n. We denote by $\mathcal{N}(P)$ the Newton diagram of P,

$$\mathcal{N}(P) = \{(\alpha, \beta) \in \mathbf{N}^{2n}; p_{\alpha,\beta} \neq 0\},$$

then we denote by $\mathrm{ord}^T(P)$ the total order of P

$$\mathrm{ord}^T(P) = \max\{|\alpha| + |\beta|\,;\, p_{\alpha,\beta} \in \mathcal{N}(P)\}.$$

The differential operator

$$h(P) = \sum_{\alpha,\beta} p_{\alpha,\beta} t^{\mathrm{ord}^T(P)-|\alpha|-|\beta|} \underline{x}^\alpha \underline{D}^\beta \in A_n[t]$$

is called the homogenization of P. If $H = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} t^k \underline{x}^\alpha \underline{D}^\beta$ is an element of $A_n[t]$, we denote by $H_{|t=1}$ the operator of A_n

$$H_{|t=1} = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta.$$

With the notations above, for all $P, Q \in A_n$ and for all homogeneous element $H \in A_n[t]$,

1. $h(PQ) = h(P)h(Q)$.
2. There exists $k, l, m \in \mathbf{N}$ such that $t^k h(P+Q) = t^l h(P) + t^m h(Q)$.
3. There exists $k \in \mathbf{N}$ such that $t^k h(H_{|t=1}) = H$.

Let $<$ be a total ordering on $\mathbf{N}^{2n}$ (not necessarily a well ordering), compatible with sums. We recall that the extension of $<$, denoted by $<^h$, is the total well ordering on $\mathbf{N}^{1+2n}$ (compatible with sums) defined by:

$$(k,\alpha,\beta) <^h (k',\alpha',\beta') \Longleftrightarrow \left\{ \begin{array}{l} k + |\alpha| + |\beta| < k' + |\alpha'| + |\beta'| \\ \text{or } \left\{ \begin{array}{l} k + |\alpha| + |\beta| = k' + |\alpha'| + |\beta'| \text{ and} \\ (\alpha,\beta) < (\alpha',\beta') \end{array} \right. \end{array} \right.$$

Since $<^h$ is a total well ordering compatible with sums, we have for all non-zero element $G = \sum_{a,\alpha,\beta} g_{(a,\alpha,\beta)} t^a \underline{x}^\alpha \underline{D}^\beta$ the notion of privileged exponent of G w.r.t. $<^h$, which we denote by $\exp_{<^h}(G)$: If $\mathcal{N}(G) = \{(a,\alpha,\beta); g_{(a,\alpha,\beta)} \neq 0\}$ denote the Newton diagram of G, then $\exp_{<^h}(G) = \max_{<^h} \mathcal{N}(G)$. Also we have for all non-zero ideal J of $A_n[t]$, the notion of Gröbner (or standard) basis of J, namely, if we denote by

$$\mathrm{Exp}_{<^h}(J) = \{\exp_{<^h}(P) \mid P \in J\},$$

then $\{P_1, \ldots, P_r\} \subseteq J$ is a standard basis of J if

$$\mathrm{Exp}_{<^h}(J) = \bigcup_{i=1}^{r} (\exp_{<^h}(P_i) + \mathbf{N}^{1+2n}).$$

We have finally a division theorem in $A_n[t]$, analogous to that in the ring of polynomials or in the Weyl algebra A_n. For more details, see [9]. Let $\pi : \mathbf{N}^{1+2n} = \mathbf{N} \times \mathbf{N}^{2n} \to \mathbf{N}^{2n}$ denote the natural projection, then we have:

1. If $P \in A_n$, then $\pi(\exp_{<^h}(h(P))) = \exp_{<}(P)$.

2. More generally, if H is an homogeneous element of $A_n[t]$, then

$$\pi(\exp_{<^h}(H)) = \pi(\exp_{<^h}(h(H_{|t=1}))) = \exp_{<}(H_{|t=1}).$$

Let I be a left ideal of A_n. We denote by $h(I)$ the homogeneous ideal of $A_n[t]$, generated by $\{h(P) \mid P \in I\}$. We call $h(I)$ the homogenized ideal of I. With these notations we have the following (see [9]):

1. $\pi(\mathrm{Exp}_{<^h}(h(I))) = \mathrm{Exp}_{<}(I)$.

2. Let $\{P_1, \ldots, P_m\}$ be a generating system of I and let $\widetilde{I}$ be the ideal generated by $\{h(P_1), \ldots, h(P_m)\}$ in $A_n[t]$. Then $\pi(\mathrm{Exp}_{<^h}(\widetilde{I})) = \mathrm{Exp}_{<}(I)$.

Let $B_\bullet(A_n)$ denote the Bernstein filtration on A_n (that is the case with $e_i = f_i = 1$ for all $i = 1, \ldots, n$). If P is a differential operator in A_n, then we denote by $\sigma^B(P)$ the principal symbol of P w.r.t. the Bernstein filtration. If I is an ideal of A_n, then we denote by $\mathrm{gr}^B(I)$ the graded ideal associated with the induced Bernstein filtration on I.

A standard basis w.r.t. the Bernstein filtration has the following interesting property:

Lemma 2.1 *Let I be a non-zero left ideal of A_n and let $\{P_1, \ldots, P_m\}$ be a family of differential operators of I. The following assertions are equivalent:*

i) $h(I) = (h(P_1), \ldots, h(P_m))$.

ii) $\mathrm{gr}^B(I) = (\sigma^B(P_1), \ldots, \sigma^B(P_m))$.

Proof. The proof is classical and uses the structure of graded algebra of $A_n[t]$ (see for details [3]). Remark that a standard basis with respect to the Bernstein filtration satisfies ii), but the converse is in general false.

3 Finiteness results

Let $L \in U$ (see 1) and consider the extension of L to $\mathbf{R} \times \mathbf{R}^{2n}$ (by abuse of notation we continue to write L and U in $\mathbf{R} \times \mathbf{R}^{2n}$), $L : \mathbf{R} \times \mathbf{R}^{2n} \to \mathbf{R}$, such that $L(a, \alpha, \beta) = \sum_{i=1}^n e_i\alpha_i + \sum_{i=1}^n f_i\beta_i$. Recall in particular that $e_i + f_i \geq 0$ for all $1 \leq i \leq n$.

Let P be a non-zero differential operator of $A_n[t]$. We define the L–order of P in the usual way (we denote this element by $\mathrm{ord}_L(P)$). If $P, Q \in A_n[t]$, then $\mathrm{ord}_L(PQ) = \mathrm{ord}_L(P) + \mathrm{ord}_L(Q)$, consequently the L–order defines a filtration on $A_n[t]$, which we shall call the L-filtration and we shall denote by $F^L_\bullet(A_n[t])$. We denote by $\sigma^L(P)$ the principal symbol of P w.r.t. the L-order, precisely, if $P = \sum p_{\alpha\beta}(t)\underline{x}^\alpha \underline{D}^\beta$, then $\sigma^L(P) = \sum_{L(\alpha,\beta)=\mathrm{ord}_L(P)} p_{\alpha\beta}(t)\underline{x}^\alpha \xi_1^{\beta_1} \cdots \xi_l^{\beta_l} D_{l+1}^{\beta_{l+1}} \cdots D_n^{\beta_n}$ with l is as defined in the introduction. If J is a non-zero homogeneous ideal of $A_n[t]$, we denote by $\mathrm{gr}^L(J)$ the graded ideal associated with the induced L–filtration on J (i.e. $\mathrm{gr}^L(J)$ is the ideal of $\mathrm{gr}^L(A_n[t])$ generated by $\{\sigma^L(P) \mid P \in J\}$). In this section we shall prove that, if the coefficients e_i, f_i vary in $\mathbf{R}$, then the set of $\mathrm{gr}^L(J)$ is finite. We shall use in the proof the Hilbert function, therefore we shall start by recalling some of its properties.

3.1 Hilbert function

Let $E \subset \mathbf{N}^{1+2n}$ such that $E + \mathbf{N}^{1+2n} = E$. We define the Hilbert function of E (and we denote it by H_E) to be the map $H_E : \mathbf{N} \longmapsto \mathbf{N}$:

$$H_E(k) = \sharp\{(a, \alpha, \beta) \in \mathbf{N}^{1+2n} \setminus E; a + |\alpha| + |\beta| = k\}, \forall k \in \mathbf{N}.$$

Let J be an homogeneous ideal of $A_n[t] = \oplus_{k\in\mathbf{N}} A_n[t]_k$, where $A_n[t]_k$ is the $\mathbf{k}$–vector space generated by the monomials $t^a \underline{x}^\alpha \underline{D}^\beta$ of total degree $a + |\alpha| + |\beta| = k$. We set $J_k = A_n[t]_k \cap J$.

Let $\prec$ be a total well ordering on $\mathbf{N}^{1+2n}$ compatible with sums, and let $E_\prec = \mathrm{Exp}_\prec(J)$.

Lemma 3.1 *For all $k \in \mathbf{N}$, we have:*

$$\dim_{\mathbf{k}}(A_n[t]_k / J_k) = \sharp\{(a, \alpha, \beta) \in \mathbf{N}^{1+2n} \setminus E_\prec; a + |\alpha| + |\beta| = k\} = H_{E_\prec}(k)$$

Proof. Let $\{P_1, \ldots, P_m\}$ be a family of homogeneous operators of J such that:

$$E_\prec = \bigcup_{i=1}^{m} (\exp_\prec(P_i) + \mathbf{N}^{1+2n}).$$

If we denote by $k_i = \mathrm{ord}^T(P_i)$, then for all $P \in A_n[t]_k$, there exists a family of homogeneous elements $Q_1, \ldots, Q_m, R$ of $A_n[t]$ such that:

1. $P = \sum_{i=1}^m Q_i P_i + R$.
2. $\mathrm{ord}^T(Q_i) = k - k_i, \mathrm{ord}^T(R) = k$.
3. If $R \neq 0$, then the Newton diagram $\mathcal{N}(R) \subset \mathbf{N}^{2n+1} \setminus E_\prec$. Thus $P \in J_k \Longleftrightarrow R = 0$.

In particular, $P + J_k = R + J_k$. This proves that the classes, modulo J_k, of the monomials $t^a \underline{x}^\alpha \underline{D}^\beta$, with $a + |\alpha| + |\beta| = k$, $(a, \alpha, \beta) \notin E_\prec$ form a basis for $A_n[t]_k / J_k$ over $\mathbf{k}$. This proves our assertion.

Let, for all $k \in \mathbf{N}$, $H_J(k) = \dim_{\mathbf{k}}(A_n[t]_k / J_k)$. This defines a map $H_J : \mathbf{N} \to \mathbf{N}$ which we call the Hilbert function of J. By Lemma 3.1, $H_J = H_{E_\prec}$ does not depend on $\prec$.

3.2 Finiteness Theorems for homogeneous ideals

Let $\mathcal{O}(\mathbf{N}^{1+2n})$ denote the set of total well ordering on $\mathbf{N}^{1+2n}$ compatibles with sums (for such an order, 0 is the smallest element, this implies in particular that $\exp_\prec(PQ) = \exp_\prec(P) + \exp_\prec(Q)$)).

Theorem 3.2 *Let J be a non-zero homogeneous ideal of $A_n[t]$. Then*

$$\{\mathrm{Exp}_\prec(J) \mid \prec \in \mathcal{O}(\mathbf{N}^{1+2n})\}$$

is a finite set.

Proof. By Lemma 3.1, it suffices to prove that the set of subsets $E \subset \mathbf{N}^{1+2n}$ such that:

1. $E + \mathbf{N}^{1+2n} = E$.
2. $H_E = H_J$.

is finite. Denote this set by $\mathcal{E}$ and assume that $\mathcal{E}$ is infinite. Given an element E of $\mathcal{E}$ and an integer $k \in \mathbf{N}$, we set

$$E_{(k)} = \{\Omega \in E; |\Omega| \leq k\}.$$

Let $k_0 \in \mathbf{N}$ be the smallest integer for which $H_J(k) < \dim_{\mathbf{k}}(A_n[t]_k)$ (such an integer exists because $J \neq (0)$). Since $\mathbf{N}^{1+2n}_{(k_0)}$ is a finite set, one of the possible choices of $E_{(k_0)}$ occurs for all E in an infinite subset $\mathcal{E}_1 = \{E_i\}_{i \geq 1}$ of $\mathcal{E}$. Thus, there are elements $\Omega_i \in \mathbf{N}^{1+2n}_{(k_0)}, 1 \leq i \leq r$ such that

$$E_{i,(k_0)} = (\cup_{i=1}^{r}(\Omega_i + \mathbf{N}^{1+2n}))_{(k_0)} \quad \text{for} \quad \text{all} \quad i \geq 1.$$

Assume, without loss of generality, that $\mathcal{E}_1 = \mathcal{E}$ and set

$$S_0 = \cup_{i=1}^{r}(\Omega_i + \mathbf{N}^{1+2n}).$$

Clearly $S_0 \subseteq E_i$ for all $i \geq 1$, on the other hand $E_i \neq E_j$ for all $i \neq j$. In particular $H_J \neq H_{S_0}$. Let consequently $k_1 > k_0$ be the smallest integer for which $H_J(k_1) < H_{S_0}(k_1)$. For all $j \geq 2$, there exists $\Theta^j \in E_j \setminus S_0$ such that $|\Theta^j| = k_1$. The set $\mathbf{N}^{1+2n}_{(k_1)}$ being finite, there is an infinite subset $\mathcal{E}_2 \subset \mathcal{E}$ and elements $\Omega_{r+i}, 1 \leq i \leq r + r_1$, in $(\mathbf{N}^{1+2n} \setminus S_0)_{(k_1)}$ such that:

$$E_{j,(k_1)} = (\cup_{i=1}^{r+r_1}(\Omega_j + \mathbf{N}^{1+2n})_{(k_1)} \quad \text{for} \quad \text{all} \quad E_j \in \mathcal{E}_2.$$

Let

$$S_1 = \cup_{i=1}^{r+r_1}(\Omega_j + \mathbf{N}^{1+2n}),$$

then $S_0 \subset S_1$. Now repeat the same argument with $\mathcal{E}_2$ and S_1,... We construct this way an infinite sequence $S_0 \subset S_1 \subset ...$ of subsets of $\mathbf{N}^{1+2n}$ with $S_i + \mathbf{N}^{1+2n} = S_i$ for all $i \geq 0$. This is impossible.

As a consequence of Theorem 3.2. we get the following result:

Theorem 3.3 *Let J be a non-zero homogeneous ideal of $A_n[t]$. Then $\{\mathrm{gr}^L(J); L \in U\}$ is a finite set.*

Proof. Fix $\prec \in \mathcal{O}(\mathbf{N}^{1+2n})$, then for any $L \in U$, denote by $\prec_L$ the total ordering on $\mathbf{N}^{1+2n}$ such that:

$$(k,\alpha,\beta) \prec_L (k',\alpha',\beta') \iff \begin{cases} k+|\alpha|+|\beta| < k'+|\alpha'|+|\beta'| \\ \text{or} \\ k+|\alpha|+|\beta| = \\ \\ = k'+|\alpha'|+|\beta'| \text{ and } \begin{cases} L(k,\alpha,\beta) < L(k',\alpha',\beta') \\ \text{or} \\ L(k,\alpha,\beta) = L(k',\alpha',\beta') \text{ and} \\ (k,\alpha,\beta) \prec (k',\alpha',\beta') \end{cases} \end{cases}$$

(Where we recall that $L(k,\alpha,\beta) = \sum_{i=1}^n e_i\alpha_i + \sum_{i=1}^n f_i\beta_i$). Clearly $\prec_L \in \mathcal{O}(\mathbf{N}^{1+2n})$. On the other hand, by 3.2, $\{\mathrm{Exp}_{\prec_L}(J) \mid L \in U\}$ is a finite set. Consequently we have only to prove that, if $E \subset \mathbf{N}^{1+2n}$ with $E + \mathbf{N}^{1+2n} = E$, then $\{\mathrm{gr}^L(J) \mid \mathrm{Exp}_{\prec_L}(J) = E, L \in U\}$ is a finite set. Fix to this end E and let $L \in U$ be such that $E = \mathrm{Exp}_{\prec_L}(J)$. Then consider a reduced standard basis $\mathcal{B} = \{Q_1, \ldots, Q_m\}$ of J w.r.t. $\prec_L$ (i.e. $\cup_{i=1}^m (\exp_{\prec_L}(Q_i) + \mathbf{N}^{1+2n}) = E$ and $\mathcal{N}(Q_i) \setminus \{\exp_{\prec_L}(Q_i)\} \subset \mathbf{N}^{1+2n} \setminus E$, for all $1 \le i \le m$, where $\mathcal{N}(Q_i)$ is the Newton diagram of Q_i). Clearly $\mathcal{B}$ is also a reduced standard basis of J w.r.t. $\prec_{L'}$, for all $L' \in U$ such that $\mathrm{Exp}_{\prec_{L'}}(J) = E$ (indeed, if $\exp_{\prec_{L'}}(Q_i) \neq \exp_{\prec_L}(Q_i)$, we would have $\exp_{\prec_{L'}}(Q_i) \notin E$). In particular, as proved in [2], Lemma 1.3.3., $\{\sigma^{L'}(Q_1), \ldots, \sigma^{L'}(Q_m)\}$ generates $\mathrm{gr}^{L'}(J)$ for all $L' \in U$ such that $\mathrm{Exp}_{\prec_{L'}}(J) = E$. Every $\mathcal{N}(Q_i)$ being finite, we have only a finite number of possibilities. This proves our assertion.

We shall finally give a bound for the cardinality of $\mathcal{O}(J) = \{\mathrm{Exp}_\theta(J); \theta \in \mathcal{O}(\mathbf{N}^{1+2n})\}$. Let for all $E \in \mathcal{O}(J)$, $J_E = (y^{\alpha_1}, \ldots, y^{\alpha_s})\mathbf{k}[y_1, \ldots, y_{2n+1}]$, where $y_1, \ldots, y_{2n+1}$ are indeterminates and $\{\alpha_1, \ldots, \alpha_s\}$ is the minimal boundary of E, that is $E = \cup_{i=1}^s (\alpha_i + \mathbf{N}^{1+2n})$ and for all $k = 1, \ldots, s$, $\alpha_k \notin \cup_{i \neq k}(\alpha_i + \mathbf{N}^{1+2n})$. Clearly $H_{J_E} = H_E$, then we have:

$$\sharp\mathcal{O}(J) = \sharp\{J_E; E \in \mathcal{O}(J)\} \le \sharp\{M \subset \mathbf{k}[y_1, \ldots, y_{2n+1}] \quad \text{monomial ideal}\,; H_M = H_J\}$$

Let $d(J)$ denote the maximal degree of the elements arising in the minimal boundaries of $\{\mathrm{Exp}_\theta(J), \theta \in \mathcal{O}(\mathbf{N}^{2n+1})\}$. If $(d_1, d_2, \ldots)$ denote the values of the Hilbert function of J, then we have:

Proposition 3.4

$$\sharp\mathcal{O}(J) \le \prod_{k=1}^{d(J)} C_{a_k - d_k}^{a_k},$$

where

$$a_k = \dim_{\mathbf{k}} A_n[t]_k = C_k^{2n+k}$$

and C_b^a is the binomial coefficient.

Proof. The number of points in E which are exponents of monomials of degree k is exactly $a_k - d_k$. This proves our assertion.

3.3 Finiteness Theorems for ideals in A_n

Let I be non-zero left ideal of A_n. The aim of this paragraph it to give for I analogous results to those of 3.2. Let to this end $<$ be a total well ordering on $\mathbf{N}^{2n}$, compatible with sums, and denote, for all $L \in U$, by $<_L$ the total ordering on $\mathbf{N}^{2n}$ such that:

$$(\alpha,\beta) <_L (\alpha',\beta') \quad \Leftrightarrow \quad \begin{cases} L(\alpha,\beta) < L(\alpha',\beta') \\ \text{or} \\ L(\alpha,\beta) = L(\alpha',\beta') \text{ and } (\alpha,\beta) < (\alpha',\beta') \end{cases}$$

Let $P \in A_n$ be a non-zero differential operator. We denote by $\exp_{<_L}(P)$ the privileged exponent of P w.r.t. $<_L$, i.e. $\exp_{<_L}(P) = \max_{<_L}\mathcal{N}(P)$ (See [2] for the main properties of the privileged exponent of an operator). We also set

$$\mathrm{Exp}_{<_L}(I) = \{\exp_{<_L}(P) \,|\, P \in I \setminus \{0\}\}.$$

Clearly $\mathrm{Exp}_{<_L}(I) + \mathbf{N}^{2n} = \mathrm{Exp}_{<_L}(I)$.

Theorem 3.5 *For a given total well ordering $<$ on $\mathbf{N}^{2n}$, compatible with sums, $\{\mathrm{Exp}_{<_L}(I) \,|\, L \in U\}$ is a finite set.*

Proof. This results follows from Theorem 3.2 as follows: firstly we remark that, with the notations of section 2, $\prec_L = <_L^h$, for the following choice of $\prec$,

$$(k,\alpha,\beta) \prec (k',\alpha',\beta') \quad \Leftrightarrow \quad \begin{cases} (\alpha,\beta) < (\alpha',\beta') \\ \text{or} \\ (\alpha,\beta) = (\alpha',\beta') \text{ et } k < k' \end{cases}$$

Now apply $\pi(\mathrm{Exp}_{<^h}(h(I))) = \mathrm{Exp}_{<}(I)$, to the order $< = <_L$.

Theorem 3.6 $\{\mathrm{gr}^L(I) \,|\, L \in U\}$ *is a finite set.*

Proof. Let $h(I)$ be the homogenized ideal of I in $A_n[t]$. The associated graded ideal $\mathrm{gr}^L(h(I))$ is an ideal of the ring $\mathrm{gr}^L(A_n[t]) \simeq (\mathrm{gr}^L(A_n))[t]$ (where $[x_i,\xi_i] = 0$ if $e_i + f_i > 0$ and $[D_i, x_i] = t^2$ if $e_i + f_i = 0$). Let

$$\phi : A_n[t] \longmapsto A_n, \phi(H) = H_{|t=1}$$

denote the deshomogenization morphism. If $L \in U$, ϕ gives rise to a morphism

$$\phi_L : \mathrm{gr}^L(A_n[t]) \longmapsto \mathrm{gr}^L(A_n) \simeq \mathrm{gr}^L(A_n[t])/(t-1).$$

Clearly $\phi(h(I)) = I$, on the other hand, for all $P \in I, \phi_L(\sigma^L(h(P))) = \sigma^L(P)$, in particular $\phi_L(\mathrm{gr}^L(h(I)) = \mathrm{gr}^L(I)$. Now apply Theorem 3.3 (remark that ϕ_L does not depend on L as far as we may identify different $\mathrm{gr}^L(A_n)$).

Finally we shall give, using Proposition 3.4., a bound for the cardinality of $\{\mathrm{Exp}_{<_L}(I) \mid L \in U\}$. Let $d(h(I))$ denote the maximal degree of the elements arising in the minimal boundaries of $\mathrm{Exp}_{<^h_L}(h(I)), L \in U$. If $(d_1, d_2, \dots)$ denote the set of values of the Hilbert function of $h(I)$, then we have:

Proposition 3.7

$$\sharp\{\mathrm{Exp}_{<_L}(I) \mid L \in U\} \leq \prod_{k=1}^{d(h(I))} C^{a_k}_{a_k - d_k},$$

where:

$$a_k = \dim_{\mathbf{k}} A_n[t]_k = C^{2n+k}_k.$$

4 The Gröbner fan

Let I be a non-zero left ideal of A_n and let $h(I)$ be the homogenized ideal of I in $A_n[t]$. The purpose of this section is to study the stability of $\mathrm{gr}^L(h(I))$ when L varies in U. For all $E \subseteq \mathbf{N}^{1+2n}$ such that $E + \mathbf{N}^{1+2n} = E$, we set:

$$U_E = \{L \in U; \mathrm{Exp}_{\prec_L}(h(I)) = E\},$$

With these notations we have the following:

Theorem 4.1 *There exists a partition $\mathcal{E}$ of U into convex rational polyhedral cones, such that for all element $\sigma \in \mathcal{E}$, $\mathrm{gr}^L(I)$ and $\mathrm{Exp}_{\prec_L}(I)$ do not depend on $L \in \sigma$. This partition is exactly the partition into the set on which both $\mathrm{gr}^L(h(I))$ and $\mathrm{Exp}_{\prec_L}(h(I))$ are fixed. Furthermore, every U_E is convex and a union of cones of $\mathcal{E}$.*

In order to prove our Theorem, we shall fix some notations and give some preliminary results. Let E be a subset of $\mathbf{N}^{2n+1}$ such that $E + \mathbf{N}^{2n+1} = E$ and let $L \in U_E$. Then consider a reduced standard basis $Q_1, \dots, Q_r$ of $h(I)$ w.r.t. $\prec_L$. As in the proof of Corollary 3.3., we can see that $Q_1, \dots, Q_r$ is also a reduced standard basis w.r.t. $\prec_{L'}$, for all $L' \in U_E$. Denote by $\sim$ the equivalence relation on U defined from $Q_1, \dots, Q_r$ by :

$$L \sim L' \Longleftrightarrow \sigma^L(Q_k) = \sigma^{L'}(Q_k) \quad \text{for} \quad \text{all} \quad k = 1, \dots, r.$$

Lemma 4.2 *$\sim$ defines on U a finite partition into convex rational polyhedral cones and U_E is a union of a part of these cones.*

Proof. Let $L_1, L_2 \in U$ such that $L_1 \sim L_2$ and let $L \in [L_1, L_2]$, also let $\theta \in [0,1]$ such that $L = \theta \cdot L_1 + (1-\theta) \cdot L_2$. Write for all $1 \leq k \leq r$, $Q_k = \sigma^{L_1}(Q_k) + R_k = \sigma^{L_2}(Q_k) + R_k$. Since for all $(\alpha, \beta) \in \mathbf{N}^{2n}$, $L(\alpha,\beta) = \theta \cdot L_1(\alpha,\beta) + (1-\theta) \cdot L_2(\alpha,\beta)$, then $\sigma^L(Q_k) = \sigma^{L_1}(Q_k) = \sigma^{L_2}(Q_k)$ by an immediate verification. On the other hand, if $L_1 \sim L_2$ and $L_1 \in U_E$, then $L_2 \in U_E$. This proves that U_E is a union of classes for $\sim$.

Proof of Theorem 4.1: We define $\mathcal{E}$ as follows: for each E we consider the restriction $\mathcal{E}_E$ to U_E of the above partition and then $\mathcal{E}$ is the finite union of the $\mathcal{E}_E$'s. On each cone of the partition, $\mathrm{gr}^L(h(I))$ and $\mathrm{Exp}_{\prec_L}(h(I))$ are fixed, and the same is true for $\mathrm{gr}^L(I)$ and $\mathrm{Exp}_{\prec_L}(I)$ because of the proof of the Theorems 3.5 and 3.6. Conversely, assume that L, L' are in the same U_E. The ideal $\mathrm{gr}^L(h(I)$ has the same E as set of priveleged exponents with respect to $\prec_L$ and $\sigma^L(Q_k), k = 1, \ldots, r$ as a reduced standard basis (let us point out that here we use the fact that $h(I)$ is homogeneous with respect to the total degree, and therefore the reduced standard basis $Q_1, \ldots, Q_r$ is also homogeneous). Therefore if $\mathrm{gr}^L(h(I)) = \mathrm{gr}^{L'}(h(I))$, we obtain $\sigma^L(Q_k) = \sigma^{L'}(Q_k)$ by the unicity of the reduced standard basis. This ends the proof of the theorem except for the convexity of U_E proved below:

Lemma 4.3 *U_E is a convex set: If $L_1, L_2 \in U_E$, then $[L_1, L_2] \subseteq U_E$.*

Proof. Let $L \in]L_1, L_2[$ and let $\theta \in]0,1[$ such that $L = \theta \cdot L_1 + (1-\theta) \cdot L_2$. For all $1 \leq k \leq r$, if $\exp_{\prec_{L_1}}(Q_k) = \exp_{\prec_{L_2}}(Q_k) = \exp_{\prec_{L_1}}(q_k(t)\underline{x}^\alpha \underline{D}^\beta)$, and if $Q_k = q_k(t)\underline{x}^\alpha \underline{D}^\beta + R_k$, then either $R_k = 0$, or $\mathrm{ord}_L(q_k(t)\underline{x}^\alpha \underline{D}^\beta) \geq \mathrm{ord}_L(R_k)$. Furthermore, if $\mathrm{ord}_L(q_k(t)\underline{x}^\alpha \underline{D}^\beta) = \mathrm{ord}_L(R_k)$, then $\mathrm{ord}_{L_i}(q_k(t)\underline{x}^\alpha \underline{D}^\beta) = \mathrm{ord}_{L_i}(R_k)$ for at least one $1 \leq i \leq 2$. In particular $\exp_{\prec_L}(Q_k) = \exp_{\prec_{L_1}}(q_k(t)\underline{x}^\alpha \underline{D}^\beta)$, which implies that $E \subseteq \mathrm{Exp}_{\prec_L}(h(I))$, and consequently, by division, that $E = \mathrm{Exp}_{\prec_L}(h(I))$, i.e. $L \in U_E$.

In the following we give some precisions about the partition:

Definition 4.4 *We say that $\mathrm{gr}^L(h(I))$ is a multihomogeneous ideal if we are in the commutative case, and $\mathrm{gr}^L(h(I))$ is homogeneous with respect to $\underline{x}, \underline{\xi}$, that is generated by monomials in $\underline{x}, \underline{\xi}$ (Since $h(I)$ is itself homogeneous, $\mathrm{gr}^L(h(I))$ is in fact generated by monomials in $t, \underline{x}, \underline{\xi}$).*

Proposition 4.5 *The set of $L \in U$ for which $\mathrm{gr}^L(h(I))$ are multihomogeneous ideals defines the open cones of dimension $2n$ of $\mathcal{E}$ (contained by construction in the part $e_i + f_i > 0$ of U).*

Proof. Let $Q_1, \ldots, Q_r$ be a reduced standard basis of $h(I)$ with respect to $\prec_L$ and let $V(L)$ be an open neighbourhood of L such that for all $L' \in V(L)$ and for all $1 \leq k \leq r$, $\sigma^{L'}(Q_k) = \sigma^L(Q_k)$. In particular, $\exp_{\prec_L}(Q_k) = \exp_{\prec_{L'}}(Q_k)$, for all $1 \leq k \leq r$. This proves that $E \subseteq \mathrm{Exp}_{\prec_{L'}}(h(I))$, and consequently that $E = \mathrm{Exp}_{\prec_{L'}}(h(I))$. Finally $V(L) \subseteq U_E$. This proves our assertion. Conversely, if L is in an open cone of U_E, then for any L' in a neighborhood of L and for all k, $\sigma^{L'}(Q_k) = \sigma^L(Q_k)$. This implies that $\sigma^L(Q_k)$ is a monomial.

Definition 4.6 *$\mathcal{E}$ is called the standard fan of $h(I)$, or, with the notations of [14], the extended Gröbner fan of I.*

Remarks 4.7 i) Let $U' \subseteq U$ be the set of linear forms $L : \mathbf{R}^{2n} \longrightarrow \mathbf{R}$ with coefficients in $\mathbf{R}_+$. Consider on $\mathbf{N}^{2n}$ the set of -total well ordering- $<_L, L \in U'$, defined as in paragraph 3.3. Then the notion of reduced standard basis of I w.r.t. $<_L$ is well defined in this case. With these notations the result of Theorem 4.1 holds if we consider the set of $\mathrm{gr}^L(I), L \in U'$. The associated fan $\mathcal{E}'$ is called the restricted Gröbner fan of I. Obviously $\mathcal{E}' = \pi'(\mathcal{E}_{U'})$, where $\mathcal{E}_{U'}$ denote the restriction of $\mathcal{E}$ on U' and π' denote the natural projection.

ii) The result of Theorem 4.1 holds for the set of $gr^L(I), L \in U$ because of the relationship between $gr^L(h(I))$ and $gr^L(I)$ given in the proof of 3.6. Nevertheless, the analogous set of U_E with respect to I is not necessarily a convex set, as it can be shown in the following example: Consider in $A_2(\mathbf{C})$ the left ideal generated by $P_1 = x_1 - x_1^2, P_2 = x_1 D_2 - x_1$. If $e_1 > 0, f_2 > 0$, then $E_L(I)$ is generated by $(2,0,0,0)$ and $(0,0,0,1)$, whereas $E_L(I)$ is generated by $(1,0,0,0)$ elsewhere.

iii) The Gröbner fan as introduced in 4.6 can be refined in order to satisfy the boundary conditions (if two strata satisfy $C_i \bigcap \overline{C_j} \neq \emptyset$, then $C_i \subset \overline{C_j}$). We can use to this end the universal standard basis of $h(I)$ which is the union of all reduced standard bases of $h(I)$.

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