Fixed points, selections and common fixed points for nonexpansive-type mappings

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Abstract

We study the existence of fixed points in the context of uniformly convex geodesic metric spaces, hyperconvex spaces and Banach spaces for single and multivalued mappings satisfying conditions that generalize the concept of nonexpansivity. Besides, we use the fixed point theorems proved here to give common fixed point results for commuting mappings.

Key-words: fixed point, selection of multifunctions, generalized nonexpansive mappings, commuting mappings, metric space, Banach space.

1 Introduction

In [26], T. Suzuki extends the concept of singlevalued nonexpansive mapping in the following way: a mapping f defined on a subset K of a Banach space is said to satisfy condition (C) if for $x, y \in K$ with $(1/2)\|x - f(x)\| \leq \|x - y\|$, then $\|f(x) - f(y)\| \leq \|x - y\|$. T. Suzuki [26] proves some basic properties and gives fixed point theorems and convergence results for mappings satisfying condition (C). Following [26], A. Razani and H. Salahifard [23] state part of T. Suzuki's [26] results in the context of a complete CAT(0) space and generalize condition (C) to the multivalued case: a multivalued mapping T defined on subset of a CAT(0) space is said to satisfy condition (C) if for each $x, y \in K$ and $u_x \in T(x)$ with $(1/2)d(x, u_x) \leq d(x, y)$ there exists $u_y \in T(y)$ such that $d(u_x, u_y) \leq d(x, y)$. This condition is used in [23] to prove a fixed point theorem for multivalued mappings and some common fixed point results. Motivated by the results in [26], J. García-Falset, E. Llorens-Fuster and T. Suzuki consider in [7] two generalizations in the singlevalued case of condition (C) giving examples and establishing fixed point results.

The purpose of this paper is to study condition (C) for multivalued mappings in the context of geodesic metric spaces (with special attention to the case of $\mathbb{R}$-trees) and Banach spaces, and condition (C) for singlevalued mappings in the context of hyperconvex spaces. After some preliminary contents in Section 2, we begin Section 3 by studying the multivalued case in geodesic spaces. We assume condition (C) for multivalued mappings as in [23] where different results in this direction were obtained for CAT(0) spaces. In our work, we derive a technical lemma (Lemma 3.2) which is a multivalued version of the key fact which is behind the main results in [7, 26]. Our results are first obtained for as general as complete uniformly convex geodesic spaces and then particularized for more precise geometries. Since CAT(0) spaces are a particular class of uniformly convex geodesic spaces, we obtain more general results than those from [23]. Moreover, thanks mainly to Lemma 3.2, we fill in a gap in the proof of the main multivalued result in [23]. We continue Section 3 by introducing a new condition for multivalued mappings in the spirit of (C). We give examples showing that this condition is actually weaker than condition (C) and prove a selection theorem in $\mathbb{R}$-trees for mappings satisfying this newly introduced condition from where a stronger fixed point result for multivalued mappings follows. This selection result resembles a very important one, see for instance [12, 25], for hyperconvex spaces (notice, see [14], that complete $\mathbb{R}$-trees are hyperconvex) although the approach here is completely different as the proof relies on very particular properties of $\mathbb{R}$-trees rather than on hyperconvexity. It is worthwhile to point out that $\mathbb{R}$-trees find a lot of applications in different areas as, for instance, the indexing of information or phylogenetics. We close Section 3 with an appendix where we study the existence of fixed points for singlevalued mappings with property (C) in hyperconvex metric spaces. It is very well-known (see [17, Chapter 13]) that nonexpansive self-mappings defined on nonempty bounded and closed hyperconvex spaces have fixed points. Therefore it is natural to wonder about this problem for mappings with condition (C). We first study the compact case providing a positive answer. For the more general case we need to introduce a new condition on the mapping under consideration. In particular it is shown that a 2-lipschitzian self-mapping with condition (C) defined on a nonempty closed and bounded hyperconvex space has a fixed point. This result is significant among the class of known results for mappings with condition (C) since it is the first one without compactness conditions for which neither the uniqueness of asymptotic centers nor anything similar to the Opial property is required (see Sections 2 and 4 for definitions). Therefore, this result follows through a completely new approach compared to those in [7, 23, 26] and implies new results even, for instance, in injective Banach spaces.

In Section 4 we revisit the classical theory of nonexpansive multivalued mappings on Banach spaces to study it under condition (C). We show the existence of fixed points for such a mapping in a Banach space with the Opial property. The method of asymptotic centers allows us to establish the same result in a uniformly convex in every direction (UCED) Banach space. Moreover, if we also assume the continuity of the mapping we can prove the existence of fixed points in a Banach space for which the asymptotic center of a bounded sequence with respect to a bounded closed convex subset is nonempty and compact, that is, a counterpart of the Kirk-Massa theorem. Finally, in Section 5, we appeal to the fixed point theorems proved in this paper in order to give some common fixed point results for commuting mappings.

2 Preliminaries

Let (X,d) be a metric space. A *geodesic path* from x to y is a mapping $c : [0,l] \subseteq \mathbb{R} \to X$ with $c(0) = x, c(l) = y$ and $d\left(c(t), c(t')\right) = |t - t'|$ for every $t, t' \in [0,l]$. The image $c\left([0,l]\right)$ of c forms a *geodesic segment* which joins x and y and is not necessarily unique. If no confusion arises, we will use $[x,y]$ to denote a geodesic segment joining x and y. (X,d) is a *(uniquely) geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x,y]$ if and only if there exists $t \in [0,1]$ such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$, and we will write $z = (1-t)x + ty$ for simplicity. A *subset K of X is convex* if it contains any geodesic segment that joins every two points of it.

In a geodesic space (X,d), the *metric $d : X \times X \to \mathbb{R}$ is convex* if for any $x, y, z \in X$ one has

$$d(x,(1-t)y + tz) \leq (1-t)d(x,y) + td(x,z) \text{ for all } t \in [0,1].$$

A geodesic space which metric is convex will be referred as a *space with convex metric.* A trivial example of a uniquely geodesic space with convex metric is a strictly convex Banach space. For more details about geodesic metric spaces one may check [2].

A geodesic space (X,d) is *uniformly convex* if for any $r > 0$ and $\epsilon \in (0,2]$ there exists

$\delta \in (0,1]$ such that if $a, x, y \in X$ with $d(x,a) \leq r$, $d(y,a) \leq r$ and $d(x,y) \geq \epsilon r$ then

$$d(\frac{1}{2}x + \frac{1}{2}y, a) \leq (1-\delta)r.$$

From the definition, it is easy to see that uniformly convex metric spaces are uniquely geodesic.
A mapping $\delta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta = \delta(r,\epsilon)$ for a given $r > 0$ and $\epsilon \in (0,2]$ is called a *modulus of uniform convexity*. The mapping δ is *monotone* (resp. *lower semi-continuous from the right*) if for every fixed ϵ it decreases (resp. is lower semi-continuous from the right) with respect to r (see also [5], [18]). CAT(0) spaces in the sense of Gromov (see [2]) are uniformly convex metric spaces with convex metric.
Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. For $x \in X$, define $r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n)$. The *asymptotic radius* of $(x_n)_{n\in\mathbb{N}}$ is given by

$$r((x_n)) = \inf \{r(x,(x_n)) : x \in X\},$$

and the *asymptotic center* of $(x_n)_{n\in\mathbb{N}}$ is the set

$$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}.$$

Throughout this paper we will denote a uniformly convex metric space with monotone (or lower semi-continuous from the right) modulus of uniform convexity as a UC space. In [5], the authors prove that every bounded sequence in a UC space has a unique asymptotic center.
A bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a complete UC space is regular if $r((x_n)) = r((x_{n_k}))$ for every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. It is known that in a Banach space every bounded sequence contains a regular subsequence (see, for instance, [17], Chapter 2, Lemma 5.2). Since the proof has a metric nature we can conclude that every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a complete UC space has a regular subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and thus every subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ has the same asymptotic center as $(x_{n_k})_{k\in\mathbb{N}}$.

Let (X,d) be a metric space. Taking $z \in X$ and $r > 0$ we denote the closed ball centered at z with radius r by $\widetilde{B}(z,r)$. Given Y a nonempty subset of X, we define the *distance of a point* $z \in X$ to Y by $\mathrm{dist}(z,Y) = \inf_{y\in Y} d(z,y)$. The *metric projection* (or *nearest point mapping*) P_Y onto Y is the mapping

$$P_Y(z) = \{y \in Y : d(z,y) = \mathrm{dist}(z,Y)\}, \text{ for every } z \in X.$$

If Y is additionally bounded, the *diameter* of Y is given by $\mathrm{diam}Y = \sup_{x,y\in Y} d(x,y)$.

In this paper we also consider the following families of sets:

$$P(X) = \{Y \subseteq X : Y \text{ is nonempty}\},$$

$$P_b(X) = \{Y \subseteq X : Y \text{ is nonempty and bounded}\},$$

$$P_{b,cv}(X) = \{Y \subseteq X : Y \text{ is nonempty, bounded and convex}\},$$

$$P_{cl,cv}(X) = \{Y \subseteq X : Y \text{ is nonempty, closed and convex}\},$$

$$P_{b,cl,cv}(X) = \{Y \subseteq X : Y \text{ is nonempty, bounded, closed and convex}\},$$

$$P_{cp}(X) = \{Y \subseteq X : Y \text{ is nonempty and compact}\},$$

$$P_{cp,cv}(X) = \{Y \subseteq X : Y \text{ is nonempty, compact and convex}\}.$$

A metric space (X, d) is *metrically convex* if for any two distinct points $x, y \in X$ and any $\alpha, \beta > 0$ such that $d(x, y) = \alpha + \beta$ there exists $z \in X$ with $d(x, z) = \alpha$ and $d(y, z) = \beta$. X has the *binary intersection property* if $\bigcap_{i\in I} \widetilde{B}_i \neq \emptyset$ for every collection of balls $(\widetilde{B}_i)_{i\in I}$ such that any two of these balls intersect.
A metric space (X, d) is *hyperconvex* if $\bigcap_{i\in I} \widetilde{B}(x_i, r_i) \neq \emptyset$ for every collection of points $(x_i)_{i\in I}$ in X and positive numbers $(r_i)_{i\in I}$ such that $d(x_i, x_j) \leq r_i + r_j$ for any $i, j \in I$. Hyperconvexity is equivalent to the binary intersection property and the metric convexity. More about hyperconvex spaces can be found in [1, 12, 25] or in Chapter 13 of [17].

Given (X, d) a metric space and $A \subseteq X$, the number $r_x(A) = \sup_{y\in A} d(x, y)$ is called the *radius of* A *relative to* $x \in X$. The *radius of* A is $r(A) = \inf_{x\in X} r_x(A)$, the *center of* A is the set $C(A) = \{x \in X : r_x(A) = r(A)\}$ and the *admissible cover of* A is defined by $\operatorname{cov}(A) = \bigcap\{\widetilde{B} : \widetilde{B} \text{ is a closed ball and } A \subseteq \widetilde{B}\}$. The set A is said to be *admissible* if $A = \operatorname{cov}(A)$. For X a hyperconvex space and $A \subseteq X$, $\operatorname{cov}(A) = \bigcap_{x\in X} \widetilde{B}(x, r_x(A))$ and $\operatorname{diam}(A) = 2r(A)$ (for details see Chapter 13 of [17]).

An $\mathbb{R}$-*tree* is a uniquely geodesic metric space X such that if $[y, x] \cap [x, z] = \{x\}$ then $[y, x] \cup [x, z] = [y, z]$ for each $x, y, z \in X$. From the definition it immediately follows that if $x, y, z \in X$, then $[x, y] \cap [x, z] = [x, w]$ for some $w \in X$. Likewise, if K is a closed and convex subset of an $\mathbb{R}$-tree X, then for every $x \in X$, $P_K(x)$ is a singleton and for any $y \in K$, $d(x, y) = d(x, P_K(x)) + d(P_K(x), y)$. A standard example of an $\mathbb{R}$-tree is $\mathbb{R}^2$ endowed with the so-called *river metric*. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, the river metric (denoted by ρ) is defined by

$$\rho(x, y) = \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1, \\ |x_2| + |y_2| + |x_1 - y_1| & \text{otherwise.} \end{cases}$$

It is known that $\mathbb{R}$-trees are CAT(0) spaces and that a metric space is a complete $\mathbb{R}$-tree if and only if it is hyperconvex and has unique geodesic segments (see [14]). More about the fixed point theory in $\mathbb{R}$-trees can be found in [4, 15, 21, 22].

In [26], T. Suzuki considered the following generalized family of nonexpansive mappings in the setting of a Banach space. We will use in the sequel the norm notation, but the same definitions also hold when working in the metric setting (naturally, the norm will be replaced by the distance).

Definition 2.1. *Let X be a Banach space, $K \in P(X)$ and $f : K \to X$. Then f satisfies condition (C) if*

$$\frac{1}{2}\|x - f(x)\| \leq \|x - y\| \Longrightarrow \|f(x) - f(y)\| \leq \|x - y\|,$$

for all $x, y \in K$.

Obviously, every nonexpansive mapping meets condition (C). We next summarize some of the basic properties proved in [26] in relation to these mappings. The proofs of these results are metric in nature so the properties also apply in the metric case. Throughout this paper we denote the set of fixed points of a mapping f by $\operatorname{Fix}(f)$.

Lemma 2.2. *Let X be a Banach space and $K \in P(X)$. Assume that the mapping $f : K \to X$ satisfies condition (C). Then for each $x, y \in K$,*

(i) *if $z \in \mathrm{Fix}(f)$, then $\|z - f(x)\| \leq \|z - x\|$, that is, f is quasinonexpansive;*

(ii) $\|f(x) - f(y)\| \leq \|x - y\|$ *or* $\|f^2(x) - f(y)\| \leq \|f(x) - y\|$;

(iii) $\|x - f(y)\| \leq 3\|f(x) - x\| + \|x - y\|$.

Using these properties, T. Suzuki [26] proves fixed point theorems for mappings satisfying condition (C).

In [7], the authors study two generalizations of condition (C) giving examples and establishing fixed point results. One of these conditions is the following.

Definition 2.3. *Let X be a Banach space, $K \in P(X)$, $f : K \to X$ and $\mu \geq 1$. The mapping f satisfies condition (E_μ) if for all $x, y \in K$,*

$$\|x - f(y)\| \leq \mu\|f(x) - x\| + \|x - y\|.$$

Lemma 2.2, (iii) yields that condition (C) implies (E_3), but Example 3 of [7] shows that (E_3) does not imply (C). Other examples for different values of μ are studied in [7].

In the next sections we will make use of the lemma below which is a special case of Proposition 2 in [9].

Lemma 2.4. *Let X be a geodesic metric space with convex metric, $\alpha \in (0, 1)$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ bounded sequences in X such that $x_{n+1} = (1 - \alpha)x_n + \alpha y_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

The following two theorems were proved in [23], but in the setting of a complete CAT(0) space. It is easy to see that these results hold in more general contexts. We will formulate the first result in the framework of a uniquely geodesic metric space.

Theorem 2.5. *Let X be a uniquely geodesic metric space and $K \in P_{cl,cv}(X)$. Suppose $f : K \to K$ satisfies condition (C) and $\mathrm{Fix}(f) \neq \emptyset$. Then $\mathrm{Fix}(f)$ is closed and convex.*

The proof of the second theorem only requires the uniqueness of the asymptotic center and the convexity of the metric. This is why we state this result under the hypothesis of a complete UC space with convex metric.

Theorem 2.6. *Let X be a complete UC space with convex metric and suppose $K \in P_{b,cl,cv}(X)$. If $f : K \to K$ satisfies condition (C) then $\mathrm{Fix}(f)$ is nonempty, closed and convex.*

In [23], the authors also extend Suzuki's [26] condition (C) to the multivalued case in the following way.

Definition 2.7. *Let X be a metric space and $K \in P(X)$. A mapping $T : K \to P(X)$ is said to satisfy condition (C) if for each $x, y \in K$ and $u_x \in T(x)$ such that*

$$\frac{1}{2}d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

The above condition is used in [23] to give a fixed point theorem for multivalued mappings and some common fixed point results.

In the rest of this paper we use condition (C) for both single and multivalued mappings with the context distinguishing between the two cases. The same also holds for other conditions we make use of.

3 Fixed points and selections in geodesic spaces

In this section we study the multivalued version of mappings with condition (C) in geodesic metric spaces. Following the singlevalued case, we introduce the next condition and prove that for $\mu = 3$ it is a generalization of condition (C).

Definition 3.1. *Let X be a metric space, $K \in P(X)$, $T : K \to P(X)$ and $\mu \geq 1$. The mapping T satisfies condition (E_μ) if for each $x, y \in K$ and $u_x \in T(x)$ there exists $u_y \in T(y)$ such that*

$$d(x, u_y) \leq \mu d(x, u_x) + d(x, y).$$

We prove next that a multivalued mapping which satisfies condition (C) also satisfies (E_3). This property will constitute a key tool in proving our results.

Lemma 3.2. *Let X be a metric space, $K \in P(X)$ and let $T : K \to P(K)$ satisfy condition (C). Then T satisfies condition (E_3).*

Proof. Let $x, y \in K$ and $u_x \in T(x)$. Because $(1/2)d(x, u_x) \leq d(x, u_x)$ there exists $v_x \in T(u_x)$ such that

$$d(u_x, v_x) \leq d(x, u_x). \tag{1}$$

We prove that either

$$\frac{1}{2}d(x, u_x) \leq d(x, y) \tag{2}$$

or

$$\frac{1}{2}d(u_x, v_x) \leq d(u_x, y) \tag{3}$$

holds. Suppose $(1/2)d(x, u_x) > d(x, y)$ and $(1/2)d(u_x, v_x) > d(u_x, y)$. Then, using (1) we obtain the following contradiction

$$d(x, u_x) \leq d(x, y) + d(y, u_x) < \frac{1}{2}d(x, u_x) + \frac{1}{2}d(u_x, v_x) \leq d(x, u_x).$$

Hence, if (2) holds, then there exists $u_y \in T(y)$ such that $d(u_x, u_y) \leq d(x, y)$, so

$$d(x, u_y) \leq d(x, u_x) + d(u_x, u_y) \leq d(x, u_x) + d(x, y).$$

If (3) holds, then there exists $u_y \in T(y)$ such that $d(v_x, u_y) \leq d(u_x, y)$. Using again (1) we have that

$$d(x, u_y) \leq d(x, u_x) + d(u_x, v_x) + d(v_x, u_y) \leq 2d(x, u_x) + d(u_x, y) \leq 3d(x, u_x) + d(x, y).$$

Thus, the inequality holds in each of the two cases and we are done. □

Definition 3.3. *Let X be a metric space, $K \in P(X)$ and $T : K \to P(X)$. We say that $(x_n)_{n\in\mathbb{N}} \subseteq K$ is an approximate fixed point sequence for the mapping T if for each $n \in \mathbb{N}$ there exists $y_n \in T(x_n)$ such that* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

The next result provides an approximate fixed point sequence for a multivalued mapping satisfying condition (C). We use this result in the rest of the paper because many of our proofs rely on it.

Proposition 3.4. *Let X be geodesic metric space with convex metric, $K \in P_{b,cv}(X)$ and $T : K \to P(K)$. If T satisfies condition (C), then T has an approximate fixed point sequence.*

Proof. Let $x_1 \in K$, $y_1 \in T(x_1)$ and take $x_2 = (1/2)x_1 + (1/2)y_1$. Then $(1/2)d(x_1, y_1) = d(x_1, x_2)$ so, by condition (C), there exists $y_2 \in T(x_2)$ such that $d(y_1, y_2) \leq d(x_1, x_2)$. Continuing in this vein, we can build the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in T(x_n)$, $x_{n+1} = (1/2)x_n + (1/2)y_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for every $n \in \mathbb{N}$. Using Lemma 2.4 we obtain that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

□

Our first fixed point result for multivalued mappings is given for self-mappings on a compact set.

Theorem 3.5. *Let X be a geodesic space with convex metric and $K \in P_{cp,cv}(X)$. Suppose $T : K \to P_{cl}(K)$ satisfies condition (C). Then* $\mathrm{Fix}(T) \neq \emptyset$.

Proof. By Proposition 3.4, there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in K such that $y_n \in T(x_n)$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$. Since K is compact, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges to some $x \in K$.
Using Lemma 3.2, we have that for all $k \in \mathbb{N}$

$$\mathrm{dist}(x_{n_k}, T(x)) \leq 3d(x_{n_k}, y_{n_k}) + d(x_{n_k}, x).$$

Taking the limit as $k \to \infty$ we obtain that $\mathrm{dist}(x, T(x)) = 0$. Since $T(x)$ is closed it follows that $x \in T(x)$. □

In the following theorem we move the compactness condition from the domain to the images of the mapping. This theorem is actually an extension of Theorem 3.2 of [23] in the context of a complete UC space with convex metric. We also remove the convexity condition on the image sets of the mapping. Moreover, we obtain our results in a simple way as a consequence of Lemma 3.2 which avoids to go through a delicate point in the proof of Theorem 3.2 of [23].

Theorem 3.6. *Let X be a complete UC space with convex metric and $K \in P_{b,cl,cv}(X)$. Suppose $T : K \to P_{cp}(K)$ satisfies condition (C). Then* $\mathrm{Fix}(T) \neq \emptyset$.

Proof. By Proposition 3.4, we can find the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in K such that $y_n \in T(x_n)$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$. As explained in Section 2, we may suppose that $(x_n)_{n\in\mathbb{N}}$ is regular (otherwise choose a regular subsequence of it). Denote the unique asymptotic center of $(x_n)_{n\in\mathbb{N}}$ by x. Let $n \in \mathbb{N}$. Applying Lemma 3.2 for x_n, x and y_n respectively it follows that there exists $z_n \in T(x)$ such that

$$d(x_n, z_n) \leq 3d(x_n, y_n) + d(x_n, x).$$

Let $(z_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(z_n)_{n\in\mathbb{N}}$ that converges to some $z \in T(x)$. Then, for each $k \in \mathbb{N}$,

$$d(x_{n_k}, z) \leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, z) \leq 3d(x_{n_k}, y_{n_k}) + d(x_{n_k}, x) + d(z_{n_k}, z).$$

Taking the superior limit as $k \to \infty$ and knowing that the asymptotic center of $(x_{n_k})_{k\in\mathbb{N}}$ is precisely x we obtain that $x = z \in T(x)$. Hence, the proof is complete. $\square$

Remark 3.7. From the above proof it is immediate that in Theorem 3.6 we can drop the convexity of the metric and assume instead that the mapping admits an approximate fixed point sequence.

In the next result we will consider the following new condition for multivalued mappings which will be shown to be weaker than condition (C).

Definition 3.8. *Let X be a metric space, $K \in P(X)$ and $T : K \to P(X)$. The mapping T satisfies condition (C') if for each $x, y \in K$ and $u_x \in T(x)$ with*

$$d(x, u_x) = \operatorname{dist}(x, T(x)) \text{ and } \frac{1}{2}d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

We prove next a selection theorem in $\mathbb{R}$-trees for multivalued mappings satisfying condition (C') and analyze afterwards the relation of (C') to (C) and (E_3) respectively.

Theorem 3.9. *Let X be an $\mathbb{R}$-tree, $K \in P(X)$ and $T : K \to P_{cl,cv}(X)$ a mapping which satisfies (C'). Then the mapping $f : K \to X$ defined by $f(x) = P_{T(x)}(x)$ for each $x \in K$ is a selection of T that satisfies condition (C).*

Proof. Notice that the properties of $\mathbb{R}$-trees (see Section 2) guarantee that f is well-defined. Let $x, y \in K$ such that $f(x) \neq f(y)$ and $(1/2)d(x, f(x)) \leq d(x, y)$. Consider $p(x) = P_{T(y)}(f(x))$ and $p(y) = P_{T(x)}(f(y))$.
First, suppose $p(x) \neq f(y)$ and $p(y) \neq f(x)$. Since $p(x)$ is the projection of $f(x)$ onto $T(y)$ it follows that

$$d(f(x), f(y)) = d(f(x), p(x)) + d(p(x), f(y)),$$

i.e., $p(x) \in [f(x), f(y)]$. Since $T(y)$ is convex, $[p(x), f(y)] \subseteq T(y)$. This implies $[f(x), f(y)] \cap [f(y), y] = \{f(y)\}$ because otherwise the minimality of $f(y)$ would be contradicted. Thus, $f(y) \in [f(x), y]$. Similarly, $f(x) \in [f(y), x]$. Then $f(x), f(y) \in [x, y]$ (otherwise supposing for example that $z \in [x, f(y)] \cap [f(y), y]$ with $z \neq f(y)$ we have that $f(x) \in [z, f(y)]$ and $f(y) \in [z, f(x)]$ which is false). Therefore, $d(f(x), f(y)) \leq d(x, y)$. In fact, $d(f(x), f(y)) = d(x, y) - \operatorname{dist}(x, T(x)) - \operatorname{dist}(y, T(y))$.
Now assume $p(x) = f(y)$. Then $d(f(x), f(y)) = \operatorname{dist}(f(x), T(y))$ and so, by condition (C'),

$$d(f(x), f(y)) = \operatorname{dist}(f(x), T(y)) \leq d(x, y).$$

Finally, suppose $p(x) \neq f(y)$ and $p(y) = f(x)$. As above, if $p(x) \neq f(y)$, we have that $f(y) \in [f(x), y]$. If $(1/2)d(y, f(y)) \leq d(x, y)$ then (C') yields that

$$d(f(x), f(y)) = \operatorname{dist}(f(y), T(x)) \leq d(x, y).$$

Otherwise, if $(1/2)d(y, f(y)) > d(x, y)$, then

$$\begin{aligned} d(f(x), f(y)) + 2d(x, y) &< d(f(x), f(y)) + d(f(y), y) = d(f(x), y) \leq d(f(x), x) + d(x, y) \\ &\leq 2d(x, y) + d(x, y). \end{aligned}$$

Consequently, $d(f(x), f(y)) \leq d(x, y)$. This completes the proof. □

Remark 3.10. Notice the similarity of the statement of this selection result with the classical selection results on hyperconvex spaces for multivalued nonexpansive mappings with admissible values (see [12, 25]).

Since complete $\mathbb{R}$-trees are CAT(0) spaces, using the above result and Theorem 2.6 we obtain the following consequence which, as we will show below, is an improvement of Theorem 3.6 for $\mathbb{R}$-trees.

Corollary 3.11. *Let X be a bounded complete $\mathbb{R}$-tree. Suppose $T : X \to P_{cl,cv}(X)$ satisfies condition (C'). Then $\operatorname{Fix}(T)$ is a nonempty complete $\mathbb{R}$-tree.*

Proof. Applying Theorem 2.6 to the selection f provided by Theorem 3.9, we obtain that $\operatorname{Fix}(f)$ is nonempty and convex (an so an $\mathbb{R}$-tree). Noticing that $\operatorname{Fix}(f) = \operatorname{Fix}(T)$ it is now clear that the conclusion follows. □

We study now the relations between conditions (C), (C') and (E_3).

Proposition 3.12. *Let K be a bounded, closed and convex subset of a complete $\mathbb{R}$-tree and $T : K \to P_{cl,cv}(K)$. The following hold:*

(i) *if T satisfies (C), then it also satisfies (C'), but the converse does not hold;*

(ii) *if T satisfies (C'), then it also satisfies (E_3), but the converse is false.*

Proof. Clearly, (C) implies (C'). To show that (C') does not imply (C) consider $\mathbb{R}^2$ with the river metric. Let

$$K = \{\{0\} \times [-9, 3]\} \cup \{[0, 2] \times \{0\}\} \cup \{\{2\} \times [-1, 0]\}.$$

and define $T : K \to P_{cl,cv}(K)$ by

$$T(x, y) = \begin{cases} \{(0, -3)\} & \text{if } x = 0 \text{ and } y \in [-9, -3], \\ \{(0, y)\} & \text{if } x = 0 \text{ and } y \in (-3, 0], \\ \{(0, -y)\} & \text{if } x = 0 \text{ and } y \in (0, 3), \\ \{\{0\} \times [-9, -3]\} & \text{if } x = 0 \text{ and } y = 3, \\ \{(x, 0)\} & \text{if } x \in (0, 2] \text{ and } y = 0, \\ \{(2, 0)\} & \text{if } x = 2 \text{ and } y \in [-1, 0). \end{cases}$$

To see that T does not satisfy (C) take $x = (0, 3), y = (2, -1), u_x = (0, -9)$. Notice that $T(y) = \{(2, 0)\}$. Then, $(1/2)d(x, u_x) = 6 = d(x, y)$ but $d(u_x, u_y) = 11 > 6$ for

$u_y = (2,0)$.
The fact that T satisfies condition (C') can be proved by an exhaustive case-by-case study. We omit the proof since this is a simple exercise. This will end the proof of (i).
To prove that (C') implies (E_3), let $x, y \in K$ and $u_x \in T(x)$. According to Theorem 3.9, the function $f : K \to K$ defined by $f(x) = P_{T(x)}(x)$ for each $x \in K$ satisfies condition (C), so, by Lemma 2.2, (iii) it also satisfies (E_3). Thus,

$$d(x, f(y)) \leq 3\text{dist}(x, T(x)) + d(x, y) \leq 3d(x, u_x) + d(x, y).$$

Since $f(y) \in T(y)$ it is clear that (E_3) holds. To show that (E_3) does not imply (C') we give a very simple example on $\mathbb{R}$ with the usual distance. This fact can also be justified via Example 3 of [7] because in the singlevalued case condition (C') is equivalent to condition (C). Set $K = [0,3]$ and define $T : K \to P_{cl,cv}(K)$ by

$$T(x) = \begin{cases} [1,3] & \text{if } x = 0, \\ \{3\} & \text{if } x \in (0,3]. \end{cases}$$

The mapping T does not satisfy (C'). Indeed, take $x = 0, y = 1$ and $u_x = 1$. Then $(1/2)d(x, u_x) \leq d(x, y)$ but $d(u_x, u_y) > d(x, y)$, where $u_y = 3$. It is also easy to see that T satisfies condition (E_3). This will complete the proof. □

The following condition for singlevalued mappings given in [7] is another natural extension of condition (C).

Definition 3.13. *Let X be a Banach space, $K \in P(X)$, $f : K \to X$ and $\lambda \in (0,1)$. The mapping f satisfies condition (C_λ) if for all $x, y \in K$,*

$$\lambda\|x - f(x)\| \leq \|x - y\| \Longrightarrow \|f(x) - f(y)\| \leq \|x - y\|.$$

For more details about this condition and its relation to conditions (C) and (E_μ) one may consult [7]. Following this idea, we introduce the next generalized version of condition (C') for multivalued mappings.

Definition 3.14. *Let X be a metric space, $K \in P(X), T : K \to P(X)$ and $\lambda \in (0,1)$. The mapping T satisfies condition (C'_λ) if for each $x, y \in K$ and $u_x \in T(x)$ with*

$$d(x, u_x) = \text{dist}(x, T(x)) \text{ and } \lambda d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

From the proof of Theorem 3.9 it is easy to see that the following result also holds.

Theorem 3.15. *Let X be an $\mathbb{R}$-tree, $K \in P(X)$ and $T : K \to P_{cl,cv}(X)$ a mapping which satisfies (C'_λ). Then the mapping $f : K \to X$ defined by $f(x) = P_{T(x)}(x)$ for each $x \in K$ is a selection of T that satisfies condition (C_λ).*

Using the results of [7] in relation to the condition (C_λ), one can further study (similarly as in the case of condition (C)) properties of multivalued mappings satisfying condition (C'_λ) and (C_λ) (defined in a similar manner).

3.1 Appendix: The hyperconvex case.

Hyperconvex metric spaces provide a very specific and interesting class of metric spaces with a large literature on fixed point results for nonexpansive mappings (see [17, Chapter 13] or [12, 25] and references therein). In particular, complete $\mathbb{R}$-trees are hyperconvex [14]. Therefore it is natural to wonder whether (singlevalued) mappings with property (C) will also have fixed points when defined from a bounded and closed hyperconvex space into itself. The goal of this appendix is to take up this question. As a result, we provide partial positive answers to it.

Although a mapping with condition (C) need not be continuous, it is shown in Theorem 2 of [26] that if T is a self-mapping on a nonempty compact and convex subset of a Banach space with condition (C) then it has a fixed point. This result follows as a consequence of Lemmas 2.2 and 2.4 in this work. In order to obtain the same result for hyperconvex metric spaces, we first need to give a meaning to convex combinations of two points in such spaces. Let H be a hyperconvex space and consider $\ell^\infty(I)$, where I stands for a certain index set, such that H can be embedded into $\ell^\infty(I)$. Then, see Chapter 13 in [17] for details, there exists a nonexpansive retraction R from $\ell^\infty(I)$ into H.

Definition 3.16. *Let H be a hyperconvex metric space and I and R as above. Then, for $x, y \in H$ and $\lambda \in [0,1]$, define*

$$(1-\lambda)x \oplus \lambda y = R((1-\lambda)x + \lambda y),$$

where $(1-\lambda)x + \lambda y$ stands for the usual convex combination in $\ell^\infty(I)$.

Notice that this definition provides a structure of segments (also called bicombing in the literature) which makes the metric convex as it is required in Lemma 2.4. In consequence, the adaptation of this lemma to this new setting (see [9, Proposition 2]) is straightforward.

Lemma 3.17. *Let H be a hyperconvex metric space and consider the bicombing given by any I and R as above. Let $\alpha \in (0,1)$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two bounded sequences in H such that $x_{n+1} = (1-\alpha)x_n \oplus \alpha y_n$ and $d(y_{n+1}, y_n) \le d(x_{n+1}, x_n)$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Theorem 2 from [26] can also be adapted in an straightforward way.

Theorem 3.18. *Let T be a self-mapping on a compact hyperconvex set H. Consider any bicombing as above on H and assume that T satisfies condition (C). Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in H with $x_1 \in H$ arbitrary and*

$$x_{n+1} = \lambda T(x_n) \oplus (1-\lambda)x_n$$

for $n \in \mathbb{N}$, where $\lambda \in [1/2, 1)$. Then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T.

Compactness in the previous theorem is only used to obtain the fixed point once it is known that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Therefore, the following corollary follows.

Corollary 3.19. *If T and $(x_n)_{n\in\mathbb{N}}$ are as above, and H is a hyperconvex metric space, not necessarily compact, then $(x_n)_{n\in\mathbb{N}}$ is a sequence of approximate fixed points for T, that is, a sequence such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.*

The next corollary follows from the fact that mappings with condition (C) are quasi-nonexpansive (see Lemma 2.2 (i)).

Corollary 3.20. *In the conditions of the previous theorem, the set of fixed points of T is hyperconvex.*

Proof. We prove first that $\mathrm{Fix}(T)$ is metrically convex. Let $x, y \in \mathrm{Fix}(T)$, $\alpha, \beta > 0$ with $d(x, y) = \alpha + \beta$. Set $M = \widetilde{B}(x, \alpha) \cap \widetilde{B}(y, \beta)$. Then M is nonempty, bounded and hyperconvex. Let $z \in T(M)$. Then there exists $v \in M$ with $T(v) = z$. By the quasinonexpansivity of T, $z \in \widetilde{B}(x, \alpha) \cap \widetilde{B}(y, \beta)$. Therefore, $T(M) \subseteq M$ and applying the above, $\mathrm{Fix}(T) \cap M \neq \emptyset$.

Next we show the binary intersection property. Let $(\widetilde{B}(x_i, r_i))_{i \in J}$ be a collection of balls with centers in $\mathrm{Fix}(f)$ and such that $\widetilde{B}(x_i, r_i) \cap \widetilde{B}(x_j, r_j) \neq \emptyset$ for all $i, j \in J$. Set $M = \bigcap_{i \in J} \widetilde{B}(x_i, r_i)$. Then M is nonempty, compact, hyperconvex and T-invariant (thanks to the quasinonexpansivity). Thus, $\mathrm{Fix}(f) \cap M \neq \emptyset$. □

To take up the noncompact case we will consider a new condition.

Definition 3.21. *Let X be a metric space and $T\colon X \to X$. Then T satisfies condition (D) if*

$$\frac{1}{2} d(x, T(x)) \geq d(x, y) \Longrightarrow d(T(x), T(y)) \leq d(x, T(x))$$

for all $x, y \in X$.

It is interesting to remark at this point that any 2-lipschitzian mapping satisfies condition (D). Notice also that this condition does not imply continuity and that it is implied by condition (C) for x, y such that $(1/2)d(x, T(x)) = d(x, y)$. This last relation explains why it is not that easy to find a mapping with condition (C) but failing condition (D). The next example shows, however, that this is possible.

Example 3.22. *Let $T : [0, 5] \to [0, 5]$ be defined as follows:*

$$T(x) = \begin{cases} 0 & \text{if } x \in [0; 2], \\ x - 2 & \text{if } x \in (2; 4], \\ 10 - 2x & \text{if } x \in (4; 4,6], \\ 0,8 & \text{if } x \in (4,6; 4,8], \\ 1 & \text{if } x \in (4,8; 5), \\ 3 & \text{if } x = 5. \end{cases}$$

It is immediate to see that T does not satisfy condition (D) by taking $x = 5$ and $y = 4,6$. A case by case analysis shows that T satisfies condition (C).

In the conjunction of conditions (C) and (D) we can adapt the classical proof of Baillon (see [1, Theorem 5]) for the existence of fixed points for nonexpansive mappings in hyperconvex spaces.

Theorem 3.23. *Let X be a nonempty bounded hyperconvex space. Suppose $T\colon X \to X$ satisfies conditions (C) and (D). Then $\mathrm{Fix}(T)$ is nonempty and hyperconvex.*

Proof. Let $\mathcal{U} = \{A \subseteq X : A \neq \emptyset, A = \mathrm{cov}(A), T(A) \subseteq A\}$ and order this family in the following way: for $U_1, U_2 \in \mathcal{U}$,

$$U_1 \leq U_2 \Longleftrightarrow U_2 \subseteq U_1.$$

The family $\mathcal{U} \neq \emptyset$ since $X \in \mathcal{U}$. Take $(U_i)_{i\in\mathbb{N}}$ an increasing chain, that is, a decreasing sequence of sets in $\mathcal{U}$. Since $U_i = \bigcap_{x\in X} \widetilde{B}(x, r_x(U_i))$ and X is hyperconvex it follows that $\bigcap_{i\in\mathbb{N}} U_i \neq \emptyset$. Because $\bigcap_{i\in\mathbb{N}} U_i$ is also T-invariant, we have an upper bound for the chain, so, by Zorn's lemma, there exists a maximal element and thus minimal with respect to the set inclusion. We shall denote this minimal element by A.

We show next that $\mathrm{cov}(T(A)) \in \mathcal{U}$. This amounts to showing that $\mathrm{cov}(T(A))$ is T-invariant. Let $y \in T(\mathrm{cov}(T(A)))$. Since $\mathrm{cov}(T(A)) \subseteq \mathrm{cov}(A) = A$, it follows that for every $x \in X$, $d(x, y) \leq r_x(T(\mathrm{cov}(T(A)))) \leq r_x(T(A))$. This implies that $y \in \mathrm{cov}(T(A))$ because $\mathrm{cov}(T(A))$ is admissible. Hence, $\mathrm{cov}(T(A)) \in \mathcal{U}$ and is at the same time a subset of A. By the minimality of A we obtain that $A = \mathrm{cov}(T(A))$ which yields that for all $x \in X$,

$$r_x(A) = r_x(T(A)). \tag{4}$$

Let $C(A)$ be the center of A. Then $C(A) = \bigcap_{x\in A} \widetilde{B}(x, r(A))$ and $C(A) \cap A \neq \emptyset$ since $r(A) = (1/2)\mathrm{diam}A$ and X is hyperconvex. We claim that $C(A) \cap A$ is also T-invariant. Take $y \in C(A) \cap A$. We want to show that $r_{T(y)}(A) = r(A)$. Let $x \in A$. Then, if $(1/2)d(y, T(y)) \leq d(x, y)$ we can apply (C) to obtain that $d(T(x), T(y)) \leq d(x, y) \leq r(A)$. Otherwise, $(1/2)d(y, T(y)) \geq d(x, y)$ and we can apply (D) to obtain that $d(T(x), T(y)) \leq d(y, T(y)) \leq r(A)$. Joining both cases, we obtain that $r_{T(y)}(T(A)) \leq r(A)$. Now it is enough to recall (4) to prove our claim.

It is now easy to see that $A \cap C(A) \in \mathcal{U}$. Using again the minimality of A we obtain that $A = A \cap C(A)$. But this yields that $\mathrm{diam}(A) = \mathrm{diam}(A \cap C(A)) \leq (1/2)\mathrm{diam}(A)$, so A is a singleton and hence $\mathrm{Fix}(T) \neq \emptyset$.

Finally, the fact that $\mathrm{Fix}(T)$ follows in the same way as in Corollary 3.20. □

The following corollary is a particular case of this theorem.

Corollary 3.24. *Let X be a nonempty bounded hyperconvex space. Suppose $T: X \to X$ is a 2-lipschitzian mapping with condition (C). Then $\mathrm{Fix}(f)$ is nonempty and hyperconvex.*

4 Fixed points in Banach spaces

The goal of this section is to revisit classical theorems for existence of fixed points for nonexpansive multivalued mappings in Banach spaces from the perspective of multivalued mappings with condition (C).

Definition 4.1. *Let X be a Banach space endowed with a linear topology τ. The space X is said to have the Opial property with respect to τ if*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for every $y \in X$, $y \neq x$ and for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X τ-convergent to x. When τ is the weak topology we will say, in short, that X has the Opial property.

Theorem 4.2. *Let X be a Banach space which has the Opial property with respect to τ. Suppose K is a bounded, convex and τ-sequentially compact subset of X and $T : K \to P_{cp}(K)$ is a mapping satisfying condition (C). Then $Fix(T) \neq \emptyset$.*

Proof. By Proposition 3.4 there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in K such that $y_n \in T(x_n)$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since K is τ-sequentially compact we may assume that $(x_n)_{n\in\mathbb{N}}$ is τ-convergent to a point $z \in K$.
Using Lemma 3.2, for each $n \in \mathbb{N}$, there exists $v_n \in T(z)$ such that

$$\|x_n - v_n\| \leq 3\|x_n - y_n\| + \|x_n - z\|.$$

By the compactness of $T(z)$, we can assume that $(v_n)_{n\in\mathbb{N}}$ converges to a point $v \in T(z)$. From the above it follows that

$$\liminf_{n\to\infty} \|x_n - v\| \leq \liminf_{n\to\infty} \|x_n - z\|.$$

From the Opial property we have that $v = z \in T(z)$ and the proof is complete. □

Remark 4.3. Notice that the class of spaces for which the preceding theorem can be applied includes the space ℓ_1 where τ is the weak star topology $\sigma(c_0, \ell_1)$ and K is a weak star compact convex subset of ℓ_1.

Now, we are going to set out some useful results concerning the asymptotic centers. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Define

$$r(K, (x_n)) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in K\},$$

and

$$A(K, (x_n)) = \{x \in K : \limsup_{n\to\infty} \|x_n - x\| = r(K, (x_n))\}.$$

The number $r(K, (x_n))$ and the (possibly empty) set $A(K, (x_n))$ are called the *asymptotic radius* and the *asymptotic center* of $(x_n)_{n\in\mathbb{N}}$ in K respectively. It should be noted that $A(K, (x_n))$ is a nonempty, weakly compact and convex set whenever K is weakly compact and convex.

Definition 4.4. *A bounded sequence is said to be regular with respect to K if each of its subsequences has the same asymptotic radius in K, and asymptotically uniform with respect to K if each of its subsequence has the same asymptotic center in K.*

Lemma 4.5. (Goebel [8], Lim [20], Kirk [13]) *Let K be a subset of a Banach space X and $(x_n)_{n\in\mathbb{N}}$ a bounded sequence in X. Then*

(i) there always exists a subsequence $(x_n)_{n\in\mathbb{N}}$ which is regular with respect to K;

(ii) if K is separable, then $(x_n)_{n\in\mathbb{N}}$ contains a subsequence which is asymptotically uniform with respect to K.

Recall that X is said to be *uniformly convex in every direction* (UCED, in short) if $\delta_z(\epsilon) > 0$ for all $\epsilon > 0$ and $z \in X$ with $\|z\| = 1$, where $\delta_z(\epsilon)$ is the modulus of convexity of X in the direction z defined by

$$\delta_z(\epsilon) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, x - y = \epsilon z \right\}.$$

Obviously, uniformly convex Banach spaces are UCED. It is known that in a UCED Banach space, the asymptotic center of a sequence with respect to a weakly compact convex set is a singleton. Hence, every regular sequence with respect to such a set is asymptotically uniform.

Theorem 4.6. *Let K be a weakly compact and convex subset of a UCED Banach space X. Suppose $T : K \to P_{cp}(K)$ is a mapping satisfying condition (C). Then $Fix(T) \neq \emptyset$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in K such that $y_n \in T(x_n)$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Without loss of generality, me may assume that $(x_n)_{n\in\mathbb{N}}$ is regular with respect to K. Let z be the unique point in the asymptotic center of $(x_n)_{n\in\mathbb{N}}$ in K. By Lemma 3.2, for each $n \in \mathbb{N}$ there exists $v_n \in T(z)$ such that

$$\|x_n - v_n\| \leq 3\|x_n - y_n\| + \|x_n - z\|.$$

From the compactness of $T(z)$ we can assume that $(v_n)_{n\in\mathbb{N}}$ converges to a point $v \in T(z)$. It follows that

$$\limsup_{n\to\infty} \|x_n - v\| \leq \limsup_{n\to\infty} \|x_n - z\|.$$

Since $(x_n)_{n\in\mathbb{N}}$ is regular we conclude that $v = z \in T(z)$. □

Dhompongsa et al. [3] have recently proved the T invariance of the asymptotic center in K of an approximate fixed point sequence for T, when T is a singlevalued mapping satisfying condition (C). We now state a result which can be seen as an adaptation of this fact to the multivalued case.

Proposition 4.7. *Let K be a weakly compact subset of a Banach space X. Suppose $T : K \to P_{cp}(K)$ satisfies condition (C) and $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence for T. Then, there exists a subsequence $(z_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that*

$$T(x) \cap A \neq \emptyset, \text{ for all } x \in A := A(K, (z_n)).$$

Proof. Since T is a self-mapping we can build a subsequence $(z_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which is regular and asymptotically uniform with respect to K. Denote $r(K, (z_n))$ by r. Taking any $x \in A$ and following the same argument as in the proof of the above theorem we obtain a sequence $(v_n)_{n\in\mathbb{N}} \subseteq T(x)$ norm convergent to a point $v \in T(x)$ such that

$$\limsup_{n\to\infty} \|x_n - v\| \leq \limsup_{n\to\infty} \|x_n - x\| = r.$$

This shows that $v \in A$, and so $T(x) \cap A \neq \emptyset$. □

Now we are ready to prove an analogous result to the Kirk-Massa theorem [16] for mappings satisfying condition (C).

Theorem 4.8. *Let K be a bounded, closed and convex subset of a Banach space X and $T : K \to P_{cp,cv}(K)$ be a continuous mapping with respect to the Pompeiu-Hausdorff distance satisfying condition (C). Suppose that each sequence in K has a nonempty and compact asymptotic center relative to K. Then $Fix(T) \neq \emptyset$.*

Proof. According to the previous proposition we can take a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that

$$T(x) \cap A \neq \emptyset, \text{ for all } x \in A := A(K, (x_n)).$$

Now we define the mapping $\tilde{T} : A \to P_{cp,cv}(A)$ by $\tilde{T}(x) = T(x)\cap A$. Since T is continuous, from Proposition 2.45 in [11] we know that the mapping $\tilde{T}$ is upper semi-continuous. Since $T(x) \cap A$ is a compact convex set we can apply the Kakutani-Bohnenblust-Karlin theorem (see [10]) to obtain a fixed point for $\tilde{T}$ and hence for T. □

Remark 4.9. Recall that a multivalued mapping $T : K \to P_b(X)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\| \text{ for all } x, y \in K,$$

where H denotes the Pompeiu-Hausdorff distance. It is worth pointing out that another natural extension of the Suzuki's condition (C) for a multivalued mapping $T : K \to P(X)$ is the following: for all $x, y \in K$

$$\frac{1}{2}\text{dist}(x, T(x)) \leq \|x - y\| \Longrightarrow H(T(x), T(y)) \leq \|x - y\|.$$

Obviously, a nonexpansive mapping meets the above condition. However, it is not clear if a mapping satisfying the above condition also satisfies (C). Still, if T takes compact values is easy to see that this new condition implies condition (C). Since in our theorems T is assumed to be compact valued, such results generalize classical fixed point theorems for multivalued mappings (see [16],[19],[20]).

5 Common fixed points

In our last section we will apply some of the fixed point theorems stated in previous sections to obtain results on the existence of common fixed point.

Definition 5.1. *Let X be a metric space and $K \in P(X)$. Suppose $f : K \to K$ and $T : K \to P(K)$. Then f and T are commuting mappings if $f(y) \in T(f(x))$ for all $x \in K$ and $y \in T(x)$.*

We start by giving a lemma that will constitute a main tool in proving our results.

Lemma 5.2. *Let X be a metric space, $K \in P(X)$, $f : K \to K$ satisfying condition (C) and with $\text{Fix}(f) \neq \emptyset$. Supppose $T : K \to P(K)$ is such that for every $x, y \in \text{Fix}(f)$, the set $P_{T(y)}(x)$ is a singleton. If f and T commute, then $P_{T(y)}(x) \in \text{Fix}(f)$ for all $x, y \in \text{Fix}(f)$.*

Proof. Let $x, y \in \text{Fix}(f)$ and denote $P_{T(y)}(x)$ by u. Because f meets condition (C) and $0 = (1/2)d(x, f(x)) \leq d(x, u)$ we obtain that $d(x, f(u)) = d(f(x), f(u)) \leq d(x, u) = \text{dist}(x, T(y))$. But $f(u) \in T(y)$ because f and T commute, $y \in \text{Fix}(f)$ and $u \in T(y)$. Hence, $f(u) = u$ and the conclusion follows. □

The following theorem is an extension of Theorem 4.2 of [23] in the setting of a UC space with convex metric. Notice that our approach is different in the second half of the proof from that of [23]. In particular, ours fills a gap in the proof of [23]. Notice also that this theorem extends some other results in the theory, see, for instance, [6, 24].

Theorem 5.3. *Let X be a complete UC space with convex metric and $K \in P_{b,cl,cv}(X)$. Suppose $f : K \to K$ and $T : K \to P_{cp,cv}(K)$ satisfy condition (C). If f and T commute, then there exists $z \in K$ such that $z = f(z) \in T(z)$.*

Proof. Using Theorem 2.6, it follows that $\mathrm{Fix}(f)$ is nonempty, closed and convex. Since the setting we work in is a UC space, the projection onto each compact and convex set is a singleton. By Lemma 5.2, $P_{T(x)}(x) \in T(x) \cap \mathrm{Fix}(f)$ for each $x \in \mathrm{Fix}(f)$ and so we can consider the mapping $T(\cdot) \cap \mathrm{Fix}(f) : \mathrm{Fix}(f) \to P_{cp}(\mathrm{Fix}(f))$. We show that this mapping satisfies condition (C). Let $x, y \in \mathrm{Fix}(f), u_x \in T(x) \cap \mathrm{Fix}(f)$ such that $(1/2)d(x, u_x) \leq d(x, y)$. Since T fulfills (C), there exists $v_y \in T(y)$ such that $d(u_x, v_y) \leq d(x, y)$. Let u_y stand for $P_{T(y)}(u_x)$. According to Lemma 5.2, $u_y \in T(y) \cap \mathrm{Fix}(f)$. It is also clear that $d(u_x, u_y) \leq d(u_x, v_y) \leq d(x, y)$. Thus, the mapping $T(\cdot) \cap \mathrm{Fix}(f) : \mathrm{Fix}(f) \to P_{cp}(\mathrm{Fix}(f))$ satisfies (C) which means, using Theorem 3.6, that there exists $z \in K$ such that $z = f(z) \in T(z)$. □

Likewise, one can prove the following result in the framework of $\mathbb{R}$-trees.

Theorem 5.4. *Let X be a bounded complete $\mathbb{R}$-tree. Suppose $f : X \to X$ and $T : X \to P_{cl,cv}(X)$ satisfy conditions (C) and (C') respectively. If f and T commute, then there exists $z \in K$ such that $z = f(z) \in T(z)$.*

Proof. According to Theorem 2.6, it follows that $\mathrm{Fix}(f)$ is nonempty, closed and convex (and so also hyperconvex). This means that $\mathrm{Fix}(f)$ is in its own turn a complete $\mathbb{R}$-tree. Since in an $\mathbb{R}$-tree the projection onto each closed and convex set is a singleton we can apply Lemma 5.2 and so $T(x) \cap \mathrm{Fix}(f) \neq \emptyset$ for each $x \in \mathrm{Fix}(f)$. Now consider the mapping $T(\cdot) \cap \mathrm{Fix}(f) : \mathrm{Fix}(f) \to P_{cl,cv}(\mathrm{Fix}(f))$. We show that this mapping satisfies condition (C'). Let $x, y \in \mathrm{Fix}(f), u_x \in T(x) \cap \mathrm{Fix}(f)$ such that $d(x, u_x) = \mathrm{dist}(x, T(x) \cap \mathrm{Fix}(f))$ and $(1/2)d(x, u_x) \leq d(x, y)$. Applying Lemma 5.2, $P_{T(x)}(x) \in T(x) \cap \mathrm{Fix}(f)$ which implies that $\mathrm{dist}(x, T(x)) = \mathrm{dist}(x, T(x) \cap \mathrm{Fix}(f))$, so $d(x, u_x) = \mathrm{dist}(x, T(x))$. Because T satisfies (C'), there exists $v_y \in T(y)$ such that $d(u_x, v_y) \leq d(x, y)$. Let u_y stand for $P_{T(y)}(u_x)$. According to Lemma 5.2, $u_y \in T(y) \cap \mathrm{Fix}(f)$. It is also clear that $d(u_x, u_y) \leq d(u_x, v_y) \leq d(x, y)$. Thus, the mapping $T(\cdot) \cap \mathrm{Fix}(f) : \mathrm{Fix}(f) \to P_{cl,cv}(\mathrm{Fix}(f))$ satisfies (C') which means, using Corollary 3.11, that there exists $z \in K$ such that $z = f(z) \in T(z)$. □

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