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SOME COMBINATORIAL REMARKS ON NORMAL FLATNESS IN ANALYTIC SPACES

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Abstract. In this article we present a combinatorial treatment of normal flatness in analytic spaces, using the idea of equimultiple standard bases. We will prove, using purely combinatorial methods, a characterization theorem for normal flatness. This will lead us to a new proof of a classical theorem on normal flatness, which can be stated by saying that normal flatness at a point along a smooth subspace is equivalent to the Hilbert function being locally constant.

Though these topics belong to classical analytic geometry, we believe that this approach is valuable, since it replaces extremely general algebraic theorems by combinatorial objects, obtaining new results and striking the combinatorial nature of the classical (and basic) ideas in the resolution of singularities.

1. Introduction

H. Hironaka introduced the concept of normal flatness in his original argument for the resolution of singularities [10]. Since then, the concept of normal flatness and the related concept of equimultiplicity —an idea which proved elusive to define properly— played a central role in the geometry of singularities in characteristic zero. After the works of H. Hironaka [11], Aroca-Hironaka-Vicente [1, 2] and B. M. Bennett [3], all of which saw the light mainly in the seventies, the whole subject of resolution in characteristic zero underwent a long period of apparent inactivity, although the study of normal flatness for its own sake was still pursued (see for instance [4, 9, 15, 16, 17], just to mention a few).

Later on, in the nineties, many mathematicians provided new versions of the resolution of singularities in characteristic zero, stressing most of the times the effective

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nature of these new proofs. Little or no use was done of concepts such as equimultiplicity and normal flatness, a fact which was actually remarked upon in some of these papers ([4] being probably a remarkable exception). The tide had turned and, apparently, the old geometric notion of normal flatness was a hard concept to compute.

The purpose of this paper is rather the contrary, and we will go into detail about the sections of the article. In the third section, we start from scratch, as we jump into technical aspects of the Weierstrass-Hironaka division Theorem. A close analysis of the algorithm, in a very particular and precise version, is given here. This version will let us state some properties essential for what follows. We are aware that it may be difficult to navigate through the section but, unfortunately, both the results and the actual proofs are needed.

In section 4 we will define normal flatness in classical terms, using graded rings, and, in section 5 we will show how normal flatness can be easily read in terms of combinatorics which are naturally attached to an analytic space. This is the essential core of the paper, where this interpretation is made explicit, with fundamental usage of the techniques from section 3.

Moreover, in section 6 we will deduce from this, using combinatorial arguments, the so called Fundamental Theorem, which explicitely states the tight relationship between normal flatness and the behaviour of the Hilbert function, displayed in section 7. The results in these two final sections are but complex analytic versions of well–known results in the algebraic case, but we have found it interesting to obtain them as a by–product of the previous combinatorial results

The basic idea goes back to an unpublished seminar of Prof. J. L. Vicente in the Faculty of Sciences at Orsay (France), in November 1982. Different approaches and parts of these theorems are sketched elsewhere, most notably in [14] and [1]. The initial situation in [14] is far more classical than the setup we present here. As a result, clear proofs in the algebraic case do not carry over so simply to the analytic case.

We are aware that some of our results overlap with previous well–known papers in the literature. Different versions of the Weierstrass-Hironaka Theorem can be found in [1, 12, 7, 6, 8] and especially [5], which is the most related with our proof, the Fundamental Theorem on section 6 can be found in [14, 13] and the relationship between Hilbert functions and flatness has been exploited, for instance, in [3, 4, 15]. All these results are also presented here for the convenience of the reader, trying to give an exposition as exhaustive and self–contained as possible. But, furthermore, some precise results (like those on section 5 and some on section 7), as well as a comprehensive and fully combinatorial treatment of the normal flatness are still missing in the existing literature. This is the gap this paper tries to fill.

2. Notations

We fix here the basic notations we will use throughout the paper. We denote by

$\mathbb{Z}_{\geq 0}$ the set of non-negative integers. When working with c-uples $A \in \mathbb{Z}^c_{\geq 0}$ we write $|A|$ for the total degree,

$$|(a_1, \dots, a_c)| = a_1 + \cdots + a_c.$$

Let $R = \mathbb{C}\{\boldsymbol{z}, \boldsymbol{w}\}$ be the ring of convergent power series in the variables $\boldsymbol{z} = \{z_1, \dots, z_c\}$ and $\boldsymbol{w} = \{w_1, \dots, w_d\}$. We will consider the following ideals of R:

$\mathfrak{m} = (\boldsymbol{z}, \boldsymbol{w})$, the maximal ideal of R.

$\mathfrak{p} = (\boldsymbol{z})$, which is a prime ideal.

$I \subset \mathfrak{p}$, an arbitrary ideal (contained in $\mathfrak{p}$).

Let W be the analytic space defined in a certain neighbourhood of the origin of $\mathbb{C}^{c+d}$ by the ideal $\mathfrak{p}$, and X the complex analytic space defined by I. Obviously, $W \subset X$.

For any element f of R we may write

$$\begin{aligned} f &= \sum_{A \in \mathbb{Z}^c_{\geq 0}} f_A(\boldsymbol{w})\, \boldsymbol{z}^A && f_A(\boldsymbol{w}) \in \mathbb{C}\{\boldsymbol{w}\} \text{ for all } A \in \mathbb{Z}^c_{\geq 0} \\ &= \sum_{(A,B) \in \mathbb{Z}^{c+d}_{\geq 0}} f_{(A,B)}\, \boldsymbol{z}^A\, \boldsymbol{w}^B && f_{(A,B)} \in \mathbb{C} \text{ for all } (A,B) \in \mathbb{Z}^{c+d}_{\geq 0}, \end{aligned}$$

and we define the supports of f with regard to $\mathfrak{p}$ and $\mathfrak{m}$, respectively, as:

$$\begin{aligned} \mathcal{E}_{\boldsymbol{z}}(f) &= \{A \in \mathbb{Z}^c_{\geq 0} \quad | \quad f_A \neq 0\} \\ \mathcal{E}_{\boldsymbol{z},\boldsymbol{w}}(f) &= \big\{(A,B) \in \mathbb{Z}^{c+d}_{\geq 0} \quad | \quad f_{(A,B)} \neq 0\big\}. \end{aligned}$$

We denote the sets of exponents of the initial forms as follows

$$\begin{aligned} \overline{u}_{\boldsymbol{z}}(f) &= \big\{A \in \mathbb{Z}^c_{\geq 0} \quad | \quad |A| = \nu_{\mathfrak{p}}(f) \text{ and } f_A(\mathbf{0}) \neq 0\big\} \\ \overline{u}_{\boldsymbol{z},\boldsymbol{w}}(f) &= \big\{(A,B) \in \mathbb{Z}^{c+d}_{\geq 0} \quad | \quad |(A,B)| = \nu_{\mathfrak{m}}(f) \text{ and } f_{(A,B)} \neq 0\big\}, \end{aligned}$$

where $\nu_{\boldsymbol{z}}(f)$ and $\nu_{\boldsymbol{z},\boldsymbol{w}}(f)$ are the usual order functions of R with respect to $\mathfrak{p}$ and $\mathfrak{m}$, respectively. Also, we write $u_{\boldsymbol{z}}(f) = \sup_{lex} \overline{u}_{\boldsymbol{z}}(f)$, where $\sup_{lex}$ means the supremum for the lexicographic order, and analogously for $u_{\boldsymbol{z},\boldsymbol{w}}(f)$.

Finally, let us put

$$\begin{aligned} u_{\boldsymbol{z}}(I) &= \{u_{\boldsymbol{z}}(f) \quad | \quad f \in I, \quad f \neq 0\} \subset \mathbb{Z}^c_{\geq 0} \\ u_{\boldsymbol{z},\boldsymbol{w}}(I) &= \big\{u_{\boldsymbol{z},\boldsymbol{w}}(f) \quad | \quad f \in I, \quad f \neq 0\big\} \subset \mathbb{Z}^{c+d}_{\geq 0}. \end{aligned}$$

Since I is an ideal, it is easy to prove the equality $u_{\boldsymbol{z}}(I) + \mathbb{Z}^c_{\geq 0} = u_{\boldsymbol{z}}(I)$ and, similarly $u_{\boldsymbol{z},\boldsymbol{w}}(I) + \mathbb{Z}^{c+d}_{\geq 0} = u_{\boldsymbol{z},\boldsymbol{w}}(I)$.

Remark 1. In principle, it could happen that $\overline{u}_{\boldsymbol{z}}(f)$ or $u_{\boldsymbol{z}}(I)$ might be empty. This would correspond to the case of non-equimultiplicity. Since normal flatness implies equimultiplicity, as it can be seen below, and we will be interested only in the case of normal flatness, we will simply assume that $u_{\boldsymbol{z}}(I) \neq \emptyset$.

Remark 2. We follow Hironaka for the definition of $u_{\boldsymbol{z}}(f)$, but it should be noted that it is possible to define $u_{\boldsymbol{z}}(f)$ as the *infimum* for the lex order of $\overline{u}_{\boldsymbol{z}}(f)$, so that $u_{\boldsymbol{z}}(f)$ is, in fact, the initial monomial of f. In this case, there would be no need of Proposition 15.

However, at several points (most notably the first part of Theorem 14, but at other points also), we need to argue by taking initial forms and considering *polynomial* orders, so one would need to prove the analog of Proposition 15 for the lex maximum. This being the case, Hironaka's definition seems best suited to our purposes.

We can write
$$u_{\boldsymbol{z}}(I) = \bigcup_{i=1}^{r} (A_i + \mathbb{Z}^c_{\geq 0})$$
as a non redundant, finite union by Dickson's Lemma. We can assume that the A_i are ordered by the graded lexicographic order.

Let us define the sets Δ_0, Δ_1, $\ldots$, Δ_r, where
$$\begin{aligned}
\Delta_0 &= \mathbb{Z}^c_{\geq 0} \setminus \bigcup_{i=1}^{r} (A_i + \mathbb{Z}^c_{\geq 0}) \\
\Delta_1 &= A_1 + \mathbb{Z}^c_{\geq 0} \\
\Delta_{j+1} &= (A_{j+1} + \mathbb{Z}^c_{\geq 0}) \setminus \bigcup_{i=1}^{j} (A_i + \mathbb{Z}^c_{\geq 0}), \quad 1 \leq j \leq r-1.
\end{aligned}$$

The appropriate setup for the normal flatness is that of graded rings. The first ring which we will introduce is the global graded ring with regard to $\mathfrak{p}$,
$$gr_{\mathfrak{p}}(R) = \bigoplus_{i \geq 0} \mathfrak{p}^i / \mathfrak{p}^{i+1} = \mathbb{C}\{\boldsymbol{w}\}[\widetilde{\boldsymbol{z}}],$$
where $\widetilde{\boldsymbol{z}} = (\widetilde{z}_1, \ldots, \widetilde{z}_c)$, with $\widetilde{z}_j = z_j + \mathfrak{p}^2$, for $1 \leq j \leq c$. For $f \in R$, let us write
$$\tilde{f} = \sum_{A \in \overline{u}_{\boldsymbol{z}}(f)} f_A(\boldsymbol{w}) \widetilde{\boldsymbol{z}}^A \in gr_{\mathfrak{p}}(R).$$

Note that $\widetilde{f}$ can be identified with the homogeneous part of f having degree $\nu_{\boldsymbol{z}}(f)$ and hence we will also call it the initial form of f w.r.t. $\boldsymbol{z}$, or the $\boldsymbol{z}$-initial form. We will also write, given the ideal $I \subset R$,
$$In_{\mathfrak{p}}(I) = \left\{ \widetilde{f} \quad | \quad f \in I \right\} \subset gr_{\mathfrak{p}}(R),$$
called the initial ideal of I with regard to $\mathfrak{p}$.

We will use analogous notations and definitions for:

- the graded ring with regard to $\mathfrak{m}$, denoted $gr_{\mathfrak{m}}(R)$,
- the ordinary initial form of f or the $\{\boldsymbol{z},\boldsymbol{w}\}$-initial form, denoted $\overline{f}$, and
- the initial ideal of I with regard to $\mathfrak{m}$, denoted $In_{\mathfrak{m}}(I)$.

Another interesting graded ring is the local graded ring with regard to the situation $I \subset \mathfrak{p}$,

$$gr_{(\mathfrak{p}/I)}(R/I) = \mathbb{C}\{\boldsymbol{w}\}[\widetilde{\boldsymbol{z}}]/In_{\mathfrak{p}}(I) = \mathbb{C}\{\boldsymbol{w}\}[\boldsymbol{z}^*], \tag{1}$$

where $z_j^* = \widetilde{z}_j + In_{\mathfrak{p}}(I)$, for $1 \le j \le c$.

3. The Local Lemmas: Variations on Weierstrass-hironaka Division

The bulk of this section consists of two auxiliary results concerning variations of Weierstrass-Hironaka division. We will give a detailed proof of both of them, although very similar results can be found, for instance, in [1, 8]. The proof in [1] is complete, but rather difficult to find in libraries, while [8] is sketchier but much easier to get. The version in [5], though the closest to our purposes, is still not exactly what we need.

Our first lemma is a specially tailored version of the original Weierstrass-Hironaka division theorem. Specifically, we need to prove the order inequalities (3) of Lemma 7 and (8) of Lemma 8, which *do not follow* from the original Weierstrass-Hironaka arguments. Before we go on to state and prove the Lemmas, it will be useful to include the precise statement of Weierstrass-Hironaka Theorem, and to discuss the differences in certain detail.

Following [1], for any linear form $L(x_1,\dots,x_c,y_1,\dots,y_d)$, let us consider a total ordering $<_L$ in $\mathbb{Z}^c_{\ge 0}$ defined in the usual way: given two monomials, we look at their L-degree breaking ties with the lexicographic ordering. Define also the operator $\nu_L(\cdot)$ as the minimal L-degree, and

$$\Delta_L = \left\{(A,B) \in \mathbb{R}^{c+d} \quad | \quad L(A,B) \ge 1\right\}.$$

Given power series

$$\{f_i \quad | \quad i = 1,\dots,r\} \subset R,$$

and a family of exponents

$$\{A_i \quad | \quad i = 1,\dots,r\} \subset \mathbb{Z}^c_{\ge 0},$$

we define

$$S(f_i, A_i) = \left\{\Delta_L \quad | \quad \exists c_i \in \mathbb{C}^* \text{ such that } \nu_L\left(\boldsymbol{z}^{A_i}\right) < \nu_L\left(f_i - c_i \boldsymbol{z}^{A_i}\right)\right\}.$$

Theorem 3. (Weierstrass-Hironaka division theorem [1]). *Given* $\{f_1,\dots,f_r\}$ *such that*

$$\bigcap_{i=1}^{r} S(f_i, A_i) \neq \emptyset,$$

for every $g \in R$ *there exist unique series* $h_0, \ldots, h_r \in R$ *such that*

$$g = h_0 + \sum_{i=1}^{r} h_i g_i.$$

Moreover, for any $\Delta_L \in \bigcap_{i=1}^{r} S(f_i, A_i)$,

$$\nu_L(h_i) \geq \nu_L(g) - \nu_L(f_i) \text{ for } i = 1, \ldots, r,$$

and

$$\mathcal{E}_{\boldsymbol{z}}\left(h_i \boldsymbol{z}^{A_i}\right) \subset \Delta_i \text{ for } i = 0, \ldots, r.$$

Remark 4. As the reader can check, the main difference between our arguments and the original ones from [1] comes from the fact that Aroca, Hironaka and Vicente used an arbitrary order given by a linear form, instead of our explicit choice of L_c and $L_{c,d}$ below. This greater generality can be achieved if we relax our condition

$$\nu_{\boldsymbol{z}}(h_i) \geq \nu_{\boldsymbol{z}}(g) - |A_i|, \text{ for } i = 1, \ldots, r;$$

in Lemma 7 (WHD1), to the milder one in Theorem 3, but our particular statement will be useful for in the sequel. Apart from this, existence is proved in a very similar way, and uniqueness is proved exactly the same way as they did (only expository details have been added).

As for convergence, our choice of ordering allows us to shorten the arguments, which in the general case are much more involved. The basic guidelines are given in [8], but even the precise details are left to the reader there. The convergence proof in [1] is complete, although rather hard to follow, being split in several parts.

Coming back to the precise statements we need, we will forget for the moment about the ideal I, and we will concentrate our efforts in the combinatorial object

$$\bigcup_{i=1}^{r}(A_i + \mathbb{Z}^c_{\geq 0}) = u(I),$$

regardless its origin. Given such a non-redundant set, let us call $t = \max\{|A_i|,\ i = 1, \ldots, r\}$, and let m be an integer, much bigger than t, say $m = 10^{10^t}$.

The following linear forms in $\mathbb{Z}^c_{\geq 0}$ and $\mathbb{Z}^{c+d}_{\geq 0}$,

$$L_c(x_1, \ldots, x_c) = \sum_{i=1}^{c}\left(1 - \frac{1}{10^{i(m+t)}}\right)x_i$$

$$L_{c,d}(x_1, \ldots, x_c; y_1, \ldots, y_d) = L_c(x_1, \ldots, x_c) + \frac{1}{10^m}\sum_{j=1}^{d} y_j.$$

are obtained by a small perturbation of the standard order form to adapt them to the set $\{A_1, \ldots, A_r\}$.

Remark 5. Informally speaking, the forms L_c and $L_{c,d}$ differ from the standard order form in less than $1/10^m$ below degree t, but are guaranteed to reach a unique maximum in the set $\{A_1, \dots, A_r\}$. Note that, since

$$A_1 < A_2 < \cdots < A_r,$$

we also have

$$L_c(A_1) < L_c(A_2) < \cdots < L_c(A_r),$$

and

$$L_{c,d}(A_1) < L_{c,d}(A_2) < \cdots < L_{c,d}(A_r).$$

We will denote, for any $f \in R$ by $\nu_c(f)$ the minimum value obtained applying L_c to $\mathcal{E}_{\boldsymbol{z}}(f)$, and by $\nu_{c,d}(f)$ the minimum value obtained when applying $L_{c,d}$ to $\mathcal{E}_{\boldsymbol{z},\boldsymbol{w}}(f)$.

The characteristics of this ordering will come particularly handy for proving the convergence on our version of Weierstrass–Hironaka and in the combinatorial characterization of normal flatness. We will begin by pointing out a simple, yet useful, property.

Lemma 6. *Let $\{g_1, \dots, g_r\}$ be a family of series in R such that*

$$u_{\boldsymbol{z}}(g_i) = A_i, \text{ and } (g_i)_{A_i,\mathbf{0}} = 1.$$

Then, for $i = 1, \dots, r$, we have

$$\nu_{c,d}(g_i) = L_c(A_i) \text{ and } \nu_{c,d}(g_i - \boldsymbol{z}^{A_i}) > L_c(A_i). \tag{2}$$

Proof. Fix i, with $1 \leq i \leq r$. We know that $\nu_{\boldsymbol{z}}(g_i) = |A_i|$. Let $A \in \mathcal{E}_{\boldsymbol{z}}(g_i)$, and assume $|A| > |A_i|$. If $|A| > t$, we have the following inequalities

$$L_c(A) > t \geq |A_i| > L_c(A_i).$$

On the other hand, if $|A| \leq t$, then $|A| - L_c(A) < 10^{-m}$, so

$$L_c(A) > |A| - \frac{1}{10^m} > |A| - 1 \geq |A_i| > L_c(A_i).$$

Suppose now that $|A| = |A_i|$. If $A <_{lex} A_i$, then $L_c(A_i) < L_c(A)$. If $A >_{lex} A_i$, then $(g_i)_A(\mathbf{0}) = 0$ and $\nu_{c,d}\big((g_i)_A\big) \geq 10^{-m}$. Thus,

$$\begin{aligned}\nu_{c,d}\big((g_i)_A \boldsymbol{z}^A\big) &= \nu_{c,d}\big((g_i)_A\big) + L_c(A)\\ &\geq \frac{1}{10^m} + L_c(A) > \frac{1}{10^m} + L_c(A_i) - \frac{1}{10^m} = L_c(A_i).\end{aligned}$$ ∎

Note that this proves the necessary condition of Theorem 3 to have a division basis.

Lemma 7. (WHD1). *Given* $\{g_1, \ldots, g_r\}$ *such that* $u_{\boldsymbol{z}}(g_i) = A_i$, *for every* $g \in R$ *there are unique series* $h_0, \ldots, h_r \in R$ *such that*

$$g = h_0 + \sum_{i=1}^{r} h_i g_i$$

verifying

$$\nu_{\boldsymbol{z}}(h_i) \geq \nu_{\boldsymbol{z}}(g) - |A_i| \qquad \textit{for } i = 1, \ldots, r; \tag{3}$$

and

$$\mathcal{E}_{\boldsymbol{z}}\left(h_i \boldsymbol{z}^{A_i}\right) \subset \Delta_i \qquad \textit{for } i = 0, \ldots, r.$$

Proof. The proof falls into three parts: existence, uniqueness and convergence of the series h_i. We will follow this order.

Existence.

We have $u_{\boldsymbol{z}}(g_i) = A_i$. If needed, dividing by the coefficient of $\boldsymbol{z}^{A_i}$, which is a unit in $\mathbb{C}\{\boldsymbol{w}\}$, we can suppose without loss of generality that the coefficient of $\boldsymbol{z}^{A_i}$ is 1. Hence we are in the hypotheses of the previous lemma.

The idea of the existence proof is to describe (almost) algorithmically the process of Weierstrass-Hironaka division to keep track of partial quotients and remainders. We can then prove that the inequality (3) holds by induction, using partial bounds in every induction step.

Let us fix the power series to be divided, $g \in \mathbb{C}\{\boldsymbol{z}, \boldsymbol{w}\}$. We will arrange the exponents of g into Δ_0, Δ_1, ..., Δ_r. If we write $A_0 = \mathbf{0}$, we will denote by $h_i^{(0)}$, for $0 \leq i \leq r$, the sum of all terms of type $g_A \boldsymbol{z}^{A - A_i}$, where $A \in \mathcal{E}_{\boldsymbol{z}}(g) \cap \Delta_i$.

We then have:

(i) On the one hand,

$$g = g^{(0)} = h_0^{(0)} + h_1^{(0)} \boldsymbol{z}^{A_1} + \cdots + h_r^{(0)} \boldsymbol{z}^{A_r} = h_0^{(0)} + \sum_{i=1}^{r} h_i^{(0)} \boldsymbol{z}^{A_i}. \tag{4}$$

(ii) On the other hand, $\nu_{\boldsymbol{z}}\big(h_i^{(0)}\big) \geq \nu_{\boldsymbol{z}}(g) - |A_i|$, for $1 \leq i \leq r$; and equality must hold for, at least, one index.

Let us write $g_i = \boldsymbol{z}^{A_i} + p_i$, with $1 \leq i \leq r$. From (2) we know that $\nu_{c,d}(p_i) > L_c(A_i)$. Equation (4) above can be rewritten as

$$g = g^{(0)} = h_0^{(0)} + \sum_{i=1}^{r} h_i^{(0)} (g_i - p_i) = h_0^{(0)} + \sum_{i=1}^{r} h_i^{(0)} g_i - \sum_{i=1}^{r} h_i^{(0)} p_i,$$

and

$$\begin{aligned}\nu_{\boldsymbol{z}}\left(-\sum_{i=1}^{r} h_i^{(0)} p_i\right) &\geq \min_{1\leq i\leq r}\bigl\{\nu_{c,d}\bigl(h_i^{(0)}\bigr)+\nu_{c,d}(p_i)\bigr\}\\ &> \min_{1\leq i\leq r}\bigl\{\nu_{c,d}\bigl(h_i^{(0)}\bigr)+L_c(A_i)\bigr\},\end{aligned}$$

because of (2).

Let us now sort out the $\boldsymbol{z}$-exponents of

$$g^{(1)} = -\sum_{i=1}^{r} h_i^{(0)} p_i$$

into $\Delta_0, \Delta_1, \ldots, \Delta_r$. For every $1 \leq i \leq r$, we will denote by $h_i^{(1)}$ the sum of all terms of type $g_A^{(1)} \boldsymbol{z}^{A-A_i}$, where $A \in \mathcal{E}_{\boldsymbol{z}}\bigl(g^{(1)}\bigr) \cap \Delta_i$. We then have

(i) The equality

$$-\sum_{i=1}^{r} h_i^{(0)} p_i = h_0^{(1)} + \sum_{i=1}^{r} h_i^{(1)} \boldsymbol{z}^{A_i},$$

holds, so

$$g = \bigl[h_0^{(0)} + h_0^{(1)}\bigr] + \sum_{i=1}^{r}\bigl[h_i^{(0)} + h_i^{(1)}\bigr] g_i - \sum_{i=1}^{r} h_i^{(1)} p_i.$$

(ii) The bound

$$\begin{aligned}\nu_{\boldsymbol{z}}\bigl(h_i^{(1)}\bigr) &\geq \nu_{\boldsymbol{z}}\bigl(g^{(1)}\bigr) - |A_i|\\ &\geq \min_{1\leq j\leq r}\bigl\{\nu_{\boldsymbol{z}}\bigl(h_j^{(0)}\bigr) + \nu_{\boldsymbol{z}}(p_j)\bigr\} - |A_i|\\ &\geq \min_{1\leq j\leq r}\bigl\{\nu_{\boldsymbol{z}}(g) - |A_j| + |A_j|\bigr\} - |A_i|\\ &= \nu_{\boldsymbol{z}}(g) - |A_i|.\end{aligned}$$

Before carrying on, we want to prove that

(i) At every induction step $\nu_{\boldsymbol{z}}\bigl(h_i^{(n)}\bigr) \geq \nu_{\boldsymbol{z}}(g) - |A_i|$.

(ii) We can formally add the sequences

$$\bigl\{h_i^{(n)}\bigr\}_{n\geq 0}$$

for every $i = 0, 1, \ldots, r$. It obviously suffices to show that we can formally add the sequences for $i = 1, \ldots, r$; and to prove this last statement, it suffices to show that the $L_{c,d}$-orders are strictly increasing.

Let us then show by induction that the $L_{c,d}$-orders are strictly increasing. The proof of the basic case,

$$\nu_{c,d}\bigl(h_i^{(1)}\bigr) > \nu_{c,d}\bigl(h_i^{(0)}\bigr),$$

is particularly easy. Indeed, to compute the terms of $h_i^{(1)}$ we can take

$$-\sum_{i=1} h_i^{(0)} p_i$$

so that every term of $h_i^{(0)}$ is multiplied by terms of $L_{c,d}$-order higher than $L_c(A_i)$. Finally, we arrange every term into Δ_i and divide it by $\boldsymbol{z}^{A_i}$. This proves that

$$\nu_{c,d}\big(h_i^{(1)}\big) > \nu_{c,d}\big(h_i^{(0)}\big),$$

for every $1 \leq i \leq r$.

Let us now prove the induction step using some formal computations:

(i) Suppose we have expresssions

$$-\sum_{i=1}^{r} h_i^{(n)} p_i = h_0^{(n+1)} + \sum_{i=1}^{r} h_i^{(n+1)} \boldsymbol{z}^{A_i},$$

so that

$$g = \sum_{j=0}^{n+1} h_0^{(j)} + \sum_{i=1}^{r} \left(\sum_{j=0}^{n+1} h_i^{(j)} \right) g_i - \sum_{i=1}^{r} h_i^{(n+1)} p_i.$$

(ii) $\nu_{\boldsymbol{z}}\big(h_i^{(n+1)}\big) \geq \nu_{\boldsymbol{z}}(g) - |A_i|$.

(iii) $\nu_{c,d}\big(h_i^{(n+1)}\big) > \nu_{c,d}\big(h_i^{(n)}\big)$.

Let us sort out the $\boldsymbol{z}$-exponents of

$$g^{(n+2)} = -\sum_{i=1}^{r} h_i^{(n+1)} p_i$$

into $\Delta_0, \Delta_1, \ldots, \Delta_r$. For $1 \leq i \leq r$, we denote by $h_i^{(n+2)}$ the sum of all terms of type $g_A^{(n+2)} \boldsymbol{z}^{A-A_i}$, where $A \in \mathcal{E}_{\boldsymbol{z}}\big(g^{(n+2)}\big)$ and $A \in \Delta_i$. We then have:

(i) The equality

$$-\sum_{i=1}^{r} h_i^{(n+1)} p_i = h_0^{(n+2)} + \sum_{i=1}^{r} h_i^{(n+2)} \boldsymbol{z}^{A_i},$$

holds, so

$$g = \sum_{j=0}^{n+2} h_0^{(j)} + \sum_{i=1}^{r} \left(\sum_{j=0}^{n+2} h_i^{(j)} \right) g_i - \sum_{i=1}^{r} h_i^{(n+2)} p_i.$$

(ii) The following bound holds

$$\begin{aligned}
\nu_{\boldsymbol{z}}\big(h_i^{(n+2)}\big) &\geq \nu_{\boldsymbol{z}}\big(g^{(n+2)}\big) - |A_i| \\
&\geq \min_{1 \leq j \leq r} \big\{ \nu_{\boldsymbol{z}}\big(h_j^{(n+1)}\big) + \nu_{\boldsymbol{z}}(p_j) \big\} - |A_i| \\
&\geq \min_{1 \leq j \leq r} \big\{ \nu_{\boldsymbol{z}}(g) - |A_j| + |A_j| \big\} - |A_i| \\
&= \nu_{\boldsymbol{z}}(g) - |A_i|,
\end{aligned}$$

(iii) and so does the bound $\nu_{c,d}\big(h_i^{(n+2)}\big) > \nu_{c,d}\big(h_i^{(n+1)}\big)$, by construction.

Since $h_i^{(n)}$ are formally convergent, we can write

$$h_i = \sum_{n=0}^{\infty} h_i^{(n)}, \qquad 0 \leq i \leq r,$$

and

$$g = h_0 + \sum_{i=1}^{r} h_i g_i$$

because the remainders

$$-\sum_{i=1}^{r} h_i^{(n+2)} p_i$$

converge formally to zero. Condition 2 of the induction step implies that

$$\nu_{\boldsymbol{z}}(h_i) \geq \nu_{\boldsymbol{z}}(g) - |A_i|, \qquad 0 \leq i \leq r,$$

as we wanted to prove. Please note that our inductive definition of the series h_i implies that

$$\mathcal{E}_{\boldsymbol{z}}\left(h_i \boldsymbol{z}^{A_i}\right) \subset \Delta_i.$$

Uniqueness.

The uniqueness condition amounts to prove that 0 only admits the trivial division. That is, if we have

$$0 = h_0 + \sum_{i=1}^{r} h_i g_i,$$

following the above process, then it must hold $h_i = 0$, for $i = 0, \ldots, r$. Uniqueness is a not completely immediate consequence of the way the quotients h_i are built.

We should mind that, if there exists some i such that there is some j with $A_i \in A_j + \mathbb{Z}^c_{\geq 0}$ then $h_i = 0$, so we will assume that this is not the case.

If we write, as above,

$$g_i = \sum (g_i)_A \, \boldsymbol{z}^A,$$

then, under our assumptions, $(g_i)_A \notin (\boldsymbol{w})R$. Indeed, if $L_c(A) \leq L_c(A_i)$ but $A \neq A_i$ then $(g_i)_A \in (\boldsymbol{w})R$. The idea of the proof, following [1] is to show that

$$(h_i)_A \in (\boldsymbol{w})R, \text{ for all } i = 0, \ldots, r \text{ and for all } A \in \mathbb{Z}^c_{\geq 0}.$$

Once this is proved, it will become clear that the same goes for any $(\boldsymbol{w})^s R$, and thus $h_i = 0$ for $i = 0, \ldots, r$. The proof will consist of a kind of double induction on $i \in \{0, 1, \ldots, r\}$ and $A \in \mathbb{Z}^c_{\geq 0}$.

Step $(0, A)$ for $A \leq A_1$. We show that $(h_0)_A \in (\boldsymbol{w})R$ when $A \leq A_1$. We have

$$(h_0)_A = -\sum_{i=1}^{r} \sum_{B+C=A} (h_i)_B\,(g_i)_C\,,$$

and, moreover $A \leq A_i$ for $i = 1, \ldots, r$. Therefore, $L_c(A) \leq L_c(A_i)$. If we had $A = A_1$ then $(h_0)_A = 0$ and we are done. Otherwise we have (as noted above) $(g_i)_C \in (\boldsymbol{w})R$ for all the non-zero terms appearing in the sum above. From this, it follows $(h_0)_A \in (\boldsymbol{w})R$.

Step $(1, \mathbf{0})$**.** As above,

$$0 = (h_0)_{A_1} + \sum_{i=1}^{r} \sum_{B+C=A_1} (h_i)_B\,(g_i)_C\,,$$

but, since we have $\mathcal{E}_{\boldsymbol{z}}\,(h_0) \subset \Delta_0$, it is clear that

$$0 = \sum_{i=1}^{r} \sum_{B+C=A_1} (h_i)_B\,(g_i)_C$$

Let us examine this sum with more detail. We have:

- A term $(h_1)_{\mathbf{0}}\,(g_1)_{A_1}$, from which we already know that $(g_1)_{A_1} \notin (\boldsymbol{w})R$.
- Terms with $i \geq 2$, where $L_c(C) < L_c(A_i)$ and hence $(g_i)_C \in (\boldsymbol{w})R$.
- Terms with $i = 1$, where $C \neq A_1$, and then $(g_1)_C \in (\boldsymbol{w})R$.

These three facts together imply that $(h_1)_{\mathbf{0}} \in (\boldsymbol{w})R$.

Induction step. Given some $A \in \mathbb{Z}^c_{\geq 0}$, let us assume the following two facts,

H1. For any $A' < A$, we have $(h_0)_{A'} \in (\boldsymbol{w})R$.

H2. For any $i = 1, .., r$ with $A_i < A$, and for any A' with $A_i + A' < A$, we have $(h_i)_{A'} \in (\boldsymbol{w})R$.

We will prove that H1 and H2 still hold for $A' = A$ and for A' such that $A_i + A' = A$, and this will finish the uniqueness proof.

Let us assume, to begin with, $A \notin \Delta_i$ for $i = 1, \ldots, r$. Then H2 holds immediately (as there is no such A'). As for H1, let us write

$$0 = (h_0)_A + \sum_{i=1}^{r} \sum_{B+C=A} (h_i)_B\,(g_i)_C\,,$$

where

(i) If $A < A_i$, then $(g_i)_A \in (\boldsymbol{w})R$.

(ii) If $A > A_i$, and we have a term $(h_i)_B\,(g_i)_C$, with $B + C = A$, either B verifies $A_i + B < A$, in which case $(h_i)_B \in (\boldsymbol{w})R$ (by H2) or $A_i + B \geq A$, in which case $C \leq A_i$. As $C \neq A_i$ (because $A \notin \Delta_i$), $(g_i)_C \in (\boldsymbol{w})R$.

This proves that $(h_0)_A = 0$.

We can then assume

$$A \in \bigcup_{i=1}^{r} \Delta_i,$$

in which case H1 holds immediately, and we only have to care about proving H2. The easiest situation here is when $A = A_k$ for some $k = 1, \ldots, r$. If this is the case, we only have to show that $(h_k)_{\mathbf{0}} \in (\boldsymbol{w})R$ (since we can assume without loss of generality that $A_i \notin A_j + \mathbb{Z}^c_{\geq 0}$ for $i \neq j$). As above, write

$$0 = (h_0)_{A_k} + \sum_{i=1}^{r} \sum_{B+C=A_k} (h_i)_B \, (g_i)_C \, ,$$

and, following the same kind of arguments, it is straightforward realizing:

(i) $(g_k)_{A_k} \notin (\boldsymbol{w})R$.

(ii) $(g_i)_C \in (\boldsymbol{w})R$ when $i \geq k$ and $C \neq A_k$, or when $i < k$ and $C \leq A_i$.

1. $(h_i)_B \notin (\boldsymbol{w})R$ when $i < k$ and $C > A_i$, from H2.

Therefore $(h_k)_{\mathbf{0}} \in (\boldsymbol{w})R$. It remains to consider the case where $A \neq A_k$ for all $k = 1, \ldots, r$. Say $A \in \Delta_l$, that is

$$l = \min\left\{i \mid 1 \leq i \leq r,\ A \in A_i + \mathbb{Z}^c_{\geq 0}\right\},$$

and write once more

$$0 = (h_0)_A = \sum_{i=1}^{r} \sum_{B+C=A} (h_i)_B \, (g_i)_C \, .$$

If $A \notin A_i + \mathbb{Z}^c_{\geq 0}$ then is once again trivial realizing that $(h_i)_B \, (g_i)_C \in (\boldsymbol{w})R$. If otherwise, then

(i) $(g_i)_C \in (\boldsymbol{w})R$ when $C \neq A_i$ and $C \leq A_i$.

(ii) $(h_i)_B \notin (\boldsymbol{w})R$ when $C > A_i$, from H2.

(iii) $(g_i)_{A_i} \notin (\boldsymbol{w})R$.

But, if $i > l$, we know that

$$\mathcal{E}_{\boldsymbol{z}}\left(h_i \boldsymbol{z}^{A_i}\right) \subset \Delta_i,$$

and since $A \notin \Delta_i$, it must be $(h_i)_{A-A_i} = 0$. Therefore, if $i = l$, then $(h_l)_{A-A_l} \in (\boldsymbol{w})R$. This finishes our proof of H2.

Convergence. We follow here the ideas in [8], adapted to our particular case. First, for any formal series f, and for $\rho > 0$ define

$$\|f\| = \sum_{(A,B)\in\mathbb{Z}_{\geq 0}^{c+d}} |f_{(A,B)}|\,\rho^{L_{c,d}(A,B)}.$$

Since

$$10^m L_{c,d}(\cdot,\cdot) > |\cdot,\cdot| > L_{c,d}(\cdot,\cdot),$$

it follows that $f \in R$ if and only if $\|f\| < \infty$ for some $\rho > 0$.

Now, let us take

$$g^{(n+1)} = -\sum_{i=1}^{r} h_i^{(n)} p_i, \tag{5}$$

as we did for the proof of existence. Also, since $p_i = g_i - \boldsymbol{z}^{A_i}$, we can write

$$g^{(n)} = h_0^{(n)} + \sum_{i=1}^{r} h_i^{(n)} \boldsymbol{z}^{A_i} = h_0^{(n)} + \sum_{i=1}^{r} h_i^{(n)} g_i + g^{(n+1)}.$$

Taking norms in the first equality,

$$\|g^{(n)}\| = \|h_0^{(n)}\| + \|h_1^{(n)}\| \cdot \rho^{L_c(A_1)} + \cdots + \|h_r^{(n)}\| \cdot \rho^{L_c(A_r)}, \tag{6}$$

and it follows that

$$\|h_0^{(n)}\| \leq \|g^{(n)}\|,$$

and

$$\|h_i^{(n)}\| \leq \|g^{(n)}\| \rho^{-L_c(A_i)}, \qquad i = 1, \ldots, r.$$

From Equation (5), we now obtain the bound

$$\|g^{(n+1)}\| \leq \|h_1^{(n)}\| \cdot \|p_1\| + \cdots + \|h_r^{(n)}\| \cdot \|p_r\|,$$

so that

$$\|g^{(n+1)}\| \leq \big(\|p_1\| \cdot \rho^{-L_c(A_1)} + \cdots + \|p_r\| \cdot \rho^{-L_c(A_r)}\big) \|g^{(n)}\|.$$

It is possible to bound the expression inside the parenthesis by ϵ. This can be done since, from our particular set-up, $\nu_{c,d}(p_i) > \nu_{c,d}(\boldsymbol{z}^{A_i}) = L_c(A_i)$, and, hence, we can choose a suitably small ρ such that $\|p_i\|/\rho^{L_c(A)}$ is arbitrarily small.

Taking sums in (6),

$$\sum_{n\geq 0} \Big(\|h_0^{(n)}\| + \|h_1\|^{(n)} \cdot \rho^{L_c(A_1)} + \cdots + \|h_r^{(n)}\| \cdot \rho^{L_c(A_r)}\Big) = \sum_{n\geq 0} \|g^{(n)}\|, \tag{7}$$

and from the previous bound we get

$$\|h_0\| + \|h_1\| \cdot \rho^{L_c(A_1)} + \cdots + \|h_r\| \cdot \rho^{L_c(A_r)} \leq \sum_{n\geq 0} \epsilon^n \|g^{(0)}\| = \frac{1}{1-\epsilon}\|g\|,$$

which shows that $h_i \in R$ for $i = 0, \ldots, r$. ∎

We can now go back now to our original situation and bring back the ideal I.

Lemma 8. (WHD2). *For every $A \in u_{\boldsymbol{z}}(I)$ there exists a series $h_{A,0}$ verifying the following properties:*

$$\mathcal{E}_{\boldsymbol{z}}(h_{A,0}) \subset \Delta_0, \quad \boldsymbol{z}^A - h_{A,0} \in I, \quad \nu_{\boldsymbol{z}}(h_{A,0}) \geq |A|. \tag{8}$$

Moreover, all series $h_{A,0}$ are convergent in an open polydisk K_σ, independently of A.

Proof. Choose any division basis $g_1, \ldots, g_r \in I$, as given by WHD1, and consider, for every A, the Weierstrass-Hironaka division

$$\boldsymbol{z}^A = h_{A,0} + \sum_{i=1}^{r} h_{A,i} g_i.$$

It follows from WHD1 that $h_{A,0}$ verifies the properties of Equation (8) for every $A \in \mathbb{Z}^c_{\geq 0}$. Now, if $g_1, \ldots, g_r$ are convergent in the closure of the polydisk K_ρ of radius ρ centered at the origin, and if $\sigma \in \mathbb{R}_+$, $0 < \sigma < \rho$, then $h_{A,i}$ is convergent in K_σ, for all $i = 0, 1, \ldots, r$, because $\boldsymbol{z}^A$ converges everywhere. This finishes the proof. ∎

We will now fix a very special division basis. Consider any division basis $\{g_1, \ldots, g_r\}$ given by WHD1, and the corresponding division basis $\{f_1, \ldots, f_r\}$ given by WHD2, defined as

$$f_i = \boldsymbol{z}^{A_i} - h_{A_i,0}.$$

Note that $\{f_1, \ldots, f_r\}$ verifies the same formal conditions as $\{g_1, \ldots, g_r\}$, i.e., $u_{\boldsymbol{z}}(f_i) = A_i$, and the coefficient of $\boldsymbol{z}^{A_i}$ is 1, for $1 \leq i \leq r$. We can redo the proof of WHD1 for this basis, and write a new division

$$\boldsymbol{z}^A = h_{A,0} + \sum_{i=1}^{r} h_{A,i} f_i, \qquad A \in u(I),$$

keeping the same notation for the common convergence disk, and even for quotients and remainders.

We thus have a family,

$$\boldsymbol{z}^A - h_{A,0} \in I, \qquad A \in u(I) \tag{9}$$

verifying the conditions of WHD2, such that

$$\boldsymbol{z}^{A_1} - h_{A_1,0}, \ldots, \boldsymbol{z}^{A_r} - h_{A_r,0}, \tag{10}$$

verify both WHD1 and WHD2.

Definition 9. We will call a family as in Equation (9) a specially prepared family, and the finite subset of Equation (10) an specially prepared set.

4. Graded Rings and Normal Flatness

We now turn to the properties of the ring $gr_{(\mathfrak{p}/I)}(R/I)$, of (1). Let us define Γ as

$$\Gamma = \left\{ (\boldsymbol{z}^*)^A \quad | \quad A \notin u_{\boldsymbol{z}}(I) \right\}.$$

This set will be extremely important in what follows, but we state here its main property.

Proposition 10. *The set Γ is a minimal generating system for $gr_{(\mathfrak{p}/I)}(R/I)$ as an $(R/\mathfrak{p})$-module.*

Proof. Since

$$gr_{(\mathfrak{p}/I)}(R/I) \simeq \mathbb{C}\{\boldsymbol{w}\}[\boldsymbol{z}^*],$$

it is clear that the set of all monomials in $\boldsymbol{z}^*$ is a generating system as a $\mathbb{C}\{\boldsymbol{w}\}$-module.

Take any $A \in u_{\boldsymbol{z}}(I)$ and consider the series $\boldsymbol{z}^A - h_{A,0}$ from an specially prepared family. Since $\nu_{\boldsymbol{z}}(h_{A,0}) \geq |A|$, the initial form of $\boldsymbol{z}^A - h_{A,0}$, taken modulo $In_{\mathfrak{p}}(I)$, gives $(\boldsymbol{z}^*)^A$ as a linear combination of elements in $\Delta_0 = u_{\boldsymbol{z}}(I)$.

We show that Γ is minimal. Suppose it is not. Then there exists an exponent $B \notin u_{\boldsymbol{z}}(I)$ such that

$$(\boldsymbol{z}^*)^B = \sum_{\substack{C \notin u_{\boldsymbol{z}}(I) \\ |C|=|B| \\ C \neq B}} \phi_C(\boldsymbol{w})(\boldsymbol{z}^*)^C$$

because the generators are homogeneous. Then,

$$\tilde{\boldsymbol{z}}^B - \sum_{\substack{C \notin u_{\boldsymbol{z}}(I) \\ |C|=|B| \\ C \neq B}} \phi_C(\boldsymbol{w})\tilde{\boldsymbol{z}}^C$$

would be the initial form of a series in I, with no exponent inside $u_{\boldsymbol{z}}(I)$. This is a contradiction. ∎

Remark 11. The ring $gr_{(\mathfrak{p}/I)}(R/I)$ is flat as an $(R/\mathfrak{p})$-module if and only if it is free. Indeed, $gr_{(\mathfrak{p}/I)}(R/I)$ is flat as an $(R/\mathfrak{p})$-module if and only if every homogeneous component is flat, because flatness of direct sums is equivalent to flatness of each direct summand. Since every homogeneous component is of finite type, flat and free are equivalent.

Remark 12. If $gr_{(\mathfrak{p}/I)}(R/I)$ is $(R/\mathfrak{p})$-flat, then the generating system Γ is in fact a basis, and vice-versa: since $gr_{\mathfrak{p}/I}(R/I)$ is a direct sum of its homogeneous components, it would suffice to show that every mininal generating systems of an $(R/\mathfrak{p})$-module of finite type is a basis. However, this property holds true, in general, for every module of finite type over a noetherian local ring (Nakayama's Lemma).

We can now introduce the two main concepts we are to work with: equimultiplicity and normal flatness.

Definition 13. If $f \in R$ is a non zero, non unit, we will say that f *is* $\mathfrak{p}$-*equimultiple* (or simply *equimultiple*) if $\nu_{\boldsymbol{z}}(f) = \nu_{\boldsymbol{z},\boldsymbol{w}}(f)$.

A basis $\{g_1, \ldots, g_s\}$ of I will be called equimultiple if every g_i is equimultiple. A basis of I will be called *standard* if the ordinary inital forms $\{\bar{g}_1, \ldots, \bar{g}_s\}$ generate the initial ideal $In_{\mathfrak{m}}(I)$.

We say that X *is normally flat along* W *at the origin* if $gr_{(\mathfrak{p}/I)}(R/I)$ is a free (equivalently flat) $(R/\mathfrak{p})$-module.

As mentioned in Remark 1, it is easy to check that if $gr_{(\mathfrak{p}/I)}(R/I)$ is a free $(R/\mathfrak{p})$-module, then $u(I)$ is necessarily non-empty, which provides a different justification for our assumption.

Also, note that the definition we give here is but the local version of the classic, geometric definition in terms of sheafs of graded algebras.

5. Combinatorial Characterization of Normal Flatness

The aim of this section is to prove the following theorem, which characterizes a geometric situation, normal flatness, in combinatorial terms. We keep the same notations as in the previous sections.

Theorem 14. (Characterization of Normal Flatness). *Under the notations we used throughout previous sections, the following conditions are equivalent:*

(i) X *is normally flat along* W *at* **0**.

(ii) I *has an equimultiple standard basis.*

Proof of $1 \Rightarrow 2$. Suppose that X is normally flat along W at **0**, i.e., that $gr_{(\mathfrak{p}/I)}(R/I)$ is free over $R/\mathfrak{p}$. Then, Γ is a basis of $gr_{(\mathfrak{p}/I)}(R/I)$. We will now show that the specially prepared set $\{f_1, \ldots, f_r\}$ given by

$$f_i = \boldsymbol{z}^{A_i} - h_{A_i,0}$$

is actually an equimultiple standard basis.

Let $f \in I$. By WHD1, we can write

$$f = h_0 + \sum_{i=1}^{r} h_i f_i,$$

where

$$\mathcal{E}_{\boldsymbol{z}}(h_0) \subset \Delta_0, \qquad \mathcal{E}_{\boldsymbol{z}}\left(h_i \boldsymbol{z}^{A_i}\right) \subset \Delta_i, \qquad \nu_{c,d}(h_i) \geq \nu_{c,d}(f) - \nu_{c,d}(f_i),$$

for $i = 1, \ldots, r$. Since $h_0 \in I$, $\mathcal{E}_{\boldsymbol{z}}(h_0) \subset \Delta_0$, and Γ is a basis of $gr_{(\mathfrak{p}/I)}(R/I)$ we must have $h_0 = 0$.

In fact, assume $h_0 \neq 0$. Then, its initial form $\tilde{h}_0$ would be 0 modulo $In_{\mathfrak{p}}(I)$, so every coefficient must be 0. Thus, we have

$$f = \sum_{i=1}^{r} h_i f_i,$$

which means that $\{f_1, \ldots, f_r\}$ is a basis of I.

Now consider $\bar{h}_i$ and $\bar{f}_i$ the ordinary initial forms of h_i and f_i, respectively. It is clear that $\bar{f}_i \in \mathbb{C}[\boldsymbol{z}]$ and

$$\bar{f}_i = \boldsymbol{z}^{A_i} + \text{lower terms in lexicographic order.}$$

If every initial form $\bar{h}_i \bar{f}_i$ has a different degree, then it is obvious that $\bar{f}$ equals the product $\bar{h}_i \bar{f}_i$ of least degree. Suppose then that there are more than one initial form $\bar{h}_i \bar{f}_i$ of minimum degree, say $\bar{h}_{i_1} \bar{f}_{i_1}, \ldots, \bar{h}_{i_s} \bar{f}_{i_s}$. Let

$$\alpha_{i_1}(\boldsymbol{w}) \boldsymbol{z}^{B_{i_1}}, \ldots, \alpha_{i_s}(\boldsymbol{w}) \boldsymbol{z}^{B_{i_s}}$$

be the greatest terms for lexicographic order in $\boldsymbol{z}$ of $\bar{h}_{i_1}, \ldots, \bar{h}_{i_s}$, respectively. Then,

$$\alpha_{i_1}(\boldsymbol{w}) \boldsymbol{z}^{B_{i_1}+A_{i_1}}, \ldots, \alpha_{i_s}(\boldsymbol{w}) \boldsymbol{z}^{B_{i_s}+A_{i_s}}$$

are the greatest degree terms for the lexicographic order with regard to $\boldsymbol{z}$ in the $\boldsymbol{z}$-forms $\bar{h}_{i_1} \bar{f}_{i_1}, \ldots, \bar{h}_{i_s} \bar{f}_{i_s}$.

This means that the term of greatest degree among them, $\alpha_{ij}(\boldsymbol{w}) \boldsymbol{z}^{B_{i_j}+A_{i_j}}$, cannot cancel out, because $\mathcal{E}_{\boldsymbol{z}}(h_i \boldsymbol{z}^{A_i}) \subset \Delta_i$. Hence, the initial forms of $h_i f_i$ of least degree cannot cancel out, and so, $\bar{f} = \sum_{i=1}^{r} \bar{h}_i \bar{f}_i$. Then, the chosen basis is standard (and equimultiple by construction). This finishes the the proof. ∎

The backwards implication of Theorem 14 builds on top of the next result, rather complicated to prove.

Proposition 15. *For every non zero $f \in I$, if $A = \min_{lex}(\mathcal{E}_{\boldsymbol{z}}(\bar{f}))$, then $A \in u(I)$ and there exists $g \in I$ equimultiple such that $u_{\boldsymbol{z}}(g) = A$.*

Proof. Let $\{g_1, \ldots, g_s\}$ be an equimultiple standard basis of I. Since $\{g_1, \ldots, g_s\}$ is an equimultiple standard basis, we have $\bar{g}_i \in \mathbb{C}[\boldsymbol{z}]$ and

$$\bar{f} = \sum_{i=1}^{s} \phi_i \bar{g}_i, \qquad \phi_i \in \mathbb{C}[\boldsymbol{z}, \boldsymbol{w}],$$

where every ϕ_i is either zero or a homogeneous polynomial in $(\boldsymbol{z}, \boldsymbol{w})$ of degree $\deg(\bar{f}) - \deg(\bar{g}_i)$.

Decomposing every non zero ϕ_i into the sum of its homogeneous components with regard to $\boldsymbol{z}$, we may write

$$\phi_i = \sum_{j=n_i}^{m_i} \phi_{i,j}, \qquad \phi_{i,n_i} \neq 0, \quad m_i \geq n_i.$$

If we had

$$\sum_{i=1}^{s} \phi_{i,n_i} \bar{g}_i = 0,$$

then

$$\sum_{i=1}^{s} \Bigl(\sum_{j=n_i+1}^{m_i} \phi_{i,j} \Bigr) \bar{g}_i = \bar{f},$$

and we can therefore substitute ϕ_i by

$$\sum_{j=n_i+1}^{m_i} \phi_{i,j},$$

whenever possible. It is possible at least for one of the ϕ_i. If we carry on with this process, we must stop at some point, because

$$0 \neq \bar{f} = \sum_{i=1}^{s} \phi_i \bar{g}_i.$$

Thus, we may suppose without loss of generality that

$$g' = \sum_{i=1}^{s} \phi_{i,n_i} \bar{g}_i \neq 0.$$

Consider now the series

$$h = \sum_{i=1}^{s} \phi_i g_i,$$

which verifies:

(i) $\bar{h} = \bar{f}$.

(ii) $\nu_{\boldsymbol{z}}(h) = \nu_{\boldsymbol{z}}\left(\bar{h}\right) = \nu_{\boldsymbol{z}}\left(\bar{f}\right)$.

To check the second property, note that, by construction,

$$\nu_{\boldsymbol{z}}(\bar{h}) = \nu_{\boldsymbol{z}}(g') = \min_{1 \leq i \leq s} \bigl\{ \nu_{\boldsymbol{z}}(\phi_i) + \nu_{\boldsymbol{z}}(g_i) \bigr\}.$$

Let i be any index, $1 \leq i \leq s$, such that there exists a term in $\phi_{i,n_i} \bar{g}_i$ that occurs also in g'. Then $\nu_{\boldsymbol{z}}(\phi_i) + \nu_{\boldsymbol{z}}(g_i) = \nu_{\boldsymbol{z}}\left(\bar{h}\right)$. If $A_i' \in \mathcal{E}_{\boldsymbol{z}}(g_i)$ and $|A_i'| > \nu_{\boldsymbol{z}}(g_i)$ we have

$$\nu_{\boldsymbol{z}}\bigl(\phi_i \boldsymbol{z}^{A_i'}\bigr) = \nu_{\boldsymbol{z}}(\phi_i) + |A_i'| > \nu_{\boldsymbol{z}}(\bar{h}).$$

If $|A_i'| = \nu_{\boldsymbol{z}}(g_i)$ and the initial term of $(g_i)_{A_i'}$ does not occur in $\bar{g}_i$ (this happens only when $(g_i)_{A_i'}(\mathbf{0}) = 0$), its total order is greater than $\nu_{\mathfrak{m}}(g_i) = \nu_{\boldsymbol{z}}(g_i)$, so no term of $\phi \cdot (g_i)_{A_i'} \cdot \boldsymbol{z}^{A_i'}$ may cancel out with any term of $\phi_i \bar{g}_i$. This proves that $\nu_{\boldsymbol{z}}(h) = \nu_{\boldsymbol{z}}\left(\bar{h}\right)$.

By definition of A, we have $|A| = \nu_{\boldsymbol{z}}(h)$. Let $B \in \mathcal{E}_{\boldsymbol{w}}\big[(\bar{h})_A\big]$ be a multi-index and

$$D = \frac{1}{B!}\frac{\partial^{|B|}}{\partial \boldsymbol{w}^B}.$$

Since $(\bar{h})_A$ is a homogeneous polynomial in $\boldsymbol{w}$, we have $D\big[(\bar{h})_A\big] = h_{A,B} = f_{A,B} \in \mathbb{C}\setminus\{0\}$, for every monomial in the initial form, and

$$\begin{aligned} D(\bar{h}) &= \sum_{A'\in\mathcal{E}_{\boldsymbol{z}}(\bar{h})} D\big[(\bar{h})_{A'}\big]\boldsymbol{z}^{A'} = \sum_{\substack{A'\in\mathcal{E}_{\boldsymbol{z}}(\bar{h}) \\ |A'|=|A|}} D\left[\left(\bar{h}\right)_{A'}\right]\boldsymbol{z}^{A'} \\ &= \sum_{\substack{A'\in\mathcal{E}_{\boldsymbol{z}}(h) \\ |A'|=|A|}} f_{A',B}\boldsymbol{z}^{A'} = \sum_{i=1}^{s} D(\phi_i)\bar{g}_i \in In_{\mathfrak{m}}(I), \end{aligned}$$

because, if $A' \in \mathcal{E}_{\boldsymbol{z}}(\bar{h})$, $|A'| > |A|$, the derivation D kills every term in $(\bar{f})_{A'}$. Moreover, it cannot happen that $|A| < |A'|$ by definition of A.

If

$$g = \sum_{i=1}^{s} D(\phi_i)g_i$$

it is then clear that $\bar{g} = D\left(\bar{h}\right) \in \mathbb{C}[z]$, $u(g) = A$; and also that g is equimultiple. This can be deduced analogously to $\nu_{\boldsymbol{z}}(h) = \nu_{\boldsymbol{z}}\left(\bar{h}\right)$. This ends the proof of the proposition. ■

Proof of $2 \Rightarrow 1$. Let $\{g_1, \dots g_s\}$ be an equimultiple standard basis of I. To prove that $gr_{\mathfrak{p}/I}(R/I)$ is free over $\mathbb{C}\{\boldsymbol{w}\}$, it suffices to show that every homogenous components is free. Let us fix then a degree $n > 0$; we want to show that the set

$$\Gamma_n = \left\{(\boldsymbol{z}^*)^A \quad | \quad A \notin u_{\boldsymbol{z}}(I), |A| = n\right\}$$

is linearly independent over $\mathbb{C}\{\boldsymbol{w}\}$.

Assume it is not, and let

$$\sum_{A\in\Gamma_n} \alpha_A(\boldsymbol{w})(\boldsymbol{z}^*)^A = 0$$

be a non-trivial relation in Γ_n. Then, there exists $0 \neq f \in I$ such that

$$\tilde{f} = \sum_{A\in\Gamma_n} \alpha_A(\boldsymbol{w})\tilde{\boldsymbol{z}}^A.$$

Obviously, the ordinary initial form $\bar{f}$ of f equals the ordinary initial form of $\tilde{f}$. Then we have just proved $\min_{lex}\big(\mathcal{E}_{\boldsymbol{z}}\big(\bar{f}\big)\big) \in u_{\boldsymbol{z}}(I)$, which is a contradiction, since $\min_{lex}\big(\mathcal{E}_{\boldsymbol{z}}\big(\bar{f}\big)\big) \in \Gamma_n$. This finishes the proof of Theorem 14. ■

6. The Fundamental Theorem of Normal Flatness

We now consider the Fundamental Theorem of normal flatness, which will be useful to relate the Hilbert function with the combinatorial objects we have defined in the preceeding section. As stated in the introduction, many of the ideas in this section as well as in the following one, are also presented elsewhere (see, for instance, [3, 2, 8]).

If normal flatness does not hold, we cannot assert that the specially prepared set $\{f_1, \dots, f_r\}$ is even a basis of I. However, there exists a basis

$$I = (f_1, \dots, f_r, g_1, \dots, g_s)$$

and an open polydisk $K \subset \mathbb{C}^c \times \mathbb{C}^d$ where f_i, g_j and $\boldsymbol{z}^A - h_{A,0}$ are all convergent, for every $A \in u_{\boldsymbol{z}}(I)$. Let K' be the projection of K onto $\mathbb{C}^d$.

We will need to consider points in a neighbourhood of the origin, but inside W. Hence, we need to translate all of our concepts and objects a bit away from the origin. In order to do that, for every $\alpha \in K'$ write $I' = (f'_1, \dots, f'_r, g'_1, \dots, g'_s)$ where

$$f'_i = f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha), \qquad g'_j = g_j(\boldsymbol{z}, \boldsymbol{w}' + \alpha).$$

I' is an ideal in the ring $R' = \mathbb{C}\{\boldsymbol{z}, \boldsymbol{w}'\}$. If we set $\mathfrak{p} = (\boldsymbol{z}) \cdot R'$ and denote with tilde the initial forms of elements in

$$gr_{\mathfrak{p}}(R') = \bigoplus_{n \geq 0} \mathfrak{p}^n / \mathfrak{p}^{n+1},$$

then $In_{\mathfrak{p}}(I')$ is an ideal of $gr_{\mathfrak{p}}(R')$. We will denote with stars the classes of elements modulo I'. Beware: these notations are actually the same as for R and I, but there is no danger of confusion. If $\alpha = \mathbf{0}$ we will of course put $\boldsymbol{w}' = \boldsymbol{w}$.

For every integer $n \geq 0$, consider the $\mathbb{C}\{\boldsymbol{w}'\}$-module of finite type

$$M_n = gr_{(\mathfrak{p}/I')}(R/I')_n.$$

Let T_α be defined by

$$\begin{aligned} T_\alpha \colon \mathbb{Z}_{\geq 0} &\to \mathbb{Z}_{\geq 0} \\ n &\mapsto T_\alpha(n) = \dim_{\mathbb{C}} M_n / (\boldsymbol{w}') M_n. \end{aligned}$$

This mapping equals the number of elements in any generating system of M_n as $\mathbb{C}\{\boldsymbol{w}'\}$-module, by Nakayama's Lemma.

Theorem 16. (Fundamental Theorem of normal flatness). *Let X be normally flat along W at $\mathbf{0}$. Then,*

(i) For every $\alpha \in K'$, X is normally flat along W at α. In other words, the normal flatness condition is open.

(ii) *The function T_α is constant in K.*

Conversely, if there exists a polydisk $K'' \subset K'$ such that T_α is constant over K'', then X is normally flat along W at $\mathbf{0}$.

Proof. Assume normal flatness at $\mathbf{0}$ and let us prove i and ii. Even if normal flatness does not hold, the expressions from WHD2

$$\boldsymbol{z}^A - h_{A,0}(\boldsymbol{z}, \boldsymbol{w}' + \alpha) = \sum_{i=1}^{r} h_{A,i}(\boldsymbol{z}, \boldsymbol{w}' + \alpha) f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha), \qquad \text{for all } A \in u_{\boldsymbol{z}}(I)$$

hold, and $\tilde{h}_{A,0}(\boldsymbol{z}, \boldsymbol{w}'+\alpha)$ is a form in $\boldsymbol{z}$ of degree $|A|$ (or zero). Thus, $\boldsymbol{z}^A - \tilde{h}_{A,0}(\boldsymbol{z}, \boldsymbol{w}'+\alpha)$ is the $\boldsymbol{z}$-initial form of $\boldsymbol{z}^A - h_{A,0}(\boldsymbol{z}, \boldsymbol{w}' + \alpha)$, so

$$(\boldsymbol{z}^*)^A - \tilde{h}_{A,0}(\boldsymbol{z}^*, \boldsymbol{w}' + \alpha) = 0$$

in $gr_{(\mathfrak{p}/I')}(R'/I')$. This proves that Γ is a (homogeneus) generating system of $gr_{(\mathfrak{p}/I')}(R'/I')$.

Suppose now that $gr_{(\mathfrak{p}/I)}(R/I)$ is free over $R/\mathfrak{p}$ and let us show that Γ is also free. Suppose further that there exists a non trivial linear combination

$$\sum_{A \in \Delta_0,\ |A|=n} \phi'_A(\boldsymbol{w}')(\boldsymbol{z}^*)^A = 0$$

where n is a fixed degree. Then, there exists $f' \in I'$ such that

$$\widetilde{(f')} = \sum_{A \in \Delta_0,\ |A|=n} \phi'_A(\boldsymbol{w}')\tilde{\boldsymbol{z}}^A.$$

Let

$$f' = \sum_{i=1}^{r} h'_i(\boldsymbol{z}, \boldsymbol{w}') f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha).$$

If $h''_i(\boldsymbol{z}, \boldsymbol{w}')$ is the truncation of $h'_i(\boldsymbol{z}, \boldsymbol{w}')$ in $\boldsymbol{z}$ up to degree $n+1$ and

$$f'' = \sum_{i=1}^{r} h''_i(\boldsymbol{z}, \boldsymbol{w}') f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha),$$

then $\widetilde{(f'')} = \widetilde{(f')}$. On the other hand, if m is the maximum order of the set of non-zero coefficients of terms in $\widetilde{(f')}$ and $h_i(\boldsymbol{z}, \boldsymbol{w}')$ is the truncation of $h''_i(\boldsymbol{z}, \boldsymbol{w}')$ up to degree $m+1$ in $\boldsymbol{w}'$, it is obvious that

$$f = \sum_{i=1}^{r} h_i(\boldsymbol{z}, \boldsymbol{w}') f_i(\boldsymbol{z}, \boldsymbol{w}' + \alpha)$$

is such that

$$\tilde{f} = \sum_{A \in \Delta_0,\ |A|=n} \phi_A(\boldsymbol{w}')\tilde{\boldsymbol{z}}^A,$$

and moreover, if $\phi'_A(\boldsymbol{w}') \neq 0$, then $\phi_A(\boldsymbol{w}') \neq 0$.

Since polynomials are convergent everywhere, we can undo the change $\boldsymbol{w} = \boldsymbol{w}' + \alpha$ in the previous expression of f, and we obtain

$$f(\boldsymbol{z}, \boldsymbol{w} - \alpha) = \sum_{i=1}^{r} h_i(\boldsymbol{z}, \boldsymbol{w} - \alpha) f_i(\boldsymbol{z}, \boldsymbol{w}),$$

whose $\mathfrak{p}$-initial form is

$$\sum_{A \in \Delta_0,\ |A|=n} \phi_A(\boldsymbol{w} - \alpha)\tilde{\boldsymbol{z}}^A,$$

whenever some $\phi_A(\boldsymbol{w} - \alpha) \neq 0$.

But this condition necessarily holds. To see why, consider the ring $S = \mathbb{C}[\boldsymbol{w}]$ and, for each $n \geq 0$, denote by $S_{\leq n}$ the set of all polynomials of degree less than or equal to n. The substitutions $\boldsymbol{w} \mapsto \boldsymbol{w} - \alpha$ induce $\mathbb{C}$-automorphisms of $S_{\leq n}$.

Since these substitutions take partial sums of $\phi_A(\boldsymbol{w})$ up to order n in partial sums of $\phi_A(\boldsymbol{w} - \alpha)$ up to order n, this proves Γ is a (minimal) free homogenous generating system of $gr_{(\mathfrak{p}/I')}(R'/I')$ and the first statement of the theorem easily follows.

Conversely, let us suppose that T_α is constant over Δ. By Proposition 10, we know that Γ is a minimal system of generators of $gr_{(\mathfrak{p}/I)}(R/I)$, so

$$T_\alpha(n) = Card(\Gamma_n),$$

for every $\alpha \in \Delta$ and $n \in \mathbb{Z}_{\geq 0}$. At the beginning of this proof, we showed that Γ is a homogeneous generating system of $gr_{(\mathfrak{p}/I')}(R'/I')$. We will show now that Γ is $\mathbb{C}\{\boldsymbol{w}\}$-free.

Suppose it is not, and let $n > 0$ be an integer such that there exists a non trivial relation

$$\sum_{A \in \Delta_0,\ |A|=n} \phi_A(\boldsymbol{w})(\boldsymbol{z}^*)^A = 0.$$

By minimality of Γ as a generating system of $gr_{(\mathfrak{p}/I)}(R/I)$, we must have $\phi_A(\mathbf{0}) = 0$ for every $A \in \Delta_0$, with $|A| = n$. Let B be an index such that $\phi_B(\boldsymbol{w}) \neq 0$ and let $K'' \subset K$ be a polydisk centered at the origin such that $\phi_A(\boldsymbol{w})$ is convergent in K'' for all $A \in \Delta_0$, with $|A| = n$. Then, there exists an $\alpha \in K''$ such that $\phi_B(\alpha) \neq 0$. Thus, the expression

$$\sum_{A \in \Delta_0,\ |A|=n} \phi_A(\boldsymbol{w}' + \alpha)(\boldsymbol{z}^*)^A$$

is well defined and $\phi_A(\boldsymbol{w}' + \alpha)$ is a unit in $\mathbb{C}\{\boldsymbol{w}'\}$. This allows us to express $(\boldsymbol{z}^*)^B$ as a linear combination, with coefficients in $\mathbb{C}\{\boldsymbol{w}'\}$, of elements in $\Gamma_n \setminus \{(\boldsymbol{z}^*)^B\}$. This would imply that

$$T_\alpha(n) < Card(\Gamma_n),$$

which is not possible. This ends our proof. ∎

7. Hilbert Funcions and Normal Flatness

Our situation for this section will be $\boldsymbol{x} \in W \subset X \subset Z$, where Z is a smooth analytic space, X is a closed subspace of Z, W is a smooth subspace of X and $\boldsymbol{x}$ a point of W. Our interest is focused in the normal flatness of X along W at $\boldsymbol{x}$. If $\mathcal{O}_X$ is the structure sheaf of X, $\mathcal{O}_{X,\boldsymbol{x}}$ is the stalk at $\boldsymbol{x}$ and $\mathfrak{m}_{X,\boldsymbol{x}}$ is the maximal ideal of $O_{X,\boldsymbol{x}}$, the Hilbert function

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$

is defined by

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}}(n) = \dim_{\mathbb{C}}\big(\mathfrak{m}^n_{X,\boldsymbol{x}}/\mathfrak{m}^{n+1}_{X,\boldsymbol{x}}\big).$$

It is also useful to consider

$$\mathcal{H}^{(1)}_{X,\boldsymbol{x}}(n) = \sum_{i=0}^{n} \dim_{\mathbb{C}}\big(\mathfrak{m}^i_{X,\boldsymbol{x}}/\mathfrak{m}^{i+1}_{X,\boldsymbol{x}}\big) = \dim_{\mathbb{C}}\big(\mathcal{O}_{X,\boldsymbol{x}}/\mathfrak{m}^{n+1}_{X,\boldsymbol{x}}\big).$$

Proposition 17. *For every $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}}(n) = Card\big\{C \in \mathbb{Z}^{c+d}_{\geq 0} \textit{ such that } |C| = n,\ C \notin u_{\boldsymbol{z},\boldsymbol{w}}(I)\big\}.$$

Proof. Consider the ring

$$G_{X,\boldsymbol{x}} = \mathbb{C}[\bar{\boldsymbol{z}}, \bar{\boldsymbol{w}}]/In_{\mathfrak{m}}(I) = \mathbb{C}[\bar{\boldsymbol{z}}^*, \bar{\boldsymbol{w}}^*],$$

where $\bar{z}^*_i = \bar{z}_i + In_{\mathfrak{m}}(I)$, $\bar{w}^*_j = \bar{w}_j + In_{\mathfrak{m}}(I)$. We just have to prove that for every $n \in \mathbb{Z}_{\geq 0}$, the set

$$\bar{\Gamma}_n = \big\{(\bar{\boldsymbol{z}}^*)^A(\bar{\boldsymbol{w}}^*)^B\ ;\ |A| + |B| = n, (A,B) \notin u_{\boldsymbol{z},\boldsymbol{w}}(I)\big\}$$

is a basis of the homogeneous component $(G_{X,\boldsymbol{x}})_n$ of degree n of $G_{X,\boldsymbol{x}}$, as a $\mathbb{C}$-vector space.

The linear independence is easy: Let

$$\sum_{\substack{(A,B)\notin u_{\boldsymbol{z},\boldsymbol{w}}(I)\\ |A|+|B|=n}} \alpha_{(A,B)}(\bar{\boldsymbol{z}}^*)^A(\bar{\boldsymbol{z}}^*)^B = 0, \qquad \alpha_{(A,B)} \in \mathbb{C}.$$

If some $\alpha_{(A,B)}$ is non zero, we have a series $0 \neq f \in I$ such that

$$\bar{f} = \sum_{\substack{(A,B)\notin u_{\boldsymbol{z},\boldsymbol{w}}(I)\\ |A|+|B|=n}} \alpha_{(A,B)}\bar{\boldsymbol{z}}^A\bar{\boldsymbol{z}}^B,$$

and this is impossible since $u_{\boldsymbol{z},\boldsymbol{w}}(f) \notin u_{\boldsymbol{z},\boldsymbol{w}}(I)$.

The fact that $\bar{\Gamma}_n$ is a generating system is a consequence of the polynomial division algorithm. This ends our proof. ∎

Theorem 18. (Normal flatness and Hilbert functions). *Suppose W is locally positive dimensional at $\boldsymbol{x}$. The following conditions are equivalent:*

(i) *X is normally flat along W at $\boldsymbol{x}$.*

(ii) $u_{\boldsymbol{z},\boldsymbol{w}}(I) = u_{\boldsymbol{z}}(I) \times \mathbb{Z}^d_{\geq 0}$.

(iii) *$\mathcal{H}^{(0)}_{X,\boldsymbol{x}}$ is constant over $W \cap \Delta$, where Δ is a polydisk in $\mathbb{C}^{c+d}$ centered at $\boldsymbol{x}$ such that every series in a certain basis of I are convergent.*

(iv) *$\mathcal{H}^{(1)}_{X,\boldsymbol{x}}$ is constant over $W \cap \Delta$, where Δ is a polydisk in $\mathbb{C}^{c+d}$ centered at $\boldsymbol{x}$ such that every series in a certain basis of I are convergent.*

Proof. If X is normally flat along W at x, then the specially prepared set

$$f_i = \boldsymbol{z}^{A_i} - h_{A_i,0}$$

is an equimultiple standard basis. Since $\bar{f}_i \in \mathbb{C}[\bar{z}]$ for $i = 1, \dots, r$, we have $u_{\boldsymbol{z}}(I) \times \mathbb{Z}^d_{\geq 0} \subset u_{\boldsymbol{z},\boldsymbol{w}}(I)$. Conversely, let (A, B) be a vertex of $u_{\boldsymbol{z},\boldsymbol{w}}(I)$. Then there exists an index i, $1 \leq i \leq r$ such that $(A, B) = u_{\boldsymbol{z},\boldsymbol{w}}(f_i) = (A_i, \mathbf{0})$. Thus, $u_{\boldsymbol{z},\boldsymbol{w}}(I) \subset u_{\boldsymbol{z}}(I) \times \mathbb{Z}^d_{\geq 0}$ and equality holds.

We will now show that the second condition implies the first. By hypothesis, we have a non redundant union

$$u_{\boldsymbol{z},\boldsymbol{w}}(I) = \bigcup_{i=1}^{r} \left[(A_i, \mathbf{0}) + \mathbb{Z}^{c+d}_{\geq 0} \right],$$

We have then that the specially prepared set $\{f_1, \dots, f_r\}$ is a standard basis. It is furthermore equimultiple because if $(A, B) \in \mathcal{E}_{\boldsymbol{z},\boldsymbol{w}}(f)$ then

$$(A, B) = (A_i, \mathbf{0}) + (A', B),$$

so $|A| \geq |A_i|$.

Statement ii also implies Statement i. First of all, we show that the function $T_{\boldsymbol{x}}$ from the Fundamental Theorem uniquely determines the function $\mathcal{H}^{(0)}_{X,\boldsymbol{x}}$. We know that $u_{\boldsymbol{z},\boldsymbol{w}}(I) = u_{\boldsymbol{z}}(I) \times \mathbb{Z}^d_{\geq 0}$. If we fix a degree n and a monomial $\boldsymbol{z}^A$ with $A \notin u_{\boldsymbol{z}}(I)$, the set of all monomials $\boldsymbol{z}^A \boldsymbol{w}^B$ of degree n with fixed $\boldsymbol{z}^A$ has the same cardinal as the set of monomials of degree $n - |A|$, that is,

$$\binom{n - |A| + d - 1}{n - |A|}.$$

Hence,

$$\mathcal{H}^{(0)}_{X,\boldsymbol{x}}(n) = \sum_{m=0}^{n} \left[\sum_{A \notin u_{\boldsymbol{z}}(I),\ |A| = m} \binom{n - m - d - 1}{n - m} \right].$$

This means that $T_{\mathbf{0}}$ determines $\mathcal{H}^{(0)}_{X,\boldsymbol{x}}$. From the Fundamental Theorem, for every $\boldsymbol{y} \in \Delta$, X is normally flat alog W in $\boldsymbol{y}$, and $T_{\boldsymbol{y}} = T_{\mathbf{0}}$. Thus, $\mathcal{H}^{(0)}_{X,\boldsymbol{x}} = \mathcal{H}^{(0)}_{X,\boldsymbol{y}}$.

Statements iii and iv are equivalent, since $\mathcal{H}^{(0)}_{X,\boldsymbol{x}}$ and $\mathcal{H}^{(1)}_{X,\boldsymbol{x}}$ uniquely determine each other.

We will now show that Statement iii implies Statement i. Suppose that $gr_{(\mathfrak{p}/I)}(R/I)$ is not free over $R/\mathfrak{p}$. There exists $0 \neq f \in I$ such that $\mathcal{E}_{\boldsymbol{z}}(\tilde{f}) \subset \Delta_0$. Let us take a specially prepared family $\{f_1, \ldots, f_r\}$ and an adapted linear form L'. The Weiestrass-Hironaka division gives

$$f = h_0 + \sum_{i=1}^{r} h_i f_i$$

where $\mathcal{E}_{\boldsymbol{z}}(h_0) \subset \Delta_0$. Following the procedure for the construction of partial quotients and remainders of WHD2, we find that $\tilde{f} = \tilde{h}_0$. So, we can suppose, without loss of generality, that $0 \neq f \in I$ is such that $\mathcal{E}_{\boldsymbol{z}}(f) \subset \Delta_0$. Then, the ordinary initial form $\bar{f}$ of f has all its $\boldsymbol{z}$-exponents in Δ_0.

Let n be the smallest integer such that there exists a series f of I of order n whose ordinary initial form has terms with exponents of $\boldsymbol{z}$ in Δ_0. Let $m < n$. If $m < \nu_{\mathfrak{m}}(I)$, we cannot have relations modulo $In_{\mathfrak{m}}(I)$ among the monomials of degree m in $(\boldsymbol{z}, \boldsymbol{w})$. Suppose then that $m \geq \nu_{\mathfrak{m}}(I)$, and let $u_{\boldsymbol{z}}(I)_m = \{B_1, \ldots, B_s\}$. We have s series

$$g_i = \boldsymbol{z}^{B_i} - h_{B_i,0}$$

such that $u_{\boldsymbol{z}}(g_i) = B_i$, for $i \leq i \leq s$. These series are all equimultiple, and $\nu_{\boldsymbol{z},\boldsymbol{w}}(g_i) = m$, for $1 \leq i \leq s$. If we write the initial forms as

$$\left.\begin{array}{rclcl} \bar{g}_1 & = & \boldsymbol{z}^{B_1} & & + \ g'_1 \\ & & \ddots & & \\ \bar{g}_s & = & & \boldsymbol{z}^{B_s} & + \ g'_s \end{array}\right\} \tag{11}$$

where $g'_i = h_{B_i,\mathbf{0}}(\boldsymbol{z}, 0)$, then $\mathcal{E}_{\boldsymbol{z}}(g'_i) \subset \Delta_0$. If we had another relation, linearly independent of the previous ones, among monomials in $(\boldsymbol{z}, \boldsymbol{w})$ of degree m, we can apply Gaussian elimination to find a non trivial linear relation modulo $In_{\mathfrak{m}}(I)$ such that all exponents of $\boldsymbol{z}$ of every term are contained in Δ_0, which is not possible. Thus,

$$\dim_{\mathbb{C}}(\mathfrak{m}^m/\mathfrak{m}^{m+1}) = \binom{c+d+m-1}{m} - Card\big(u(I)_m\big).$$

Let now be

$$\phi(\boldsymbol{z}^*, \boldsymbol{w}^*) = \sum_{|C|+|D|=n,\ C \notin u(I)} \alpha_{C,D} (\bar{\boldsymbol{z}}^*)^C (\bar{\boldsymbol{w}}^*)^D = 0$$

a non trivial relation, and let $f \in I$ be such that $\bar{f} = \phi(\boldsymbol{z}, \boldsymbol{w})$. Then, we can write

$$f = h_0 + \sum_{i=1}^{r} h_i f_i.$$

Again by construction of quotients and remainders, we must have $\bar{f} = \bar{h}_0$ and we can suppose that $\mathcal{E}_{\boldsymbol{z}}(f) \subset \Delta_0$. Then,

$$\nu_{\boldsymbol{z}}(f) \leq \min\big\{|C| \; ; \; C \in \mathcal{E}_{\boldsymbol{z}}\big(\phi(\boldsymbol{z}, \boldsymbol{w})\big)\big\} \leq n.$$

If we had $\nu_{\boldsymbol{z}}(f) = n$ we would find a coefficient $\alpha_{C,\mathbf{0}} \neq 0$ in ϕ, which is not possible since $C \notin u_{\boldsymbol{z}}(I)$. We have then $\nu_{\boldsymbol{z}}(f) < m$. Let us write

$$\tilde{f} = \sum_{|A| = \nu_{\boldsymbol{z}}(f)} \phi_A(\boldsymbol{w}) \boldsymbol{z}^A.$$

Taking $\boldsymbol{y} \in W \cap \Delta$ such that $\phi_A(\boldsymbol{y}) \neq 0$ for a certain $A \in \mathbb{Z}_{\geq 0}$ with $|A| = \nu_{\boldsymbol{z}}(f)$, we find a non trivial relation modulo $In_{\mathfrak{m}_{X,\boldsymbol{y}}}(I')$, where I' is the ideal obtained by the substitution $\boldsymbol{w}' = \boldsymbol{w} + \boldsymbol{y}$ in a convergent basis in Δ of I. This relation depends only on $\bar{\boldsymbol{z}}^*$ and has all its coefficients in Δ_0. Using the same notations as above, with $m = \nu_{\boldsymbol{z}}(f)$, the relations in Equation (11) are preserved in degree m when we substitute $\boldsymbol{w}$ by $\boldsymbol{w} + \boldsymbol{y}$. This proves that the dimension of the homogeneous component of degree m in the graded ring falls by the substitution $\boldsymbol{w} \mapsto \boldsymbol{w} + \boldsymbol{y}$, which contradicts the fact that $\mathcal{H}^{(0)}_{X,\boldsymbol{x}}$ is constant. ∎

8. Conclusions and Open Problems

Many things about normal flatness, specially as for its behaviour in the resolution process is concerned are still unknown.

Some questions regarding this can be solved. For instance, we know that the strict transform of a variety, after blowing up an equimutiple subvariety can be computed, in a fairly easy and direct way, using the equations of an equimultiple basis (the reader is referred to [18] for this and some other technical results), but there are still many questions to be answered.

Some of them, which we consider interesting, are the following:

- What is a sufficient condition for normal flatness to hold after a blowing up?
- What happens when normal flatness does not hold after a blowing up?
- How do the Newton diagrams and Hilbert functions evolve after blowing ups?

We hope that the results contained in this paper and the techniques proposed may contribute to shed some light in these fascinating problems.

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