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Massless particles in three-dimensional Lorentzian warped products

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The model of a massless relativistic particle with curvature-dependent Lagrangian is well known in $(d+1)$ -dimensional Minkowski space. For other gravitational fields less rigid than those with constant (zero) curvature only a few results are known. In this paper, we give a geometric approach in order to solve the field equations associated with that Lagrangian in the setting of an interesting three-dimensional background, namely, a three-dimensional warped product with Lorentzian fibers. When some rigidity conditions are imposed to the fiber (constant Gauss curvature), the trajectories can be totally described. Several examples help us clarify this. © 2007 American Institute of Physics. [DOI: 10.1063/1.2409522]

I. INTRODUCTION

A natural variational problem can be defined on a space of suitable curves Γ in a semi-Riemannian manifold (\bar{M}, \bar{g}) , associated with the Lagrangian

$$\mathcal{L}: \Gamma \rightarrow \mathbb{R}, \quad \mathcal{L}(\bar{\gamma}) = \int \bar{\kappa} ds, \quad (1)$$

where $\bar{\kappa}$ is the Frenet first curvature of $\bar{\gamma}$, that measures the *total curvature* of these curves in (\bar{M}, \bar{g}) . Concerning this, we found the following in the literature.

1. In the Euclidean plane \mathbb{R}^2 , the total curvature action of any closed curve is not arbitrary, but an integer multiple of 2π , $\mathcal{L}(\bar{\gamma}) = 2\pi i(\bar{\gamma})$, where the integer number $i(\bar{\gamma})$ is the rotation number of $\bar{\gamma}$. Moreover, a classical result due to Whitney and Grauert ensures that the rotation number characterizes the regular homotopy class of closed curves. Hence, the functional \mathcal{L} is constant on any regular homotopy class of closed curves. This fact also holds true for curves connecting to fixed points of the Euclidean plane and with the same direction in these boundary points.
2. Another classical result, due to Fenchel,⁹ states that the total curvature action of any closed curve in the Euclidean space satisfies the inequality $\mathcal{L}(\bar{\gamma}) \geq 2\pi$, and the equality holds if, and only if, $\bar{\gamma}$ is a convex plane curve. In particular, every minimum of the total curvature functional, on closed curves, in the Euclidean three space is a plane curve, and so, a trivial one. This holds not only for minima but also for critical points. In fact, the critical points of $\mathcal{L}: \Gamma \rightarrow \mathbb{R}$ (where Γ is the space of closed curves, or curves connecting two points with given

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tangent vectors at these two points) in the Euclidean n -space are just the plane curves and so they are again trivial.¹

From a physical point of view, Lagrangians are a very powerful tool to introduce mathematical models. The following nonexhaustive, but illustrative, list displays some of the uses of Lagrangians in physics.

1. Models of n -dimensional relativistic objects with rigidity attract considerable attention. Some of the most remarkable examples are point particles ($n=0$),^{15,17} strings ($n=1$),^{8,18} membranes ($n=2$),¹⁶ etc. The action functionals of these models depend on the external curvature of the $(n+1)$ -dimensional world surfaces swept by the n -dimensional objects in the process of evolution. The starting point was the model of a relativistic string with rigidity proposed by Polyakov.¹⁸ In a d -dimensional Lorentz space (P^d, g) we have the bosonic conformal string theory associated to the combined Nambu-Goto-Polyakov (NGP) action,³

$$S = \mathcal{W} + \lambda \int dA = \int (\|H\|^2 + R + \lambda) dA,$$

where H and R are the mean and the extrinsic curvature of the surface, respectively, and $\mathcal{W} = \int (\|H\|^2 + R) dA$ is the Willmore action on surfaces. This Willmore action is invariant under conformal changes of the metric g in P^d ,²⁵ and it was generalized to m -dimensional submanifolds.⁷ The critical points of \mathcal{W} are called the *Willmore surfaces* in (P^d, g) . When (P^d, g) is flat, the action S is given by $S = \int (\|H\|^2 + \lambda)$, and it coincides with the classical NGP action.¹¹

2. A $(2+1)$ -dimensional space-time offers new possibilities which are not present in any higher-dimensional case. In fact, thanks to the Abelian nature of the spatial rotation group $SO(2)$, and the topology of many-particle configuration space, the spin of relativistic particles can be an arbitrary real number, and a generalized statistics, intermediate between Bose and Fermi statistics, is also possible.²⁶ The considerable interest to the field models of such particles, called *anyons*, is due to their application to different planar physical phenomena: the fractional quantum Hall effect, high- T_c superconductivity, and description of the physical processes in the presence of cosmic strings.²⁷ In particular, $(2+1)$ -dimensional models of relativistic particles described by the action A depending linearly on the world-trajectory curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ have been studied,²⁸

$$A = - \int (m + \lambda \bar{\kappa} + \mu \bar{\tau}) ds,$$

where m is a parameter with the dimension of mass, whereas λ and μ are dimensionless parameters. This model contains only spin internal degrees of freedom, its classical and quantum spectra contain tachyonic states, and depending on the relation between the parameters λ and μ , may also have massive and massless states.

3. Plyushchay²¹ proved the particularity of the case of the linear dependence on the first curvature,

$$A_1 = - \int (m + \lambda \bar{\kappa}) ds. \quad (2)$$

In fact, only spin internal degrees of freedom are present on the system, whereas any other dependence also implies the existence of radial-like degrees of freedom. The model of a particle described by the action A_1 with $m=0$,^{4,22,29} which is essentially given by \mathcal{L} ,²⁰ corresponds to a massless particle moving classically at a super-relativistic velocity. Nevertheless, this model has been shown not to contradict relativity.²²

4. On the other hand, in $(2+1)$ dimensions, the relativistic particle with torsion is described by the action

$$A_2 = - \int (m + \mu \bar{\tau}) ds,$$

and the spectrum of the model contains massive, massless, and tachyonic states.²³ The interest is the appearance of the action of this model under consideration of the problem describing high- T_c superconductivity,¹⁹ which allows us to use it as the basis for constructing models for the relativistic anyon (in the massive sector the spin of the quantum states can take fractional values).

5. The action depending quadratically on the curvature of the particle path was considered not only in several papers,^{2,16,24} but also from a geometric point of view concerning the problem of the *elastic rod*.^{5,6,12,13}

A warped product $\bar{M} = B \times_f F$ of the semi-Riemannian manifolds (B, g_B) (the *base*) and (F, g_F) (the *fiber*) with $f: B \rightarrow \mathbb{R}$ a given positive smooth function (the *warping function*) is an interesting geometric tool to easily model some spaces. The *warped metric* is defined as $g_M = g_B + f^2 g_F$.¹⁴ For example, a surface of revolution \bar{M} in \mathbb{R}^3 can be obtained by rotation around the OZ axis of an arclength parametrized curve in the OXZ plane, $\alpha(t) = (x(t), z(t))$, $x(t) > 0$, $t \in I = (-\varepsilon, \varepsilon)$. If θ is the rotation angle, then the induced metric on \bar{M} is $ds^2 = dt^2 + x^2(t) d\theta^2$. But dt^2 ($d\theta^2$) is the standard metric on I (the unit circle S^1), and hence \bar{M} can be viewed as the warped product $\bar{M} = I \times_{x(t)} S^1$, with $f(t) = x(t)$ as the warping function. The standard space-time models of the universe are warped products (the Robertson-Walker space-time is $\bar{M} = I \times_f S$, where S is a three-dimensional connected manifold of constant curvature), as are the simplest models of neighborhoods of stars and black holes.

Inspired by all this information, we have considered the problem of finding critical points of the total curvature Lagrangian \mathcal{L} on $\bar{M} = I \times_f M$, that is, trajectories of massless particles in the setting of three-dimensional warped products with Lorentzian fiber M , for curves contained in the fibers.

The content of this paper is as follows. In Sec. II, we obtain the expression of the Euler-Lagrange equation corresponding to the action \mathcal{L} for clamped (or closed) spacelike curves $\bar{\gamma}$ in any three-dimensional semi-Riemannian manifold, obtaining the motion equation (5). Then, in Sec. III, we adapt this equation to our case, bearing in mind that we are handling a manifold which is a warped product, which means that we can use some very well-known formulas that can be found in Ref. 14. In this way, the Euler-Lagrange equation can be reduced to a pair of ordinary differential equations whose unknown is the curvature function κ of the original curve γ in M . Section IV is devoted to study the case when the Gaussian curvature of the given Lorentz surface is a constant along the original curve, since this condition allows us to completely solve the motion equations (14) and (15). In particular, we remark the important case when the given Lorentz surface (M, g) has constant Gaussian curvature G , because we can completely describe all critical points of the action \mathcal{L} . In Sec. V, general methods to construct explicit examples are explained, following some ideas that can be found in Ref. 10. Finally, in Sec. VI, we consider the situation if timelike curves are considered instead.

II. THE EULER-LAGRANGE EQUATION

The problem of finding the critical points of the Lagrangian \mathcal{L} can be considered in a more general setting, namely, by considering a family of Frenet curves in a three-dimensional semi-Riemannian manifold.

Let (\bar{M}, \bar{g}) be a three-dimensional semi-Riemannian manifold. Let $\bar{\gamma}$ be an arclength parametrized spacelike curve in \bar{M} admitting a Frenet system $\{\bar{T} = \bar{\gamma}', \bar{N}, \bar{B}\}$ with curvature and torsion functions $\bar{\kappa}$ and $\bar{\tau}$, respectively (the timelike case is considered in the last section). We have the Frenet equations

$$\bar{\nabla}_{\bar{\gamma}} \bar{T} = \varepsilon_2 \bar{\kappa} \bar{N},$$

$$\bar{\nabla}_{\bar{\gamma}}\bar{N} = -\varepsilon_1\bar{\kappa}\bar{T} + \varepsilon_3\bar{\tau}\bar{B},$$

$$\bar{\nabla}_{\bar{\gamma}}\bar{B} = -\varepsilon_2\bar{\tau}\bar{N}, \quad (3)$$

where $\bar{\nabla}$ is the Levi-Civita connection of \bar{g} , $\varepsilon_1 = \bar{g}(\bar{T}, \bar{T}) = 1$, $\varepsilon_2 = \bar{g}(\bar{N}, \bar{N})$, and $\varepsilon_3 = \bar{g}(\bar{B}, \bar{B})$, $\varepsilon_i = \pm 1$, $i = 1, 2, 3$. Given a suitable space of curves Γ , say, clamped or closed, we are interested in studying the critical points of the total curvature Lagrangian,

$$\mathcal{L}: \Gamma \rightarrow \mathbb{R}, \quad \bar{\gamma} \mapsto \mathcal{L}(\bar{\gamma}) = \int_{\bar{\gamma}} \bar{\kappa}. \quad (4)$$

For this end, we take a tangent vector at $\bar{\gamma} \in \Gamma$, which is nothing but a vector field W along the curve $\bar{\gamma}$. Also, it defines in \bar{M} a variation of $\bar{\gamma}$ in Γ . Next, if $\Phi = \Phi(t, r): [a, b] \times (-\varepsilon, \varepsilon) \rightarrow \bar{M}$ is the corresponding variation of $\bar{\gamma}$, $\Phi(t, 0) = \bar{\gamma}(t)$, with variational field $W = W(t) = (\partial\Phi/\partial r)_{(t,0)}$, the critical points of the variational problem are those curves $\bar{\gamma} \in \Gamma$ such that

$$\delta\mathcal{L}(\bar{\gamma})[W] = 0, \quad \forall W \in T_{\bar{\gamma}}\Gamma.$$

Some standard arguments, involving integration by parts, let us compute the first derivative of \mathcal{L} ,

$$\delta\mathcal{L}(\bar{\gamma})[W] = \int_{\bar{\gamma}} \bar{g}(\Omega(\bar{\gamma}), W) ds + [\mathcal{B}(\bar{\gamma}, W)]_a^b,$$

where $\Omega(\bar{\gamma})$ and $\mathcal{B}(\bar{\gamma}, W)$ denote the Euler-Lagrange and the boundary operators, respectively, which, in this case, are given by

$$\Omega(\bar{\gamma}) = \bar{\nabla}_{\bar{\gamma}}^2\bar{N} + \varepsilon_1\bar{\kappa}'\bar{T} + \varepsilon_1\bar{\kappa}\bar{\nabla}_{\bar{\gamma}}\bar{T} + \bar{R}(\bar{N}, \bar{T})\bar{T},$$

$$\mathcal{B}(\bar{\gamma}, W) = \bar{g}(\bar{\nabla}_{\bar{\gamma}}W, \bar{N}) - \bar{g}(W, \bar{\nabla}_{\bar{\gamma}}\bar{N}) - \varepsilon_1\bar{\kappa}\bar{g}(W, \bar{T}),$$

where \bar{R} stands for the curvature operator of \bar{g} and $\bar{\kappa}'$ is the derivative of the function $\bar{\kappa}$ with respect to the arclength parameter. Now, if we set $V = \partial\Phi/\partial t = v\bar{T}$, we have $\bar{\nabla}_{\bar{\gamma}}W = \partial_r(\log v)\bar{T} + \bar{\nabla}_W\bar{T}$. This formula shows that if the curves in variation Φ are in Γ , then $W(a) = W(b) = 0$, $\bar{\nabla}_W\bar{T}(a) = \bar{\nabla}_W\bar{T}(b) = 0$, and consequently $[\mathcal{B}(\bar{\gamma}, W)]_a^b = 0$. By using Eq. (3), the Euler-Lagrange operator becomes

$$\Omega(\bar{\gamma}) = -\varepsilon_2\varepsilon_3\bar{\tau}'\bar{N} + \varepsilon_3\bar{\tau}'\bar{B} + \bar{R}(\bar{N}, \bar{T})\bar{T},$$

where $\bar{\tau}'$ is the derivative of $\bar{\tau}$ with respect to the arclength. Finally a curve $\bar{\gamma}$ in Γ is a critical point of \mathcal{L} if, and only if, it satisfies the Euler-Lagrange equation

$$\bar{R}(\bar{N}, \bar{T})\bar{T} = \varepsilon_2\varepsilon_3\bar{\tau}'\bar{N} - \varepsilon_3\bar{\tau}'\bar{B}. \quad (5)$$

III. SOLVING THE EULER-LAGRANGE EQUATION

Now we adapt the general case of Sec. II to a warped product $\bar{M} = I \times_f M$, where I is an interval endowed with the metric εdt^2 (ε is 1 or -1), (M, g) is an oriented Lorentzian surface (the fiber), and $f: I \rightarrow \mathbb{R}$ a positive smooth function. We set the warping metric $\bar{g} = \varepsilon dt^2 + f^2 g$. Thus, (\bar{M}, \bar{g}) is a three-dimensional manifold of index 1 or 2, according to ε be 1 or -1 , respectively. For a given spacelike unit speed curve $\gamma(s)$ in the surface M , let us denote by $\{T, N\}$ its Frenet frame and by

κ its geodesic curvature. A positive orthonormal Frenet frame along $\gamma_t(s)=(t, \gamma(s))$ in (\bar{M}, \bar{g}) is $\{\bar{T}=\gamma_t, \bar{\xi}=(1/f)N, \bar{\partial}_t\}$. Note that $\varepsilon_1=\bar{g}(\bar{T}, \bar{T})=1$, $\bar{g}(\bar{\xi}, \bar{\xi})=-1$, $\bar{g}(\bar{\partial}_t, \bar{\partial}_t)=\varepsilon$, and it is easy to see that the gradient of f is given by

$$\nabla f = \varepsilon \dot{f} \bar{\partial}_t. \quad (6)$$

Next, we compute the Frenet system of γ_t . Let $\bar{\nabla}$ be the Levi-Civita connection of (\bar{M}, \bar{g}) . We have

$$\bar{\nabla}_{\bar{T}} \bar{T} = \varepsilon_2 \bar{\kappa} \bar{N},$$

$$\bar{\nabla}_{\bar{T}} \bar{N} = -\bar{\kappa} \bar{T} + \varepsilon_3 \bar{\tau} \bar{B},$$

$$\bar{\nabla}_{\bar{T}} \bar{B} = -\varepsilon_2 \bar{\tau} \bar{N},$$

where $\bar{\kappa}$ and $\bar{\tau}$ are the curvature and torsion of γ_t in (\bar{M}, \bar{g}) , respectively. In our case, some well-known formulas in warped spaces allow us to express the geometry of (\bar{M}, \bar{g}) in terms of the warping function f and the geometries of $(I, \varepsilon dt^2)$ and (M, g) (Ref. 14, p. 206). Thus, after some computations, we obtain

$$\bar{\nabla}_{\bar{T}} \bar{T} = -\frac{\kappa}{f} \bar{\xi} - \frac{\varepsilon \dot{f}}{f} \bar{\partial}_t. \quad (7)$$

Note that the vector $\bar{\nabla}_{\bar{T}} \bar{T}$ is lightlike if, and only if, $\varepsilon \dot{f}^2(t) - \kappa^2(s) = 0$. We will always assume that the curve γ_t is a spacelike Frenet curve, and hence we have

$$\varepsilon \dot{f}^2(t) - \kappa^2(s) \neq 0 \quad \text{for any } t \in I \text{ and any } s. \quad (8)$$

First, let us suppose that γ_t is a geodesic. From Eq. (7), we have $\kappa=0$ and $\dot{f}(t)=0$, that is, γ is a geodesic and the slice $M_t=\{t\} \times M$ is critical. In that case, the Euler-Lagrange equation reduces trivially and γ_t is a critical point of \mathcal{L} .

Now, we assume that $\dot{f}(t)=0$ for a certain $t \in I$. If γ is a geodesic, we already know that γ_t is a geodesic on \bar{M} . Next, if $\dot{f}(t)=0$ and $\kappa \neq 0$, simple computations let us obtain

$$\nabla f = 0, \quad \bar{\nabla}_{\bar{T}} \bar{T} = -\frac{\kappa}{f} \bar{\xi}, \quad \bar{\kappa} = \frac{\kappa}{f},$$

$$\bar{N} = \bar{\xi}, \quad \bar{B} = \bar{\partial}_t, \quad \bar{\nabla}_{\bar{T}} \bar{\xi} = -\frac{\kappa}{f} \bar{T},$$

$$\bar{\tau} = 0, \quad \bar{\nabla}_{\bar{T}} \bar{\partial}_t = \frac{\dot{f}}{f} \bar{T} = 0.$$

In this case, the Euler-Lagrange equation reduces to $\bar{R}(\bar{N}, \bar{T})\bar{T} = 0$, or equivalently (Ref. 14, p. 210), $0 = R^M(N, T)T = GN$, where R^M is the curvature operator of (M, g) and G is the Gauss curvature of M (recall that $G = -g(R^M(T, N)N, T)$). As a consequence, we have the following.

Theorem 3.1: *Assume that $t \in I$ is such that $\dot{f}(t)=0$. Let $\gamma(s)$ be a unit spacelike curve in M such that either γ is a geodesic or $G(\gamma(s))=0$ for any s (that is, γ is a curve of parabolic points). Then, γ_t is a critical point of \mathcal{L} .*

From now on, we assume $\dot{f} \neq 0$. After some computations, recalling O'Neill's book (Ref. 14, p. 206), we obtain

$$\bar{N} = \frac{-\varepsilon_2 \kappa \xi - \varepsilon \varepsilon_2 \dot{f} \partial_t}{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}}, \quad \bar{B} = \frac{\dot{f} \xi + \kappa \partial_t}{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}}, \tag{9}$$

$$\bar{\kappa} = \frac{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}}{f}, \quad \bar{\tau} = \frac{\dot{f} \kappa'}{f(\varepsilon \dot{f}^2 - \kappa^2)}, \quad \varepsilon_2 \varepsilon_3 = -\varepsilon. \tag{10}$$

In order to solve the Euler-Lagrange equation (5), we have to distinguish several cases.

Case 1: Assume that γ is a geodesic in the fiber M , i.e., $\kappa=0$. According to Eq. (8), $\dot{f} \neq 0$. If we set $\sigma = \text{sign}(\dot{f})$, Eqs. (5), (7), (9), and (10) become

$$\bar{R}(\partial_t, \bar{T})\bar{T} = 0, \quad \bar{T} = \frac{1}{f}T, \quad \varepsilon_2 = \varepsilon, \quad \bar{N} = -\sigma \partial_t, \quad \bar{B} = \sigma \xi, \quad \bar{\kappa} = \frac{|\dot{f}|}{f}, \quad \bar{\tau} = 0.$$

Once again, the well-known formulas in Ref. 14, p. 210, gives $0 = \bar{R}(\partial_t, \bar{T})\bar{T} = \varepsilon f \ddot{f} \partial_t$. Then, we have obtained the following.

Theorem 3.2: *Let γ be a unit spacelike geodesic in M . Let $t \in I$ such that $\dot{f}(t) \neq 0$ and $\ddot{f}(t) = 0$. Then, γ_t is a nongeodesic critical point of \mathcal{L} .*

Case 2: Now, we suppose that $\kappa > 0$ is a constant. Once again, from Eqs. (9) and (10) we obtain the following information:

$$\bar{N} = \frac{-\kappa \xi - \varepsilon \dot{f} \partial_t}{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}}, \quad \bar{\kappa} = \frac{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}}{f}, \quad \bar{\tau} = 0.$$

Thus, the Euler-Lagrange equation reduces to $0 = \bar{R}(\bar{N}, \bar{T})\bar{T}$, and by using the formulas for warped products,

$$0 = \frac{1}{\sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}} (\kappa(\varepsilon \dot{f}^2 - G(\gamma(s)))\xi - \dot{f} \ddot{f} \partial_t).$$

Therefore, $\dot{f}(t) = 0$ and $G((\gamma(s)) - \varepsilon \dot{f}^2(t)) = 0$. Therefore, we can state the following result.

Theorem 3.3: *Let us assume that the curvature function κ of γ is a positive constant. Let $t \in I$ such that $\dot{f}(t) \neq 0$ and $\ddot{f}(t) = 0$. If in addition $G(\gamma(s)) = \varepsilon \dot{f}^2(t)$ for any s , then γ_t is a critical point of \mathcal{L} .*

Case 3: Finally, we assume that the curvature κ is not a constant and the current slice is not critical, that is, $\dot{f}(t) \neq 0$ for $t \in I$. By using some formulas in O'Neill's book¹⁴, we obtain

$$\bar{R}(\bar{N}, \bar{T})\bar{T} = \frac{\varepsilon_2 \kappa (\varepsilon \dot{f}^2 - G \circ \gamma)}{f^2 \sqrt{\varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}} \xi + \frac{\varepsilon_2 \dot{f} \ddot{f}}{\sqrt{f \varepsilon_2(\varepsilon \dot{f}^2 - \kappa^2)}} \partial_t, \tag{11}$$

where $G \circ \gamma$ means the Gauss curvature of M along γ . From Eqs. (9)–(11) we obtain the equations

$$\frac{\varepsilon_2 \kappa (\varepsilon \dot{f}^2 - G \circ \gamma)}{f^2} = -\varepsilon_3 \kappa \bar{\tau}' - \varepsilon_3 \dot{f} \bar{\tau}', \quad \frac{\varepsilon_2 \dot{f} \ddot{f}}{f} = -\varepsilon \varepsilon_3 \dot{f} \bar{\tau}' - \varepsilon_3 \kappa \bar{\tau}'.$$

Since $-\varepsilon_2 \varepsilon_3 = \varepsilon$, we easily obtain the following equations:

$$\bar{\tau}^2 = \frac{\kappa^2(\dot{f}^2 - \varepsilon G \circ \gamma) - \varepsilon f \dot{f}^2 \ddot{f}}{f^2(\kappa^2 - \varepsilon \dot{f}^2)}, \quad (12)$$

$$\bar{\tau}' = \frac{\varepsilon \kappa f(\dot{f} \ddot{f} - \dot{f}^2 + \varepsilon G \circ \gamma)}{f^2(\kappa^2 - \varepsilon \dot{f}^2)}. \quad (13)$$

By recalling the expression of $\bar{\tau}$, bearing in mind Eq. (10), we have

$$(\kappa')^2 = \left(\frac{\dot{f}^2 - \varepsilon G \circ \gamma}{\dot{f}^2} \kappa^2 - \varepsilon f \dot{f} \ddot{f} \right) (\kappa^2 - \varepsilon \dot{f}^2). \quad (14)$$

Now, we compute the derivative of $\bar{\tau}$, and by Eqs. (13) and (14), we have

$$k'' = 2 \frac{\dot{f}^2 - \varepsilon G \circ \gamma}{\dot{f}^2} \kappa^3 + \left(\frac{\varepsilon \dot{f}^2 - G \circ \gamma}{f} - \varepsilon \dot{f} - 2\varepsilon f \dot{f} \ddot{f} \right) \kappa. \quad (15)$$

IV. RIGIDITY CONDITIONS

We first assume that the Gaussian curvature of M along γ is a *constant*. Then, Eq. (15) can be integrated by multiplying by $2\kappa'$,

$$(k')^2 = \frac{\dot{f}^2 - \varepsilon G \circ \gamma}{\dot{f}^2} \kappa^4 + \left(\frac{\varepsilon \dot{f}^2 - G \circ \gamma}{f} - \varepsilon \dot{f} - 2\varepsilon f \dot{f} \ddot{f} \right) \kappa^2 + A, \quad (16)$$

where A is an integration constant. Now, Eq. (14) compared with Eq. (16) gives

$$A = f \dot{f}^2 \ddot{f}, \quad (17)$$

$$\varepsilon G \circ \gamma = \dot{f}^2 - f \ddot{f}. \quad (18)$$

Next, we insert Eq. (18) into Eq. (14), obtaining

$$(k')^2 = \frac{f \ddot{f}}{\dot{f}^2} (\kappa^2 - \varepsilon \dot{f}^2)^2. \quad (19)$$

Since we are assuming that the curvature function κ of γ is not constant, then $\ddot{f} > 0$. Next, we solve Eq. (19) by discussing the value of ε .

$$\varepsilon = 1, \quad \kappa(s) = \dot{f} \tanh(s \sqrt{f \ddot{f}} + \dot{f}c), \quad (20)$$

$$\varepsilon = -1, \quad \kappa(s) = \dot{f} \tan(s \sqrt{f \ddot{f}} + \dot{f}c), \quad (21)$$

where c is an integration constant. Therefore, we have proved the following.

Theorem 4.1: Fix a point $t \in I$. We assume

1. $\dot{f}(t) \neq 0$, $\ddot{f}(t) > 0$;
2. $G \circ \gamma$ is a constant function and $\varepsilon G \circ \gamma = \dot{f}^2(t) - f(t) \ddot{f}(t)$; and
3. the curvature of γ is given by either Eq. (20) or Eq. (21).

Then, the curve γ_t is a critical point of \mathcal{L} .

Next, we assume that the Gaussian curvature G of the whole surface M is constant. There is no loss of generality if we assume $G=0, 1, -1$. We recall that $dS^2(1)$ is the two-dimensional de Sitter space of Gaussian curvature $G=1$, and $AdS^2(-1)$ is the two-dimensional anti de Sitter space of Gaussian curvature $G=-1$. We denote by M_δ the space $dS^2(1)$ or $AdS^2(-1)$ according to $\delta = \pm 1$, respectively. Now, let \bar{M}_δ be the warped product of an open interval and M_δ . From Theorems 3.1, 3.2, 3.3, and 4.1 we have the following.

Corollary 4.1: A unit spacelike curve $\gamma_t \in \Gamma$ in \bar{M}_δ is a critical point of \mathcal{L} if, and only if, one of the following statements holds:

1. $\dot{f}(t)=0$ and γ is a geodesic;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t)=0$, and γ is a geodesic;
3. $\dot{f}^2(t)=\varepsilon\delta>0$, $\ddot{f}(t)=0$, and $\kappa \neq 0$ is constant; and
4. $\dot{f}(t) \neq 0$, $\ddot{f}(t)>0$, $\varepsilon\delta=\dot{f}^2(t)-f(t)\ddot{f}(t)$, and the curvature function is given by Eq. (20) or Eq. (21), according to $\varepsilon=1$ or $\varepsilon=-1$, respectively.

Next, let \mathbb{L}^2 be the Lorentz-Minkowski 2-plane.

Corollary 4.2: A unit spacelike curve $\gamma_t \in \Gamma$ in $\bar{M}=I \times_f \mathbb{L}^2$ is a critical point of \mathcal{L} if, and only if, one of the following statements holds:

1. $\dot{f}(t)=0$ and κ is any smooth function;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t)=0$, and γ is a geodesic;
3. $0 \neq \dot{f}^2(t)=f(t)\ddot{f}(t)$, $\ddot{f}(t)>0$, and the curvature function is given by Eq. (20) or Eq. (21), according to $\varepsilon=1$ or $\varepsilon=-1$, respectively.

V. EXAMPLES

Some of the ideas of Examples 5.1, 5.2, and 5.3 can be found in Ref. 10.

Example 5.1: In \mathbb{L}^3 with usual metric $ds^2=dx^2+dy^2-dz^2$, let $\alpha:(a,b) \subset \mathbb{R} \rightarrow \mathbb{L}^3$, $\alpha(u)=(x(u), 0, z(u))$ be a timelike unit curve with $x(u)>0$ for any $u \in (a,b)$. Let M be the revolution surface around the z axis whose profile curve is α . A local parametrization of M is given by

$$X:(a,b) \times S^1 \rightarrow \mathbb{L}^3, \quad X(u, \theta) = (x(u)\cos \theta, x(u)\sin \theta, z(u)),$$

for any $(u, e^{i\theta}) \in (a,b) \times S^1$. A unit spacelike normal vector field to M is

$$N(u, \theta) = (z'(u)\cos \theta, z'(u)\sin \theta, x'(u)) \quad \text{for any } (u, e^{i\theta}) \in (a,b) \times S^1.$$

Since α is timelike, we have a Lorentzian surface whose Gaussian curvature is not a constant, in general. In fact, it is easy to obtain the following expression for the Gaussian curvature of M :

$$G(u, \theta) = \frac{z'(u)(z'(u)x''(u) - x'(u)z''(u))}{x(u)} \quad \text{for any } (u, e^{i\theta}) \in (a,b) \times S^1.$$

Now, we pick a point $u \in (a,b)$. The curve γ given by $e^{i\theta} \in S^1 \mapsto \gamma(\theta)=X(u, \theta)$ is nothing but the orbit of point $\alpha(u)$ by a subgroup of orthochronal isometries of \mathbb{L}^3 . Therefore, both the Gaussian curvature of M along γ and the geodesic curvature κ of γ are constant functions. Now, we choose $f:I \rightarrow \mathbb{R}$ a positive smooth function such that there is a point $t \in I$ and suitable $\varepsilon = \pm 1$ satisfying

$$\dot{f}(t) \neq 0, \quad \ddot{f}(t)=0, \quad \dot{f}^2(t) = \varepsilon \frac{z'(u)(z'(u)x''(u) - x'(u)z''(u))}{x(u)}.$$

According to Theorem 3.3, the curve γ_t is a critical point of the action \mathcal{L} defined on $\bar{M}=I \times_f M$.

As a remark, the surface $dS^2(1)$ can be obtained in this way by choosing the curve $\alpha:\mathbb{R} \rightarrow \mathbb{L}^3$ defined by $\alpha(u)=(\cosh(u), 0, \sinh(u))$.

Example 5.2: In \mathbb{L}^3 with usual metric $ds^2=dx^2+dy^2-dz^2$, let $\alpha:(a,b)\subset\mathbb{R}\rightarrow\mathbb{L}^3$, $\alpha(u)=(x(u),y(u),0)$ be a spacelike unit curve with $x(u)>0$ for any $u\in(a,b)$. We consider the surface M in \mathbb{L}^3 given by the local parametrization

$$X:(a,b)\times\mathbb{R}\rightarrow\mathbb{L}^3, \quad X(u,\theta)=(x(u)\cosh\theta,y(u),x(u)\sinh\theta),$$

for any $(u,\theta)\in(a,b)\times\mathbb{R}$. Clearly, we are using the subgroup of isometries H of \mathbb{L}^3 defined by $H=\{f_\theta:\theta\in\mathbb{R}\}$, where each f_θ is given by

$$f_\theta(x,y,z)=(x,y,z)\begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}.$$

The unit spacelike normal vector field to M is

$$N(u,\theta)=(y'(u)\cosh\theta,-x'(u),y'(u)\sinh\theta) \quad \text{for any } (u,\theta)\in(a,b)\times\mathbb{R}.$$

Since α is spacelike, we have a Lorentzian surface whose Gaussian curvature is not constant in general. The Gaussian curvature of M is given by

$$G(u,\theta)=\frac{y'(u)(x'(u)y''(u)-y'(u)x''(u))}{x(u)} \quad \text{for any } (u,\theta)\in(a,b)\times\mathbb{R}.$$

Now, we pick a point $u\in(a,b)$. The timelike curve γ given by $\gamma(\theta)=X(u,\theta)$ is nothing but the orbit of point $\alpha(u)$ by the group H . Therefore, both the Gaussian curvature of M along γ and the geodesic curvature κ of γ are constant functions. Now, we choose $f:I\rightarrow\mathbb{R}$ a positive smooth function such that there is a point $t\in I$ and suitable $\varepsilon=\pm 1$ satisfying

$$\dot{f}(t)\neq 0, \quad \ddot{f}(t)=0, \quad \dot{f}^2(t)=\varepsilon\frac{y'(u)(x'(u)y''(u)-y'(u)x''(u))}{x(u)}.$$

But as it will be noticed in the next section, Theorem 3.3 can be rewritten for timelike curves γ . Therefore, the curve γ_t is a critical point of \mathcal{L} .

As a remark, the surface $dS^2(1)$ can be obtained by choosing the curve $\alpha:\mathbb{R}\rightarrow\mathbb{L}^3$, $\alpha(u)=(\cos(u),\sin(u),0)$.

Example 5.3: In \mathbb{L}^3 with usual metric $ds^2=dx^2+dy^2-dz^2$, we consider a unit timelike curve $\alpha:(a,b)\rightarrow\mathbb{L}^3$, $\alpha(u)=(0,y(u),z(u))$, with $y(u)\neq z(u)$ for any $u\in(a,b)$. We define the surface M in \mathbb{L}^3 given by

$$X:(a,b)\times\mathbb{R}\rightarrow\mathbb{L}^3,$$

$$X(u,\theta)=\left(\theta(y(u)-z(u)),y(u)-\frac{\theta^2}{2}(y(u)-z(u)),z(u)-\frac{\theta^2}{2}(y(u)-z(u))\right),$$

for any $(u,\theta)\in(a,b)\times\mathbb{R}$. Clearly, the surface M is a revolution surface around the axis L of equations $x=0$ and $y=z$, whose profile curve is α . The group $H=\{f_\theta:\theta\in\mathbb{R}\}$ of rotations around axis L consists of the isometries given by

$$f_\theta(x,y,z)=(x,y,z)\begin{pmatrix} 1 & -\theta & -\theta \\ \theta & 1-\theta^2/2 & -\theta^2/2 \\ -\theta & \theta^2/2 & 1+\theta^2/2 \end{pmatrix}.$$

Since α is timelike, the unit normal vector field

$$N(u, \theta) = \left(\theta(y'(u) - z'(u)), -z'(u) + \frac{\theta^2}{2}(-y'(u) + z'(u)), -y'(u) + \frac{\theta^2}{2}(-y'(u) + z'(u)) \right)$$

is spacelike for any (u, θ) . Hence, the surface M is Lorentzian, with Gaussian curvature

$$G(u, \theta) = \frac{(z'(u) - y'(u))(y'(u)z''(u) - y''(u)z'(u))}{y(u) - z(u)},$$

for any $(u, \theta) \in (a, b) \times \mathbb{R}$.

Now, we pick a point $u \in (a, b)$. The curve γ in M defined by $\gamma(\theta) = X(u, \theta)$ is nothing but the orbit of point $\alpha(u)$ by the group H . Therefore, both the Gaussian curvature of M along γ and the geodesic curvature κ of γ are constant functions. Now, we choose $f: I \rightarrow \mathbb{R}$ a positive smooth function such that there is a point $t \in I$ and suitable $\varepsilon = \pm 1$ satisfying

$$\dot{f}(t) \neq 0, \quad \ddot{f}(t) = 0, \quad \dot{f}^2(t) = \varepsilon \frac{(z'(u) - y'(u))(y'(u)z''(u) - y''(u)z'(u))}{y(u) - z(u)}.$$

According to Theorem 3.3, the curve γ_t is a critical point of \mathcal{L} .

Example 5.4: We consider the above Example 5.1, 5.2, or 5.3.

Case 1: Let $\alpha: (a, b) \rightarrow \mathbb{L}^3$ be a unit timelike curve, $\alpha(u) = (x(u), 0, z(u))$. Assume that at $u_0 \in (a, b)$ $z'(u_0)x''(u_0) - x'(u_0)z''(u_0) = 0$ holds.

Case 2: Let $\alpha: (a, b) \rightarrow \mathbb{L}^3$ be a unit spacelike curve, $\alpha(u) = (x(u), y(u), 0)$, and suppose that $x'(u_0)y''(u_0) - y'(u_0)x''(u_0) = 0$ holds at a suitable point $u_0 \in (a, b)$.

Case 3: Let $\alpha: (a, b) \rightarrow \mathbb{L}^3$ be a unit timelike curve, $\alpha(u) = (0, y(u), z(u))$, such that $y'(u_0)z''(u_0) - y''(u_0)z'(u_0) = 0$ is satisfied at the point $u_0 \in (a, b)$.

Then, in all three cases, we have that the Gauss curvature satisfies $G(u_0, \theta) = 0$. Now, we consider the curve γ given by $\gamma(\theta) = X(u_0, \theta)$. We choose a positive function $f: I \rightarrow \mathbb{R}$ admitting a critical point $t_0 \in I$. Given $\varepsilon = \pm 1$, according to Theorem 3.1, the curve γ_{t_0} is a critical point of \mathcal{L} .

Example 5.5: Define f_1, f_2, f_3 to be the smooth functions given by $f_1: (0, \infty) \rightarrow \mathbb{R}$, $f_1(t) = \sinh(t)$, $f_2: (0, \infty) \rightarrow \mathbb{R}$, $f_2(t) = \cosh(t)$, and $f_3: \mathbb{R} \rightarrow \mathbb{R}$, $f_3(t) = e^{at+b}$, where $a, b \in \mathbb{R}$ with $a \neq 0$. It is easy to check the following identities:

$$(\dot{f}_1(t))^2 - f_1(t)\ddot{f}_1(t) = 1, \quad (\dot{f}_2(t))^2 - f_2(t)\ddot{f}_2(t) = -1, \quad (\dot{f}_3(t))^2 - f_3(t)\ddot{f}_3(t) = 0,$$

$$\dot{f}_i(t) \neq 0, \quad \ddot{f}_i(t) > 0,$$

for any $i=1, 2, 3$ and any t . Now, we consider the following particular cases:

- for $\varepsilon=1$, $M = dS^2(1)$, or $\varepsilon=-1$ and $M = AdS^2(-1)$, warping function f_1 ;
- for $\varepsilon=-1$, $M = dS^2(1)$, or $\varepsilon=1$ and $M = AdS^2(-1)$, warping function f_2 ; and
- for $\varepsilon = \pm 1$, $M = \mathbb{L}^2$, warping function f_3 .

Then, all slices M_t in the warped product $\bar{M} = \mathbb{R} \times_f M$ with metric $\bar{g} = \varepsilon dt^2 + f^2 g_M$ contain critical points γ_t of \mathcal{L} coming from curves γ whose curvature functions κ are given by Eq. (20) or Eq. (21).

VI. TIMELIKE CURVES

Now, we assume that γ is a unit timelike curve with curvature κ in the Lorentzian surface (M, g) . Then, the Frenet equations associated with γ are now given by

$$\nabla_T T = \kappa N, \quad \nabla_T N = \kappa T \tag{22}$$

(the normal vector N is now spacelike). Also, given $\varepsilon = \pm 1$, the metric \bar{g} is formally the same as in Sec. III, and for the rest of the geometric tools we keep the same notation (\bar{R} , \bar{N} , and so on). In particular, the Euler-Lagrange equation (5) is formally the same.

Next, we consider the Lorentzian metric $g^o = -g$ on M . Obviously, γ is a *spacelike curve* for g^o . Let ∇^o , N^o , κ^o , and G^o denote the Levi-Civita connection of g^o , the normal vector along the curve, the curvature function of the curve, and the Gaussian curvature of the surface, respectively. Simple computations show

$$\nabla^o = \nabla, \quad N^o = N, \quad \kappa^o = -\kappa, \quad G^o = -G. \quad (23)$$

Next, we consider the warped product manifold as in Sec. III, but taking $-\varepsilon$ and calling the warped metric \bar{g}^o . Thus, $\bar{g}^o = -\bar{g}$, and therefore, the Levi-Civita connections of \bar{g}^o and \bar{g} coincide. As a consequence, the curvature operator is the same. Moreover, the Frenet systems of γ_t in (\bar{M}, \bar{g}) and in (\bar{M}, \bar{g}^o) also agree. In this way we reduce the case of timelike curves to that of spacelike curves. This readily proves Theorems 3.1, 3.2, 3.3, and 4.1 for a timelike curve γ .

Next, we assume that the Gaussian curvature G of the surface M is constant. As in Sec. IV, we denote M_δ the spaces $dS^2(1)$ or $AdS^2(-1)$ according to $\delta = \pm 1$, respectively. Now, let \bar{M}_δ be the warped product of an open interval and M_δ . We have the following.

Corollary 6.1: A unit timelike curve $\gamma_t \in \Gamma$ in \bar{M}_δ is a critical point of \mathcal{L} if, and only if, one of the following statements holds:

1. $\dot{f}(t) = 0$ and γ is a geodesic;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$, and γ is a geodesic;
3. $\dot{f}^2(t) = \varepsilon \delta > 0$, $\ddot{f}(t) = 0$, and $\kappa \neq 0$ is constant; and
4. $\dot{f}(t) \neq 0$, $\ddot{f}(t) > 0$, $\varepsilon \delta = \dot{f}^2(t) - f(t)\ddot{f}(t)$, and the curvature function is given by

$$\varepsilon = 1, \quad \kappa(s) = -\dot{f} \tan(s\sqrt{\dot{f}\ddot{f} + \dot{f}c}), \quad (24)$$

$$\varepsilon = -1, \quad \kappa(s) = -\dot{f} \tanh(s\sqrt{\dot{f}\ddot{f} + \dot{f}c}), \quad (25)$$

where c is an integration constant.

Next, let \mathbb{L}^2 be the Lorentz-Minkowski 2-plane.

Corollary 6.2: A unit timelike curve $\gamma_t \in \Gamma$ of \mathcal{L} in $\bar{M} = I \times_f \mathbb{L}^2$ is a critical point of \mathcal{L} if, and only if, one of the following statements holds:

1. $\dot{f}(t) = 0$ and κ is any smooth function;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$, and γ is a geodesic; and
3. $0 \neq \dot{f}^2(t) = f(t)\ddot{f}(t)$, $\ddot{f}(t) > 0$, and the curvature function is given by Eq. (24) or Eq. (25), according to $\varepsilon = 1$ or $\varepsilon = -1$, respectively.

Remark 6.1: In case the fiber M is a Riemannian surface and $\varepsilon = -1$, then Eqs. (14) and (15) are also obtained up to some minor sign changes. Nevertheless, in such a space-time Einstein's equations for gravitational field in three-dimensional space-time have only trivial solutions unless one introduces some complicated matter density.

VII. CONCLUSIONS

In this paper we have considered the dynamics associated with the total curvature of a particle path, which is described by the action \mathcal{L} given by Eq. (4) in three-dimensional warped spaces with Lorentzian fiber. First, in Sec. II we have obtained the Euler-Lagrange equation of motion (5) in more general backgrounds, namely, three-dimensional semi-Riemannian spaces. Then, in Sec. III, we have particularized to the case of warped spaces with a Lorentzian fiber, and there we have obtained solutions to the motion equation which lie in a critical slice when the particle path of the fiber is a geodesic, or a curve constituted by parabolic points (Theorem 3.1). If we assume that the slice is not critical, but corresponds to an inflection value t of the warping function f , then we

obtain a trajectory of the model in two cases: when γ in the fiber M is a geodesic (Theorem 3.2), or it is a curve of constant curvature $\kappa > 0$ (i.e., a circle) and the Gauss curvature of the fiber satisfies $G = \varepsilon \dot{f}^2(t)$ along γ (Theorem 3.3). In Sec. IV, we consider the case when the slice is neither critical nor corresponding to an inflexion value of the warping function. When the Gauss curvature of the fiber along γ satisfies Eq. (18), we obtain the particle path by means of its curvature given by Eq. (20) or Eq. (21). In a higher rigidity case (when the Gauss curvature G of the fiber is a constant), the fiber can be considered one of the classical spaces L^2 , dS^2 , or AdS^2 , and the particle paths can be completely characterized (Corollaries 4.1 and 4.2). We have included also several examples to easily show particle trajectories. In Sec. VI we have seen that timelike curves in the fiber can be considered instead of spacelike ones.

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