Some examples of compact composition operators on H^2

Pascal Lefèvre, Daniel Li,
Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. *We construct, in an essentially explicit way, various composition operators on H^2 and study their compactness or their membership in the Schatten classes. We construct: non-compact composition operators on H^2 whose symbols have the same modulus on the boundary of $\mathbb{D}$ as symbols whose composition operators are in various Schatten classes S_p with $p > 2$; compact composition operators which are in no Schatten class but whose symbols have the same modulus on the boundary of $\mathbb{D}$ as symbols whose associated composition operators are in S_p for every $p > 2$.*

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1 Introduction

Compactness of composition operators on H^2 was first studied in 1968 by H. Schwartz in his doctoral dissertation [10], and refined in 1973 by J. Shapiro and P. Taylor [13], who discovered the role played by the classical angular derivative, and refined the compactness problem by asking which composition operators belong to various Schatten classes S_p. In particular, they showed in [13] that $C_\phi \in S_2$, the Hilbert-Schmidt class, if and only if $\int_{\partial\mathbb{D}} (1 - |\phi^*|)^{-1} < +\infty$, where ϕ^* denotes the radial limit function of ϕ. In this paper, we show that for the larger class S_p with $p > 2$, the situation is completely different: we prove in Theorem 4.2, that for every $p > 2$, there exist two symbols ϕ_1 and ϕ_2 having the same modulus on $\partial\mathbb{D}$ and such that C_{ϕ_1} is not compact on H^2, but C_{ϕ_2} is in the Schatten class S_p.

An amusing feature of the theory of composition operators is that, whereas sophisticated necessary and sufficient conditions for the composition operators $C_\phi \colon H^2 \to H^2$ to belong to the Schatten classes $S_p = S_p(H^2)$ have been known for more than twenty years ([7], [8]), either in terms of the Nevanlinna counting function, or in terms of the pull-back measure m_ϕ, yet explicit and concrete examples are lacking. For example, D. Sarason posed in 1988 the question whether there existed a compact composition operator $C_\phi\colon H^2 \to H^2$ which was in *no* Schatten class, and the (affirmative) answer only came in 1991, by T. Carroll and C. Cowen ([1]). Their example, based on the Riemann mapping Theorem was not completely explicit. Moreover, their construction used a difficult and delicate argument, with estimates of the hyperbolic metric for certain domains, due to Hayman (see however [15] and [3]). We shall see in this paper that Luecking's criterion [7] for pullback measures leads to very concrete examples of composition operators in various Schatten classes.

In Section 3, we give a necessary condition, Proposition 3.4, on the Carleson function ρ_ϕ in order for the composition operator C_ϕ to be in S_p, as well as a general construction of symbols ϕ with control on their Carleson function.

In Section 4, we construct, for every $p > 2$, symbols ϕ_1 and ϕ_2 having the same modulus on $\partial\mathbb{D}$ such that C_{ϕ_1} is not compact on H^2, but C_{ϕ_2} is in the Schatten class S_p (Theorem 4.2).

In Section 5, we revisit an example of J. Shapiro and P. Taylor ([13], § 4) to show that for every $p_0 > 0$, there exists a symbol ϕ such that the composition operator $C_\phi\colon H^2 \to H^2$ is in the Schatten class S_p for every $p > p_0$, but not in S_{p_0} (Theorem 5.1), and also that for every $p_0 > 0$, there exists a symbol ϕ such that $C_\phi\colon H^2 \to H^2$ is in the Schatten class S_{p_0}, but not in S_p, for $p < p_0$ (Theorem 5.4). Moreover, there exists a symbol ϕ such that $C_\phi\colon H^2 \to H^2$ is compact but in no Schatten class S_p for $p < \infty$ (Theorem 5.6) and there exists a symbol ψ, whose boundary values ψ^* have the same modulus as those ϕ^* of ϕ on $\partial\mathbb{D}$, but for which $C_\psi\colon H^2 \to H^2$ is in S_p for every $p > 2$ (Theorem 5.7).

After our work was completed, we became aware of the papers [15] and [3]; in [3], M. Jones gives another proof of the theorem of Carroll and Cowen (our Theorem 5.6), and Y. Zhu gives also another proof of this theorem, as well as a proof of our Theorem 5.1. However, our proofs are different and lead to further results: see Theorem 4.2.

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2 Notation

Throughout this paper, the notation $f \approx g$ will mean that there are two constants $0 < c < C < +\infty$ such that $cf(t) \leq g(t) \leq Cf(t)$ (for t sufficiently near of a specified value), and the notation $f(t) \lesssim g(t)$, when t is in the neighbourhood of some value t_0, will have the same meaning as $g = O(f)$.

We shall denote by $\mathbb{D}$ the open unit disc of the complex plane: $\mathbb{D} = \{z \in \mathbb{C}\,;\ |z| < 1\}$, and by $\mathbb{T} = \partial\mathbb{D}$ its boundary: $\mathbb{T} = \{z \in \mathbb{C}\,;\ |z| = 1\}$. We shall denote by m the normalized Lebesgue measure on $\mathbb{T}$.

For every analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$, the composition operator C_ϕ is the map $f \mapsto f \circ \phi$. By Littlewood's subordination principle (see [2], Theorem 1.7), every composition operator maps every Hardy space H^p ($p > 0$) into itself, and is continuous on H^p.

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ is the set

$$W(\xi, h) = \{z \in \mathbb{D}\,;\ |z| \geq 1 - h \quad \text{and} \quad |\arg(z\bar{\xi})| \leq h\}.$$

For every finite positive measure μ on $\mathbb{D}$, one sets:

$$\rho_\mu(h) = \sup_{\xi \in \mathbb{T}} \mu[W(\xi, h)].$$

We shall call this function ρ_μ the *Carleson function* of μ.

When $\phi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$, and $\mu = m_\phi$ is the measure defined on $\mathbb{D}$, for every Borel set $B \subseteq \mathbb{D}$, by:

$$m_\phi(B) = m(\{\xi \in \mathbb{T}\,;\ \phi^*(\xi) \in B\}),$$

where ϕ^* is the boundary values function of ϕ, we shall denote ρ_{m_ϕ} by ρ_ϕ. In this case, we shall say that ρ_ϕ is the Carleson function of ϕ.

For $\alpha \geq 1$, we shall say that μ is an *α-Carleson measure* if $\rho_\mu(h) \lesssim h^\alpha$. For $\alpha = 1$, μ is merely said to be a *Carleson measure*.

The Carleson Theorem (see [2], Theorem 9.3) asserts that, for $1 \leq p < \infty$ (actually, for $0 < p < \infty$), the canonical inclusion $j_\mu\colon H^p \to L^p(\mu)$ is bounded if and only if μ is a Carleson measure. Since every composition operator C_ϕ is continuous on H^p, it defines a continuous map $j_\phi\colon H^p \to L^p(\mu_\phi)$; hence every pull-back measure μ_ϕ is a Carleson measure.

When $C_\phi\colon H^2 \to H^2$ is compact, one has, as it is easy to see:

$$|\phi^*| < 1 \qquad \textit{a.e.} \text{ on } \partial\mathbb{D}. \tag{2.1}$$

Hence, we shall only consider in this paper symbols ϕ for which (2.1) is satisfied (which is the case, as we said, when C_ϕ is compact on H^2).

B. MacCluer ([9], Theorem 1.1) has shown (assuming condition (2.1)) that C_ϕ is compact on H^p if and only if $\rho_\phi(h) = o\,(h)$, as h goes to 0.

Note that, in this paper, we shall not work, most often, with exact inequalities, but with inequalities up to constants. It follows that we shall not actually work with true Carleson windows $W(\xi, h)$ (or Luecking sets, defined below), but with distorted Carleson windows:

$$\tilde{W}(\xi, h) = \{z \in \mathbb{D}\,;\ |z| \geq 1 - ah \quad \text{and} \quad |\arg(z\bar{\xi})| \leq bh\},$$

where $a, b > 0$ are given constants. Since, for a given symbol ϕ, one has:

$$m_\phi\big(W(\xi, c\,h)\big) \leq m_\phi\big(\tilde{W}(\xi, h)\big) \leq m_\phi\big(W(\xi, C\,h)\big)$$

for some constants $c = c(a, b)$ and $C = C(a, b)$ which only depend on a and b, that will not matter for our purpose.

3 Preliminaries

3.1 Luecking sets and Carleson windows

We shall begin by recalling the characterization, due to D. Luecking ([7]), of the composition operators on H^2 which belong to the Schatten classes. Let, for every integer $n \geq 1$ and $0 \leq j \leq 2^n - 1$:

$$R_{n,j} = \Big\{ z \in \mathbb{D}\,;\ 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \Big\}$$

be the *Luecking sets.*

The result is:

Theorem 3.1 (Luecking [7]) *For every $p > 0$, the composition operator C_ϕ, assuming condition* (2.1)*, is in the Schatten class S_p if and only if:*

$$\sum_{n\geq 0} 2^{np/2} \bigg(\sum_{j=0}^{2^n-1} \big[m_\phi(R_{n,j}) \big]^{p/2} \bigg) < +\infty. \tag{3.1}$$

Majorizing $m_\phi(R_{n,j})$ by $m_\phi\big(W(\mathrm{e}^{2^{-n}(2j+1)i\pi}, 2^{-n})\big) \leq \rho_\phi(2^{-n})$, one gets:

Corollary 3.2 *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map, with condition* (2.1)*, and assume that m_ϕ is an α-Carleson measure, with $\alpha > 1$. Then $C_\phi \in S_p$ for every $p > \frac{2}{\alpha - 1}\cdot$*

Proof. Since $\rho_\phi(h) \lesssim h^\alpha$, one gets:

$$\sum_{n\geq 0} 2^{np/2} \bigg(\sum_{j=0}^{2^n-1} \big[m_\phi(R_{n,j}) \big]^{p/2} \bigg) \lesssim \sum_{n\geq 0} 2^{np/2}.2^n.(2^{-n\alpha})^{p/2} = \sum_{n\geq 0} 2^{n[1-(\alpha-1)p/2]},$$

which is $< +\infty$ since $1 - (\alpha - 1)\frac{p}{2} < 0$. $\square$

In order to get this corollary, we majorized crudely $m_\phi(R_{n,j})$ by $\rho_\phi(2^{-n})$. Actually, we shall see, through the results of this paper, that we lose too much with this majorization. Nevertheless, if we replace the Luecking sets by the *dyadic Carleson windows*:

$$W_{n,j} = \Big\{ z \in \mathbb{D}\,;\ 1 - 2^{-n} \leq |z| < 1\,, \quad \frac{2j\pi}{2^n} \leq \arg(z) < \frac{2(j+1)\pi}{2^n} \Big\},$$

$(j = 0, 1, \ldots, 2^n - 1,\ n = 1, 2, \ldots)$, we have the same behaviour:

Proposition 3.3 *Let μ be a finite positive measure on the open unit disk $\mathbb{D}$ and let $\alpha > 0$. Then the following assertions are equivalent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \big(\mu(R_{n,j}) \big)^\alpha < +\infty$;

(b) $\displaystyle\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \big(\mu(W_{n,j}) \big)^\alpha < +\infty.$

Proof. It is clear that (b) implies (a) since $R_{n,j} \subset W_{n,j}$ for all n and j (we already used this in the proof of Corollary 3.2).

For the proof of the converse implication, we shall need the following sets, for positive integers l, n with $l \geq n$, and $j \in \{0, 1, \ldots, 2^n - 1\}$:

$$H_{l,n,j} = \Big\{k \in \{0, 1, \ldots, 2^l - 1\}\,;\ \frac{j}{2^n} \leq \frac{k}{2^l} < \frac{j+1}{2^n}\Big\}.$$

It is clear that we have, for every n and j:

$$W_{n,j} = \bigcup_{l \geq n} \bigcup_{k \in H_{l,n,j}} R_{l,k},$$

and

$$\mu(W_{n,j}) = \sum_{l \geq n} \sum_{k \in H_{l,n,j}} \mu(R_{l,k})\,.$$

We shall first treat the case $\alpha \leq 1$, where we can use, for $x_1, x_2, \ldots, x_N \geq 0$:

$$(x_1 + x_2 + \cdots + x_N)^\alpha \leq x_1^\alpha + x_2^\alpha + \cdots + x_N^\alpha\,.$$

We have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \big(\mu(W_{n,j})\big)^\alpha &= \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \Big(\sum_{l \geq n} \sum_{k \in H_{l,n,j}} \mu(R_{l,k})\Big)^\alpha \\
&\leq \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \sum_{l \geq n} \sum_{k \in H_{l,n,j}} \big(\mu(R_{l,k})\big)^\alpha \\
&= \sum_{l=1}^{\infty} \sum_{k=0}^{2^l-1} \big(\mu(R_{l,k})\big)^\alpha \sum_{(n,j)\,:\, n \leq l\,;\, k \in H_{l,n,j}} 2^{n\alpha}
\end{aligned}$$

Observe that, for every $n \leq l$, there is only one j such that $k \in H_{l,n,j}$. Since we have

$$\sum_{n=1}^{l} 2^{n\alpha} \leq C_\alpha 2^{l\alpha},$$

we get:

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} 2^{n\alpha} \big(\mu(W_{n,j})\big)^\alpha \leq C_\alpha \sum_{l=1}^{\infty} \sum_{k=0}^{2^l-1} 2^{l\alpha} \big(\mu(R_{l,k})\big)^\alpha\,,$$

and (a) implies (b) in the case $\alpha \leq 1$.

If $\alpha > 1$, we can use Hölder's inequality. Let β be the conjugate exponent of α. Choose a such that $1 < a < 2 < a^\beta$. Then:

$$\begin{aligned}
\mu(W_{n,j}) &= \sum_{l \geq n\,:\, k \in H_{l,n,j}} \mu(R_{l,k}) \\
&\leq \Big(\sum_{l \geq n\,:\, k \in H_{l,n,j}} a^{-l\beta}\Big)^{1/\beta} \Big(\sum_{l \geq n\,:\, k \in H_{l,n,j}} a^{l\alpha} \big(\mu(R_{l,k})\big)^\alpha\Big)^{1/\alpha}
\end{aligned}$$

Using that $|H_{l,n,j}| = 2^{l-n}$, we get:

$$\Bigg[\sum_{l\geq n\,:\,k\in H_{l,n,j}} a^{-l\beta}\Bigg]^{1/\beta} = \Bigg[\sum_{l\geq n} 2^{l-n}a^{-l\beta}\Bigg]^{1/\beta} = \left(2^{-n}\frac{(2a^{-\beta})^n}{1-2a^{-\beta}}\right)^{1/\beta} = C_\beta a^{-n}.$$

Therefore we have:

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} 2^{n\alpha}\big(\mu(W_{n,j})\big)^{\alpha} &\leq \sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} 2^{n\alpha}C_\beta^\alpha a^{-n\alpha}\sum_{l\geq n\,:\,k\in H_{l,n,j}} a^{l\alpha}\big(\mu(R_{l,k})\big)^{\alpha}\\
&= C_\beta^\alpha\sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1}\big(\mu(R_{l,k})\big)^{\alpha}a^{l\alpha}\sum_{(n,j)\,:\,n\leq l\,,k\in H_{l,n,j}}(2/a)^{n\alpha}\\
&\lesssim \sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1}\big(\mu(R_{l,k})\big)^{\alpha}a^{l\alpha}(2/a)^{l\alpha}\\
&= \sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1}2^{l\alpha}\big(\mu(R_{l,k})\big)^{\alpha}.
\end{aligned}$$

We have hence proved that (a) implies (b) for $\alpha > 1$ and therefore Proposition 3.3 follows. □

As a corollary we prove a necessary condition that ρ_ϕ must satisfy when C_ϕ is in the Schatten class S_p.

Proposition 3.4 *Let $\phi\colon \mathbb{D}\to\mathbb{D}$ be an analytic self-map. If the composition operator $C_\phi\colon H^2\to H^2$ is in the Schatten class S_p for some $p>0$, then, as h goes to 0:*

$$\rho_\phi(h) = o\left(h\Big(\log\frac{1}{h}\Big)^{-2/p}\right). \tag{3.2}$$

Proof. Thanks to Luecking's characterization and the equivalence in Proposition 3.3, we have, for the pullback mesure m_ϕ:

$$\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1}2^{np/2}\big(m_\phi(W_{n,j})\big)^{p/2} < +\infty. \tag{3.3}$$

Observe that, for $h=2^{-n}$ each window $W(\xi,h)$ is contained in the union of at most three of the $W_{n,j}$'s; hence:

$$\big(\rho_\phi(2^{-n})\big)^{p/2} \leq \Big(3\max_{0\leq j\leq 2^n-1} m_\phi(W_{n,j})\Big)^{p/2} \leq 3^{p/2}\sum_{j=0}^{2^n-1}\big(m_\phi(W_{n,j})\big)^{p/2},$$

and (3.3) yields:

$$\sum_{n=1}^{\infty}\big(\rho_\phi(2^{-n})\big)^{p/2}2^{np/2} < +\infty\,.$$

Hence, setting:

$$\gamma_n = \sum_{n/2\le k\le n} \big(\rho_\phi(2^{-k})\big)^{p/2} 2^{kp/2}\,, \tag{3.4}$$

we have:

$$\lim_{n\to\infty} \gamma_n = 0\,. \tag{3.5}$$

Now, by using [5], Theorem 4.19, we get a constant $C > 0$ such that, for $k \le n$:

$$C\rho_\phi(2^{-k}) \ge 2^{n-k}\rho_\phi(2^{-n})\,,$$

and so:

$$C^{p/2}\gamma_n \ge (n/2)\Big(\frac{\rho_\phi(2^{-n})}{2^{-n}}\Big)^{p/2}\,. \tag{3.6}$$

To finish the proof, it remains to consider, for every $h \in (0, 1/2)$, the integer n such that $2^{-n-1} < h \le 2^{-n}$; then (3.5) and (3.6) give:

$$\lim_{h\to 0^+} \Big(\frac{\rho_\phi(h)}{h}\Big)^{p/2} \log(1/h) = 0,$$

as announced. □

Remark. We can also deduce Corollary 3.2 from the following result.

Proposition 3.5 *If μ is a β-Carleson probability measure on $\mathbb{D}$, with $\beta > 2$, then the Poisson integral $\mathcal{P}\colon L^2(\mathbb{T}) \to L^2(\mu)$ is in the Schatten class S_p for any $p > 2/(\beta - 1)$.*

Proof. We may assume that $p \le 2$ since $S_{p_1} \subseteq S_{p_2}$ when $p_1 \le p_2$. To have $\mathcal{P} \in S_p$, it suffices then to have $\sum_{n\in\mathbb{Z}} \|\mathcal{P}(e_n)\|_{L^2(\mu)}^p < +\infty$, where $e_n(\mathrm{e}^{it}) = \mathrm{e}^{int}$ (see [4], Proposition 1.b.16, page 40, for example).

But $(\mathcal{P}e_n)(z) = z^n$ for $n \ge 0$ and $(\mathcal{P}e_n)(z) = \bar{z}^{|n|}$ for $n \le -1$. Hence:

$$\sum_{n\in\mathbb{Z}} \|\mathcal{P}(e_n)\|_{L^2(\mu)}^p \le 2\sum_{n=0}^{\infty} \Big(\int_{\mathbb{D}} |z|^{2n}\, d\mu\Big)^{p/2}.$$

But

$$\begin{aligned}
\int_{\mathbb{D}} |z|^{2n}\, d\mu &= \int_0^1 2nt^{2n-1}\mu(|z| \ge t)\, dt \\
&= 2n\int_0^1 (1-x)^{2n-1}\mu(|z| \ge 1-x)\, dx \\
&\lesssim n\int_0^1 (1-x)^{2n-1}\frac{1}{x}\, x^\beta\, dx \\
&\quad \text{since } \{|z| \ge 1-x\} \text{ can be split in } O(1/x) \text{ windows } W(a,x) \\
&= n\int_0^1 (1-x)^{2n-1}x^{\beta-1}\, dx \lesssim n.n^{-\beta}.
\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{Z}} \|\mathcal{P}(e_n)\|^p_{L^2(\mu)} \lesssim \sum_{n\geq 1} \frac{1}{n^{(\beta-1)p/2}},$$

which is finite since $(\beta-1)p/2 > 1$. $\square$

3.2 A general contruction

In this subsection, we are going to describe a general way to construct symbols with some prescribed conditions. A particular case of this construction has been used in [5], Theorem 4.1. We also shall use it in [6].

Let

$$f(t) = \sum_{k=0}^{\infty} a_k \cos(kt)$$

be an even, non-negative, 2π-periodic continuous function, vanishing at the origin: $f(0) = 0$, and such that:

$$f \text{ is } \textit{strictly} \text{ increasing on } [0,\pi]. \tag{3.7}$$

The Hilbert transform (or conjugate function) $\mathcal{H}f$ of f is:

$$\mathcal{H}f(t) = \sum_{k=1}^{\infty} a_k \sin(kt).$$

We shall assume moreover that, as t tends to zero:

$$\big(\mathcal{H}f\big)'(t) = o\,(1/t^2). \tag{3.8}$$

Let now $F\colon \mathbb{D} \to \Pi^+ = \{\Re\mathrm{e}\, z > 0\}$ be the analytic function whose boundary values are

$$F^*(\mathrm{e}^{it}) = f(t) + i\mathcal{H}f(t). \tag{3.9}$$

One has:

$$F(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad |z| < 1.$$

We define:

$$\Phi(z) = \exp\big(-F(z)\big), \qquad z \in \mathbb{D}. \tag{3.10}$$

Since f is non-negative, one has

$$\Re\mathrm{e}\, F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) P_z(t)\, dt > 0,$$

so that $|\Phi(z)| < 1$ for every $z \in \mathbb{D}$: Φ is an analytic self-map of $\mathbb{D}$, and $|\Phi^*| = \exp(-f) < 1$ *a.e.* . Note that the assumption $f(0) = 0$ means that $\Phi^*(1) = 1$; we then have $\|\Phi\|_\infty = 1$, which is necessary for the non-compactness of C_Φ.

An example is $\Phi(z) = \exp\big(-(1-z)/2\big)$, for which $f(t) = \sin^2(t/2)$.

Lemma 3.6 *Assume that f and $\mathcal{H}f$ are $\mathcal{C}^1$ functions. Then, the Carleson function ρ_Φ of Φ is not $o\,(h)$ when h goes to 0, and so the composition operator $C_\Phi\colon H^2 \to H^2$ is not compact.*

Note that the hypothesis of the lemma holds, for example, when:

$$\sum_{k=0}^{\infty} k\,|a_k| < +\infty. \tag{3.11}$$

Proof. The non-compactness of C_Φ follows immediately from the "angular derivative" condition ([11], Theorem 3.5: see (4.5) in the remark at the end of the next section). But we shall give a proof using the Carleson function, in order to illustrate the methods to be used later on.

It will be enough to minorize $\mu_\Phi[W(1,h)]$. Since f and $\mathcal{H}f$ are $\mathcal{C}^1$, we have $|f(t)| \leq C|t|$ and $|\mathcal{H}f(t)| \leq C|t|$ for some positive constant C. Now, if $|t| \leq h/C$, we see that

$$|\Phi^*(\mathrm{e}^{it})| = \mathrm{e}^{-f(t)} \geq \mathrm{e}^{-C|t|} \geq \mathrm{e}^{-h} \geq 1-h,$$

and $|\arg\Phi^*(\mathrm{e}^{it})| = |\mathcal{H}f(t)| \leq C|t| \leq h$; hence $\Phi^*(\mathrm{e}^{it}) \in W(1,h)$. This means that

$$m_\Phi[W(1,h)] \geq m(\{\mathrm{e}^{it}\,;\ |t| \leq h/C\}) \geq \frac{1}{\pi C}\,h,$$

and the lemma follows. $\square$

We shall now perturb Φ by considering

$$M(z) = \exp\Big(-\frac{1+z}{1-z}\Big), \qquad |z|<1, \tag{3.12}$$

and

$$\phi(z) = M(z)\Phi(z), \qquad |z|<1. \tag{3.13}$$

One has:

$$\phi^*(\mathrm{e}^{it}) = \mathrm{e}^{-f(t)}\,\mathrm{e}^{-i(\mathcal{H}f(t)+\cot\frac{t}{2})}.$$

We will now, according to the various choices of f, study the behaviour of ϕ with respect to the Carleson windows.

We set:

$$\gamma(t) = \mathcal{H}f(t) + \cot\frac{t}{2} = \frac{2}{t} + r(t), \tag{3.14}$$

where the derivative of the odd function r satisfies $r'(t) = o\,(1/t^2)$.

Now, we have:

Lemma 3.7 *When $h>0$ goes to 0, one has:*

$$hf^{-1}(h) \lesssim \rho_\phi(h) \lesssim hf^{-1}(2h). \tag{3.15}$$

Proof. Let $a = \mathrm{e}^{i\theta} \in \mathbb{T}$, $|\theta| \leq \pi$. We may assume that $0 < h \leq h_0 \leq 1/2$, for some h_0 small enough.

We have to analyze the set of t's such that $\phi^*(\mathrm{e}^{-it}) = \mathrm{e}^{-f(t)}\mathrm{e}^{i\gamma(t)} \in W(a,h)$, which imposes two constraints. Without loss of generality, we may analyze only the set of positive t's, *i.e.* $0 < t \leq \pi$.

Modulus constraint. We must have $|\phi^*(\mathrm{e}^{-it})| \geq 1-h$, *i.e.* $\mathrm{e}^{-f(t)} \geq 1-h$, or $f(t) \leq \log \frac{1}{1-h}$, which is $\leq 2h$ since $h \leq 1/2$. Hence we must have:

$$0 < t \leq f^{-1}(2h). \tag{3.16}$$

Argument constraint. We must have $|\gamma(t) - \theta| \leq h \mod 2\pi$, *i.e.*, since we have assumed that $t > 0$:

$$\gamma(t) \in \bigcup_{n\geq 0} [\theta - h + 2n\pi, \theta + h + 2n\pi] = \bigcup_{n \geq 0} J_n(h). \tag{3.17}$$

Since $\gamma(t) \to +\infty$ as $t \overset{>}{\to} 0$, and since we already have $t \leq f^{-1}(2h)$ by (3.16), we know that $\gamma(t) > 2\pi$ for h small enough; hence we have $\gamma(t) \in J_n(h)$ only for $n \geq N_h$, where the integer N_h goes to infinity when h goes to 0; in particular, we may assume that, for h small enough, we have $\gamma(t) \in J_n(h)$ for $n \geq 1$ only.

Let $I_n(h) = \gamma^{-1}\big(J_n(h)\big)$. Since $\gamma(t) = \frac{2}{t} + r(t)$, and $r'(t) = o\,(1/t^2)$, $\gamma(t)$ is decreasing for $0 < t \leq h_0$, for h_0 small enough. Hence, for h small enough:

$$I_n(h) = \big[\gamma^{-1}(\theta + h + 2n\pi), \gamma^{-1}(\theta - h + 2n\pi)\big].$$

Since $\gamma(t) = \frac{2}{t} + o\,(1/t)$ and $|\gamma'(t)| = \frac{2}{t^2} + o\,(1/t^2)$, one has:

$$\frac{c_1}{n} \leq \min I_n(h) \leq \max I_n(h) \leq \frac{c_2}{n}\,, \tag{3.18}$$

where c_1 and c_2 are two universal positive constants. By the mean-value theorem, we get that:

$$\begin{aligned} 2\pi m\big(I_n(h)\big) &= \gamma^{-1}(\theta - h + 2n\pi) - \gamma^{-1}(\theta + h + 2n\pi) \\ &= 2h|(\gamma^{-1})'(\xi_n)| = \frac{2h}{|\gamma'(t_n)|}\,, \end{aligned}$$

for some $\xi_n \in J_n(h)$ and $t_n \in I_n(h)$. But, (3.18) ensures that $\frac{c_1}{n} \leq t_n \leq \frac{c_2}{n}$ and, since $|\gamma'(t)| = \frac{2}{t^2} + o(1/t^2)$, we get that:

$$m\big(I_n(h)\big) \approx \frac{h}{n^2}\cdot \tag{3.19}$$

Now:

1) Assume that $\phi^*(\mathrm{e}^{-it}) \in W(a,h)$. By (3.17), (3.18) and (3.16), we must have $t \in I_n(h)$ with $\frac{c_1}{n} \leq t \leq f^{-1}(2h)$. Hence, if n_0 is the integer part of $\frac{c_1}{f^{-1}(2h)}$,

we must have $n \geq n_0$. Now, (3.19) shows that:

$$m\big(\{t \in]0,\pi]\,;\ \phi^*(\mathrm{e}^{-it}) \in W(a,h)\}\big) \leq \sum_{n \geq n_0} m\big(I_n(h)\big)$$
$$\lesssim \sum_{n \geq n_0} \frac{h}{n^2} \lesssim \frac{h}{n_0} \lesssim h f^{-1}(2h).$$

2) We want to minorize, as it suffices, $m\big(\{t \in]0,\pi]\,;\ \phi^*(\mathrm{e}^{it}) \in W(1,h)\}\big)$. Let n_1 be the integer part of $\frac{c_2}{f^{-1}(h)} + 1$; we have:

$$t \in \bigcup_{n \geq n_1} I_n(h) \qquad \Longrightarrow \qquad \phi^*(\mathrm{e}^{it}) \in W(1,h) \tag{3.20}$$

because $t \in I_n(h)$ for $n \geq n_1$ implies $t \leq \frac{c_2}{n_1} \leq f^{-1}(h)$, so that $|\phi^*(\mathrm{e}^{it})| = \mathrm{e}^{-f(t)} \geq \mathrm{e}^{-h} \geq 1 - h$, and the modulus constraint for $\phi^*(\mathrm{e}^{it})$ is automatically satisfied. Since the argument constraint is satisfied by construction, as t belongs to some $I_n(h)$, this proves (3.20).

As a consequence, we have, using (3.19):

$$\rho_\phi(h) \geq m_\phi\big(W(1,h)\big) \geq \sum_{n \geq n_1} m\big(I_n(h)\big) \gtrsim \sum_{n \geq n_1} \frac{h}{n^2} \gtrsim \frac{h}{n_1} \gtrsim h f^{-1}(h),$$

and this ends the proof of Lemma 3.7. □

4 Composition operators with symbol of same modulus

J. Shapiro and P. Taylor ([13], Theorem 3.1) characterized Hilbert-Schmidt composition operators C_ϕ on H^2 (*i.e.* $C_\phi \in S_2$), and this characterization depends only on the modulus of ϕ^* on $\partial\mathbb{D}$. It follows that if ϕ_1 and ϕ_2 are two symbols such that $|\phi_1^*| = |\phi_2^*|$, and C_{ϕ_2} is Hilbert-Schmidt, then C_{ϕ_1} is also Hilbert-Schmidt, and in particular, compact.

It appears that this is a limiting case, as we shall see in Theorem 4.2.

Actually, in view of Corollary 3.2, we may formulate this problem in the following way. Assume that the composition operator C_{ϕ_1} is not compact on H^2, and that $|\phi_1^*| = |\phi_2^*|$; for which values of α, can m_{ϕ_2} be an α-Carleson measure?

Note that if m_{ϕ_2} is an α-Carleson measure, we necessarily must have $\alpha \leq 2$. Indeed, if $\alpha > 2$, Corollary 3.2 implies that $C_{\phi_2} \in S_2$, and hence $C_{\phi_1} \in S_2$ as well.

When $\alpha < 2$, the situation is very different, and one has:

Theorem 4.1 *For every α with $1 < \alpha < 2$, there exist two symbols ϕ_1 and ϕ_2 having the same modulus on $\partial\mathbb{D}$ and such that $\rho_{\phi_1}(h) \approx h$, but $\rho_{\phi_2}(h) \approx h^\alpha$.*

It follows from Corollary 3.2 that:

Theorem 4.2 *For every $p > 2$, there exist two symbols ϕ_1 and ϕ_2 having the same modulus on $\partial\mathbb{D}$ and such that C_{ϕ_1} is not compact on H^2, but C_{ϕ_2} is in the Schatten class S_p.*

Proof of Theorem 4.1. In [5], Theorem 4.1, we proved a particular case of this result, corresponding to $\alpha = 3/2$. We took there $\phi_1(z) = \frac{1+z}{2}$. This function behaves as $\exp\big(-[\sin^2(t/2)] + i\mathcal{H}[\sin^2(t/2)]\big)$, and to prove Theorem 4.1, we shall just change the power 2 of $\sin(t/2)$.

We shall use the following lemma, whose proof will be postponed.

Lemma 4.3 *For $0 < \beta < 2$, let $f(t) = |\sin\frac{t}{2}|^\beta$. Then:*

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos kt\,,$$

with:

$$c_k < 0 \quad \textit{for all } k \geq 1; \tag{4.1}$$

and

$$c_k = O\Big(\frac{1}{k^{\beta+1}}\Big)\cdot \tag{4.2}$$

In particular, for $\beta > 1$, the series $\sum_{k\geq 1} kc_k$ is convergent, and $\sum_{k\geq 1} kc_k < 0$.

Taking $\beta = \frac{1}{\alpha-1}$, which is > 1, and $f(t) = |\sin(t/2)|^\beta$ as in Lemma 4.3, observe that f satisfies the assumptions (3.11) and (3.7) of the general construction of the Subsection 3.2 (note that for $\beta \geq 2$, f is a $\mathcal{C}^2$ function, and hence, we have (3.11) directly, without using Lemma 4.3). With the notation of that subsection (see (3.9), (3.10), and (3.12)), set:

$$\phi_1 = \Phi \qquad \text{and} \qquad \phi_2 = M\Phi.$$

One has:

$$|\phi_1^*| = |\phi_2^*| \quad a.e.$$

Lemma 3.6 and MacCluer's theorem show that C_{ϕ_1} is not compact on H^2. On the other hand, since $f^{-1}(h) \approx h^{1/\beta}$, Lemma 3.7 shows that $\rho_{\phi_2}(h) \approx h^\alpha$, with $\alpha = 1 + \frac{1}{\beta} \in]1, 2[$.

That ends the proof of Theorem 4.1. □

Proof of Lemma 4.3. Before beginning the proof, it should be remarked that for $\beta = 2$ (*i.e.* $\alpha = 3/2$, which is the case processed in [5], Theorem 4.1), we have a trivial situation: $\sin^2\frac{t}{2} = \frac{1}{2} - \frac{1}{2}\cos t$.

Proof of (4.1). For $0 < p < 1$, we have the well-known binomial expansion, for $-1 \leq x \leq 1$:

$$(1-x)^p = 1 - \sum_{k=1}^{\infty} \alpha_k x^k,$$

with:

$$\alpha_k = \frac{p(1-p)\cdots(k-1-p)}{k!} > 0.$$

Taking $x = \cos t$ and $p = \beta/2$, we get:

$$2^{\beta/2}\Big|\sin\frac{t}{2}\Big|^{\beta} = 1 - \sum_{k=1}^{\infty} \alpha_k (\cos t)^k. \tag{4.3}$$

Now, we know that:

$$(\cos t)^k = \sum_{j=0}^{k} b_{j,k} \cos(k-2j)t\,,$$

with $b_{j,k} > 0$. Substituting this in (4.3), grouping terms, and dividing by $2^{\beta/2}$, we get (4.1).

Proof of (4.2). We shall separate the cases $\beta = 1$, $0 < \beta < 1$, and $1 < \beta < 2$.

For $\beta = 1$, one has, in an explicit way:

$$\Big|\sin\frac{t}{2}\Big| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos kt}{4k^2-1} = c_0 + \sum_{k=1}^{\infty} c_k \cos kt\,.$$

Assume $0 < \beta < 1$. Since $\frac{\sin t/2}{t/2} > 0$ on $[0,\pi]$, we can write:

$$\Big(\frac{\sin t/2}{t/2}\Big)^{\beta} = 1 + t^2 u(t)\,,$$

where u is a $\mathcal{C}^\infty$ function on $[0,\pi]$. Then:

$$c_k = \frac{2}{2^\beta \pi}\int_0^\pi t^\beta \cos kt\,dt + \frac{2}{2^\beta \pi}\int_0^\pi t^{\beta+2} u(t)\cos kt\,dt.$$

The second integral is $O(k^{-2})$, as easily seen by making two integrations by parts. The first one writes:

$$\begin{aligned}\int_0^\pi t^\beta \cos kt\,dt &= -\frac{\beta}{k}\int_0^\pi t^{\beta-1}\sin kt\,dt \\ &= -\frac{\beta}{k}\int_0^{k\pi}\Big(\frac{x}{k}\Big)^{\beta-1}\sin x\,\frac{dx}{k} \\ &\sim -\frac{\beta}{k^{\beta+1}}\int_0^{+\infty} x^{\beta-1}\sin x\,dx.\end{aligned}$$

This last integral is convergent and positive. Hence, since $\beta + 1 < 2$:

$$c_k \sim -\delta k^{-(\beta+1)},$$

where δ is a positive constant.

Before continuing, let us observe that we have similarly, due to the vanishing of the integrated terms:

$$\int_0^\pi \Big(\sin\frac{t}{2}\Big)^\sigma \sin(2k+1)\frac{t}{2}\,dt = O(k^{-(\sigma+1)}) + O(k^{-2}) = O(k^{-(\sigma+1)}), \tag{4.4}$$

for $0 < \sigma < 1$.

Assume now $1 < \beta < 2$. We have:

$$\begin{aligned}
\frac{\pi}{2} c_k &= \int_0^\pi f(t)\cos kt\,dt = -\frac{1}{k}\int_0^\pi f'(t)\sin kt\,dt \\
&= -\frac{\beta}{2k}\int_0^\pi \big(\sin(t/2)\big)^{\beta-1}\cos(t/2)\sin(kt)\,dt \\
&= -\frac{\beta}{4k}\Big[\int_0^\pi \Big(\sin\frac{t}{2}\Big)^{\beta-1}\sin(2k+1)\frac{t}{2}\,dt \\
&\qquad\qquad + \int_0^\pi \Big(\sin\frac{t}{2}\Big)^{\beta-1}\sin(2k-1)\frac{t}{2}\,dt\Big] \\
&= \frac{1}{k}\,O(k^{-\beta}) = O(k^{-(\beta+1)}),
\end{aligned}$$

in view of 4.4, applied with $\sigma = \beta - 1 \in]0,1[$.

This ends the proof of Lemma 4.3. $\square$

Remark. The following question arises naturally: does $C_\phi \in S_p$ for some $p < \infty$ imply that μ_ϕ is α-Carleson for some $\alpha > 1$? We shall see in the next section (Remark 2 after the proof of Proposition 5.3) that the answer is negative.

Another question, related to our work, has been raised by K. Kellay: given a compact composition operator $C_\phi\colon H^2 \to H^2$, and another symbol $\psi\colon \mathbb{D} \to \mathbb{D}$, is the composition operator $C_{\phi\psi}\colon H^2 \to H^2$ still compact? This is the case if $\phi\psi$ is univalent. In fact, the compactness of C_ϕ implies ([11], Theorem 3.5) that:

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = +\infty; \tag{4.5}$$

hence, since $|\phi(z)\psi(z)| \leq |\phi(z)|$ for every $z \in \mathbb{D}$, we have:

$$\lim_{|z|\to 1} \frac{1-|\phi(z)\psi(z)|}{1-|z|} = +\infty,$$

which implies the compactness of $C_{\phi\psi}$, thanks to the univalence of $\phi\psi$ ([11], Theorem 3.2).

In [5], Proposition 4.2, we proved a related result: let $\phi_1, \phi_2 \colon \mathbb{D} \to \mathbb{D}$ be univalent analytic self-maps such that $|\phi_1^*| \leq |\phi_2^*|$ on $\partial\mathbb{D}$; if $C_{\phi_2} \colon H^2 \to H^2$ is compact, and ϕ_2 vanishes at some point $a \in \mathbb{D}$, then $C_{\phi_1} \colon H^2 \to H^2$ is also compact. Note that the vanishing condition for ϕ_2 in that result is automatic (since ϕ_2 must have a fixed point $a \in \mathbb{D}$ because the composition operator C_{ϕ_2} is compact, see [11], page 84, § 5.5, Corollary), but the univalence condition for ϕ_1 and ϕ_2 cannot be dropped. In fact, take $\phi_1(z) = \exp\big(-(1-z)/2\big)$ and $\phi_2(z) = z\,M(z)\,\exp\big(-(1-z)/2\big)$ (where $M(z)$ is defined by (3.12)). One has $|\phi_1^*| = |\phi_2^*|$ on $\partial\mathbb{D}$, and ϕ_1 is univalent (if $\phi_1(z) = \phi_1(w)$, $k = 0$ is the only integer such that $\frac{z}{2} - \frac{w}{2} = 2k\pi i$, since $|z| + |w| \leq 2$). But ϕ_1 is the function Φ defined by (3.10), with $f(t) = \sin^2(t/2)$. Since f and $\mathcal{H}f$ are $\mathcal{C}^1$, Lemma 3.6 says that C_{ϕ_1} is not compact. On the other hand, one has $f^{-1}(h) \approx h^{1/2}$, so Lemma 3.7 gives the compactness of C_{ϕ_2} (and even, $C_{\phi_2} \in S_p$ for every $p > 4$, by Corollary 3.2).

It should be pointed out that, however, ϕ_1 cannot be written $\phi_1 = \phi_2\psi$ for some analytic self-map $\psi \colon \mathbb{D} \to \mathbb{D}$.

5 Composition operators in Schatten classes

In [13], Theorem 4.2, J. Shapiro and P. Taylor constructed a family of composition operators $C_{\phi_\theta} \colon H^2 \to H^2$, indexed by a parameter $\theta > 0$ such that C_{ϕ_θ} is always compact, but C_{ϕ_θ} is Hilbert-Schmidt if and only if $\theta > 2$. In this section, we shall slightly modify the symbol ϕ_θ, and shall study the membership of C_{ϕ_θ} in the Schatten classes S_p. In [6], we study on which Hardy-Orlicz spaces H^Ψ these composition operators C_{ϕ_θ} are compact.

Theorem 5.1 *For every $p_0 > 0$, there exists an analytic self-map $\phi \colon \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\phi \colon H^2 \to H^2$ is in the Schatten class S_p for every $p > p_0$, but not in S_{p_0}.*

Proof. We shall use the same function as J. Shapiro and P. Taylor in [13], § 4, with slight modifications. This modified function will be easier to analyze.

Let $\theta > 0$.

For $\mathfrak{Re}\, z > 0$, $\log z$ will be the principal determination of the logarithm. Let, for $\varepsilon > 0$:

$$V_\varepsilon = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0 \text{ and } |z| < \varepsilon\}. \tag{5.1}$$

and consider, for $\varepsilon > 0$ small enough:

$$f_\theta(z) = z(-\log z)^\theta, \qquad z \in V_\varepsilon. \tag{5.2}$$

Lemma 5.2 *For $\varepsilon > 0$ small enough, one has $\mathfrak{Re}\, f_\theta(r\mathrm{e}^{i\alpha}) > 0$ for $0 < r < \varepsilon$ and $|\alpha| < \pi/2$. Moreover, one has $\mathfrak{Re}\, f_\theta^*(z) > 0$ for all $z \in \partial V_\varepsilon \setminus \{0\}$.*

Proof. Actually, f_θ can be defined on $\overline{V_\varepsilon} \setminus \{0\}$, and we shall do that.

Let, for $|\alpha| \leq \pi/2$ and $0 < r \leq \varepsilon$:

$$Z_\alpha = \big(-\log(r\mathrm{e}^{i\alpha})\big)^\theta = \Big(\log\frac{1}{r} - i\alpha\Big)^\theta .$$

One has $f_\theta(r\mathrm{e}^{i\alpha}) = r\mathrm{e}^{i\alpha}|Z_\alpha|\mathrm{e}^{i\arg Z_\alpha}$, so that:

$$\mathfrak{Re}\, f_\theta(r\mathrm{e}^{i\alpha}) = r\,|Z_\alpha|\cos(\alpha + \arg Z_\alpha).$$

On the other hand,

$$\arg Z_\alpha = -\theta \arctan\frac{\alpha}{\log 1/r}\,\cdot$$

Since $\arctan x \geq x/2$ for $0 \leq x \leq 1$, we get, for $0 < r \leq \varepsilon \leq \mathrm{e}^{-\pi/2}$:

$$\begin{aligned}|\alpha + \arg Z_\alpha| &\leq |\alpha|\Big(1 - \frac{\theta}{\alpha}\arctan\frac{\alpha}{\log 1/r}\Big)\\ &\leq \frac{\pi}{2}\Big(1 - \frac{\theta}{2\log 1/r}\Big) = \Upsilon_r.\end{aligned}$$

Therefore, for $0 < r \leq \varepsilon$ and $|\alpha| \leq \pi/2$:

$$\mathfrak{Re}\, f_\theta(r\mathrm{e}^{i\alpha}) \geq (\cos\Upsilon_r)\, r\Big(\log\frac{1}{r}\Big)^\theta > 0,$$

as announced in Lemma 5.2. □

Let now g_θ be the conformal mapping from $\mathbb{D}$ onto V_ε, which maps $\mathbb{T} = \partial\mathbb{D}$ onto ∂V_ε, and with $g_\theta(1) = 0$ and $g_\theta'(1) = -\varepsilon/4$. Explicitly, g_θ is the composition of the following maps: a) $\sigma\colon z \mapsto -z$ from $\mathbb{D}$ onto itself; b) $\gamma\colon z \mapsto \frac{z+i}{1+iz} = \frac{z+\bar{z}+i(1-|z|^2)}{|1+iz|^2}$ from $\mathbb{D}$ onto $P = \{\mathfrak{Im}\, z > 0\}$; c) $s\colon z \mapsto \sqrt{z}$ from P onto $Q = \{\mathfrak{Re}\, z > 0\,, \mathfrak{Im}\, z > 0\}$; d) $\gamma^{-1}\colon z \mapsto \frac{z-i}{1-iz}$ from Q onto $V = \{|z| < 1\,, \mathfrak{Re}\, z > 0\}$, and e) $h_\varepsilon\colon z \mapsto \varepsilon z$ from V onto V_ε.

Let:

$$\phi_\theta = \exp(-f_\theta \circ g_\theta). \tag{5.3}$$

By Lemma 5.2, the analytic function ϕ_θ maps $\mathbb{D}$ into $\mathbb{D}$. Moreover, one has $|\phi_\theta^*| < 1$ on $\partial\mathbb{D} \setminus \{1\}$.

Now Theorem 5.1 will follow from the following proposition.

Proposition 5.3 *With the above notation, $C_{\phi_\theta}\colon H^2 \to H^2$ is compact for every $\theta > 0$ and, moreover, is in the Schatten class S_p if and only if $p > \frac{4}{\theta}\cdot$*

Hence, given $p_0 > 0$, if we choose $\theta_0 = \frac{4}{p_0}$, we get that $C_{\phi_{\theta_0}}$ is in S_p if and only if $p > p_0$. □

Proof of Proposition 5.3 The compactness was proved in [13], Theorem 4.2; indeed, setting $h_\theta = f_\theta \circ g_\theta$, J. Shapiro and P. Taylor used the symbol $\phi_\theta = 1 - h_\theta$

and proved that $\lim_{t\to 0} |\phi_\theta'(\mathrm{e}^{it})| = +\infty$: see equation (4.4) in [13] (note that, in order to deduce the compactness from this equality, J. Shapiro and P. Taylor had to prove a theorem: Theorem 2.4 in [13], whose proof needs to use Gabriel's Theorem). Since our symbol is $\phi_\theta = \exp(-h_\theta)$, and $\exp\big(h_\theta(\mathrm{e}^{it})\big) \to 1$ as $t \to 0$, it follows that the derivatives have the same behaviour when $t \to 0$. However, we are going to recover this result by another method. For convenience, we shall write the boundary values $f_\theta^*, g_\theta^*, \ldots$ of the different analytic functions $f_\theta, g_\theta, \ldots$ in the same way as the analytic functions, without the exponent *.

Note that, by Lemma 5.2, $|\phi_\theta|$ is far from 1 when $g_\theta(z)$ belongs to the half-circle $\{\varepsilon\mathrm{e}^{i\alpha}\,;\ |\alpha| \leq \pi/2\}$. Hence, we only have to study the case where $g_\theta(\mathrm{e}^{it}) = it$, $-\varepsilon \leq t \leq \varepsilon$. Moreover, since $f_\theta(it) = it\big(\log\frac{1}{|t|} - i\,\mathrm{sgn}\,(t)\frac{\pi}{2}\big)^\theta$, one has

$$|f_\theta(it)| \approx |t|\Big(\log\frac{1}{|t|}\Big)^\theta,$$

so that $|\phi_\theta\big(g_\theta^{-1}(it)\big)|$ is far from 1 when t is away from 0. Therefore, it suffices to study what happens when t is in a neighbourhood of 0.

Note also that g_θ is bi-Lipschitz in a neighbourhood of 1 (so $g_\theta(\mathrm{e}^{it}) \approx it$ when $|t| \leq \pi/2$), so we may forget it, and only consider the measure of the t's for which $f_\theta(it)$ belongs to the suitable sets. Moreover, for convenience, we only write the proof for $t > 0$.

Since

$$\begin{aligned} f_\theta(it) &= it\Big(\log\frac{1}{t}\Big)^\theta\bigg[\Big(1 - \frac{i\pi/2}{\log 1/t}\Big)^\theta\bigg] \\ &= it\Big(\log\frac{1}{t}\Big)^\theta\bigg[1 - \frac{i\pi\theta/2}{\log 1/t} + o\Big(\frac{1}{\log 1/t}\Big)\bigg], \end{aligned} \tag{5.4}$$

one has:

$$\mathfrak{Re}\,f_\theta(it) \approx t\Big(\log\frac{1}{t}\Big)^{\theta-1} \tag{5.5}$$

$$\mathfrak{Im}\,f_\theta(it) \approx t\Big(\log\frac{1}{t}\Big)^{\theta}. \tag{5.6}$$

Now, we have $\exp\big(-f_\theta(it)\big) \in W(e^{-i\alpha}, h)$ $(0 \leq \alpha < 2\pi)$ if and only if

$$\mathfrak{Re}\,f_\theta(it) \lesssim h \tag{5.7}$$

$$|\mathfrak{Im}\,f_\theta(it) - \alpha| \lesssim h. \tag{5.8}$$

But, when $e^{-i\alpha} \not\approx 1$, this cannot happen for h small enough (since t goes to 0 as h goes to 0). Essentially, we only have to consider the case $\mathrm{e}^{-i\alpha} = 1$, for which one has: when $t > 0$ goes to 0, $|\mathfrak{Re}\,f_\theta(it)| \lesssim |\mathfrak{Im}\,f_\theta(it)|$, and hence $\exp\big(-f_\theta(it)\big) \in W(1,h)$ if and only if $t(\log 1/t)^\theta \lesssim h$, *i.e.* $t \lesssim h/(\log 1/h)^\theta$. Actually, we cannot assume $\alpha = 0$, and we have to be more precise, and must do the following reasoning. When (5.7) is satisfied, one has $t(\log 1/t)^{\theta-1} \lesssim h$; then $t \lesssim h/(\log 1/h)^{\theta-1}$ and hence:

$$t\Big(\log\frac{1}{t}\Big)^\theta \lesssim \frac{h}{(\log 1/h)^{\theta-1}}\Big(\log\frac{1}{h}\Big)^\theta = h\log\frac{1}{h}\cdot$$

It follows that the condition (5.8) implies that:

$$0 \leq \alpha \lesssim h \log \frac{1}{h} \cdot \tag{5.9}$$

Therefore condition (5.8) implies that $t(\log 1/t)^\theta \lesssim h \log 1/h$, which gives:

$$t \lesssim \frac{h \log 1/h}{(\log 1/h)^\theta} = \frac{h}{(\log 1/h)^{\theta-1}},$$

i.e. $t(\log 1/t)^{\theta-1} \lesssim h$: condition (5.7) is satisfied (up to a constant factor for h).

Since, by (5.9), condition (5.8) is satisfied when $\alpha - h \lesssim t(\log 1/t)^\theta \lesssim (\alpha+h)$ and implies that $-(\alpha+h) \lesssim t(\log 1/t)^\theta \lesssim (\alpha+h)$, it follows that the set of t's such that (5.7) and (5.8) are satisfied has, since $\log(\alpha+h) \approx \log h$, a measure $\approx h/(\log 1/h)^\theta$.

We then have proved that:

$$\rho_{\phi_\theta}(h) \approx \frac{h}{(\log 1/h)^\theta}. \tag{5.10}$$

Since $\rho_{\phi_\theta}(h) = o\,(h)$, MacCluer's criterion gives the compactness of C_{ϕ_θ}.

Now, we shall examine when C_{ϕ_θ} is in the Schatten class S_p, and for that we shall use Luecking's theorem (Theorem 3.1). We have to analyze the behaviour of ϕ_θ with respect to the Luecking sets $R_{n,j}$; actually, for convenience, we shall work with $R'_{n,j} = R_{n,2^n-1-j}$.

We have to consider the modulus and the argument constraints.

Modulus constraint. The condition $\frac{h}{2} \leq \Re\mathrm{e}\, f_\theta(it) < h$ writes:

$$\frac{h}{2} \lesssim t\Big(\log \frac{1}{t}\Big)^{\theta-1} \lesssim h, \tag{5.11}$$

and reads as:

$$\frac{h/2}{(\log 1/h)^{\theta-1}} \lesssim t \lesssim \frac{h}{(\log 1/h)^{\theta-1}}\,, \tag{5.12}$$

or, when $h = h_n = 2^{-n}$:

$$\frac{2^{-(n+1)}}{(n+1)^{\theta-1}} \lesssim t \lesssim \frac{2^{-n}}{n^{\theta-1}}. \tag{5.13}$$

More precisely, one must have $t \in I_n = [a_n, b_n]$, with $a_n \approx \frac{2^{-(n+1)}}{(n+1)^{\theta-1}}$, $b_n \approx \frac{2^{-n}}{n^{\theta-1}}$, and $|I_n| \approx \frac{2^{-n}}{n^{\theta-1}}$.

Argument constraint. We are looking for the set J_n of the indices $j = 0, 1, \ldots, 2^n - 1$ for which we have $\exp\big(-f_\theta(it)\big) \in R'_{n,j}$. We have must have both the modulus constraint $t \in I_n$ and:

$$jh_n \lesssim t\Big(\log \frac{1}{t}\Big)^\theta \lesssim (j+1)h_n, \tag{5.14}$$

which implies, for $j \geq 1$, since one has (5.11):

$$j \lesssim \log \frac{1}{t} \lesssim 2(j+1),$$

and hence, by (5.12) and (5.13):

$$\frac{n}{2} \log 2 \lesssim j \lesssim n \log 2,$$

i.e. $j \approx n$. The constant coefficients are not relevant here, and hence this estimation means that j can only take $O(n)$ values.

On the other hand, when the modulus constraint (5.11) is satisfied, one has $\log(1/t) \approx \log(1/h_n)$, and the argument constraint (5.14) is equivalent, with $h = h_n$, to:

$$\frac{jh}{(\log 1/h)^\theta} \lesssim t \lesssim \frac{(j+1)h}{(\log 1/h)^\theta}.$$

The length of the corresponding interval is $\approx h/\big(\log(1/h)\big)^\theta$, which is equal, when $h = h_n = 2^{-n}$, to $2^{-n}/n^\theta$.

It follows hence that $m_{\phi_\theta}(R'_{n,j}) \approx 2^{-n}/n^\theta$ for exactly $O(n)$ values of j, and otherwise $m_{\phi_\theta}(R'_{n,j}) = 0$; therefore:

$$\sum_{n=0}^{\infty} 2^{np/2} \sum_{j=0}^{2^n-1} \big[m_{\phi_\theta}(R'_{n,j})\big]^{p/2} \approx \sum_{n=1}^{\infty} 2^{np/2} n \Big(\frac{2^{-n}}{n^\theta}\Big)^{p/2} = \sum_{n=1}^{\infty} \frac{1}{n^{\theta\frac{p}{2}-1}},$$

which is finite if and only if $\frac{\theta p}{2} - 1 > 1$, *i.e.* $p > 4/\theta$. □

Remark 1. Proposition 5.3 shows that, for these symbols ϕ_θ, the necessary condition (3.2) of Proposition 3.4 is not sharp; in fact, we have proved in (5.10) that $\rho_{\phi_\theta}(h) \approx h/(\log 1/h)^\theta$; hence (3.2) gives $\theta > 2/p$ when $C_{\phi_\theta} \in S_p$, even though we must have $\theta > 4/p$.

Remark 2. These composition operators answer negatively the question asked at the end of Section 4, since, by (5.10), the measure m_{ϕ_θ} is α-Carleson for no $\alpha > 1$, though C_{ϕ_θ} is in S_p for every $p > 4/\theta$.

Remark 3. Our proof of Theorem 4.1 fails when $\alpha = 2$ (*i.e.* $\beta = 1$). However, we are going to see that in this case, the composition operator C_{ϕ_1} of this Theorem 4.1 is in S_p for every $p > 4$.

Indeed, for $\beta = 1$, one has, in an explicit way:

$$\Big|\sin\frac{t}{2}\Big| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos kt}{4k^2-1} = c_0 + \sum_{k=1}^{\infty} c_k \cos kt\,,$$

with $c_k = O(k^{-2})$. Then:

$$\mathcal{H}f(t) = \sum_{k=1}^{\infty} \frac{\sin kt}{4k^2 - 1} = \frac{1}{4} u(t) + v(t),$$

where:

$$u(t) = \sum_{k=1}^{\infty} \frac{\sin kt}{k^2}\,,$$

and v is $\mathcal{C}^2$, since $\hat{v}(k) = O(k^{-4})$. We have:

$$u'(t) = \sum_{k=1}^{\infty} \frac{\cos kt}{k} = -\log\Big(2\sin\frac{t}{2}\Big), \qquad 0 < t < \pi,$$

(note that $u \notin \mathrm{Lip}\,1$, but u is in the Zygmund class). It follows, when $t > 0$ goes to zero, that:

$$\mathcal{H}f(t) \approx -t\log\Big(2\sin\frac{t}{2}\Big) \approx t\log\frac{1}{t}\cdot$$

Since the modulus of continuity of $\mathcal{H}f$ at 0 is $\approx t\log(1/t)$ (see also [16], Chapter III, Theorem (13.30) for a general result), it follows that the argument condition on ϕ_1 is $|t| \lesssim \frac{h}{\log(1/h)}$ and that $\rho_{\phi_1}(h) = O\big(\frac{h}{\log(1/h)}\big)$. Then C_{ϕ_1} is compact. It is actually in S_p for every $p > 4$. In fact, when $t > 0$ goes to zero, $\sin(t/2) \sim t/2$, and hence, with the notation of the Subsection 3.2, one has $|\Phi^*(t)| \sim t$, whereas $\arg\big(\Phi^*(t)\big) \approx t/\log(1/t)$. We have then the same conditions as in the proof of Proposition 5.3, when $\theta = 1$. Hence $C_{\phi_1} \in S_p$ for every $p > 4$. $\square$

Remark 4. We shall characterize in [6] the Orlicz functions Ψ for which these composition operators C_{ϕ_θ} are compact.

In Theorem 5.1, we get, for any $p_0 > 0$, a composition operator which is in S_p if and only if $p > p_0$. We can modify slightly this operator so as to belong in S_p if and only if $p \geq p_0$.

Theorem 5.4 *For every $p_0 > 0$, there exists an analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\phi\colon H^2 \to H^2$ is in the Schatten class S_{p_0}, but not in S_p, for $p < p_0$.*

Proof. We shall use the same method as in Theorem 5.1, but by replacing the function f_θ by this modified function:

$$\tilde{f}_\theta(z) = z(-\log z)^\theta\big[\log(-\log z)\big]^q, \tag{5.15}$$

where $q > \theta/2$ ($q = \theta$, for example).

Call $\tilde{\phi}_\theta$ the corresponding self-map of $\mathbb{D}$.

We shall not give all the details since they are mostly the same as in the proof of Theorem 5.1.

We have first to check that:

Lemma 5.5 *For $\varepsilon > 0$ small enough, one has $\Re\mathrm{e}\, \tilde{f}_\theta(z) > 0$ for every $z \in V_\varepsilon$. Moreover $|\tilde{\phi}_\theta^*| < 1$ a.e. on $\partial\mathbb{D}$.*

Proof. Write $z = r\mathrm{e}^{i\alpha}$ with $0 < r < \varepsilon$ and $|\alpha| \leq \pi/2$. One has:

$$\tilde{f}_\theta(z) = r\mathrm{e}^{i\alpha}\Big(\log\frac{1}{r} - i\alpha\Big)^\theta \Big(\log\Big(\log\frac{1}{r} - i\alpha\Big)\Big)^q.$$

But:

$$\Big(\log\frac{1}{r} - i\alpha\Big)^\theta = \Big(\log\frac{1}{r}\Big)^\theta \Big(1 - \frac{i\alpha\theta}{\log 1/r} + o\Big(\frac{1}{(\log 1/r)}\Big)\Big);$$

and, on the other hand:

(5.16)

$$\begin{aligned}\log\Big(\log\frac{1}{r} - i\alpha\Big) &= \log\sqrt{\Big(\log\frac{1}{r}\Big)^2 + \alpha^2} + i\arg\Big(\log\frac{1}{r} - i\alpha\Big)\\ &= \Big(\log\log\frac{1}{r}\Big)\big(1 + o\,(1/\log(1/r))\big) - i\arctan\frac{\alpha}{\log 1/r}\\ &= \Big(\log\log\frac{1}{r}\Big)\big(1 + o\,(1/\log(1/r))\big) - i\frac{\alpha}{\log 1/r}\big(1 + o\,(1)\big),\end{aligned}$$

and

(5.17)

$$\begin{aligned}&\Big[\log\Big(\log\frac{1}{r} - i\alpha\Big)\Big]^q\\ &= \Big(\log\log\frac{1}{r}\Big)^q \Big[\Big[1 + o\Big(\frac{1}{\log 1/r}\Big)\Big] - i\frac{1}{\log 1/r\log\log 1/r}\big(\alpha q + o\,(1)\big)\Big]\\ &= \Big(\log\log\frac{1}{r}\Big)^q \Big[1 + o\Big(\frac{1}{\log 1/r}\Big)\Big].\end{aligned}$$

Hence, for $\varepsilon > 0$ small enough:

$$\begin{aligned}\Re\mathrm{e}\, \tilde{f}_\theta(z) &= r(\log 1/r)^\theta(\log\log 1/r)^q\Big[\cos\alpha + \frac{\theta\alpha\sin\alpha}{\log 1/r} + o\Big(\frac{1}{\log 1/r}\Big)\Big]\\ &\geq r(\log 1/r)^\theta(\log\log 1/r)^q\Big[\cos\alpha + \frac{\theta(\alpha\sin\alpha - 1/4)}{\log 1/r}\Big].\end{aligned}$$

That gives the result since, on the one hand, $|\theta(\alpha\sin\alpha - 1/4)/\log 1/r|$ is $\leq \sqrt{2}/4$ for $\varepsilon > 0$ small enough and $\cos\alpha \geq \sqrt{2}/2$ when $|\alpha| \leq \pi/4$, and, on the other hand, when $|\alpha| \geq \pi/4$, one has

$$\cos\alpha + \theta(\alpha\sin\alpha - 1/4) \geq \theta\big(\pi\sqrt{2}/8 - 1/4\big) > 0.$$

That ends the proof of the lemma. $\square$

Now, by (5.16), for $t > 0$ going to zero:

$$\log\Big(\log\frac{1}{t} - i\frac{\pi}{2}\Big) = \Big[\log\log\frac{1}{t} + O\Big(\frac{1}{(\log t)^2}\Big)\Big] + i\,O\Big(\frac{1}{\log 1/t}\Big).$$

Therefore:

$$\begin{aligned}\Re\mathrm{e}\left[\log\Big(\log\frac{1}{t}-i\frac{\pi}{2}\Big)\right]^q &\approx \Big(\log\log\frac{1}{t}\Big)^q\\ \Im\mathrm{m}\left[\log\Big(\log\frac{1}{t}-i\frac{\pi}{2}\Big)\right]^q &\approx \frac{(\log\log 1/t)^{q-1}}{\log 1/t}\,,\end{aligned}$$

and, using (5.4) (or (5.17), with $\alpha=\pi/2$), we get:

$$\begin{aligned}\Re\mathrm{e}\,\tilde f_\theta(it) &\approx t\Big(\log\frac{1}{t}\Big)^\theta\frac{(\log\log 1/t)^q}{\log 1/t} = t\Big(\log\frac{1}{t}\Big)^{\theta-1}\Big(\log\log\frac{1}{t}\Big)^q\\ \Im\mathrm{m}\,\tilde f_\theta(it) &\approx t\Big(\log\frac{1}{t}\Big)^\theta\Big(\log\log\frac{1}{t}\Big)^q.\end{aligned}$$

Hence the modulus constraint gives, with $h=h_n=2^{-n}$:

$$t\approx\frac{h}{\big(\log 1/h\big)^{\theta-1}\big(\log\log 1/h\big)^{q-1}}\,,\tag{5.18}$$

and the argument constraint:

$$jh\lesssim t\Big(\log\frac{1}{t}\Big)^\theta\Big(\log\log\frac{1}{t}\Big)^q\lesssim (j+1)h.$$

One gets:

$$j\approx\log\frac{1}{t}\approx\log\frac{1}{h}\approx n,$$

and:

$$m_{\tilde\phi_\theta}(R'_{n,j})\approx\frac{h}{\big(\log 1/h)^\theta\big(\log\log 1/h\big)^q}\approx\frac{2^{-n}}{n^\theta(\log n)^q}.$$

It follows that Luecking's criterion becomes:

$$\sum_{n=1}^\infty 2^{np/2}n\frac{2^{-np/2}}{n^{\theta p/2}(\log n)^{qp/2}}=\sum_{n=1}^\infty\frac{1}{n^{\theta p/2-1}(\log n)^{qp/2}}.$$

This series converges if and only if $p>4/\theta$ or else $p=4/\theta$, since then $qp/2=q(4/\theta)/2=q/(\theta/2)>1$. Hence $C_{\tilde\phi_\theta}\in S_p$ if and only if $p\geq 4/\theta$, and that proves Theorem 5.4. □

In [1], T. Carroll and C. Cowen showed that there exist compact composition operators on H^2 which are in no Schatten class S_p for $p<\infty$ (see also [15] and [3]). We shall give an explicit example of such an operator.

Theorem 5.6 *There exist compact composition operators $C_\phi\colon H^2\to H^2$ which are in no Schatten class S_p with $p<\infty$.*

Proof. The proof follows the lines of those of Theorem 5.1 and Theorem 5.4, but with, instead of f_θ or $\tilde{f}_\theta$:

$$f(z) = z \log(-\log z). \tag{5.19}$$

For $\varepsilon > 0$ small enough, we have $\mathfrak{Re}\, f(z) > 0$, so that the corresponding function ϕ sends $\mathbb{D}$ into itself; moreover, one has $|\phi^*| < 1$ *a.e.* on $\partial\mathbb{D}$. Indeed, if $z = r\mathrm{e}^{i\alpha} \in V_\varepsilon$, then $-\log z = R\,\mathrm{e}^{i\beta}$, with

$$\begin{aligned} R &= \sqrt{(\log 1/r)^2 + \alpha^2} = \log(1/r) + o\,(\log 1/r) \\ \beta &= -\arctan\frac{\alpha}{\log 1/r} = -\frac{\alpha}{\log 1/r} + o\,\big(1/\log(1/r)\big). \end{aligned}$$

Then $\log(-\log z) = \log R + i\beta = (\log R)\big(1 + \frac{i\beta}{\log R}\big)$ and

$$\mathfrak{Re}\, f(z) = r(\log R)\Big(\cos\alpha - \frac{\beta\sin\alpha}{\log R}\Big).$$

But

$$\cos\alpha - \frac{\beta\sin\alpha}{\log R} = \cos\alpha + \frac{\alpha\sin\alpha}{(\log 1/r)(\log\log 1/r)} + o\,\big(1/(\log 1/r)(\log\log 1/r)\big)$$

and we see, as in the proof of Lemma 5.5 that this quantity is, for $\varepsilon > 0$ small enough, greater than a positive constant, uniformly for $|\alpha| \leq \pi/2$.

Now, one has, as above, for t going to zero:

$$\begin{aligned} \mathfrak{Re}\, f(it) &\approx \frac{|t|}{\log 1/|t|} \\ \mathfrak{Im}\, f(it) &\approx t\log\log\frac{1}{|t|}. \end{aligned}$$

It follows that C_ϕ is compact since $\rho_\phi(h) \approx h/\log\log 1/h$: indeed, we have to control the two conditions:

$$0 < \frac{|t|}{\log 1/|t|} \lesssim h \tag{5.20}$$

$$\alpha - h \lesssim t\log\log\frac{1}{|t|} \lesssim \alpha + h\,, \tag{5.21}$$

for $0 \leq \alpha < 2\pi$. When $\alpha \not\approx 0$, condition (5.21) cannot happen for h small enough, because of condition (5.20); more precisely, (5.20) and (5.21) imply that

$$0 \leq \alpha \lesssim h + h\log\frac{1}{h}\log\log\frac{1}{h} \lesssim h\log\frac{1}{h}\log\log\frac{1}{h}, \tag{5.22}$$

and then (5.21) implies:

$$-h \lesssim t\log\log\frac{1}{|t|} \lesssim h\log\frac{1}{h}\log\log\frac{1}{h};$$

a fortiori:

$$|t| \log\log \frac{1}{|t|} \lesssim h \log \frac{1}{h} \log\log \frac{1}{h},$$

and then:

$$|t| \lesssim h \log \frac{1}{h}\cdot$$

Therefore condition (5.21) implies condition (5.20). Since, (5.22) implies that

$$\log(\alpha + h) \approx \log h,$$

we get that the measure of the t's satisfying (5.21) is $\approx h \log\log 1/h$: (5.21) is implied by $-h \leq t \log\log 1/|t| \leq \alpha + h$ and implies that $-(\alpha + h) \leq t \log\log 1/|t| \leq (\alpha + h)$.

Hence, we have got that $\rho_\phi(h) \approx h/\log\log 1/h$, showing that C_ϕ is compact.

By Proposition 3.4, C_ϕ is in no Schatten class S_p, $p > 0$.

In order to make more transparent the behaviour of the measure m_ϕ, we shall give a direct proof, using Luecking's criterion.

For testing Luecking's criterion, we must have, assuming $t > 0$:

$$t \approx h \log \frac{1}{h}$$

and:

$$jh \lesssim t \log\log \frac{1}{t} \lesssim (j+1)h,$$

so $j \approx \log 1/h \log\log 1/h \approx n \log n$, and:

$$m_\phi(R'_{n,j}) \approx \frac{h}{\log\log 1/h} \approx \frac{2^{-n}}{\log n}\cdot$$

Therefore Luecking's condition becomes:

$$\sum_{n=1}^{\infty} 2^{np/2} \left[n \log n \left(\frac{2^{-np/2}}{(\log n)^{p/2}} \right) \right] = \sum_{n=1}^{\infty} \frac{n}{(\log n)^{p/2-1}} = +\infty.$$

It follows that $C_\phi \notin S_p$. □

As a corollary, we get:

Theorem 5.7 *There exist two symbols ϕ and ψ of same modulus $|\phi^*| = |\psi^*|$ on $\mathbb{T}$ such that C_ϕ is compact on H^2, but in no Schatten class S_p for $p < \infty$, whereas C_ψ is in S_p for every $p > 2$.*

Proof. Set

$$\psi = \phi M,$$

where ϕ is the function used in Theorem 5.6, and M is the singular inner function given by (3.12). Since $\Re\mathrm{e}\, f(it) \approx t/\log(1/t)$, Lemma 3.7 gives, if one sets $\tilde{f}(t) = t/\log(1/t)$:

$$\rho_\psi(h) \approx h \tilde{f}^{-1}(h) \approx h^2 \log(1/h).$$

It follows that m_ψ is an α-Carleson measure for any $\alpha < 2$. Hence, given any $p > 2$, and choosing an $\alpha < 2$ such that $p > 2/(\alpha - 1)$, it follows from Corollary 3.2 that $C_\psi \in S_p$. $\square$

A funny question about Schatten classes is: for which Orlicz functions Ψ is the above composition operator C_ϕ in the Orlicz-Schatten class S_Ψ?

6 Questions

We shall end this paper with some comments and questions.

1. G. Pisier recently suggested that one should study more carefully the approximation numbers of composition operators. Since an operator T on a Hilbert space is in the Schatten class S_p if and only if $\sum_{n=1}^{\infty} a_n^p < +\infty$, where a_n is the n^{th} approximation number of T, our present work may be seen as a, very partial, contribution to that study.

Note that Luecking's proof of its trace-class theorem does not make explicit mention of singular numbers, but relies instead on an interpolation argument.

2. It is clear, by our results, that the membership in S_p for the composition operator C_ϕ cannot be characterized in terms of the growth of the Carleson function ρ_ϕ. But we have given a sufficient condition in Corollary 3.2, and a necessary one in Proposition 3.4. Are these two conditions sharp? Can they be improved?

3. Do there exist two symbols ϕ_1 and ϕ_2 having the same modulus on $\partial\mathbb{D}$ such that C_{ϕ_1} is not compact on H^2, but C_{ϕ_2} is in S_p for every $p > 2$?

4. (K. Kellay) If $C_\phi \colon H^2 \to H^2$ is compact (or even in S_p, with $p > 2$) and $\psi \colon \mathbb{D} \to \mathbb{D}$ is analytic, is $C_{\phi\psi} \colon H^2 \to H^2$ compact?

References

[1] T. Carroll and C. C. Cowen, Compact composition operators not in the Schatten classes, J. Oper. Th. 26 (1991), 109–120

[2] P. Duren, Theory of H^p Spaces, Dover (2000)

[3] M. Jones, Compact composition operators not in the Schatten classes, Proc. Amer. Math. Soc. 134 (2006), 1947–1953

[4] H. König, Eigenvalue Distribution of Compact Operators, Birkhäuser (1986)

[5] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *preprint*, math.FA/0610905, *to appear* in Memoirs Amer. Math. Soc.

[6] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Compact composition operators on H^2 and Hardy-Orlicz spaces, *preprint*

[7] D. H. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. 73 (1987), 345–368

[8] D. H. Luecking and K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992), 1127–1145

[9] B. MacCluer, Compact composition operators on $H^p(B_N)$, Michigan Math. J. 32 (1985), 237–248

[10] H. J. Schwartz, Composition operators on H^p, Thesis, University of Toledo (1969).

[11] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext. Tracts in Mathematics, Springer-Verlag, New York (1993).

[12] J. H. Shapiro, The essential norm of a composition operator, Annals of Math., 125 (1987), 375–404

[13] J. H. Shapiro and P. D. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators On H^2, Indiana Univ. Math. J. 23 (1973), 471–496

[14] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge Studies in Advanced Mathematics, 25. Cambridge University Press, Cambridge, 1991

[15] Y. Zhu, Geometric properties of composition operators belonging to Schatten classes, Int. J. Math. Math. Sci. 26 (2001), 239–248

[16] A. Zygmund, Trigonometric Series, Vol. I and II, Second Edition, Cambridge Univ. Press (1993)

Pascal Lefèvre, *Université d'Artois,*
Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
Faculté des Sciences Jean Perrin,
Rue Jean Souvraz, S.P. 18,
62 307 LENS Cedex, FRANCE
pascal.lefevre@euler.univ-artois.fr

Daniel Li, *Université d'Artois,*
Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
Faculté des Sciences Jean Perrin,
Rue Jean Souvraz, S.P. 18,
62 307 LENS Cedex, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec, *Université des Sciences et Technologies de Lille,*
Laboratoire Paul Painlevé U.M.R. CNRS 8524,
U.F.R. de Mathématiques,
59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
queff@math.univ-lille1.fr

Luis Rodríguez-Piazza, *Universidad de Sevilla,*
Facultad de Matemáticas, Departamento de Análisis Matemático,
Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es