On some random thin sets of integers

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Abstract

We show how different random thin sets of integers may have different behaviour. First, using a recent deviation inequality of Boucheron, Lugosi and Massart, we give a simpler proof of one of our results in Some new thin sets of integers in Harmonic Analysis, Journal d'Analyse Mathématique *86 (2002), 105–138, namely that there exist $\frac{4}{3}$-Rider sets which are sets of uniform convergence and $\Lambda(q)$-sets for all $q < \infty$, but which are not Rosenthal sets. In a second part, we show, using an older result of Kashin and Tzafriri that, for $p > \frac{4}{3}$, the p-Rider sets which we had constructed in that paper are almost surely not of uniform convergence.*

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1 Introduction

It is well-known that the Fourier series $S_n(f,x) = \sum_{-n}^{n} \hat{f}(k)\mathrm{e}^{ikx}$ of a 2π-periodic continuous function f may be badly behaved: for example, it may diverge on a prescribed set of values of x with measure zero. Similarly, the Fourier series of an integrable function may diverge everywhere. But it is equally well-known that, as soon as the spectrum $Sp\,(f)$ of f (the set of integers k at which the Fourier coefficients of f do not vanish, *i.e.* $\hat{f}(k) \neq 0$) is sufficiently "lacunary", in the sense of Hadamard *e.g.*, then the Fourier series of f is absolutely convergent if f is continuous and almost everywhere convergent if f is merely integrable (and in this latter case $f \in L^p$ for every $p < \infty$). Those facts have given birth to the theory of thin sets Λ of integers, initiated by Rudin [15]: those sets Λ such that, if $Sp\,(f) \subseteq \Lambda$ (we shall write $f \in \mathscr{B}_\Lambda$ when f is in some Banach function space $\mathscr{B}$ contained in $L^1(\mathbb{T})$) and $Sp\,(f) \subseteq \Lambda$), then $S_n(f)$, or f itself, is better behaved than in the general case. Let us for example recall that the set Λ is said to be:

- a *p-Sidon set* $(1 \leq p < 2)$ if $\hat{f} \in l_p$ (and not only $\hat{f} \in l_2$) as soon as f is continuous and $Sp\,(f) \subseteq \Lambda$; this amounts to an "*a priori* inequality" $\|\hat{f}\|_p \leq C\|f\|_\infty$, for each $f \in \mathscr{C}_\Lambda$; the case $p = 1$ is the celebrated case of Sidon (= 1-Sidon) sets;
- a *p-Rider set* $(1 \leq p < 2)$ if we have an *a priori* inequality $\|\hat{f}\|_p \leq C\,[\![f]\!]$, for every trigonometric polynomial with spectrum in Λ; here $[\![f]\!]$ is the so-called

Pisier norm of $f = \sum \hat{f}(n)e_n$, where $e_n(x) = \mathrm{e}^{inx}$, *i.e.* $[\![f]\!] = \mathbb{E}\,\|f_\omega\|_\infty$, where $f_\omega = \sum \varepsilon_n(\omega)\hat{f}(n)e_n$, (ε_n) being an *i.i.d.* sequence of centered, ± 1-valued, random variables defined on some probability space (a Rademacher sequence), and where $\mathbb{E}$ denotes the expectation on that space; this apparently exotic notion (weaker than p-Sidonicity) turned out to be very useful when Rider [12] reformulated a result of Drury (proved in the course of the result that the union of two Sidon set sets is a Sidon set) under the form: 1-Rider sets and Sidon sets are the same (in spite of some partial results, it is not yet known whether a p-Rider set is a p-Sidon set: see [5] however, for a partial result);
- a *set of uniform convergence* (in short a UC-set) if the Fourier series of each $f \in \mathscr{C}_\Lambda$ converges uniformly, which amounts to the inequality $\|S_n(f)\|_\infty \leq C\|f\|_\infty$, $\forall f \in \mathscr{C}_\Lambda$; Sidon sets are UC, but the converse is false;
- a $\Lambda(q)$-*set*, $1 < q < \infty$, if every $f \in L^1_\Lambda$ is in fact in L^q, which amounts to the inequality $\|f\|_q \leq C_q\|f\|_1$, $\forall f \in L^1_\Lambda$. Sidon sets are $\Lambda(q)$ for every $q < \infty$ (and even $C_q \leq C\sqrt{q}$); the converse is false, except when we require $C_q \leq C\sqrt{q}$ ([11]);
- a *Rosenthal set* if every $f \in L^\infty_\Lambda$ is almost everywhere equal to a continuous function. Sidon sets are Rosenthal, but the converse in false.

This theory has long suffered from a severe lack of examples: those examples were always, more or less, sums of Hadamard sets, and in that case the banachic properties of the corresponding $\mathscr{C}_\Lambda$-spaces were very rigid. The use of random sets (in the sense of the selectors method) of integers has significantly changed the situation (see [8], and our paper [9]). Let us recall more in detail the notation and setting of our previous work [9]. The method of selectors consists in the following: let $(\varepsilon_k)_{k\geq 1}$ be a sequence of independent, $(0,1)$-valued random variables, with respective means δ_k, defined on a probability space Ω, and to which we attach the random set of integers $\Lambda = \Lambda(\omega)$, $\omega \in \Omega$, defined by $\Lambda(\omega) = \{k \geq 1\,;\ \varepsilon_k(\omega) = 1\}$.

The properties of $\Lambda(\omega)$ of course highly depend on the δ_k's, and roughly speaking the smaller the δ_k's, the better $\mathscr{C}_\Lambda$, L^1_Λ, … . In [7], and then, in a much deeper way, in [9], relying on a probabilistic result of J. Bourgain on ergodic means, and on a deterministic result of F. Lust-Piquard ([10]) on those ergodic means, we had randomly built new examples of sets Λ of integers which were both: locally thin from the point of view of harmonic analysis (their traces on big segments $[M_n, M_{n+1}]$ of integers were uniformly Sidon sets); regularly distributed from the point of view of number theory, and therefore globally big from the point of view of Banach space theory, in that the space $\mathscr{C}_\Lambda$ contained an isomorphic copy of the Banach space c_0 of sequences vanishing at infinity. More precisely, we have constructed subsets $\Lambda \subseteq \mathbb{N}$ which are thin in the following respects: Λ is a UC-set, a p-Rider set for various $p \in [1, 2[$, a $\Lambda(q)$-set for every $q < \infty$, and large in two respects: the space $\mathscr{C}_\Lambda$ contains an isomorphic copy of c_0, and, most often, Λ is dense in the integers equipped with the Bohr topology.

Now, taking δ_k bigger and bigger, we had obtained sets Λ which were less and less thin (p-Sidon for every $p > 1$, q-Rider, but s-Rider for no $s < q$, s-Rider for every $s > q$, but not q-Rider), and, in any case $\Lambda(q)$ for every $q < \infty$, and such that $\mathscr{C}_\Lambda$ contains a subspace isomorphic to c_0. In particular, in Theorem II.7, page 124, and Theorem II.10, page 130, we take respectively $\delta_k \approx \frac{\log k}{k}$ and $\delta_k \approx \frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}}$, where $\alpha = \frac{2(p-1)}{2-p}$ is an increasing function of $p \in [1,2)$, and which becomes ≥ 1 as p becomes $\geq 4/3$. The case $\delta_k = \frac{1}{k}$ would correspond (randomly) to Sidon sets (*i.e.* 1-Sidon sets).

After the proofs of Theorem II.7 and Theorem II.10, we were asking two questions:

1) (p. 129) Our construction is very complicated and needs a second random construction of a set E inside the random set Λ. Is it possible to give a simpler proof?

2) (p. 130) In Theorem II.10, can we keep the property for the random set Λ to be a UC-set, with high probability, when $\alpha > 1$ (equivalently when $p > \frac{4}{3}$)?

The goal of this work is to answer affirmatively the first question (relying on a recent deviation inequality of Boucheron, Lugosi and Massart [1]) and negatively the second one (relying on an older result of Kashin and Tzafriri [3]). This work is accordingly divided into three parts. In Section 2, we prove a (one-sided) concentration inequality for norms of Rademacher sums. In Section 3, we apply the concentration inequality to get a substantially simplified proof of Theorem II.7 in [9]. Finally, in Section 4, we give a (stochastically) negative answer to question 2 when $p > \frac{4}{3}$: almost surely, Λ will not be a UC-set; here, we use the above mentionned result of Kashin and Tzafriri [3] on the non-UC character of big random subsets of integers.

2 A one-sided inequality for norms of Rademacher sums

Let E be a (real or complex) Banach space, $v_1, \dots, v_n$ be vectors of E, $X_1, \dots, X_n$ be independent, real-valued, centered, random variables, and let $Z = \| \sum_1^n X_j v_j \|$.

If $|X_j| \leq 1$ *a.s.*, it is well-known (see [6]) that:

$$\mathbb{P}\left(|Z - \mathbb{E}\,(Z)| > t\right) \leq 2 \exp\left(- \frac{t^2}{8 \sum_1^n \|v_j\|^2} \right), \quad \forall t > 0. \tag{2.1}$$

But often, the "strong" l_2-norm of the n-tuple $v = (v_1, \dots, v_n)$, namely $\|v\|_{strong} = (\sum_{j=1}^n \|v_j\|^2)^{1/2}$, is too large for (2.1) to be interesting, and it is advisable to work with the "weak" l_2-norm of v, defined by:

$$\sigma = \|v\|_{weak} = \sup_{\varphi \in B_{E^*}} \left(\sum_1^n |\varphi(v_j)|^2 \right)^{1/2} = \sup_{\sum |a_j|^2 \leq 1} \Big\| \sum_1^n a_j v_j \Big\|, \tag{2.2}$$

where B_{E^*} denotes the closed unit ball of the dual space E^*.

If $(X_j)_j$ is a standard gaussian sequence ($\mathbb{E}\, X_j = 0, \mathbb{E}\, X_j^2 = 1$), this is what Maurey and Pisier suceeded in doing, using either the Itô formula or the rotational invariance of the X_j's; they proved the following (see [8], Chapitre 8, Théorème I.4):

$$\mathbb{P}\left(|Z - \mathbb{E}\, Z| > t\right) \leq 2 \exp\Big(-\frac{t^2}{C\sigma^2}\Big), \quad \forall t > 0, \tag{2.3}$$

where σ is as in (2.2), and C is a numerical constant, *e.g.* $C = \pi^2/2$.

To the best of our knowledge, no inequality as simple and direct as (2.3) is available for non-gaussian (*e.g.* for Rademacher variables) variables, although several more complicated deviation inequalities are known: see *e.g.* [2], [6].

For the applications to Harmonic analysis which we have in view, where we use the so-called "selectors method", we precisely need an analogue of (2.3), in the non-gaussian, uniformly bounded (and centered) case; we shall prove that at least a one-sided version of (2.3) holds in this case, by showing the following result, which is interesting for itself.

Theorem 2.1 *With the previous notations, assume that* $|X_j| \leq 1$ *a.s.. Then, we have the one-sided estimate:*

$$\mathbb{P}\left(Z - \mathbb{E}\, Z > t\right) \leq \exp\Big(-\frac{t^2}{C\sigma^2}\Big), \quad \forall t > 0, \tag{2.4}$$

where $C > 0$ *is a numerical constant (*$C = 32$*, for example).*

The proof of (2.4) will make use of a recent deviation inequality due to Boucheron, Lugosi and Massart [1]. Before stating this inequality, we need some notation.

Let $X_1, \ldots, X_n$ be independent, real-valued random variables (here, we temporarily forget the assumptions of the previous Theorem), and let $(X_1', \ldots, X_n')$ be an independent copy of $(X_1, \ldots, X_n)$.

If $f \colon \mathbb{R}^n \to \mathbb{R}$ is a given measurable function, we set $Z = f(X_1, \ldots, X_n)$ and $Z_i' = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$, $1 \leq i \leq n$. With those notations, the Boucheron-Lugosi-Massart Theorem goes as follows:

Theorem 2.2 *Assume that there is some constant* $a, b \geq 0$*, not both zero, such that:*

$$\sum_{i=1}^{n} (Z - Z_i')^2 \mathbb{1}_{(Z > Z_i')} \leq aZ + b \quad a.s. \tag{2.5}$$

Then, we have the following one-sided deviation inequality:

$$\mathbb{P}\left(Z > \mathbb{E}\, Z + t\right) \leq \exp\Big(-\frac{t^2}{4a\,\mathbb{E}\, Z + 4b + 2at}\Big), \quad \forall t > 0. \tag{2.6}$$

Proof of Theorem 2.1. We shall in fact use a very special case of Theorem 2.2, the case when $a = 0$; but, as the three fore-named authors remark, this special case is already very useful, and far from trivial to prove! To prove (2.4), we are going to check that, for $f(X_1, \ldots, X_n) = \| \sum_1^n X_j v_j \| = Z$, the assumption

(2.5) holds for $a=0$ and $b=4\sigma^2$. In fact, fix $\omega\in\Omega$ and denote by $I=I_\omega$ the set of indices i such that $Z(\omega)>Z'_i(\omega)$. For simplicity of notation, we assume that the Banach space E is real. Let $\varphi=\varphi_\omega\in E^*$ such that $\|\varphi\|=1$ and $Z=\varphi\big(\sum_{j=1}^n X_j v_j\big)=\sum_{j=1}^n X_j\varphi(v_j)$.

For $i\in I$, we have $Z'_i(\omega)=Z'_i\geq\varphi\big(\sum_{j\neq i}X_jv_j+X'_iv_i\big)$, so that $0\leq Z-Z'_i\leq\sum_{j=1}^n X_j\varphi(v_j)-\sum_{j\neq i}X_j\varphi(v_j)-X'_i\varphi(v_i)=(X_i-X'_i)\varphi(v_i)$, implying $(Z-Z'_i)^2\leq 4|\varphi(v_i)|^2$. By summing those inequalities, we get:

$$\begin{aligned}\sum_{i=1}^n(Z-Z'_i)^2\mathbb{1}_{(Z>Z'_i)}&=\sum_{i\in I}(Z-Z'_i)^2\leq 4\sum_{i\in I}|\varphi(v_i)|^2\leq 4\sum_{i=1}^n|\varphi(v_i)|^2\leq 4\sigma^2\\&=0.Z+4\sigma^2.\end{aligned}$$

Let us observe the crucial role of the "conditioning" $Z>Z'_i$ when we want to check that (2.5) holds. Now, (2.4) is an immediate consequence of (2.6). □

3 Construction of 4/3-Rider sets

We first recall some notations of [9]. Ψ_2 denotes the Orlicz function $\Psi_2(x)=\mathrm{e}^{x^2}-1$, and $\|\ \|_{\Psi_2}$ is the corresponding Luxemburg norm. If A is a finite subset of the integers, Ψ_A denotes the quantity $\|\sum_{n\in A}e_n\|_{\Psi_2}$, where $e_n(t)=\mathrm{e}^{int}$, $t\in\mathbb{R}/2\pi\mathbb{Z}=\mathbb{T}$, and $\mathbb{T}$ is equipped with its Haar measure m. Λ will always be a subset of the positive integers $\mathbb{N}$. Recall that Λ is *uniformly distributed* if the ergodic means $A_N(t)=\frac{1}{|\Lambda_N|}\sum_{n\in\Lambda_N}e_n(t)$ tend to zero as $N\to\infty$, for each $t\in\mathbb{T}$, $t\neq 0$. Here, $\Lambda_N=\Lambda\cap[1,N]$. If Λ is uniformly distributed, $\mathscr{C}_\Lambda$ contains c_0, and if $\mathscr{C}_\Lambda$ contains c_0, Λ cannot be a Rosenthal set (see [9]). According to results of J. Bourgain (see [9]) and F. Lust-Piquard ([10]), respectively, a random set Λ corresponding to selectors of mean δ_k with $k\delta_k\to\infty$ is almost surely uniformly distributed and if a subset E of a uniformly distributed set Λ has positive upper density in Λ, *i.e.* if $\limsup_N\frac{|E\cap[1,N]|}{|\Lambda\cap[1,N]|}>0$, then $\mathscr{C}_E$ contains c_0, and E is non-Rosenthal.

In [9], we had given a fairly complicated proof of the following theorem (labelled as Theorem II.7):

Theorem 3.1 *There exists a subset Λ of the integers, which is uniformly distributed, and contains a subset E of positive integers with the following properties:*

1) E is a $\frac{4}{3}$-Rider set, but is not q-Rider for $q<4/3$, a UC-set, and a $\Lambda(q)$-set for all $q<\infty$;

2) E is of positive upper density inside Λ; in particular, $\mathscr{C}_E$ contains c_0 and E is not a Rosenthal set.

We shall show here that the use of Theorem 2.1 allows a substantially simplified proof, which avoids a double random selection. We first need the following simple lemma.

Lemma 3.2 *Let A be a finite subset of the integers, of cardinality $n \geq 2$; let $v = (e_j)_{j\in A}$, considered as an n-tuple of elements of the Banach space $E = L^{\Psi_2} = L^{\Psi_2}(\mathbb{T}, m)$, and let σ be its weak l_2-norm. Then:*

$$\sigma \leq C_0 \sqrt{\frac{n}{\log n}}\,, \tag{3.1}$$

where C_0 is a numerical constant.

Proof. Let $a = (a_j)_{j\in A}$ be such that $\sum_{j\in A} |a_j|^2 = 1$. Let $f = f_a = \sum_{j\in A} a_j e_j$, and $M = \|f\|_\infty$. By Hölder's inequality, we have $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{M}{\sqrt{p}\, M^{2/p}}$ for $2 < p < \infty$. Since $M \leq \sqrt{n}$, we get $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{\sqrt{n}}{\sqrt{p}\, n^{1/p}} \leq C\sqrt{\frac{n}{\log n}}$. By Stirling's formula, $\|f\|_{\Psi_2} \approx \sup_{p>2} \frac{\|f\|_p}{\sqrt{p}}$, so the lemma is proved, since $\sigma = \sup_a \|f_a\|_{\Psi_2}$ □

We now turn to the shortened proof of Theorem 3.1.

Let $I_n = [2^n, 2^{n+1}[$, $n \geq 2$; $\delta_k = c\frac{n}{2^n}$ if $k \in I_n$ $(c > 0)$.

Let $(\varepsilon_k)_k$ be a sequence of "selectors", *i.e.* independent, $(0,1)$-valued, random variables of expectation $\mathbb{E}\,\varepsilon_k = \delta_k$, and let $\Lambda = \Lambda(\omega)$ be the random set of positive integers defined by $\Lambda = \{k \geq 1\,;\ \varepsilon_k = 1\}$. We set also $\Lambda_n = \Lambda \cap I_n$ and $\sigma_n = \mathbb{E}\,|\Lambda_n| = \sum_{k\in I_n} \delta_k = cn$.

We shall now need the following lemma (the notation Ψ_A is defined at the beginning of the section).

Lemma 3.3 *Almost surely, for n large enough:*

$$\frac{c}{2} n \leq |\Lambda_n| \leq 2cn\ ; \tag{3.2}$$

$$\Psi_{\Lambda_n} \leq C'' |\Lambda_n|^{1/2}\ . \tag{3.3}$$

Proof : (3.2) is the easier part of Lemma II.9 in [9]. To prove (3.3), we recall an inequality due to G. Pisier [11]: if (X_k) is a sequence of independent, centered and square-integrable, random variables of respective variances $V(X_k)$, we have:

$$\mathbb{E}\Big\| \sum_k X_k e_k \Big\|_{\Psi_2} \leq C_1 \Big(\sum_k V(X_k) \Big)^{1/2}. \tag{3.4}$$

Applying (3.4) to the centered variables $X_k = \varepsilon_k - \delta_k$, we get, assuming $c \leq 1$:

$$\mathbb{E}\Big\| \sum_{k\in I_n} (\varepsilon_k - \delta_k) e_k \Big\|_{\Psi_2} \leq C_1 \Big(\sum_{k\in I_n} \delta_k (1-\delta_k) \Big)^{1/2} \leq C_1 \Big(\sum_{k\in I_n} \delta_k \Big)^{1/2} \leq C_1 \sqrt{n}.$$

Now, set $Z_n = \big\| \sum_{k\in I_n} (\varepsilon_k - \delta_k) e_k \big\|_{\Psi_2}$. Let λ be a fixed real number > 1, and C_0 be as in Lemma 3.2. Applying Theorem 2.1 with $C = 32$, and $t_n = \lambda\sqrt{32 C_0^2 n}$, we get, using Lemma 3.2:

$$\mathbb{P}\,(Z_n - \mathbb{E}\,Z_n > t_n) \leq \exp\Big(-\frac{t_n^2}{32\sigma^2} \Big) \leq \exp\Big(-\frac{32\lambda^2 C_0^2 n \log n}{32 C_0^2 n} \Big) = n^{-\lambda^2}.$$

By the Borel-Cantelli Lemma, we have almost surely, for n large enough:

$$Z_n \leq \mathbb{E}\, Z_n + t_n \leq (C_1 + 4C_0\lambda)\sqrt{n} = C_2\sqrt{n}.$$

For such ω's and n's, it follows that:

$$\begin{aligned}\Psi_{\Lambda_n} = \Big\| \sum_{k\in I_n} \varepsilon_k e_k \Big\|_{\Psi_2} &\leq Z_n + \Big\| \sum_{k\in I_n} \delta_k e_k \Big\|_{\Psi_2} \leq Z_n + \frac{n}{2^n} \Big\| \sum_{k\in I_n} e_k \Big\|_{\Psi_2} \\ &\leq C_2\sqrt{n} + \frac{n}{2^n} C_0 \frac{2^n}{\sqrt{\log 2^n}} =: C_3\sqrt{n},\end{aligned}$$

because, with the notations of Lemma 3.2, we have:

$$\Big\| \sum_{k\in I_n} e_k \Big\|_{\Psi_2} \leq \sqrt{|I_n|}\sigma \leq 2^{n/2} C_0 \frac{2^{\frac{n}{2}}}{\sqrt{\log 2^n}}\cdot$$

This ends the proof of Lemma 3.3, because we know that $n \leq \frac{2}{c}|\Lambda_n|$ for large n, almost surely, and therefore $\Psi_{\Lambda_n} \leq C_3\sqrt{\frac{2}{c}}|\Lambda_n|^{1/2} =: c''|\Lambda_n|^{1/2}$, *a.s.*. □

We now prove Theorem 3.1 as follows: let us fix a point $\omega \in \Omega$ in such a way that $\Lambda = \Lambda(\omega)$ is uniformly distributed and that Λ_n verifies (3.2) and (3.3) for $n \geq n_0$; this is possible from [9] and from Lemma 3.3. We then use a result of the third-named author ([13]), asserting that there is a numerical constant $\delta > 0$ such that each finite subset A of $\mathbb{Z}^*$ contains a quasi-independent subset B such that $|B| \geq \delta\big(\frac{|A|}{\Psi_A}\big)^2$ (recall that a subset Q of $\mathbb{Z}$ is said to be quasi-independent if, whenever $n_1, \ldots, n_k \in Q$, the equality $\sum_{j=1}^k \theta_j n_j = 0$ with $\theta_j = 0, -1, +1$ holds only when $\theta_j = 0$ for all j). This allows us to select inside each Λ_n a quasi-independent subset E_n such that:

$$|E_n| \geq \delta\Big(\frac{|\Lambda_n|}{\Psi_{\Lambda_n}}\Big)^2 \geq \frac{\delta}{c''^2}|\Lambda_n| =: \delta'|\Lambda_n|\,. \tag{3.5}$$

A combinatorial argument (see [9], p. 128–129) shows that, if $E = \cup_{n>n_0} E_n$, then each finite $A \subset E$ contains a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta|A|^{1/2}$. By [13], E is a $\frac{4}{3}$-Rider set. The set E has all the required properties. Indeed, it follows from Lemma 3.2, a) that $|E \cap [1, N]| \geq \delta(\log N)^2$. If now E is p-Rider, we must have $|E \cap [1, N]| \leq C(\log N)^{\frac{p}{2-p}}$; therefore $2 \leq \frac{p}{2-p}$, so $p \geq 4/3$. The fact that E is both UC and $\Lambda(q)$ is due to the local character of these notions, and to the fact that the sets $E \cap [2^n, 2^{n+1}[= E_n$ are by construction quasi-independent (as detailed in [9]). On the other hand, since each E_n is approximately proportional to Λ_n, E is of positive upper density in Λ. Now Λ is uniformly distributed (by Bourgain's criterion: see [9], p. 115). Therefore, by the result of F. Lust-Piquard ([10], and see Theorem I.9, p. 114 in [9]), $\mathscr{C}_E$ contains c_0, which prevents E from being a Rosenthal set. □

4 p-Rider sets, with $p > 4/3$, which are not UC-sets

Let $p \in]\frac{4}{3}, 2[$, so that $\alpha = \frac{2(p-1)}{2-p} > 1$. As we mentioned in the Introduction, the random set $\Lambda = \Lambda(\omega)$ of integers in Theorem II.10 of [9] corresponds to selectors ε_k with mean $\delta_k = c\frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}}\cdot$ We shall prove the following:

Theorem 4.1 *The random set Λ corresponding to selectors of mean $\delta_k = c\frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}}$ has almost surely the following properties:*

a) Λ is p-Rider, but q-Rider for no $q < p$;

b) Λ is $\Lambda(q)$ for all $q < \infty$;

c) Λ is uniformly distributed; in particular, it is dense in the Bohr group and $\mathscr{C}_\Lambda$ contains c_0;

d) Λ is **not** *a UC-set.*

Remark. This supports the conjecture that p-Rider sets with $p > 4/3$ are not of the same nature as p-Rider sets for $p < 4/3$ (see also [4], Theorem 3.1. and [5]).

The novelty here is *d)*, which answers in the negative a question of [9] and we shall mainly concentrate on it, although we shall add some details for *a)*,*b)*, *c)*, since the proof of Theorem II.10 in [9] is too sketchy and contains two small misprints (namely $(*)$ and $(**)$, p. 130).

Recall that the UC-constant $U(E)$ of a set E of positive integers is the smallest constant M such that $\|S_N f\|_\infty \leq M\|f\|_\infty$ for every $f \in \mathscr{C}_E$ and every non-negative integer N, where $S_N f = \sum_{-N}^{N} \hat{f}(k)e_k$. We shall use the following result of Kashin and Tzafriri [3]:

Theorem 4.2 *Let $N \geq 1$ be an integer and $\varepsilon'_1, \ldots, \varepsilon'_N$ be selectors of equal mean δ. Set $\sigma(\omega) = \{k \leq N\,;\ \varepsilon'_k(\omega) = 1\}$. Then:*

$$\mathbb{P}\left(U\big(\sigma(\omega)\big) \leq \gamma \log\left(2 + \frac{\delta N}{\log N}\right)\right) \leq \frac{5}{N^3}, \tag{4.1}$$

where γ is a positive numerical constant.

We now turn to the proof of Theorem 4.1. As in [9], we set, for a fixed $\beta > \alpha$:

$$M_n = n^{\beta n}\ ; \quad \Lambda_n = \Lambda \cap [1, n]\ ; \quad \Lambda_n^* = \Lambda \cap [M_n, M_{n+1}[. \tag{4.2}$$

We need the following technical lemma, whose proof is postponed (and is needed only for *a)*, *b)*, *c)*).

Lemma 4.3 *We have almost surely for large n*

$$|\Lambda_{M_n}| \approx n^{\alpha+1}\ ; \qquad |\Lambda_n^*| \approx n^\alpha. \tag{4.3}$$

Observe that, for $k \in \Lambda_n^*$, one has:

$$\delta_k = c\,\frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}} \gg \frac{(n\log n)^\alpha}{M_{n+1}(\log n)^{\alpha+1}} = \frac{n^\alpha}{M_{n+1}\log n} =: \frac{q_n}{N_n}\,,$$

where $N_n = M_{n+1} - M_n$ is the number of elements of the support of Λ_n^* (note that $N_n \sim M_{n+1}$), and where q_n is such that

$$q_n \approx \frac{n^\alpha}{\log n}\,\cdot \tag{4.4}$$

We can adjust the constants so as to have $\delta_k \geq q_n/N_n$ for $k \in \Lambda_n^*$. Now, we introduce selectors (ε''_k) independent of the ε_j's, of respective means $\delta''_k = q_n/(N_n\delta_k)$. Then the selectors $\varepsilon'_k = \varepsilon_k\varepsilon''_k$ have means $\delta'_k = q_n/N_n$ for $k \in \Lambda_n^*$, and we have $\delta_k \geq \delta'_k$ for each $k \geq 1$.

Let $\Lambda' = \{k\,;\ \varepsilon'_k = 1\}$ and ${\Lambda'}_n^* = \Lambda' \cap [M_n, M_{n+1}[$. It follows from (4.1) and the fact that $U(E+a) = U(E)$ for any set E of positive integers and any non-negative integer a that:

$$\mathbb{P}\Big(U({\Lambda'}_n^*) \leq \gamma\log\Big(2 + \frac{q_n}{\log N_n}\Big)\Big) \leq 5N_n^{-3}.$$

By the Borel-Cantelli Lemma, we have almost surely $U({\Lambda'}_n^*) > \gamma\log\big(2 + \frac{q_n}{\log N_n}\big)$ for n large enough. But we see from (4.3) and (4.2) that:

$$\frac{q_n}{\log N_n} \approx \frac{n^\alpha}{(\log n)(n\log n)} = \frac{n^{\alpha-1}}{(\log n)^2}\,,$$

and this tends to infinity since $\alpha > 1$. This shows that Λ' is almost surely non-UC. And due to the construction of the ε'_k's, we have: $\Lambda \supseteq \Lambda'$ almost surely. This of course implies that Λ is not a UC-set either (almost surely), ending the proof of *d)* in Theorem 4.1. $\square$

We now indicate a proof of the lemma. Almost surely, $|\Lambda_{M_n}|$ behaves for large n as:

$$\begin{aligned}\mathbb{E}\,(|\Lambda_{M_n}|) &= \sum_1^{M_n} \frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}} \approx \int_{e^2}^{M_n} \frac{(\log t)^\alpha}{t(\log\log t)^{\alpha+1}}dt \\ &= \int_2^{\log M_n} \frac{x^\alpha dx}{(\log x)^{\alpha+1}} \approx \frac{1}{(\log n)^{\alpha+1}} \int_2^{\log M_n} x^\alpha dx \approx \frac{(\log M_n)^{\alpha+1}}{(\log n)^{\alpha+1}} \approx n^{\alpha+1}.\end{aligned}$$

Similarly, $|\Lambda_n^*|$ behaves almost surely as:

$$\begin{aligned}\int_{M_n}^{M_{n+1}} \frac{(\log t)^\alpha}{t(\log\log t)^{\alpha+1}}dt &= \int_{\log M_n}^{\log M_{n+1}} \frac{x^\alpha}{(\log x)^{\alpha+1}}dx \approx \frac{1}{(\log n)^{\alpha+1}} x^\alpha dx \\ &\approx \frac{1}{(\log n)^{\alpha+1}}(\log M_{n+1} - \log M_n)(\log M_n)^\alpha \\ &\approx \frac{1}{(\log n)^{\alpha+1}}\log n(n\log n)^\alpha \approx n^\alpha.\end{aligned}$$

$\square$

To finish the proof, we shall use a lemma of [9] (recall that a *relation of length* n in $A \subseteq \mathbb{Z}^*$ is a $(-1,0,+1)$-valued sequence $(\theta_k)_{k\in A}$ such that $\sum_{k\in A}\theta_k\,k = 0$ and $\sum_{k\in A}|\theta_k| = n$):

Lemma 4.4 *Let $n \geq 2$ and M be integers. Set*

$$\Omega_n(M) = \{\omega \mid \Lambda(\omega) \cap [M, \infty[\text{ contains at least a relation of length } n\}.$$

Then:

$$\mathbb{P}\left[\Omega_n(M)\right] \leq \frac{C^n}{n^n} \sum_{j>M} \delta_j^2 \sigma_j^{n-2},$$

where $\sigma_j = \delta_1 + \ldots + \delta_j$, and C is a numerical constant.

In our case, with $M = M_n$, this lemma gives :

$$\begin{aligned}
\mathbb{P}\left[\Omega_n(M)\right] &\ll \frac{C^n}{n^n} \sum_{j>M} \frac{(\log j)^{2\alpha}}{j^2 (\log\log j)^{2\alpha+2}} \left[\frac{(\log j)^{\alpha+1}}{(\log\log j)^{\alpha+1}}\right]^{n-2} \\
&\ll \frac{C^n}{n^n} \int_M^\infty \frac{(\log t)^{(\alpha+1)n+2\alpha}}{(\log\log t)^{(\alpha+1)n+2\alpha+2}} \frac{dt}{t^2}
\end{aligned}$$

and an integration by parts (see [9], p. 117–118) now gives:

$$\begin{aligned}
\mathbb{P}\left[\Omega_n(M)\right] &\ll \frac{C^n}{n^n} \frac{1}{M} \frac{(\log M)^{(\alpha+1)n+2\alpha}}{(\log\log M)^{(\alpha+1)n+2\alpha+2}} \\
&\ll \frac{C^n}{n^n} \frac{1}{n^{\beta n}} \frac{(n \log n)^{(\alpha+1)n+2\alpha}}{(\log n)^{(\alpha+1)n+2\alpha+2}} \ll \frac{n^{2\alpha} C^n}{n^{(\beta-\alpha)n} (\log n)^2};
\end{aligned}$$

then the assumption $\beta > \alpha$ (which reveals its importance here!) shows that $\sum_n \mathbb{P}\left[\Omega_n(M_n)\right] < \infty$, so that, almost surely $\Lambda(\omega) \cap [M_n, \infty[$ contains no relation of length n, for $n \geq n_0$. Having this property at our disposal, we prove (exactly as in [9], p. 119–120) that Λ is p-Rider. It is not q-Rider for $q < p$, because then $|\Lambda_{M_n}| \ll (\log M_n)^{\frac{q}{2-q}} \ll (n \log n)^{\frac{q}{2-q}}$, whereas (4.3) of Lemma 4.3 shows that $|\Lambda_{M_n}| \gg n^{\alpha+1}$, with $\alpha + 1 = \frac{p}{2-p} > \frac{q}{2-q}$. This proves *a*). Conditions *b),c)* are clearly explained in [9]. □

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