The Rayleigh–Taylor condition for the evolution of irrotational fluid interfaces

Antonio Cordoba[a], **Diego Cordoba**[b,1], **and Francisco Gancedo**[c]

[a]Department of Mathematics, Universidad Autonoma de Madrid, Carretera de Colmenar Viejo, Km. 15, 28049 Madrid, Spain; [b]Instituto de Ciencias Matematicas, Consejo Superior de Investigaciones Cientificas, c/Serrano 123, 28006 Madrid, Spain; and [c]Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637

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For the free boundary dynamics of the two-phase Hele-Shaw and Muskat problems, and also for the irrotational incompressible Euler equation, we prove existence locally in time when the Rayleigh–Taylor condition is initially satisfied for a 2D interface. The result for water waves was first obtained by Wu in a slightly different scenario (vanishing at infinity), but our approach is different because it emphasizes the active scalar character of the system and does not require the presence of gravity.

Euler | Hele–Shaw–Muskat | incompressible | well-possedness

There are several interesting problems in fluid mechanics regarding the evolution of the interface between two fluids [the Hele-Shaw cell (1, 2) and the Muskat problem (3)] or between a fluid and vacuum or another fluid with zero density, as in models of water waves. In all of them the first important question to be asked is to guarantee local existence, usually within the chain of Sobolev spaces. However, such a result turns out to be false for general initial data. First, Rayleigh (4), Taylor (5), and Saffman and Taylor (2), and later Beale et al. (6), Wu (7, 8), Christodoulou and Lindblad (9), Ambrose (10), Lindblad (11), Ambrose and Masmoudi (12), Coutand and Shkoller (13), Córdoba and Gancedo (14), Shatah and Zeng (15) and Zhang and Zhang (16) gave a condition that must be satisfied to have a solution locally in time; namely, the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh–Taylor condition.

In refs. 17 and 18 we have obtained local existence in the 2D case: for the Hele-Shaw and Muskat problems, our result addresses the more difficult case when the two fluids have different densities and viscosities; for water waves, we give a different proof of the important theorem of Wu (7) where gravity plays a crucial role in the sign of the Rayleigh–Taylor condition. In our proof, however, we consider the two cases, with or without gravity, and with initial data always satisfying the Rayleigh–Taylor condition.

When that condition is not imposed initially there are several cases where ill-posedness has been proved. Let us point out the works of Ebin (19, 20), Caflisch and Orellana (21), Siegel et al. (22), and Córdoba and Gancedo (14).

We regard these models as transport equations for the density, considered as an active scalar, with a divergence-free velocity field given by Darcy's law (Hele-Shaw and Muskat) or Bernoulli's law (irrotational incompressible Euler equation). It follows that the vorticity is then a delta distribution at the interface multiplied by an amplitude. The dynamics of that interface is governed by the Birkhoff–Rott integral of the amplitude from which we may subtract any component in the tangential direction without modifying its evolution (see ref. 23). We treat the case without surface tension which leads to equality of the pressure on the free boundary, and in both problems it is assumed that the initial interface does not self-intersect. We quantify that property by imposing that the arc-chord quotient be initially strictly positive. It is part of the evolution problem to check carefully that such a positivity prevails for a short time (see ref. 24), as does the Rayleigh–Taylor condition, although depending, in both cases, on the initial data.

1. Equations

The free boundary is given by the discontinuity of the densities and the viscosities (in the case of the free boundary for the irrotational incompressible Euler equation, the viscosity is zero) of the fluids

$$(\mu,\rho)(x_1,x_2,t)=\begin{cases}(\mu^1,\rho^1), & x\in\Omega^1(t)\\ (\mu^2,\rho^2), & x\in\Omega^2(t)=\mathbb{R}^2-\Omega^1(t),\end{cases}$$

and μ^1, μ^2, ρ^1, ρ^2 ($\mu^1\neq\mu^2$ and $\rho^1\neq\rho^2$) are constants.

Let the free boundary be parameterized by

$$\partial\Omega^j(t)=\{z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t)):\alpha\in\mathbb{R}\}$$

where

$$(z_1(\alpha+2k\pi,t),z_2(\alpha+2k\pi,t))=(z_1(\alpha,t)+2k\pi,z_2(\alpha,t)), \quad \textbf{[1]}$$

with the initial data $z(\alpha,0)=z_0(\alpha)$. We will study also the case of a closed curve:

$$(z_1(\alpha+2k\pi,t),z_2(\alpha+2k\pi,t))=(z_1(\alpha,t),z_2(\alpha,t)). \quad \textbf{[2]}$$

We consider that the velocity $v=v(x_1,x_2,t)$ is irrotational, i.e., $\omega=\nabla\times v=0$, in the interior of each domain Ω^j ($j=1,2$). Therefore, the vorticity ω has its support on the curve $z(\alpha,t)$ and it has the form

$$\omega(x,t)=\varpi(\alpha,t)\delta(x-z(\alpha,t))$$

i.e., ω is a measure defined by

$$<\omega,\eta>=\int\varpi(\alpha,t)\eta(z(\alpha,t))d\alpha,$$

with $\eta(x)$ a test function.

Then $z(\alpha,t)$ evolves with a velocity field coming from the Biot–Savart law, which can be explicitly computed; it is given by the Birkhoff–Rott integral of the amplitude ϖ along the interface curve:

$$BR(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta, \quad \textbf{[3]}$$

where PV denotes principal value (see ref. 25).

We have

$$v^2(z(\alpha,t),t)=BR(z,\varpi)(\alpha,t)+\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t),$$

$$v^1(z(\alpha,t),t)=BR(z,\varpi)(\alpha,t)-\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \quad \textbf{[4]}$$

where $v^j(z(\alpha,t),t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside Ω^j and $BR(z,\varpi)(\alpha,t)$ is given by Eq. **3**. It provides us with the velocity field at the interface from which we can subtract any term in the

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[1]To whom correspondence should be addressed. E-mail: dcg@imaff.cfmac.csic.es.

tangential direction without modifying the geometric evolution of the curve

$$z_t(\alpha,t)=BR(z,\varpi)(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t). \quad [5]$$

A wise choice of $c(\alpha,t)$ namely:

$$c(\alpha,t)=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)d\alpha - \int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{|\partial_\alpha z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta, \quad [6]$$

allows us to accomplish the fact that the length of the tangent vector to $z(\alpha,t)$ only depends on the variable t:

$$A(t)=|\partial_\alpha z(\alpha,t)|^2.$$

Next, to close the system we apply Darcy's law or, respectively, Bernoulli's law for the case of Euler equations, which leads to an equation relating the parametrization $z(\alpha,t)$ with the amplitude $\varpi(\alpha,t)$.

1.1. Darcy's Law. Darcy's law is the following momentum equation for the velocity v

$$\frac{\mu}{\kappa}v=-\nabla p-(0,g\rho), \quad [7]$$

where p is the pressure, μ is the dynamic viscosity, κ is the permeability of the medium, ρ is the liquid density, and g is the acceleration due to gravity. Together with the incompressibility condition $\nabla\cdot v=0$, Eq. **7** implies the identity

$$p^2(z(\alpha,t),t)=p^1(z(\alpha,t),t),$$

(see also ref. 17). Let us introduce the following notation:

$$[\mu v](\alpha,t)=(\mu^2v^2(z(\alpha,t),t)-\mu^1v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t).$$

Then, taking the limit in Darcy's law we obtain

$$\begin{aligned}\frac{[\mu v](\alpha,t)}{\kappa}&=-(\nabla p^2(z(\alpha,t),t)-\nabla p^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\&\quad-\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t)\\&=-\partial_\alpha(p^2(z(\alpha,t),t)-p^1(z(\alpha,t),t))-\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t)\\&=-\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),\end{aligned}$$

which gives us

$$\frac{\mu^2+\mu^1}{2\kappa}\varpi(\alpha,t)+\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)=-\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),$$

so that

$$\varpi(\alpha,t)=-A_\mu 2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t). \quad [8]$$

where $A_\mu=\frac{\mu_2-\mu_1}{\mu_2+\mu_1}$.

1.2. Bernoulli's Law. In this section we deduce the evolution equation for the amplitude of vorticity $\varpi(\alpha,t)$ from Bernoulli's law. Let us consider an irrotational flow satisfying the Euler equations

$$\rho(v_t+v\nabla v)=-\nabla p-(0,\mathrm{g}\rho),$$

and the incompressibility condition $\nabla\cdot v=0$, and let ϕ be such that $v(x,t)=\nabla\phi(x,t)$ for $x\neq z(\alpha,t)$. Then, we have the expression

$$\rho\left(\phi_t(x,t)+\frac{1}{2}|v(x,t)|^2+\mathrm{g}x_2\right)+p(x,t)=0.$$

From the Biot–Savart law, for $x\neq z(\alpha,t)$, we get

$$\phi(x,t)=\frac{1}{2\pi}PV\int\arctan\left(\frac{x_2-z_2(\beta,t)}{x_1-z_1(\beta,t)}\right)\varpi(\beta,t)d\beta.$$

Let us define

$$\Pi(\alpha,t)=\phi^2(z(\alpha,t),t)-\phi^1(z(\alpha,t),t),$$

where again $\phi^j(z(\alpha,t),t)$ denotes the limit obtained approaching the boundary in the normal direction inside Ω^j. It is clear that

$$\begin{aligned}\partial_\alpha\Pi(\alpha,t)&=(\nabla\phi^2(z(\alpha,t),t)-\nabla\phi^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\&=(v^2(z(\alpha,t),t)-v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\&=\varpi(\alpha,t),\end{aligned}$$

therefore,

$$\int_{-\pi}^{\pi}\varpi(\alpha,t)d\alpha=0.$$

Now, we observe that

$$\begin{aligned}\phi^2(z(\alpha,t),t)&=IT(z,\varpi)(\alpha,t)+\frac{1}{2}\Pi(\alpha,t),\\\phi^1(z(\alpha,t),t)&=IT(z,\varpi)(\alpha,t)-\frac{1}{2}\Pi(\alpha,t).\end{aligned} \quad [9]$$

where

$$IT(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int\arctan\left(\frac{z_2(\alpha,t)-z_2(\beta,t)}{z_1(\alpha,t)-z_1(\beta,t)}\right)\varpi(\beta,t)d\beta.$$

Then, using Bernoulli's law inside each domain and taking limits approaching the common boundary, one finds

$$\rho^j\left(\phi^j_t(z(\alpha,t),t)+\frac{1}{2}|v^j(z(\alpha,t),t)|^2+\mathrm{g}z_2(\alpha,t)\right)+p^j(z(\alpha,t),t)=0,$$

and since

$$p^1(z(\alpha,t),t)=p^2(z(\alpha,t),t)$$

(see also ref. 18), we get

$$[\rho\phi_t](\alpha,t)+\frac{\rho^2}{2}|v^2(z(\alpha,t),t)|^2-\frac{\rho^1}{2}|v^1(z(\alpha,t),t)|^2+(\rho^2-\rho^1)\mathrm{g}z_2(\alpha,t)=0 \quad [10]$$

where we have introduced the following notation:

$$[\rho\phi_t](\alpha,t)=\rho^2\phi^2_t(z(\alpha,t),t)-\rho^1\phi^1_t(z(\alpha,t),t).$$

Then, it is clear that $\phi^j_t(z(\alpha,t),t)=\partial_t(\phi^j(z(\alpha,t),t))-z_t(\alpha,t)\cdot\nabla\phi^j(z(\alpha,t),t)$, and using Eq. **9** we find

$$\begin{aligned}[\rho\phi_t](\alpha,t)&=\frac{\rho^2+\rho^1}{2}\Pi_t(\alpha,t)+(\rho^2-\rho^1)\partial_t(IT(z,\varpi)(\alpha,t))\\&\quad-z_t(\alpha,t)\cdot(\rho^2v^2(z(\alpha,t),t)-\rho^1v^1(z(\alpha,t),t)).\end{aligned}$$

Eqs. **4** and **5** in Eq. **10** give us

$$\begin{aligned}\Pi_t(\alpha,t)&=-2A_\rho\partial_t(IT(z,\varpi)(\alpha,t))+c(\alpha,t)\varpi(\alpha,t)\\&\quad+A_\rho|BR(z,\varpi)(\alpha,t)|^2+2A_\rho c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)\\&\quad-A_\rho\frac{|\varpi(\alpha,t)|^2}{4|\partial_\alpha z(\alpha,t)|^2}-2A_\rho\mathrm{g}z_2(\alpha,t).\end{aligned} \quad [11]$$

where $A_\rho = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}$. Easily we find the identity:

$$\partial_\alpha \partial_t (IT(z,\varpi)(\alpha,t)) = \partial_t (BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t))$$
$$= \partial_t(BR(z,\varpi)(\alpha,t))\cdot\partial_\alpha z(\alpha,t) + BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)$$
$$+c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^2 z(\alpha,t)+\partial_\alpha c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)$$

Then taking a derivative in Eq. **11** and using the identity above we get the desired formula for ϖ, which in the case $A_\rho = 1$, i.e., $\rho_1 = 0$, reads as follows

$$\varpi_t(\alpha,t) = -2\partial_t BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - \partial_\alpha\left(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\right)(\alpha,t)$$
$$+\partial_\alpha(c\varpi)(\alpha,t) + 2c(\alpha,t)\partial_\alpha BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) + 2g\partial_\alpha z_2(\alpha,t). \quad \textbf{[12]}$$

That is, we have obtained the standard water waves model where g is the acceleration due to gravity.

2. Rayleigh–Taylor Condition

Our next step is to find the formula for the difference of the gradients of the pressure in the normal direction:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t))\cdot\partial_\alpha^\perp z(\alpha,t).$$

2.1. Darcy's Law. Applying Darcy's law and approaching the boundary, we get

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + g(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t). \quad \textbf{[13]}$$

2.2. Bernoulli's Law. We will consider the case $A_\rho = 1$, which yields $-\nabla p(x,t) = 0$ inside $\Omega^1(t)$ and therefore $\nabla p^1(z(\alpha,t),t) = 0$. Next, we define the Lagrangian coordinates for the free boundary with the velocity v^2

$$Z_t(\gamma,t) = v^2(Z(\gamma,t),t))$$
$$Z(\gamma,0) = z_0(\gamma).$$

We have the same curve with different parameterizations $Z(\gamma,t) = z(\alpha(\gamma,t),t)$ and two equations for the velocity of the curve, namely,

$$Z_t(\gamma,t) = z_t(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha z(\alpha,t)$$
$$= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha z(\alpha,t) \quad \textbf{[14]}$$

and another one given by

$$Z_t(\gamma,t) = BR(z,\varpi)(\alpha,t) + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t). \quad \textbf{[15]}$$

Next, we introduce the function (see refs. 6 and 12)

$$\varphi(\alpha,t) = \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} - c(\alpha,t)|\partial_\alpha z(\alpha,t)|, \quad \textbf{[16]}$$

and we observe that the dot product of Eqs. **15** and **14** with the tangential vector gives

$$\alpha_t(\gamma,t) = \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}.$$

Taking a time derivative in Eq. **15** yields

$$Z_{tt}(\gamma,t)\cdot\partial_\alpha^\perp z(\alpha,t)$$
$$= (\partial_t BR(z,\varpi)(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha BR(z,\varpi)(\alpha,t))\cdot\partial_\alpha^\perp z(\alpha,t)$$
$$+\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}(\partial_\alpha z_t(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha^2 z(\alpha,t))\cdot\partial_\alpha^\perp z(\alpha,t)$$

and therefore

$$\frac{\sigma(\alpha,t)}{\rho^2} = (\partial_t BR(z,\varpi)(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}\partial_\alpha BR(z,\varpi)(\alpha,t))\cdot\partial_\alpha^\perp z(\alpha,t)$$
$$+\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\left(\partial_\alpha z_t(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}\partial_\alpha^2 z(\alpha,t)\right)$$
$$\cdot\partial_\alpha^\perp z(\alpha,t) + g\partial_\alpha z_1(\alpha,t). \quad \textbf{[17]}$$

3. Local Existence

Our main results consist on the existence of a positive time τ (depending on the initial conditions) for which we have a solution of the periodic Muskat problem (Eqs. **5**, **6**, and **8**) and of the free boundary of the irrotational incompressible Euler equations in vacuum (Eqs. **5**, **6**, and **12**) during the time interval $[0,\tau]$ so long as the initial data belong to $H^k(\mathbb{T})$ for k sufficiently large, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha),0) - \nabla p^1(z_0(\alpha),0))\cdot\partial_\alpha^\perp z_0(\alpha) > 0,$$

where p^j denote the pressure in Ω^j and the function $\mathcal{F}(z)$, which measures the arc-chord condition (see ref. 24), is defined by

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{|\beta|}{|z(\alpha,t)-z(\alpha-\beta,t)|} \quad \forall\alpha,\beta\in(-\pi,\pi), \quad \textbf{[18]}$$

with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|}.$$

Theorem 3.1. *Let $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 3$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha),0) - \nabla p^1(z_0(\alpha),0))\cdot\partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $\tau > 0$ so that there is a solution to Eqs. **5**, **6**, *and* **8** *in $C^1([0,\tau];H^k(\mathbb{T}))$ with $z(\alpha,0) = z_0(\alpha)$.*

Theorem 3.2. *Let $z_0(\alpha)\in H^k(\mathbb{T})$, $\varpi_0(\alpha)\in H^{k-1}$, and $\varphi(\alpha,0) = \varphi_0(\alpha)\in H^{k-\frac{1}{2}}$ defined in Eq.* **16** *for $k\geq 4$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, $g\geq 0$, and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha),0) - \nabla p^1(z_0(\alpha),0))\cdot\partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $\tau > 0$ so that there is a solution to Eqs. **5**, **6**, *and* **12** *with $z(\alpha,t)\in C^1([0,\tau];H^k(\mathbb{T}))$, $\varpi(\alpha,t)\in C^1([0,\tau];H^{k-1}(\mathbb{T}))$ for $z(\alpha,0) = z_0(\alpha)$ and $\varpi(\alpha,0) = \varpi_0(\alpha)$.*

Remark 3.1. Notice that the parametrization is defined by properties Eqs. **1** or **2**. But in the Hele-Shaw and Muskat problems we can show easily that

$$\int_{-\pi}^{\pi}\sigma(\alpha,t)d\alpha = 0$$

for a closed curve, making impossible the task of prescribing a sign as in the Rayleigh–Taylor condition.

4. Sketch of the Proof

First, we consider the operator $T(u)(\alpha) = 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha)$ associated to a smooth H^3 curve z satisfying the arc-chord condition. T is a smoothing compact operator in Sobolev space whose adjoint T^*, acting on u, is described in terms of the Cauchy integral of u along the curve z, and whose real eigenvalues have an absolute value strictly less than one (see ref. 26).

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In our proof it is crucial to get control of the norm of the inverse operators $(I - \xi T)^{-1}$, $|\xi| \le 1$. The arguments rely on the boundedness properties of the Hilbert transforms associated to $C^{1,\alpha}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, Hopf maximum principle, and Harnack inequalities (see refs. 27 and 28).

We then provide upper bounds for the amplitude of the vorticity, the Birkhoff–Rott integral, the parametrization of the curve, the arc-chord condition, and the Rayleigh–Taylor condition, namely:

4.1. A Priori Estimates for Theorem 3.1.

$$\|\varpi\|_{H^k} \le \exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}\big),$$

$$\|BR(z,\varpi)\|_{H^k} \le \exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}\big),$$

$$\frac{d}{dt}\|z\|^2_{H^k}(t) \le -\frac{\kappa}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2}\partial^k_\alpha z(\alpha,t) \cdot \Lambda\big(\partial^k_\alpha z\big)(\alpha,t)d\alpha + \exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^k}\big),$$

and

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \le \exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\big)$$

where the operator Λ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and $\sigma(\alpha,t)$ is the difference of the gradients of the pressure in the normal direction. The first inequality in our list above is where we use the precise control of the norm of the inverse operator. The third estimate depends crucially on the positive sign of the Rayleigh–Taylor condition and the chosen well-adapted parametrization. The second and the fourth follow from the standard Sobolev embedding.

We then define the quantity $E(t)$ for the Muskat problem given by

$$E(t) = \|z\|^2_{H^k}(t) + \|\mathcal{F}(z)\|^2_{L^\infty}(t)$$

to get

$$\frac{d}{dt}E(t) \le -\frac{\kappa}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2}\partial^k_\alpha z(\alpha,t) \cdot \Lambda\big(\partial^k_\alpha z\big)(\alpha,t)d\alpha + \exp C(E(t)). \quad \textbf{[19]}$$

4.2. A Priori Estimates for Theorem 3.2.

In the case of Bernoulli's law our previous comments can be applied in the following setting: For some universal constant q

$$\|BR(z,\varpi)\|_{H^k} \le C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^k}\big)^q,$$

$$\|z_t\|_{H^k} \le C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^k}\big)^q,$$

$$\|\varpi_t\|_{H^k} \le C\big(\exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}\big)\big)\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+2}} + \|\varpi\|^2_{H^{k+1}} + \|\varphi\|^2_{H^{k+1}}\big)^q,$$

$$\|\varpi\|_{H^k} \le C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^{k-1}} + \|\varphi\|^2_{H^k}\big)^q.$$

We define the quantity $E(t)$ in this case by

$$E(t) = \|z\|^2_{H^{k-1}}(t) + \int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{\rho^2|\partial_\alpha z(\alpha,t)|^2}\big|\partial^k_\alpha z(\alpha,t)\big|^2 d\alpha + \|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|\varpi\|^2_{H^{k-2}}(t) + \|\varphi\|^2_{H^{k-\frac{1}{2}}}(t).$$

If we take

$$m(t) = \min_{\alpha\in\mathbb{T}}\sigma(\alpha,t)$$

and using that

$$\big\|\partial^4_\alpha z\big\|^2_{L^2}(t) = \int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{\sigma(\alpha,t)}\big|\partial^4_\alpha z(\alpha,t)\big|^2 d\alpha \le \frac{1}{m(t)}\int_{\mathbb{T}}\sigma(\alpha,t)\big|\partial^4_\alpha z(\alpha,t)\big|^2 d\alpha,$$

we obtain

$$\frac{d}{dt}E(t) \le \frac{1}{m^p(t)}C\exp(CE(t)), \quad \textbf{[20]}$$

with $p \in \mathbb{N}$.

4.3. The Evolution of the Rayleigh–Taylor Condition.

It is clear that in the evolution of the quantities $E(t)$ in both contour dynamics problems in Eqs. **19** and **20**, everything depends on the sign of $\sigma(\alpha,t)$. Therefore, to integrate the systems, it is crucial to study the evolution of

$$m(t) = \min_{\alpha\in\mathbb{T}}\sigma(\alpha,t).$$

Let us then introduce the Rayleigh–Taylor condition in a definition of energy as follows:

$$E_{RT}(t) = E(t) + \frac{1}{m(t)}.$$

Similar arguments (see ref. 29) allow us to accomplish the fact that $m'(t) = \sigma_t(\alpha_t,t)$ for almost all t. Since we can use Eqs. **13** and **17** and the above estimates, it is possible to control $\|\sigma_t(\alpha_t,t)\|_{L^\infty}$ by means of $E_{RT}(t)$. It yields finally

$$\frac{d}{dt}E_{RT}(t) \le C\exp(CE_{RT}(t)),$$

for C a universal constant.

4.4. Existence.

To conclude the existence proof we introduce regularized evolution equations (allowing us to take limits) satisfying uniformly the above a priori estimates and for which the local existence follows by standard arguments. Furthermore, in the case of the Hele-Shaw and Muskat problem, to take advantage of the positivity of σ, we have to use a pointwise inequality satisfied by the nonlocal operator Λ (see ref. 29).

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