

# New Relationships involving the Mean Curvature of Slant Submanifolds in $S$ -space-forms

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(2000 Mathematics Subject Classification: 53C25, 53C40).

(Keywords and sentences:  $S$ -space-form, Slant Immersion, Mean Curvature Vector, Ricci Curvature, Shape Operator).

**Abstract.** Relationships between the Ricci curvature and the squared mean curvature and between the shape operator associated with the mean curvature vector and the sectional curvature function for slant submanifolds of an  $S$ -space-form are proved, particularizing them to invariant and anti-invariant submanifolds tangent to the structure vector fields.

## 1 Introduction.

In words of B.-Y. Chen, to “find simple relationships between the main extrinsic invariant and the main intrinsic invariants of a submanifold” is one of the basic problems in the theory of submanifolds ([7]). In this way, he established a relationship between sectional curvature function and the shape operator for submanifolds in real space-forms [7] and another relationship between the Ricci curvature and the squared mean curvature [8]. Corresponding relationships have been proved in [13] for slant submanifolds of Sasakian space-forms.

Slant immersions in complex geometry were defined by B.-Y. Chen as a natural generalization of both holomorphic and totally real immersions [9]. Recently, A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [11] and slant submanifolds of Sasakian manifolds have been studied in [2]. For a general view about slant submanifolds, the survey written by A. Carriazo ([3]) can be consulted. On the other hand, for manifolds with an  $f$ -structure, D.E. Blair has introduced  $S$ -manifolds as the analogue of the Kaehler structure in the almost complex case and of Sasakian

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<sup>1</sup>The authors are partially supported by the PAI project (Junta de Andalucía, Spain, 2004).

structure in the almost contact case [1] and we have defined and begun with the study of slant submanifolds in such  $S$ -manifolds [4,5,6].

The purpose of this paper is to obtain similar relationships to Chen's ones mentioned above, generalizing and improving in some sense the ones proved in [13], for slant submanifolds in  $S$ -space-forms. To this end, after reviewing, for later use, necessary details about  $S$ -manifolds and slant submanifolds in Section 2, we devote Section 3 to get an inequality between Ricci curvature and squared mean curvature vector and discuss the equality case. Finally, in Section 4 we establish an inequality between the shape operator associated with the mean curvature vector and the sectional curvature function for slant submanifolds of  $S$ -space forms. In both cases, we particularize these inequalities for invariant and anti-invariant submanifolds tangent to the structure vector fields.

## 2 Preliminaries.

Let  $(\widetilde{M}, g)$  be a Riemannian manifold and denote by  $T\widetilde{M}$  the Lie algebra of vector fields in  $\widetilde{M}$ .  $\widetilde{M}$  is said to be a metric  $f$ -manifold if there exist a  $(1, 1)$  tensor field  $f$ ,  $s$  global unit vector fields  $\xi_1, \dots, \xi_s$  (called structure vector fields) and  $s$  1-forms  $\eta_1, \dots, \eta_s$  on  $\widetilde{M}$  such that

$$f^2 X = -X + \sum_{\alpha=1}^s \eta_\alpha(X) \xi_\alpha, \quad g(X, \xi_\alpha) = \eta_\alpha(X),$$

$$f \xi_\alpha = 0, \quad \eta_\alpha \circ f = 0$$

and

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^s \eta_\alpha(X) \eta_\alpha(Y),$$

for any  $X, Y \in T\widetilde{M}$  and  $\alpha = 1, \dots, s$ . Let  $F$  denote the fundamental 2-form in  $\widetilde{M}$  given by  $F(X, Y) = g(X, fY)$ , for any  $X, Y \in T\widetilde{M}$ . The  $f$ -structure  $f$  is said to be normal if  $[f, f] + 2 \sum_{\alpha} \xi_\alpha \otimes d\eta_\alpha = 0$ , where  $[f, f]$  is the Nijenhuis torsion of  $f$ .  $\widetilde{M}$  is called an  **$S$ -manifold** if the structure is normal and  $F = d\eta_\alpha$ , for any  $\alpha = 1, \dots, s$ .

Given an  $S$ -manifold  $\widetilde{M}$ , a plane section  $\pi$  in  $T_p \widetilde{M}$  is called an  $f$ -section if it is spanned by  $X$  and  $fX$ , where  $X$  is a unit tangent vector field orthogonal to the distribution  $\mathcal{M}$  spanned by the structure vector fields. The sectional curvature  $K(\pi)$  of an  $f$ -section  $\pi$  is called  $f$ -sectional curvature. An  $S$ -manifolds is said to be an  **$S$ -space-form** if it has constant  $f$ -sectional curvature  $c$  and then, it is denoted by  $\widetilde{M}(c)$ .

Now, let  $M$  be a submanifold isometrically immersed in an  $S$ -manifold  $\widetilde{M}$ . Let  $TM$  be the Lie algebra of vector fields in  $M$  and  $T^\perp M$  the set of all vector fields normal to  $M$ . We denote by  $\sigma$  the second fundamental form of  $M$  and by  $A_V$  the shape operator associated with any  $V \in T^\perp M$ . They are related by the equation  $g(\sigma(X, Y), V) = g(A_V X, Y)$ , for any  $X, Y \in TM$  and any  $V \in T^\perp M$ . The mean curvature vector  $H$  is defined by  $H = (1/\dim(M))\text{trace}(\sigma)$ . The submanifold is said to be minimal if  $H$  vanishes identically and it is said to be totally geodesic if  $\sigma(X, Y) = 0$ , for any  $X, Y \in TM$ . Moreover, the **relative null space** of  $M$  is defined by:

$$\mathcal{N} = \{X \in TM : \sigma(X, Y) = 0, \text{ for all } Y \in TM\}.$$

For any  $X \in TM$ , we put  $fX = TX + NX$ , where  $TX$  (resp.  $NX$ ) is the tangential (resp. normal) component of  $fX$ . The submanifold  $M$  is said to be invariant if  $N$  is identically zero, that is, if  $fX \in TM$ , for any  $X \in TM$  and it is said to be anti-invariant if  $T$  is identically zero, that is, if  $fX \in T^\perp M$ , for any  $X \in TM$ .

From now on, we suppose that the structure vector fields are tangent to  $M$  and we denote by  $n + s$  (resp.  $2m + s$ ) the dimension of  $M$  (resp.  $\widetilde{M}$ ). Hence, if we denote by  $\mathcal{L}$  the orthogonal distribution to  $\mathcal{M}$  in  $TM$ , we can write the orthogonal direct decomposition  $TM = \mathcal{L} \oplus \mathcal{M}$ .

It is well-known that

$$\sigma(X, \xi_\alpha) = -NX, \quad (2.1)$$

for any  $X \in TM$  and any  $\alpha = 1, \dots, s$ . In particular,  $\sigma(\xi_\alpha, \xi_\beta) = 0$ , for any  $\alpha, \beta = 1, \dots, s$ .

Now, given a local orthonormal basis

$$\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}, \xi_1, \dots, \xi_s\}$$

of  $T\widetilde{M}$ , such that  $\{e_1, \dots, e_n\}$  is a local orthonormal basis of  $\mathcal{L}$ , we can write the mean curvature vector  $H$  and the squared norms of  $T$  and  $\sigma$  by:

$$H = \frac{1}{n+s} \sum_{i=1}^n \sigma(e_i, e_i), \quad (2.2)$$

$$\|T\|^2 = \sum_{i,j=1}^n g^2(e_i, Te_j), \quad (2.3)$$

$$\|\sigma\|^2 = \sum_{i=1}^n \|\sigma(e_i, e_i)\|^2 + 2 \sum_{1 \leq i < j \leq n} \|\sigma(e_i, e_j)\|^2 + 2 \sum_{i=1}^n \sum_{\alpha=1}^s \|\sigma(e_i, \xi_\alpha)\|^2. \quad (2.4)$$

The curvature tensor field  $R$  of a submanifold  $M$  of an  $S$ -space-form  $\widetilde{M}(c)$  satisfies

$$\begin{aligned}
R(X, Y, Z, W) = & g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) + \\
& + \sum_{\alpha, \beta} (g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + \\
& + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)) + \\
& + \frac{c+3s}{4}(g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)) + \\
& + \frac{c-s}{4}(F(X, W)F(Y, Z) - F(X, Z)F(Y, W) - 2F(X, Y)F(Z, W)), \quad (2.5)
\end{aligned}$$

for any  $X, Y, Z, W \in T\widetilde{M}$  (see, for references, [1,10]).

The scalar curvature  $\tau$  of  $M$  is defined by

$$\tau = \frac{1}{2} \sum_{i \neq j} K(e_i \wedge e_j) + \sum_{i=1}^n \sum_{\alpha=1}^s K(e_i \wedge \xi_\alpha), \quad (2.6)$$

where  $K(X \wedge Y)$  denotes the sectional curvature of  $M$  associated with the plane section spanned by  $X, Y \in TM$ .

From (2.1)-(2.6), we obtain the following relation between the scalar curvature and the mean curvature of  $M$  [4]:

$$2\tau = (n+s)^2 \|H\|^2 - \|\sigma\|^2 + n(n-1) \frac{c+3s}{4} + 2ns + \frac{3(c-s)}{4} \|T\|^2. \quad (2.7)$$

If for each nonzero vector  $X \in T_p M - \mathcal{M}_p$ , we consider the angle  $\theta(X)$  between  $fX$  and  $T_p M$ , then the submanifold is said to be  $\theta$ -slant [6] if such angle is a constant, which is independent on the choice of  $p \in M$  and  $X \in T_p M - \mathcal{M}_p$ . The angle  $\theta$  of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant submanifolds tangent to the structure vector fields are slant submanifolds with slant angle  $\theta = 0$  and  $\theta = \pi/2$ , respectively. A slant immersion which is not invariant nor anti-invariant is called a proper slant immersion.

In [6], we have proved that a  $\theta$ -slant submanifold  $M$  of a metric  $f$ -manifold  $\widetilde{M}$  satisfies

$$g(TX, TY) = \cos^2 \theta g(fX, fY),$$

for any  $X, Y \in TM$ . Moreover, it is easy to show that, using a local orthonormal basis  $\{e_1, \dots, e_{n+s}\}$  of  $TM$ ,

$$\sum_{j=1}^{n+s} g^2(e_i, fe_j) = \cos^2 \theta (1 - \sum_{\alpha=1}^s \eta_\alpha^2(e_i)), \quad (2.8)$$

for any  $i = 1, \dots, s$ .

### 3 The Ricci curvature for slant immersions.

Let  $M$  be an  $(n+s)$ -dimensional isometrically immersed submanifold in a  $(2m+s)$ -dimensional  $S$ -space-form  $\widetilde{M}(c)$ , tangent to the structure vector fields. In this section, we want to study the Ricci curvature of unit vector fields in  $M$ , normal to the structure vector fields, when  $M$  is a slant submanifold and to relate them with the mean curvature vector. For this reason, we shall assume that  $n \geq 2$ , because it is known that there are not proper slant submanifolds of dimension  $1+s$  ([5]). Throughout the section, we consider local orthonormal basis of  $T\widetilde{M}(c)$ ,

$$\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}, e_{2m+1} = \xi_1, \dots, e_{2m+s} = \xi_s\}, \quad (3.1)$$

such that  $\{e_1, \dots, e_n\}$  is a local orthonormal basis of  $\mathcal{L}$ . First, we have the following general result:

**Theorem 3.1.** *Let  $M$  be an  $(n+s)$ -dimensional submanifold  $\widetilde{M}(c)$ , tangent to the structure vector fields. Then,*

$$4\text{Ric}(U) \leq (n+s)^2\|H\|^2 + (n-1)(c+3s) + \|TU\|^2(3c+s), \quad (3.2)$$

for any unit vector field  $U \in \mathcal{L}$ .

*Proof.* We choose a local orthonormal basis of  $T\widetilde{M}(c)$  as in (3.1) and such that  $e_1 = U$ . Then:

$$\begin{aligned} \|\sigma\|^2 &= \frac{1}{2}(n+s)^2\|H\|^2 + 2 \sum_{1 \leq i < j \leq n} \|\sigma(e_i, e_j)\|^2 + 2 \sum_{i=1}^n \sum_{\alpha=1}^s \|\sigma(e_i, \xi_\alpha)\|^2 + \\ &+ \frac{1}{2} \sum_{i=1}^n \|\sigma(e_i, e_i)\|^2 - \sum_{1 \leq i < j \leq n} g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \end{aligned} \quad (3.3)$$

Thus, from (2.7) and (3.3), we have:

$$\begin{aligned} \frac{1}{4}(n+s)^2\|H\|^2 &= \tau - \frac{1}{8}n(n-1)(c+3s) - \frac{3}{8}\|T\|^2(c-s) - ns + \\ &+ \sum_{1 \leq i < j \leq n} \|\sigma(e_i, e_j)\|^2 + \sum_{i=1}^n \sum_{\alpha=1}^s \|\sigma(e_i, \xi_\alpha)\|^2 + \\ &+ \frac{1}{4} \sum_{i=1}^n \|\sigma(e_i, e_i)\|^2 - \frac{1}{2} \sum_{1 \leq i < j \leq n} g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \end{aligned} \quad (3.4)$$

On the other hand, since from (2.6),

$$\tau = \text{Ric}(U) + \sum_{2 \leq i < j \leq n} K(e_i \wedge e_j) + \sum_{i=2}^n \sum_{\alpha=1}^s K(e_i \wedge \xi_\alpha)$$

and, by using (2.5), we get

$$\begin{aligned} \sum_{2 \leq i < j \leq n} K(e_i \wedge e_j) &= \frac{1}{8}n(n-1)(c+3s) + \frac{3}{4} \left( \frac{\|T\|^2}{2} - \|TU\|^2 \right) (c-s) + \\ &+ \sum_{2 \leq i < j \leq n} (g(\sigma(e_i, e_i), \sigma(e_j, e_j)) - \|\sigma(e_i, e_j)\|^2) \end{aligned}$$

and

$$\sum_{i=2}^n \sum_{\alpha=1}^s K(e_i \wedge \xi_\alpha) = (n-1)s - \sum_{i=2}^n \sum_{\alpha=1}^s \|\sigma(e_i, \xi_\alpha)\|^2,$$

then, substituting into (3.4) and taking into account (2.1), we obtain:

$$\begin{aligned} \text{Ric}(U) &= \frac{1}{4}(n+s)^2\|H\|^2 + \frac{1}{4}(n-1)(c+3s) + \frac{1}{4}\|TU\|^2(3c+s) - \\ &- \frac{1}{2} \sum_{2 \leq i < j \leq n} g(\sigma(e_i, e_i), \sigma(e_j, e_j)) + \frac{1}{2} \sum_{i=2}^n g(\sigma(U, U), \sigma(e_i, e_i)) - \\ &- \sum_{i=2}^n \|\sigma(U, e_i)\|^2 - \frac{1}{4} \sum_{i=1}^n \|\sigma(e_i, e_i)\|^2. \end{aligned} \quad (3.5)$$

Now, if we put  $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$ , for any  $i, j = 1, \dots, n$  and  $r = n+1, \dots, 2m$ , (3.5) becomes

$$\begin{aligned} \text{Ric}(U) &= \frac{1}{4}(n+s)^2\|H\|^2 + \frac{1}{4}(n-1)(c+3s) + \frac{1}{4}\|TU\|^2(3c+s) - \\ &- \sum_{r=n+1}^{2m} \left[ \frac{1}{4} \left( \sigma_{11}^r - \sum_{i=2}^n \sigma_{ii}^r \right)^2 + \sum_{i=2}^n (\sigma_{1i}^r)^2 \right], \end{aligned} \quad (3.6)$$

which completes the proof.  $\square$

Observe that, if we also put  $\sigma_{1(2m+\alpha)}^r = g(\sigma(U, \xi_\alpha), e_r) = -g(NU, e_r)$ , for any  $\alpha = 1, \dots, s$  and  $r = n+1, \dots, 2m$ , we have

$$\begin{aligned} \text{Ric}(U) &= \frac{1}{4}(n+s)^2\|H\|^2 + \frac{1}{4}n(c+3s) + \frac{1}{4}(3\|TU\|^2 - 1)(c-s) - \\ &- \sum_{r=n+1}^{2m} \left[ \frac{1}{4} \left( \sigma_{11}^r - \sum_{i=2}^n \sigma_{ii}^r \right)^2 + \sum_{i=2}^n (\sigma_{1i}^r)^2 + \sum_{\alpha=1}^s (\sigma_{1(2m+\alpha)}^r)^2 \right] \end{aligned} \quad (3.7)$$

and, consequently:

$$4\text{Ric}(U) \leq (n+s)^2\|H\|^2 + n(c+3s) + (3\|TU\|^2 - 1)(c-s) \quad (3.8).$$

When  $s = 1$ , this upper bound is the one obtained in [13] for submanifolds of Sasakian space-forms tangent to the structure vector field. However, it is easy to show that it is worse (in the sense of higher) than the one of (3.2). Moreover, both upper bounds are equal if and only if  $\|NU\|^2 = 0$ , that is, if and only if, from (2.1),  $\sigma(U, \xi_\alpha) = 0$ , for any  $\alpha = 1, \dots, s$ . If it is the case, their common value is  $(n+s)^2\|H\|^2 + n(c+3s) + 2(c-s)$ . Then, we can prove the following theorem.

**Theorem 3.2.** *Let  $M$  be an  $(n+s)$ -dimensional minimal submanifold of  $\widetilde{M}(c)$ , tangent to the structure vector fields. Then, a unit vector field  $U$  in  $\mathcal{L}$  satisfies the equality case of (3.8) if and only if  $U$  lies in the relative null space of  $M$ . Moreover, in this case:*

$$4\text{Ric}(U) = n(c+3s) + 2(c-s).$$

*Proof.* If  $U \in \mathcal{L}$  is a unit vector field satisfying the equality case of (3.8), then it also satisfies the equality case of (3.2) and so,  $\sigma(U, \xi_\alpha) = 0$ , for any  $\alpha = 1, \dots, s$ . Furthermore, choosing a local orthonormal basis of  $T\widetilde{M}(c)$  as in (3.1) such that  $e_1 = U$ , from (3.7) we get  $\sigma_{1i}^r = 0$ , for any  $i = 2, \dots, n$ ,  $r = n+1, \dots, 2m$  and

$$\sigma_{11}^r = \sum_{i=2}^n \sigma_{ii}^r,$$

for any  $r = n+1, \dots, 2m$ . But, since  $H = 0$ ,

$$\sigma_{11}^r = - \sum_{i=2}^n \sigma_{ii}^r,$$

for any  $r = n+1, \dots, 2m$ , that is,  $\sigma_{11}^r = 0$ . Thus,  $U \in \mathcal{N}$ .

Conversely, if  $U \in \mathcal{N}$ , choosing a local orthonormal basis of  $T\widetilde{M}(c)$  as in (3.1) with  $e_1 = U$ , we have that  $\sigma_{1i}^r = 0$  and  $\sigma_{1(2m+\alpha)}^r = 0$ , for any  $i = 1, \dots, n$ ,  $\alpha = 1, \dots, s$ ,  $r = n+1, \dots, 2m$ . Again, since  $H = 0$ , we obtain that  $\sigma_{22}^r + \dots + \sigma_{nn}^r = 0$ , for any  $r = n+1, \dots, 2m$ . Then, from (3.7) we get the equality case of (3.8). Finally, from (2.1) we complete the proof.  $\square$

Now, observe that if the equality case of (3.8) holds for all unit vector fields  $U \in \mathcal{L}$ , the equality case of (3.2) is true for these vector fields too. Thus,

from (2.1),  $NU = 0$  for any  $U \in \mathcal{L}$  and  $M$  is an invariant submanifold. Then, it is easy to show that it is minimal. Consequently, making use of Theorem 3.2,  $U \in \mathcal{N}$ , for any  $U \in \mathcal{L}$  and  $M$  is totally geodesic. The converse result is a straightforward computation. So, we have proved the following corollary of Theorem 3.2.

**Corollary 3.1.** *Let  $M$  be an  $(n+s)$ -dimensional minimal submanifold of  $\widetilde{M}(c)$ , tangent to the structure vector fields. Then, the equality case of (3.8) holds for all unit vector field in  $\mathcal{L}$  if and only if  $M$  is a totally geodesic submanifold.*

Next, we are going to study the equality case of (3.2). To this end, we recall that an  $(n+s)$ -dimensional submanifold of an  $S$ -manifold, tangent to the structure vector fields, is said to be a totally  $f$ -geodesic submanifold (resp., totally  $f$ -umbilical) if the distribution  $\mathcal{L}$  is totally geodesic (resp., totally umbilical), that is, if  $\sigma(X, Y) = 0$  (resp.,  $\sigma(X, Y) = g(X, Y)V$ ), being

$$V = \frac{n+s}{n}H),$$

for any  $X, Y \in \mathcal{L}$  ([12]). Then, we can prove the following theorem.

**Theorem 3.3.** *Let  $M$  be an  $(n+s)$ -dimensional ( $n \geq 2$ ) submanifold of  $\widetilde{M}(c)$ , tangent to the structure vector fields. Then, the equality case of (3.2) holds for all unit vector field in  $\mathcal{L}$  if and only if either  $M$  is a totally  $f$ -geodesic submanifold or  $n = 2$  and  $M$  is a totally  $f$ -umbilical submanifold.*

*Proof.* If the equality case of (3.2) is true for any unit vector field  $U \in \mathcal{L}$ , then, by choosing local orthonormal basis of  $T\widetilde{M}(c)$  as in (3.1) and since  $e_1$  can be chosen to be any arbitrary unit vector fields in  $\mathcal{L}$ , from (3.6) we get

$$2\sigma_{ii}^r = \sigma_{11}^r + \cdots + \sigma_{nn}^r, \quad i = 1, \dots, n,$$

$$\sigma_{ij}^r = 0, \quad i \neq j,$$

for any  $r = n+1, \dots, 2m$ . Thus, we have two cases, namely either  $n = 2$  or  $n > 2$ . In the first case,  $\sigma_{11}^r = \sigma_{22}^r$ , for any  $r$  and  $M$  is a totally  $f$ -umbilical submanifold, while in the second case  $\sigma_{ii}^r = 0$ ,  $i = 1, \dots, n$  and  $M$  is a totally  $f$ -geodesic submanifold. The converse part is a straightforward computation.  $\square$

The above results correspond to that one proved by B.-Y. Chen in [8] for submanifolds in real space-forms. Moreover, they imply the following theorem for a slant submanifold isometrically immersed in an  $S$ -space-form.



**Theorem 3.4.** *Let  $M$  be an  $(n + s)$ -dimensional ( $n \geq 2$ )  $\theta$ -slant submanifold of an  $S$ -space-form  $\widetilde{M}(c)$ . Then:*

(i) *For each unit vector field  $U \in \mathcal{L}$ , we have:*

$$4\text{Ric}(U) \leq (n + s)^2 \|H\|^2 + (n - 1)(c + 3s) + \cos^2 \theta(3c + s). \quad (3.9)$$

(ii) *The equality case of (3.9) holds for all unit vector field in  $\mathcal{L}$  if and only if either  $M$  is a totally  $f$ -geodesic submanifold or  $n = 2$  and  $M$  is a totally  $f$ -umbilical submanifold.*

*Proof.* For any unit vector field  $U \in \mathcal{L}$ , by using a local orthonormal basis of  $T\widetilde{M}(c)$  as in (3.1), such that  $e_1 = U$ , we get from (2.8) that

$$\|TU\|^2 = \cos^2 \theta$$

and so, from (3.2) we have (3.9). Rest of the proof is similar to that of Theorem 3.3.  $\square$

In particular, if  $M$  is an invariant submanifold (then, it is a minimal manifold too), we have:

**Theorem 3.5.** *Let  $M$  be an  $(n + s)$ -dimensional ( $n \geq 2$ ) invariant submanifold of an  $S$ -space-form  $\widetilde{M}(c)$  tangent to the structure vector fields. Then:*

(i) *For each unit vector field  $U \in \mathcal{L}$ , we have:*

$$4\text{Ric}(U) \leq n(c + 3s) + 2(c - s). \quad (3.10)$$

(ii) *The equality case of (3.10) holds for all unit vector field in  $\mathcal{L}$  if and only if either  $M$  is a totally  $f$ -geodesic submanifold or  $n = 2$  and  $M$  is a totally  $f$ -umbilical submanifold.*

Finally, if  $M$  is an anti-invariant submanifold, we obtain:

**Theorem 3.6.** *Let  $M$  be an  $(n + s)$ -dimensional ( $n \geq 2$ ) anti-invariant submanifold of an  $S$ -space-form  $\widetilde{M}(c)$  tangent to the structure vector fields. Then:*

(i) *For each unit vector field  $U \in \mathcal{L}$ , we have:*

$$4\text{Ric}(U) \leq (n + s)^2 \|H\|^2 + (n - 1)(c + 3s). \quad (3.11)$$

(ii) *The equality case of (3.11) holds for all unit vector field in  $\mathcal{L}$  if and only if either  $M$  is a totally  $f$ -geodesic submanifold or  $n = 2$  and  $M$  is a totally  $f$ -umbilical submanifold.*

These results improve those ones proved in [13] for slant submanifolds of a Sasakian space-form (case  $s = 1$ ).

## 4 The shape operator in slant submanifolds.

Let  $M$  be an  $(n+s)$ -dimensional  $\theta$ -slant submanifold of a  $(2m+s)$ -dimensional  $S$ -space-form  $\widetilde{M}(c)$ . Let  $p \in M$  and a number

$$b > \frac{c+3s}{4} + \frac{3(c-s)}{4} \cos^2 \theta$$

such that the sectional curvature of  $M$ ,  $K \geq b$  at  $p$ . Throughout this section, we consider an orthonormal basis of the tangent space  $T_p(\widetilde{M})$ ,

$$\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+s}, e_{n+s+1}, \dots, e_{2m+s}\}$$

with  $\{e_1, \dots, e_{n+s}\}$  being an orthonormal basis of  $T_p(M)$ ,  $e_{n+s+1}$  parallel to the mean curvature vector at  $p$  and  $e_1, \dots, e_{n+s}$  diagonalizing the shape operator  $A_{n+s+1}$ . As above, we put  $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$ , for any  $i, j = 1, \dots, n+s$  and  $r = n+s+1, \dots, 2m+s$ . Then, we have

$$A_{n+s+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+s} \end{pmatrix}, \quad (4.1)$$

where  $a_i = \sigma_{ii}^{n+s+1}$ ,  $i = 1, \dots, n+s$ . Moreover, if  $r = n+s+2, \dots, 2m+s$ ,  $A_r = (\sigma_{ij}^r)$ ,  $i, j = 1, \dots, n+s$  and so,

$$\text{trace} A_r = \sum_{i=1}^{n+s} \sigma_{ii}^r = 0, \quad (4.2)$$

since

$$\sum_{i=1}^{n+s} \sigma_{ii}^r = g((n+s)H, e_r) = 0,$$

for any  $r = n+s+2, \dots, 2m+s$  because  $H$  is parallel to  $e_{n+s+1}$ .

Now, for  $i \neq j$ ,  $i, j = 1, \dots, n+s$ , we define

$$u_{ij} = a_i a_j = \sigma_{ii}^{n+s+1} \sigma_{jj}^{n+s+1} = u_{ji}$$

and from (2.5), (2.8) and by using (4.1), we obtain, for  $X = Z = e_i$ ,  $Y = W = e_j$ :

$$u_{ij} \geq b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta - \sum_{r=n+s+2}^{2m+s} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \quad (4.3)$$

Taking into account these formulas, we can prove the following technical lemma for  $u_{ij}$ ,  $i, j = 1, \dots, n+s$ ,  $i \neq j$ .

**Lemma 4.1.** (i) For any fixed  $i \in \{1, \dots, n+s\}$ :

$$\sum_{i \neq j} u_{ij} \geq (n+s-1) \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right) > 0.$$

(ii)  $u_{ij} \neq 0$ .

(iii) For distinct  $i, j, k \in \{1, \dots, n+s\}$ :

$$a_i^2 = \frac{u_{ij}u_{ik}}{u_{jk}}.$$

(iv) For a fixed  $k$ ,  $1 \leq k \leq \left\lceil \frac{n+s}{2} \right\rceil$ :

$$\sum_{i=1}^k \sum_{j=k+1}^{n+s} u_{ij} \geq k(n-k+s) \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right).$$

(v)  $u_{ij} > 0$ .

*Proof.* First, from (4.2) and (4.3), a straightforward computation gives (i). Now, if  $u_{ij} = 0$ , then  $a_i = 0$  or  $a_j = 0$ . If  $a_i = 0$ , we have  $u_{ik} = 0$ , for any  $k \in \{1, \dots, n+s\}$ ,  $k \neq i$ . Thus,

$$\sum_{j \neq i} u_{ij} = 0,$$

which contradicts to (i). Now, (iii) is direct from the definition of  $u_{ij}$ ,  $u_{ik}$  and  $u_{jk}$ . Again, a straightforward computation using (4.2) and (4.3) gives (iv). Finally, if we suppose that  $u_{1(n+s)} < 0$ , from (iii) we get  $u_{1i}u_{i(n+s)} < 0$ , for  $1 < i < n+s$ . Without loss of generality, we may assume

$$u_{12}, \dots, u_{1k}, u_{(k+1)(n+s)}, \dots, u_{(n+s-1)(n+s)} > 0,$$

$$u_{1(k+1)}, \dots, u_{1(n+s-1)}, u_{2(n+s)}, \dots, u_{k(n+s)} < 0,$$

for some  $k$  such that:

$$\left\lceil \frac{n+s-1}{2} + 1 \right\rceil \leq k \leq n+s-1.$$

If  $k = n+s-1$ , then  $u_{1(n+s)} + u_{2(n+s)} + \dots + u_{(n+s-1)(n+s)} < 0$ , which contradicts (i). Consequently,  $k < n+s-1$ . From (iii), we obtain

$$a_{n+s}^2 = \frac{u_{i(n+s)}u_{t(n+s)}}{u_{it}} > 0,$$

where  $2 \leq i \leq k$ ,  $k+1 \leq t \leq n+s-1$  and so,  $u_{it} < 0$ . This implies that

$$\sum_{i=1}^k \sum_{t=k+1}^{n+s} u_{it} = \sum_{i=2}^k \sum_{t=k+1}^{n+s-1} u_{it} + \sum_{i=1}^k u_{i(n+s)} + \sum_{t=k+1}^{n+s} u_{1t} < 0,$$

which contradicts to (iv) and completes the proof.  $\square$

In the following theorem, we establish a relationship between the shape operator associated with the mean curvature vector and the sectional curvature for slant submanifolds in an  $S$ -space-form.

**Theorem 4.1.** *Let  $M$  be an  $(n+s)$ -dimensional  $\theta$ -slant submanifold isometrically immersed in a  $(2m+s)$ -dimensional  $S$ -pace-form  $\widetilde{M}(c)$ . If at a point  $p \in M$  there exists a number*

$$b > \frac{c+3s}{4} + \frac{3(c-s)}{4} \cos^2 \theta$$

such that the sectional curvature of  $M$ ,  $K \geq b$  at  $p$ , then the shape operator associated with the mean curvature vector satisfies

$$A_H > \frac{n+s-1}{n+s} \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right) I_{n+s},$$

at  $p$ , where  $I_{n+s}$  is the identity map.

*Proof.* Let  $p \in M$  and a number

$$b > \frac{c+3s}{4} + \frac{3(c-s)}{4} \cos^2 \theta$$

such that the sectional curvature  $K \geq b$  at  $p$ . We choose an orthonormal basis of  $T_p(\widetilde{M})$  as in the beginning of the section. Then, from the above lemma, we observe that  $a_1, \dots, a_{n+s}$  have the same sign. We assume that  $a_j > 0$ , for any  $j \in \{1, \dots, n+s\}$ . Thus:

$$\sum_{j \neq i} u_{ij} = a_i(a_1 + \dots + a_{n+s}) - a_i^2 \geq (n+s-1) \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right). \quad (4.4)$$

From (4.1) and (4.4), we get, for any  $i = 1, \dots, n+s$ :

$$\begin{aligned} a_i(n+s)|H| &\geq (n+s-1) \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right) + a_i^2 > \\ &> (n+s-1) \left( b - \frac{c+3s}{4} - \frac{3(c-s)}{4} \cos^2 \theta \right). \end{aligned}$$

This completes the proof.  $\square$

For the particular cases of invariant and anti-invariant submanifolds (cases  $\theta = 0$  and  $\theta = \pi/2$ , respectively), we have the following two theorems.

**Theorem 4.2.** *Let  $M$  be an  $(n + s)$ -dimensional invariant submanifold isometrically immersed in a  $(2m + s)$ -dimensional  $S$ -space-form  $\widetilde{M}(c)$  tangent to the structure vector fields. If at a point  $p \in M$  there exists a number  $b > c$  such that the sectional curvature of  $M$ ,  $K \geq b$  at  $p$ , then the shape operator associated with the mean curvature vector satisfies at  $p$ :*

$$A_H > \frac{n + s - 1}{n + s}(b - c)I_{n+s}.$$

**Theorem 4.3.** *Let  $M$  be an  $(n + s)$ -dimensional anti-invariant submanifold isometrically immersed in a  $(2m + s)$ -dimensional  $S$ -space-form  $\widetilde{M}(c)$  tangent to the structure vector fields. If at a point  $p \in M$  there exists a number*

$$b > \frac{c + 3s}{4}$$

*such that the sectional curvature of  $M$ ,  $K \geq b$  at  $p$ , then the shape operator associated with the mean curvature vector satisfies at  $p$ :*

$$A_H > \frac{n + s - 1}{n + s} \left( b - \frac{c + 3s}{4} \right) I_{n+s}.$$

These results should be compared with those ones proved in [13] for Sasakian space-forms (case  $s = 1$ ).

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