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SEMI-KOLMOGOROV MODELS FOR PREDATION WITH INDIRECT EFFECTS IN RANDOM ENVIRONMENTS

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Abstract. In this work we study semi-Kolmogorov models for predation with both the carrying capacities and the indirect effects varying with respect to randomly fluctuating environments. In particular, we consider one random semi-Kolmogorov system involving random and essentially bounded parameters, and one stochastic semi-Kolmogorov system involving white noise and stochastic parameters defined upon a continuous-time Markov chain. For both systems we investigate the existence and uniqueness of solutions, as well as positiveness and boundedness of solutions. For the random semi-Kolmogorov system we also obtain sufficient conditions for the existence of a global random attractor.

1. **Introduction.**

Kolmogorov's predator-prey model refers to the general model describing dynamics of interacting populations:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= xf(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= yg(x,y),\end{aligned}$$

where f and g denote the respective per capita growth rates of the two species satisfying $\partial_y f(x,y) < 0$ and $\partial_x g(x,y) > 0$, respectively. One simple example of Kolmogorov-type system is the following plakton food web model

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}t} &= Z(-\nu_z + c_3 P),\\ \frac{\mathrm{d}P}{\mathrm{d}t} &= P(c_1 I_0 - c_2 Z - \nu_p P),\end{aligned}$$

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where I_0 is the amount of nutrient, Z is the nutrient content of zooplankton, P is the nutrient content of phytoplankton, ν_z and ν_p are the morality rates of the zooplankton and phytoplankton, respectively, c_1 is the phytoplankton's nutrient yield from consuming resources, c_2 accounts for losses in phytoplankton population due to zooplankton consumption, and c_3 is the zooplankton's nutrient yield from consuming phytoplankton. The study of phytoplankton predator-prey systems is fundamental to mathematical biology and ecology, due to their simple experimental structure and their wide applications to aquatic food chains, water quality control, regulating carbon dioxide uptake, etc.

Usually when there are two groups of preys with different sizes, the predator prefers to predate the preys in the group with smaller size and the other group with larger size takes advantages of it. This results in predator-prey systems with indirect effects. More precisely, indirect effect refers to species interactions which can occur through chains of direct species interaction, such as predation or interference competition. The studies of indirect effects are of great importance to the biology and ecology communities, as they can link the population dynamics of species that do not interact directly (see [4, 5, 6, 17, 27, 31, 32] and references therein).

Denote by $P_1(t)$ and $P_2(t)$ the nutrient contents of two groups of phytoplankon with different sizes, and index all the parameters according to their group. Taking into account the indirect effects, we obtain the following plankton food web model (see e.g. [12],[13]):

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = Z(-\nu_z + c_{3,1}P_1 + c_{3,2}P_2), \tag{1}$$

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = P_1[c_{1,1}I_0 - (c_{2,1} + c_{3,1})Z - \nu_{p,1}P_1 - \nu_{p,1}P_2] - m_1P_1Z, \tag{2}$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = P_2[c_{1,2}I_0 - (c_{2,2} + c_{3,2})Z - \nu_{p,2}P_1 - \nu_{p,2}P_2] + m_2P_1Z, \tag{3}$$

where the terms m_1P_1Z and m_2P_1Z describe the indirect effect caused by the zooplankton preferring to prey the smaller group of phytoplankton (P_1) over the larger group of phytoplankton (P_2).

Notice that system (1) - (3) is not exactly of Kolmogorov type because of the structure of equation (3). We thus classify it as a semi-Kolmogorov system. It has been pointed out in various occasions that the presence of indirect effects serves as a regulator of predator-prey systems (see, e.g., [12]) and helps coexistence. This motivates the study of the following general semi-Kolmogorov system for predation with indirect effects:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)[-b_1 + a_{12}y(t) + a_{13}z(t)], \tag{4}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)[b_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)] - m_1x(t)y(t), \tag{5}$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = z(t)[b_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)] + m_2x(t)y(t), \tag{6}$$

where all the parameters $a_{i,j}(i, j = 1, 2, 3)$, $b_j(j = 1, 2, 3)$, $m_j(j = 1, 2)$ are nonnegative.

It has been well understood that the carrying capacities b_j ($j = 1, 2, 3$) are often subject to environmental noise. In [8], the authors studied a stochastic semi-Kolmogorov system obtained from perturbing each b_j by a white noise, and a random semi-Kolmogorov system obtained from making the parameters b_j ($j = 1, 2, 3$) random and essentially bounded. Conditions for the existence of a random attractor for each system have been investigated. On the other hand, indirect effects are also subject to environmental fluctuations, since the changes in nutrition resources resulted from changes in the environment can affect the behavior of the populations. A nonautonomous semi-Kolmogorov model was introduced and analyzed in [7], where periodic forcing was considered to describe seasonal changes of the environment. The nonautonomous model is capable to describe complex and even chaotic oscillations as described in the biological literature (see for example [19]). In particular the authors proved the existence of a pullback attractor and provided an estimate of its dimension by numerical experiments.

In this work we will study the semi-Kolmogorov system in randomly fluctuating environments, with both the carrying capacities (represented by b_j, $i = 1, 2, 3$) and the indirect effects (represented by m_j, $i = 1, 2$) varying with respect to the environments. In particular, we will consider two systems: (1) A random semi-Kolmogorov system which consists of random differential equations resulting from making the parameters b_j's and m_j's random and essentially bounded, i.e., perturbed by real noise. (2) A stochastic semi-Kolmogorov system which consists of stochastic differential equations with regime switching resulting from modeling the random environments by a continuous-time Markov chain. For both systems we will investigate the existence and uniqueness of solutions, as well as positiveness and boundedness of solutions. For the random semi-Kolmogorov system we also obtain sufficient conditions for the existence of a global random attractor. Note that studying these two systems requires two different set of techniques, one based on the theory of random dynamical systems, and the other based on the theory of stochastic analysis. The rest of this paper is organized as follows. In Section 2 we will set up and analyze the random semi-Kolmogorov system, in Section 3 we will set up and analyze the stochastic semi-Kolmogorov system and in Section 4 we will provide some closing remarks.

2. **A random semi-Kolmogorov model.** In this section we will consider a random semi-Kolmogorov model with random carrying capacities and random indirect effects, in the sense that the parameters b_j's and m_j's are both perturbed by random environmental influences modeled by the paths of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precisely we will study the following system of random differential equations, or, in another words, a system of non-autonomous differential equations with random parameters:

$$\frac{\mathrm{d}x(t,\omega)}{\mathrm{d}t} = x(t)[-b_1(\theta_t\omega) + a_{12}y(t) + a_{13}z(t)], \tag{7}$$

$$\frac{\mathrm{d}y(t,\omega)}{\mathrm{d}t} = y(t)[b_2(\theta_t\omega) - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)] - m_1(\theta_t\omega)x(t)y(t), \tag{8}$$

$$\frac{\mathrm{d}z(t,\omega)}{\mathrm{d}t} = z(t)[b_3(\theta_t\omega) - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)] + m_2(\theta_t\omega)x(t)y(t), \tag{9}$$

where $b_j(\theta_t\omega)$, $(j = 1,2,3)$ and $m_j(\theta_t\omega)$, $(j = 1,2)$ are continuous and essentially bounded, i.e.,

$$b_j(\theta_t\omega) \in B_j \cdot [1-\sigma_j, 1+\sigma_j], \quad B_j > 0, \quad 0 < \sigma_j < 1, \quad j = 1,2,3,$$
$$m_j(\theta_t\omega) \in M_j \cdot [1-\epsilon_j, 1+\epsilon_j], \quad M_j > 0, \quad 0 < \epsilon_j < 1, \quad j = 1,2.$$

Bounded noise can be modeled in various ways (see [2]). For example, given a (driving) stochastic process Z_t such as an Ornstein-Uhlenbeck process, the stochastic process $\zeta(Z_t) := \zeta_0\Big(1 - 2\sigma\frac{Z_t}{1+Z_t^2}\Big)$, where ζ_0 and σ are positive constants with $\sigma \in (0,1)$, takes values in the interval $\zeta_0[1-\sigma, 1+\sigma]$ and tends to peak around $\zeta_0(1\pm\sigma)$. It is thus suitable for a noisy switching scenario. In the theory of random dynamical systems the driving noise process $Z_t(\omega)$ is replaced by a canonical driving system $\theta_t\omega$. This simplification allows a better understanding of the path-wise approach to model noise: a system influenced by stochastic processes for each single realization ω can be interpreted as wandering along a path $\theta_t\omega$ in Ω and thus may provide additional statistical/geological information to the modeler.

2.1. **Preliminaries.** In this subsection we first present some concepts (from [1]) related to general random dynamical systems (RDSs) and random attractors that we require in the sequel.

Let $(X, \|\cdot\|_X)$ be a separable Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\mathcal{F}$ is the $\sigma-$algebra of measurable subsets of Ω (called "events") and $\mathbb{P}$ is the probability measure. To connect the state ω in the probability space Ω at time 0 with its state after a time of t elapses, we define a flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on Ω with each θ_t being a mapping $\theta_t : \Omega \to \Omega$ that satisfies

(1) $\theta_0 = \mathrm{Id}_\Omega$,
(2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,
(3) the mapping $(t,\omega) \mapsto \theta_t\omega$ is measurable and
(4) the probability measure $\mathbb{P}$ is preserved by θ_t, i.e., $\theta_t\mathbb{P} = \mathbb{P}$.

This set-up establishes a time-dependent family θ that tracks the noise, and $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a *metric dynamical system* [1].

Definition 2.1. A stochastic process $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ is said to be a continuous random dynamical system (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space X if $\varphi : [0,+\infty) \times \Omega \times X \to X$ is $(\mathcal{B}[0,+\infty) \times \mathcal{F} \times \mathcal{B}(X),\ \mathcal{B}(X))$–measurable, and for each $\omega \in \Omega$,

(i) the mapping $\varphi(t,\omega) : X \to X$, $x \mapsto \varphi(t,\omega)x$ is continuous for every $t \geq 0$;
(ii) $\varphi(0,\omega)$ is the identity operator on X;
(iii) (cocycle property) $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\varphi(s,\omega)$ for all $s,t \geq 0$.

Definition 2.2. (i) A random set K is a measurable subset of $X\times\Omega$ with respect to the product $\sigma-$algebra $\mathcal{B}(X) \times \mathcal{F}$.
The $\omega-$section of a random set K is defined by

$$K(\omega) = \{x : (x,\omega) \in K\}, \quad \omega \in \Omega.$$

When a set $K \subset X \times \Omega$ possesses closed or compact $\omega-$sections, then it is a random set provided that the mapping $\omega \mapsto d(x, K(\omega))$ is measurable for every $x \in X$, see [14]. Then K will be said to be a closed or compact random set, respectively.

(ii) A random set $K(\omega)$ is said to be bounded if $K(\omega)$ is bounded for a.e. $\omega \in \Omega$.

(iii) A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{x\in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{t\in\mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

In what follows we use $\mathcal{D}(X)$ to denote the set of all tempered random sets of X.

Definition 2.3. A random set $\Gamma(\omega) \subset X$ is called a random absorbing set in $\mathcal{D}(X)$ if for any $K \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$, there exists $T_K(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega) \subset \Gamma(\omega), \quad \forall t \geq T_K(\omega).$$

Definition 2.4. Let $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space X and let $\mathcal{A}(\omega)(\subset X)$ be a random set. Then $\mathcal{A}(\omega)$ is called a global random $\mathcal{D}$ attractor (or pullback $\mathcal{D}$ attractor) for $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ if $\omega \mapsto \mathcal{A}(\omega)$ satisfies

(i) (random compactness) $\mathcal{A}(\omega)$ is a compact set of X for a.e. $\omega \in \Omega$;

(ii) (invariance) for a.e. $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega);$$

(iii) (attracting property) for any $K \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \text{dist}_X(\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega), A(\omega)) = 0,$$

where

$$\text{dist}_X(G,H) = \sup_{g\in G} \inf_{h\in H} \|g-h\|_X$$

is the Hausdorff semi-metric for $G, H \subseteq X$.

Proposition 1. [10, 18] *Let $\Gamma \in \mathcal{D}(X)$ be an absorbing set for the continuous random dynamical system $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ which is closed and satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega)\Gamma(\theta_{-t_n}\omega)$ has a convergent subsequence in X when $t_n \to \infty$. Then the cocycle φ has a unique global random attractor with component subsets*

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq T_\Gamma(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega)\Gamma(\theta_{-t}\omega)}.$$

If the pullback absorbing set is positively invariant, i.e., $\varphi(t,\omega)\Gamma(\omega) \subset \Gamma(\theta_t\omega)$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} \overline{\varphi(t,\theta_{-t}\omega)\Gamma(\theta_{-t}\omega)}.$$

Remark 1. When the state space $X = \mathbb{R}^d$ as in this paper, the asymptotic compactness follows trivially. Note that the random attractor is path-wise attracting in the pullback sense, but does not need to be path-wise attracting in the forward sense, although it is forward attracting in probability, due to some possible large deviations, see e.g., Arnold [1].

2.2. **Properties of solutions.** In this subsection we will prove the existence and uniqueness of solutions, as well as positiveness and boundedness of solutions, to system (7) – (9). For conveninice, denote by $\mathbf{u}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$.

Theorem 2.5. *For any $\omega \in \Omega$, $t_0 \in \mathbb{R}$ and $\boldsymbol{u}_0 := (x(t_0), y(t_0), z(t_0)) \in \mathbb{R}^3$, the system (7)-(9) admits a unique nonnegative and bounded solution $\boldsymbol{u}(\cdot; t_0, \omega, \boldsymbol{u}_0) \in C([t_0, \infty), \mathbb{R}^3_+)$ with $\boldsymbol{u}(t_0; t_0, \omega, \boldsymbol{u}_0) = \boldsymbol{u}_0$ provided that*

$$M_1(1-\epsilon_1) - M_2(1+\epsilon_2) + a_{21} - a_{12} > 0, \qquad \text{and} \qquad a_{31} - a_{13} > 0. \tag{10}$$

Moreover the solution generates a random dynamical system $\varphi(t,\omega)(\cdot)$ defined as

$$\varphi(t,\omega)\boldsymbol{u}_0 = \boldsymbol{u}(t; 0, \omega, \boldsymbol{u}_0), \quad \forall t \geq t_0, \boldsymbol{u}_0 \in \mathbb{R}^3_+,\ \omega \in \Omega.$$

Proof. Using the identification $\mathbf{u}(t) = (x(t), y(t), z(t))$, the system (7)-(9) can be written as

$$\dot{\mathbf{u}}(t) = A(\theta_t\omega) \cdot \mathbf{u} + \mathbf{f}(\theta_t\omega, \mathbf{u}) + \mathbf{g}(\mathbf{u}),$$

where

$$A(\theta_t\omega) = \begin{pmatrix} -b_1(\theta_t\omega) & 0 & 0 \\ 0 & b_2(\theta_t\omega) & 0 \\ 0 & 0 & b_3(\theta_t\omega) \end{pmatrix},$$

and $\mathbf{f} : \Omega \times \mathbb{R}^3_+ \to \mathbb{R}^3$ and $\mathbf{g} : \mathbb{R}^3_+ \to \mathbb{R}^3$ are given by

$$\mathbf{f}(\theta_t\omega, \mathbf{u}) = \begin{pmatrix} 0 \\ -m_1(\theta_t\omega)xy \\ m_2(\theta_t\omega)xy \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} a_{12}xy + a_{13}xz \\ -a_{21}xy - a_{22}y^2 - a_{23}yz \\ -a_{31}xz - a_{32}yz - a_{33}z^2 \end{pmatrix}.$$

Since each $b_j(\theta_t\omega)$ is bounded, the operator A generates an evolution system on $\mathbb{R}^3$. Moreover, due to the boundedness of $m_i(\theta_t\omega)$, the function $\mathbf{f}$ is locally Lipschitz and the function $\mathbf{g}$ is continuously differentiable in $\mathbb{R}^3$. Hence the system (7)-(9) possesses a unique local solution.

To prove the positiveness of solution we observe that each solution has to take value 0 before it reaches a negative value. Then since the plane $x = 0$ and $y = 0$ are invariant and on the plane $z = 0$ the vector field is tangent or point inward $\mathbb{R}^3_+$, we conclude that the set $\mathbb{R}^3_+$ is positively invariant. Then for any $\mathbf{u}_0 \in \mathbb{R}^3_+$, the solution $\mathbf{u}(\cdot; \omega, \mathbf{u}_0) \in \mathbb{R}^3_+$ for $t \in [0, \infty)$.

In order to prove the boundedness of solutions we consider the norm

$$\|\mathbf{u}(t)\|_1 := S(t) = x(t) + y(t) + z(t).$$

Then by equations (7), (8) and (9) we have

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= -b_1(\theta_t\omega)x + b_2(\theta_t\omega)y + b_3(\theta_t\omega)z - a_{22}y^2 - a_{33}z^2 - (a_{23} + a_{32})yz \\ &\quad -[m_1(\theta_t\omega) - m_2(\theta_t\omega) + a_{21} - a_{12}]xy - (a_{31} - a_{13})xz \\ &\leq -b_1(1-\sigma_1)x + b_2(1+\sigma_2)y + b_3(1+\sigma_3)z - a(y+z)(x+y+z) \\ &\leq (y+z)[b - aS(t)], \end{aligned} \tag{11}$$

where

$$\begin{aligned} a &:= \min\{a_{22}, a_{33}, a_{23} + a_{32}, \\ &\qquad M_1(1-\epsilon_1) - M_2(1+\epsilon_2) + a_{21} - a_{12}, a_{31} - a_{13}\}, \qquad (12) \\ b &:= \max\{b_2(1+\sigma_2), b_3(1+\sigma_3)\}. \qquad (13) \end{aligned}$$

(1) If $S(t_0) \in \{S(t) : S(t) \geq \frac{b}{a}\}$ then $S(t)$ will be non-increasing.
(2) If $S(t)$ enters the complementary region $\{S(t) : S(t) < \frac{b}{a}\}$ at some time $t_1 \in \mathbb{R}$, then

$$y + z \leq S(t) < \frac{b}{a}, \quad \text{for all } t \geq t_1.$$

And hence

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} < \frac{b}{a} \cdot (b - aS(t)), \quad \text{for all } t \geq t_1. \tag{14}$$

These imply that

$$S(t) \leq \frac{b}{a} + S(t_1)e^{bt_1} \text{ for all } t \geq t_1,$$

and therefore $\|\mathbf{u}(t)\|_1$ is bounded and the local solution can be extended to a global solution $\mathbf{u}(\cdot; t_0, \omega, \mathbf{u}_0) \in C^1([t_0, \infty), \mathbb{R}^3)$.

It is straightforward to check the cocycle property

$$\mathbf{u}(t + t_0; t_0, \omega, v_0) = \mathbf{u}(t; 0, \theta_{t_0}\omega, \mathbf{u}_0)$$

for all $t_0 \in \mathbb{R}, t \geq t_0, \omega \in \Omega, \mathbf{u}_0 \in \mathbb{R}^3_+$. This allows us to define a mapping $\varphi(t, \omega)(\cdot)$, which will be our random dynamical system, as

$$\varphi(t, \omega)\mathbf{u}_0 = \mathbf{u}(t; 0, \omega, \mathbf{u}_0), \quad \forall t \geq 0, \ \mathbf{u}_0 \in \mathbb{R}^3_+, \ \omega \in \Omega. \tag{15}$$

Since the functions $\mathbf{f}(\theta_t\omega, \mathbf{u})$ and $\mathbf{g}(\mathbf{u})$ are continuous in $\mathbf{u}$, the mapping $\mathbf{u} : [0, \infty) \times \Omega \times \mathbb{R}^3_+ \to \mathbb{R}^3_+$, defined by $(t; \omega, \mathbf{u}_0) \mapsto \mathbf{u}(t; \omega, \mathbf{u}_0)$ is $(\mathcal{B}[0, \infty) \times \mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^3_+), \mathcal{B}(\mathbb{R}^3_+))$-measurable. It then follows directly that (7)–(9) generate a continuous random dynamical system $\varphi(t, \omega)(\cdot)$ defined by (15).

□

From now on, we will simply write $\mathbf{u}(t; \omega, \mathbf{u}_0)$ instead of $\mathbf{u}(t; 0, \omega, \mathbf{u}_0)$. Also in what follows, when $\omega \in \Omega$ fixed, we will not mention explicitly the random parameter and will write $\mathbf{u}(t; \omega, v_0)$ as $\mathbf{u}(t)$ in short.

2.3. **Existence of global random attractors.** The main goal of this subsection is to prove the existence of a random attractor for the random dynamical system (RDS) $\varphi(t, \omega)(\cdot)$ generated by the solution to system (7) – (9). To this end, we first prove in the following lemma that the RDS $\varphi(t, \omega)(\cdot)$ has a tempered random bounded absorbing set.

Lemma 2.6. *Assume that*

$$M_1(1 - \epsilon_1) - M_2(1 + \epsilon_2) + a_{21} - a_{12} > 0 \qquad \text{and} \qquad a_{31} - a_{13} > 0.$$

Then for each $\omega \in \Omega$, there exists a tempered bounded closed random absorbing set $\mathcal{B}(\omega) \in \mathcal{D}(\mathbb{R}^3_+)$ of the random dynamical system $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ such that for any $K \in \mathcal{D}(\mathbb{R}^3_+)$ and each $\omega \in \Omega$, there exists $T_K(\omega) > 0$ yielding

$$\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega) \subset \mathcal{B}(\omega), \quad \forall t \geq T_K(\omega).$$

More precisely, for a given $\delta > 0$, the set $\mathcal{B}(\omega)$ can be chosen as the deterministic set

$$\mathcal{B}_\delta := \left\{ (x, y, z) \in \mathbb{R}^3_+ : x + y + z \leq \frac{b}{a} + \delta \right\}$$

for all $\omega \in \Omega$, i.e. $\mathcal{B}(\omega) = \mathcal{B}_\delta$ for all $\omega \in \Omega$ and where a and b are defined as in (12) and (13) respectively.

Proof. For any $\delta > 0$, we consider the set $\mathcal{B}_\delta$ as defined above. We start by proving that $\mathcal{B}_\delta$ is invariant.

(a) For any solution of (7)–(9) starting from a point inside the set

$$\mathcal{B}_0 := \left\{(x, y, z) \in \mathbb{R}^3_+ : x + y + z \le b/a\right\},$$

using that $\dot{S}(t) \le 0$ on $x + y + z = b/a$, inequality (14) and the positive invariance of $\mathbb{R}^3_+$ we have that $S(t) \le b/a$ for all $t \ge t_0$. This means that $\mathcal{B}_0$ is positively invariant.

Moreover, using inequality (14), for any $S(t) \in \mathcal{B}_0$ we have

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} \le S(t)(b - aS(t)).$$

Then, integrating the Bernoulli type inequality (2.3), we deduce

$$S(t) \le \frac{bS(t_0)}{aS(t_0) + (b - aS(t_0))e^{-b(t-t_0)}},$$

which implies that

$$\lim_{t\to\infty} S(t) \le \frac{b}{a}, \quad \text{and} \quad \lim_{t_0\to-\infty} S(t) \le \frac{b}{a}$$

(b) If a solution starts inside the set $\mathcal{B}_\delta \setminus \mathcal{B}_0$, then inequality (2.3) ensures that $\dot{S}(t) \le 0$ and, as a consequence, the solution cannot leave the set $\mathcal{B}_\delta$.

From the arguments (a) and (b) we conclude that the set $\mathcal{B}_\delta$ is positively invariant for any $\delta \ge 0$.

(c) It remains to study the case $S(t) \ge b/a + \delta$ since we have already proved that $\mathcal{B}_\delta$ is positively invariant. Similar to inequality (11) we can write

$$\begin{aligned}\dot{S}(t) \quad &\le \quad -b_1(1-\sigma_1)x + (y+z)[b - aS(t)] \le -b_1(1-\sigma_1)x - a\delta(y+z)\\ &\le \quad -\min\{b_1(1-\sigma_1), a\delta\} \cdot S(t).\end{aligned}$$

Integrating the previous inequality, and making the dependence on the random parameter explicit (for better representation of subsequent computations), we obtain

$$S(t, \omega) \le S_0(\omega)e^{-\min\{b_1(1-\sigma_1), a\delta\}\cdot(t-t_0)}, \tag{16}$$

where $S_0(\omega) := S(t_0, \omega)$. Replacing ω by $\theta_{-t}\omega$ in (16) gives

$$S(t; \theta_{-t}\omega, S_0(\theta_{-t}\omega)) \le \sup_{v\in K(\theta_{-t}\omega)} \|v\| \cdot e^{-\min\{b_1(1-\sigma_1), a\delta\}\cdot(t-t_0)}.$$

If we denote the solution of system (7)–(9) satisfying $\mathbf{u}(0; \omega, \mathbf{u}_0) = \mathbf{u}_0$ as $\mathbf{u}(t; \omega, \mathbf{u}_0) = \varphi(t, \omega)\mathbf{u}_0$, then for any $\mathbf{u}_0 := \mathbf{u}_0(\theta_{-t}\omega) \in K(\theta_{-t}\omega)$, we can write

$$\|\varphi(t, \theta_{-t}\omega)\mathbf{u}_0\|_1 = \|\mathbf{u}(t; \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega)\|_1 \le S(t; \theta_{-t}\omega, S_0(\theta_{-t}\omega)).$$

Using the three steps above we deduce the existence of a time $T_K(\omega)$ such that when $t > T_K$, $\varphi(t, \theta_{-t}\omega)\mathbf{u}_0 \in \mathcal{B}_\delta$ for all $\mathbf{u}_0 \in K(\theta_{-t}\omega)$, i.e., $\mathcal{B}_\delta$ is a compact absorbing set for any $\delta > 0$ and absorbs all tempered random sets of $\mathbb{R}^3_+$, in particular all bounded sets of $\mathbb{R}^3_+$. □

Finally by Proposition 1, Lemma 2.6, and Remark 1 we obtain immediately the following theorem:

Theorem 2.7. *The random dynamical system generated by system* (7)–(9) *possesses a global random attractor provided that*

$$M_1(1-\epsilon_1) - M_2(1+\epsilon_2) + a_{21} - a_{12} > 0 \qquad \textit{and} \qquad a_{31} - a_{13} > 0.$$

3. **A stochastic semi-Kolmogorov model.** In this section we will introduce a stochastic system for the semi-Kolmogorov model in random environments. This stochastic system is set up based on the assumption that the random environments can be modeled by a Markov chain. In fact, it has been observed that the switching between different environments is memoryless and the waiting time for the next switch is exponentially distributed (see [11], [20], [35]). Hence it is sensible to model the random environments and other random factors by a continuous-time Markov chain $\alpha(t)$, $t \geq 0$, that takes values in a finite state space $\mathcal{S} = \{1, 2, \dots, m\}$.

Stochastic ecological population models in Markovian environments have been studied extensively lately (see, e.g., [3, 15, 16, 22, 23, 29, 33, 34, 35] and references therein). Yet all the studies up to date are on Kolmogorov type systems such as Lotka-Volterra systems. In this section we will investigate a 3-dimensional stochastic semi-Kolmogorov model, in the sense that two equations are in the Kolmogorov format but one equation is not. This allows the consideration of important indirect effects, but at the same time increases the complexity of the analysis involved.

Let the Markov chain $\alpha(\cdot)$ be generated by the *transition rate matrix* Q, whose elements, q_{ij}, are the transition rates that represent the derivatives with respect to time of the transition probabilities between states i and j in $\mathcal{S}$. Mathematically we have

$$\text{Prob}\,(\alpha(t+\Delta t) = j | \alpha(t) = i) = \delta_{ij} + q_{ij}\Delta t + o(\Delta t),$$

where δ_{ij} is the Kronecher Symbol, $q_{ij} \geq 0$ for $i, j = 1, 2, \cdots, m$ with $i \neq j$, and the values q_{ii} are such that the rows of Q have sum 0, i.e.,

$$\sum_{j=1}^{m} q_{ij} = 0, \qquad \text{for any } i \in \mathcal{S}.$$

In this work we will focus on effects of Markovian environments on carrying capacities b_j's and indirect effects m_j's, i.e., we assume all the direct competition coefficients a_{ij}'s are constant. The resulting stochastic semi-Kolmogorov model in Markovian environments can be described by the following system of stochastic differential equations with regime switching:

$$\begin{aligned} \mathrm{d}x(t) &= x(t)[-b_1(\alpha(t)) + a_{12}y(t) + a_{13})z(t)]\mathrm{d}t \\ &\quad + \mu_1(\alpha(t))x(t) \circ \mathrm{d}w_1(t), & (17) \\ \mathrm{d}y(t) &= y(t)[b_2(\alpha(t)) - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)]dt \\ &\quad - m_1(\alpha(t))x(t)y(t)\mathrm{d}t + \mu_2(\alpha(t))y(t) \circ \mathrm{d}w_2(t), & (18) \\ \mathrm{d}z(t) &= z(t)[b_3(\alpha(t)) - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)]\mathrm{d}t \\ &\quad + m_2(\alpha(t))x(t)y(t)\mathrm{d}t + \mu_3(\alpha(t))z(t) \circ \mathrm{d}w_3(t), & (19) \end{aligned}$$

where $w(\cdot) = (w_1(\cdot), w_2(\cdot), w_3(\cdot))$ is a three dimensional standard Brownian Motion. Here we assume that both the Markov chain $\alpha(\cdot)$ and the Brownian motion $w(\cdot)$ are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that $\alpha(\cdot)$ and $w(\cdot)$ are independent. Without loss of generality, we also assume that the initial conditions

$\alpha(0)$, $(x(0), y(0), z(0))$ are deterministic. Furthermore, throughout this section we assume that the direct competition among same species are strictly positive, i.e.,

$$a_{ii} > 0, \quad a_{ij} \geq 0, \qquad i, j = 1, 2, 3. \tag{20}$$

3.1. **Preliminaries.** Notice that system (17) – (19) is equivalent to the following stochastic differential system in the Itô sense (for simplicity of notation, we write $\alpha(t)$ as α in short):

$$\begin{aligned}
\mathrm{d}x(t) &= x(t)[-b_1(\alpha) + \frac{1}{2}\mu_1^2(\alpha) + a_{12}y(t) + a_{13}z(t)]\mathrm{d}t \\
&\quad +\mu_1(\alpha)x(t)\mathrm{d}w_1(t), & (21)\\
\mathrm{d}y(t) &= y(t)\left[b_2(\alpha) + \frac{1}{2}\mu_2^2(\alpha) - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)\right]\mathrm{d}t \\
&\quad -m_1(\alpha)x(t)y(t)\mathrm{d}t + \mu_2(\alpha)y(t)\mathrm{d}w_2(t), & (22)\\
\mathrm{d}z(t) &= z(t)\left[b_3(\alpha) + \frac{1}{2}\mu_3^2(\alpha) - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)\right]\mathrm{d}t \\
&\quad +m_2(\alpha)x(t)y(t)\mathrm{d}t + \mu_3(\alpha)z(t)\mathrm{d}w_3(t). & (23)
\end{aligned}$$

Denote by $\mathbf{u}(t) = (x(t), y(t), z(t))$, $\mathbf{w}(t) = (w_1(t), w_2(t), w_3(t))$,

$$\begin{aligned}
\mathbf{f}(\mathbf{u}, \alpha) &= \begin{pmatrix} x[-b_1(\alpha) + \frac{1}{2}\mu_1^2(\alpha) + a_{12}y + a_{13}z] \\ y\left[b_2(\alpha) + \frac{1}{2}\mu_2^2(\alpha) - a_{21}x - a_{22}y - a_{23}z\right] - m_1(\alpha)xy \\ z\left[b_3(\alpha) + \frac{1}{2}\mu_3^2(\alpha) - a_{31}x - a_{32}y - a_{33}z\right] + m_2(\alpha)xy \end{pmatrix} \\
&:= (f_1(\mathbf{u}, \alpha), f_2(\mathbf{u}, \alpha), f_3(\mathbf{u}, \alpha)), & (24)
\end{aligned}$$

and

$$F(\mathbf{u}, \alpha) = \begin{pmatrix} \mu_1(\alpha)x & 0 & 0 \\ 0 & \mu_2(\alpha)y & 0 \\ 0 & 0 & \mu_3(\alpha)z \end{pmatrix}.$$

Then system (21)–(23) can be written in a simpler form as

$$d\mathbf{u}(t) = \mathbf{f}(\mathbf{u}, \alpha)\mathrm{d}t + F(\mathbf{u}, \alpha)\mathrm{d}\mathbf{w}(t). \tag{25}$$

For future convenience we introduce the following generalized Itô's Formula (see [26]).

Proposition 2. *For any function $V : \mathbb{R}^3 \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}$ such that $V(\cdot, t, \alpha)$ is twice continuously differentiable with respect to variable $\boldsymbol{u}$ for each $\alpha \in \mathcal{S}$ and $t \in \mathbb{R}_+$, we have*

$$\mathbb{E}V(\boldsymbol{u}(\tau_2), \tau_2, \alpha(\tau_2)) = \mathbb{E}V(\boldsymbol{u}(\tau_1), \tau_2, \alpha(\tau_1)) + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathscr{L}V(\boldsymbol{u}(t), t, \alpha(t))\, dt,$$

for any stopping time $0 \leq \tau_1 \leq \tau_2 < \infty$, where

$$\begin{aligned}
\mathscr{L}V &= V_t(\boldsymbol{u}, t, \alpha) + V_{\boldsymbol{u}}(\boldsymbol{u}, t, \alpha)\boldsymbol{f}(\boldsymbol{u}, \alpha) \\
&\quad + \frac{1}{2}\mathit{Trace}[F^T(\boldsymbol{u}, \alpha)V_{\boldsymbol{uu}}(\boldsymbol{u}, t, \alpha)F(\boldsymbol{u}, \alpha)] + \sum_{\beta=1}^{m} q_{\alpha\beta}V(\boldsymbol{u}, t, \beta), & (26)
\end{aligned}$$

and

$$
\begin{aligned}
V_t(\boldsymbol{u},t,\alpha) &= \frac{\partial}{\partial t}V(\boldsymbol{u},t,\alpha),\\
V_{\boldsymbol{u}}(\boldsymbol{u},t,\alpha) &= \left(\frac{\partial}{\partial x}V(\boldsymbol{u},t,\alpha),\frac{\partial}{\partial y}V(\boldsymbol{u},t,\alpha),\frac{\partial}{\partial z}V(\boldsymbol{u},t,\alpha)\right),\\
V_{\boldsymbol{uu}}(\boldsymbol{u},t,\alpha) &= \begin{pmatrix} \frac{\partial^2}{\partial x^2}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial x\partial y}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial x\partial z}V(\boldsymbol{u},t,\alpha)\\ \frac{\partial^2}{\partial y\partial x}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial y^2}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial y\partial z}V(\boldsymbol{u},t,\alpha)\\ \frac{\partial^2}{\partial z\partial x}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial z\partial y}V(\boldsymbol{u},t,\alpha) & \frac{\partial^2}{\partial z^2}V(\boldsymbol{u},t,\alpha) \end{pmatrix}.
\end{aligned}
$$

3.2. **Properties of solutions.** In this subsection we will discuss the existence and uniqueness of solutions, as well as the positiveness and boundedness of solutions, to system (25).

Theorem 3.1. *For any initial condition $\boldsymbol{u}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3_+$ and any $\alpha_0 = \alpha(0) \in \mathcal{S}$, system* (25) *admits a unique solution $\boldsymbol{u}(t)$ that will remain in $\mathbb{R}^3_+$ almost surely, i.e., $\boldsymbol{u}(t) \in \mathbb{R}^3_+$ for any $t \geq 0$ with probability* 1.

Proof. Since the coefficients of system (25) are locally Lipschitz, then according to Theorem A2 in [22], system (25) has a unique local solution up to a blow-up time, τ_b, for any $\mathbf{u}_0 \in \mathbb{R}^3_+$. In order to obtain a global solution, we next prove that $\tau_b = \infty$ a.s.

The proof follows a similar idea to those used in [22] and [23]. Let $k_0 > 0$ be a positive integer, large enough, such that

$$x(0) \in \left(\frac{1}{k_0}, k_0\right), \quad y(0) \in \left(\frac{1}{k_0}, k_0\right), \quad z(0) \in \left(\frac{1}{k_0}, k_0\right).$$

For any $k \geq k_0$ we define the sequence of "stopping times", $\{\tau_k\}$, by

$$\tau_k := \inf\left\{t \in [0,\tau_b) :\ x(t) \notin \left(\frac{1}{k}, k\right) \text{ or } y(t) \notin \left(\frac{1}{k}, k\right) \text{ or } z(t) \notin \left(\frac{1}{k}, k\right)\right\}.$$

Clearly the sequence $\{\tau_k\}_{k=1,2,\cdots}$ is increasing. Denote by $\tau_\infty = \lim_{k\to\infty}\tau_k$, then $\tau_\infty \leq \tau_b$. We next show that $\tau_\infty = \infty$ a.s.

Consider the following Lyapunov function on $\mathbb{R}^3_+ \times \mathcal{S}$

$$V(\mathbf{u},\alpha) = c_1(x - 1 - \ln x) + c_2(y - 1 - \ln y) + c_3 z,$$

where $c_j > 0$ for $j = 1, 2, 3$. Clearly $V(\mathbf{u},\alpha) \geq 0$ for any $\mathbf{u} \in \mathbb{R}^3_+$ and $\alpha \in \mathcal{S}$, and by using (26) we obtain

$$
\begin{aligned}
\mathscr{L}V(\mathbf{u},\alpha) &= c_1\left(1 - \frac{1}{x}\right)f_1(\mathbf{u},\alpha) + c_2\left(1 - \frac{1}{y}\right)f_2(\mathbf{u},\alpha) + c_3 f_3(\mathbf{u},\alpha)\\
&\quad + \frac{1}{2}c_1\mu_1^2(\alpha) + \frac{1}{2}c_2\mu_2^2(\alpha), \qquad (27)
\end{aligned}
$$

where functions f_1, f_2, f_3 are defined as in (24). Simplifying (27) gives

$$
\begin{aligned}
\mathscr{L}V(\mathbf{u},\alpha) &= -c_2a_{22}y^2 - c_3a_{33}z^2 - (c_3a_{31} - c_1a_{13})xz - (c_3a_{32} + c_2a_{23})yz\\
&\quad -(c_2a_{21} + c_2m_1(\alpha) - c_1a_{12} - c_3m_2(\alpha))xy\\
&\quad +c_1x(-b_1 + \frac{1}{2}\mu_1^2) + c_2y(b_2 + \frac{1}{2}\mu_2^2) + c_3z(b_3 + \frac{1}{2}\mu_3^2) - b_1c_1 - b_2c_2.
\end{aligned}
$$

Denote by $\underline{m} := \min_{\alpha\in\mathcal{S}}\{m_j(\alpha) : j = 1, 2\}$, $\bar{m} := \max_{\alpha\in\mathcal{S}}\{m_j(\alpha) : j = 1, 2\}$, and pick c_1, c_2, c_3 such that

$$c_2 a_{21} + c_2 \underline{m} - c_1 a_{12} - c_3 \bar{m} \geq 0, \quad c_3 a_{31} - c_1 a_{13} \geq 0.$$

Then we have

$$\mathscr{L}V(\mathbf{u},\alpha) \leq c_1 x(-b_1 + \frac{1}{2}\mu_1^2) + c_2 y(b_2 + \frac{1}{2}\mu_2^2) + c_3 z(b_3 + \frac{1}{2}\mu_3^2) - b_1 c_1 - b_2 c_2,$$

and consequently there exists a positive constant γ_1 such that

$$\mathscr{L}V(\mathbf{u},\alpha) \leq \gamma_1(1 + x + y + z).$$

On the other hand, letting $\gamma_2 = \min\{c_1, c_2, c_3\}$, it is straightforward to show that for any $(x, y, z) \in \mathbb{R}^3_+$

$$\begin{aligned} x + y + z &\leq 4 + 2\left[(x - 1 - \ln x) + (y - 1 - \ln y) + z\right] \\ &\leq 4 + \frac{2}{\gamma_2} V(\mathbf{u},\alpha). \end{aligned}$$

And hence

$$\mathscr{L}V(\mathbf{u},\alpha) \leq \gamma_3[1 + V(\mathbf{u},\alpha)], \tag{28}$$

where

$$\gamma_3 = \max\{5\gamma_1, 2\gamma_1/\gamma_2\}.$$

Now, arguing by contradiction, suppose that there exist $T > 0$ and $\varepsilon > 0$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \varepsilon.$$

As a consequence there exists $K \geq k_0$ such that

$$\mathbb{P}(\tau_k \leq T) > \varepsilon, \qquad \text{for any } k \geq K.$$

Then by the generalized Itô Lemma and (28) we have for any $k \geq K$ that

$$\begin{aligned} V(\mathbf{u}(T\wedge\tau_k), \alpha(T\wedge\tau_k)) &= V(\mathbf{u}_0, \alpha_0) + \int_0^{T\wedge\tau_k} \mathscr{L}V(\mathbf{u}(s), \alpha(s))\mathrm{d}s \\ &\quad + c_1 \int_0^{T\wedge\tau_k} (x(s) - 1)\mu_1(\alpha(s))\mathrm{d}w_1(s) \\ &\quad + c_2 \int_0^{T\wedge\tau_k} (y(s) - 1)\mu_2(\alpha(s))\mathrm{d}w_2(s) \\ &\quad + c_3 \int_0^{T\wedge\tau_k} z(s)\mu_3(\alpha(s))\mathrm{d}w_3(s). \end{aligned} \tag{29}$$

Taking expectation of (29) and using (28) give

$$\begin{aligned} \mathbb{E}V(\mathbf{u}(T\wedge\tau_k), \alpha(T\wedge\tau_k)) &= V(\mathbf{u}_0, \alpha_0) + \mathbb{E}\int_0^{T\wedge\tau_k} \mathscr{L}V(\mathbf{u}(s), \alpha(s))\mathrm{d}s \\ &\leq V(x_0, \alpha_0) + \mathbb{E}\int_0^{t\wedge\tau_k} \gamma_3[1 + V(\mathbf{u}(s), \alpha(s))]\mathrm{d}s \\ &\leq \gamma_4 + \gamma_3 \int_0^{T\wedge\tau_k} \mathbb{E}V(\mathbf{u}(s), \alpha(s))\mathrm{d}s. \end{aligned}$$

where

$$\gamma_4 = V(\mathbf{u}_0, \alpha_0) + \gamma_3 T.$$

It follows immediately from Gronwall's inequality that

$$\mathbb{E}V(\mathbf{u}(T\wedge\tau_k),\alpha(T\wedge\tau_k))\le\gamma_4 e^{\gamma_3 T}.$$

Observe that

$$\mathbb{E}V(\mathbf{u}(T\wedge\tau_k),\alpha(T\wedge\tau_k))\ge\mathbb{E}V(\mathbf{u}(\tau_k),\alpha(\tau_k))I_{\{\tau_k<T\}}$$

while

$$V(\mathbf{u}(\tau_k),\alpha(\tau_k))\ge\gamma_2\max\left\{(k-1-\ln k),(\frac{1}{k}-1-\ln k),k,\frac{1}{k}\right\}:=\gamma_5(k).$$

Hence we conclude that

$$\gamma_4 e^{\gamma_3 T}\ge\gamma_5(k),$$

where $\gamma_5(k)$ goes to ∞ as $k\to\infty$. This contradicts the assumption that there exist $T>0$ and $\varepsilon>0$ such that $\mathbb{P}(\tau_\infty\le T)>\varepsilon$, and implies that

$$\lim_{k\to\infty}\tau_k=\infty=\tau_b\quad a.s.$$

i.e., $\mathbf{u}(t)\in\mathbb{R}^3_+$ a.s. for all $t\ge 0$. □

The above theorem states that for any initial condition $\mathbf{u}_0\in\mathbb{R}^3_+$ and $\alpha_0\in\mathcal{S}$, system (25) has a unique solution $\mathbf{u}(t)$ that stays in $\mathbb{R}^3_+$ almost surely. Next we will investigate the boundedness of the solution $\mathbf{u}(t)$. In particular, we consider the *uniformly boundedness in mean* defined as follows.

Definition 3.2. A stochastic process $\mathbf{u}(t)=(x(t),y(t),z(t))\in\mathbb{R}^3_+$ is said to be uniformly bounded in mean if there exists a positive constant M such that

$$\limsup_{t\to\infty}\mathbb{E}[\|\mathbf{u}(t)\|_1]=\limsup_{t\to\infty}\mathbb{E}[x(t)+y(t)+z(t)]\le M.$$

Theorem 3.3. *For any $\boldsymbol{u}_0=(x_0,y_0,z_0)\in\mathbb{R}^3_+$ and $\alpha_0\in\mathcal{S}$, the solution to system* (25) *is uniformly bounded in mean, provided that*

$$a_{31}>a_{13}\quad\textit{and}\quad a_{21}-a_{12}>\bar{m}-\underline{m},\tag{30}$$

where

$$\bar{m}:=\max_{\alpha\in\mathcal{S}}\{m_j(\alpha):j=1,2\},\quad \underline{m}:=\min_{\alpha\in\mathcal{S}}\{m_j(\alpha):j=1,2\}.$$

Proof. To unify notations to be used in the sequel, we first set

$$\bar{b}=\max_{\alpha\in\mathcal{S}}\{b_j(\alpha):j=1,2,3\},\quad \underline{b}=\min_{\alpha\in\mathcal{S}}\{b_j(\alpha):j=1,2,3\},$$

$$\bar{\mu}=\max_{\alpha\in\mathcal{S}}\{\mu_j(\alpha):j=1,2,3\},\quad \underline{\mu}=\min_{\alpha\in\mathcal{S}}\{\mu_j(\alpha):j=1,2,3\}.$$

Since the solution is positive, by using the assumptions (30) we have

$$\begin{aligned}\mathrm{d}v(t) \quad = \quad & x(t)\left[-b_1(\alpha)+\frac{1}{2}\mu_1^2(\alpha)+(a_{12}-a_{21}+m_2(\alpha)-m_1(\alpha))y\right]\mathrm{d}t\\ &+y(t)\left[b_2(\alpha)+\frac{1}{2}\mu_2^2(\alpha)-a_{22}y-(a_{23}+a_{32})z\right]\mathrm{d}t\\ &+z(t)\left[b_3(\alpha)+\frac{1}{2}\mu_3^2(\alpha)-a_{33}z+(a_{13}-a_{31})x\right]\mathrm{d}t+\mu_1(\alpha)x(t)\mathrm{d}w_1(t)\\ &+\mu_2(\alpha)y(t)\mathrm{d}w_2(t)+\mu_3(\alpha)z(t)\mathrm{d}w_3(t)\end{aligned}$$

$$
\begin{aligned}
&\leq x(t)\left[-b_1(\alpha)+\frac{1}{2}\mu_1^2(\alpha)\right]\mathrm{d}t+\mu_1(\alpha)x(t)\mathrm{d}w_1(t)\\
&\quad +y(t)\left[b_2(\alpha)+\frac{1}{2}\mu_2^2(\alpha)-a_{22}y\right]\mathrm{d}t+\mu_2(\alpha)y(t)\mathrm{d}w_2(t)\\
&\quad +z(t)\left[b_3(\alpha)+\frac{1}{2}\mu_3^2(\alpha)-a_{33}z\right]\mathrm{d}t+\mu_3(\alpha)z(t)\mathrm{d}w_3(t)\\
&\leq x(t)(-\underline{b}+\frac{1}{2}\mu^2)\mathrm{d}t+\mu x(t)\mathrm{d}w_1(t)\\
&\quad +y(t)\left(\bar{b}+\frac{1}{2}\mu^2-a_{22}y\right)\mathrm{d}t+\mu y(t)\mathrm{d}w_2(t)\\
&\quad +z(t)\left(\bar{b}+\frac{1}{2}\mu^2-a_{33}z\right)\mathrm{d}t+\mu z(t)\mathrm{d}w_3(t),
\end{aligned}
$$

where

$$\mu := \max\{|\bar{\mu}|, |\underline{\mu}|\}.$$

Then by comparison of SDE, we have $0 \leq v(t) \leq \bar{x}(t)+\bar{y}(t)+\bar{z}(t)$, a.s., for $t \geq 0$, where $\bar{x}(t)$, $\bar{y}(t)$ and $\bar{z}(t)$ satisfy the following equations respectively

$$\mathrm{d}\bar{x}(t) = \bar{x}(t)(-\underline{b}+\frac{1}{2}\mu^2)\mathrm{d}t+\mu\bar{x}(t)\mathrm{d}w_1(t), \quad \bar{x}(0)=x_0, \tag{31}$$

$$\mathrm{d}\bar{y}(t) = \bar{y}(t)\left(\bar{b}+\frac{1}{2}\mu^2-a_{22}\bar{y}(t)\right)\mathrm{d}t+\mu\bar{y}(t)\mathrm{d}w_2(t), \quad \bar{y}(0)=y_0, \tag{32}$$

$$\mathrm{d}\bar{z}(t) = \bar{z}(t)\left(\bar{b}+\frac{1}{2}\mu^2-a_{33}\bar{z}(t)\right)\mathrm{d}t+\mu\bar{z}(t)\mathrm{d}w_3(t), \quad \bar{z}(0)=z_0. \tag{33}$$

Solving equations (31) – (33) (see e.g. [24]) to obtain

$$
\begin{aligned}
\bar{x}(t) &= x_0 e^{-\underline{b}t+\mu w_1(t)},\\
\bar{y}(t) &= \frac{y_0 e^{\bar{b}t+\mu w_2(t)}}{1+a_{22}y_0\int_0^t e^{\bar{b}s+\mu w_2(s)}\mathrm{d}s},\\
\bar{z}(t) &= \frac{z_0 e^{\bar{b}t+\mu w_2(t)}}{1+a_{33}z_0\int_0^t e^{\bar{b}s+\mu w_3(s)}\mathrm{d}s},
\end{aligned}
$$

and it is straightforward to check that

$$
\begin{aligned}
\limsup_{t\to\infty}\mathbb{E}[\|\mathbf{u}\|_1] &\leq \limsup_{t\to\infty}\{\mathbb{E}[\bar{x}(t)]+\mathbb{E}[\bar{y}(t)]+\mathbb{E}[\bar{z}(t)]\}\\
&\leq \left(\bar{b}+\frac{1}{2}\mu^2\right)\left(\frac{1}{a_{22}}+\frac{1}{a_{33}}\right).
\end{aligned}
$$

This completes the proof. □

4. **Closing remarks.** In this paper we studied two different semi-Kolmogorov systems under random environments – first a system of random ordinary differential equations with random parameters, and second a system of stochastic ordinary differential equations with regime switching. For both system we obtain the existence, uniqueness and positiveness of solutions. We have also proved that the solution to each system is bounded under certain assumptions. It is worth mentioning that although the two systems have different set-up, the sufficient conditions to have bounded solutions are consistent with each other. This provides important information on the intrinsic critical parameters a_{13}, a_{31}, a_{12}, a_{21}, m_1 and m_2, regardless

the underlying model. We only proved the existence of a global random attractor for the random system. We leave the nontrivial construction of global random attractors for the stochastic system in future work.

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