# IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION WITH UNBOUNDED DELAY

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Abstract. In this paper, we prove the local and global existence and attractivity of mild solutions for stochastic impulsive neutral functional differential equations with infinite delay, driven by fractional Brownian motion.

## 1. Introduction

The theory of impulsive differential equations has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medical biology, economical systems etc. One can find detailed information in [3, 5, 14, 21, 22, 23, 24, 28] and references therein.

Neutral differential equations arise in many areas of applied mathematics and, for this reason, these equations have received much attention over the last few decades. Some good literature for ordinary neutral functional differential equations are the books of Benchohra et al. [5], Graef *et al.* [14], Lakshmikantham et al. [20] and the references therein. On the other hand, for partial neutral functional differential equations we refer the reader to Balachandran [4], Benchohra *et al.* [6], Jiang [17], Hale [15] and Hernandez [18].

The existence of neutral stochastic functional differential equation driven by a fractional Brownian motion have attracted great interest of researchers. For example, Boufoussi and Hajji [9] analyzed the existence and uniqueness of mild solutions for a neutral stochastic differential equation with finite delay,driven by a fractional Brownian motion in a Hilbert space, and established some sufficient conditions ensuring the exponential decay to zero in mean square for the mild solution. In [12] Caraballo and Diop, studied the existence and uniqueness of mild solutions to neutral stochastic delay functional integro-differential equations perturbed by a fractional Brownian motion. The existence and stability of second order stochastic differential equations driven by a fractional Brownian motion has been examined by Revathi *et al.* [34].

Recently, Boudaoui *et al.* [8] and Ren *et al.* [32] proved the existence of mild solutions to stochastic impulsive evolution equations with time delays driven by

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fractional Brownian motion by using a Krasnoselski-Schaefer type fixed point theorem. The existence of integral solution of non-densely impulsive neutral stochastic differential equation was studied by Ren *et. al.* [31]

In this paper, our main objective is to establish sufficient conditions for the local and global existence and attractivity of mild solutions to the following first order neutral stochastic impulsive functional equation with time delays:

$$d[y(t)-g(t,y_t)] = [Ay(t)+f(t,y_t)]dt+\sigma(t)dB_Q^H(t), \quad t\in J:=[0,T], \tag{1.1}$$

$$\Delta y|_{t=t_k} = y(t_k^+)-y(t_k) = I_k(y(t_k)), \; k=1,\dots,m, \tag{1.2}$$

$$y(t)=\phi(t)\in\mathcal{D}_{\mathcal{F}_0}, \quad \text{for a.e. } t\in J_0=(-\infty,0], \tag{1.3}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t\geq 0\}$, $B_Q^H$ is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H\in(1/2,1)$, and with respect to a complete probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ furnished with a family of right continuous and increasing $\sigma$-algebras $\{\mathcal{F}_t, t\in J\}$ satisfying $\mathcal{F}_t\subset\mathcal{F}$. The impulse times $t_k$ satisfy $0=t_0<t_1<t_2<\dots,t_m<T$ (if $T=\infty$, $t_k$ satisfies $0=t_0<t_1<t_2<\dots,t_m<\cdots$ ). As for $y_t$, we mean the segment solution which is defined in the usual way, that is, if $y(\cdot,\cdot):(-\infty,T]\times\Omega\to\mathcal{H}$, then for any $t\geq 0$, $y_t(\cdot,\cdot):(-\infty,0]\times\Omega\to\mathcal{H}$ is given by:

$$y_t(\theta,\omega)=y(t+\theta,\omega), \text{ for } \theta\in(-\infty,0], \; \omega\in\Omega.$$

Before describing the properties fulfilled by the operators $f,g,\sigma$ and $I_k$, we need to introduce some notation and describe some spaces.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato [16].

**Definition 1.1.** *$\mathcal{D}_{\mathcal{F}_0}$ is a linear space of a family of $\mathcal{F}_0$-measurable functions from $(-\infty,0]$ into $\mathcal{H}$ endowed with a norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms.*

- **(A-1):** *If $y:(-\infty,T]\longrightarrow\mathcal{H}$, $T>0$, is such that $y_0\in\mathcal{D}_{\mathcal{F}_0}$, then for every $t\in[0,T)$ the following conditions hold:*
  - **(i):** $y_t\in\mathcal{D}_{\mathcal{F}_0}$,
  - **(ii):** $\|y(t)\|\leq\mathcal{L}\|y_t\|_{\mathcal{D}_{\mathcal{F}_0}}$,
  - **(iii):** $\|y_t\|_{\mathcal{D}}\leq K(t)\sup\{\|y(s)\|: 0\leq s\leq t\}+N(t)\|y_0\|_{\mathcal{D}_{\mathcal{F}_0}}$,

    *where $\mathcal{L}>0$ is a constant; $K,N:[0,\infty)\longrightarrow[0,\infty)$, $K$ is continuous, $N$ is locally bounded and $K,N$ are independent of $y(\cdot)$.*
- **(A-2):** *For the function $y(\cdot)$ in $(A-1)$, $y_t$ is a $\mathcal{D}_{\mathcal{F}_0}$-valued function on $[0,T)$.*
- **(A-3):** *The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.*

Denote

$$\widetilde{K}=\sup\{K(t): t\in J\} \text{ and } \widetilde{N}=\sup\{N(t): t\in J\}.$$

Now, for a given $T>0$, we define

$$\mathcal{D}_{\mathcal{F}_T}=\Big\{y\colon(-\infty,T]\times\Omega\to\mathcal{H},\; y_k\in C(J_k,\mathcal{H}) \text{ for } k=1,\dots m,\; y_0\in\mathcal{D}_{\mathcal{F}_0}, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k)=y(t_k^-),\; k=1,\cdots,m, \text{ and } \sup_{t\in[0,T]}E(|y(t)|^2)<\infty\Big\},$$

endowed with the norm

$$\|y\|_{\mathcal{D}_{\mathcal{F}_T}}=\|\phi\|_{\mathcal{D}_{\mathcal{F}_0}}+\sup_{0\leq s\leq T}(E\|y(s)\|^2)^{\frac{1}{2}},$$

where $y_k$ denotes the restriction of $y$ to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = (-\infty, 0]$.

Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$.

Let $\mathcal{K}$ be another real separable Hilbert and suppose that $B_Q^H$ is a $\mathcal{K}$-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator $Q$ (see next section for more details), and let us denote by $L(\mathcal{K}, \mathcal{H})$ the space of all bounded, continuous linear operators from $\mathcal{K}$ into $\mathcal{H}$.

Then we assume that $g : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$, $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ and $\sigma : J \to L_Q^0(\mathcal{K}, \mathcal{H})$. Here, $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all $Q$-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$, which will be also defined in the next section.

As for the impulse functions, we will assume that $I_k \in C(\mathcal{H}, \mathcal{H})$ $(k = 1, \ldots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$.

The plan of this paper is as follows. In Section 2 we introduce notations, definitions, and preliminary facts which are useful throughout the paper. In Section 3 we prove existence of mild solutions for problem (1.1)-(1.3). Our approach to prove the local existence of mild solutions is based on a fixed point theorem of Burton and Kirk ([10]) for the sum of a contraction map and a completely continuous one. The global existence and uniqueness of mild solutions are discussed in Section 4 by using the Banach fixed point theorem. In Section 5 we provides sufficient conditions for the attractivity of mild solutions to problem (1.1)-(1.3). Finally, in Section 6, an example is given to demonstrate the applicability of our results.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which will be used throughout this paper. In particular, we consider fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper.

Recall that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete probability space furnished with a family of right continuous increasing $\sigma$-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

**Definition 2.1.** *Given $H \in (0, 1)$, a continuous centered Gaussian process $\beta^H = \{\beta^H(t), t \in \mathbb{R}\}$, with the covariance function*

$$R_H(t, s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), t, s \in \mathbb{R}$$

*is called a two$-$sided one$-$dimensional fractional Brownian motion, and $H$ is the Hurst parameter.*

Now we aim at introducing the Wiener integral with respect to the one-dimensional $\beta^H$.

Let $T > 0$ and denote by $\Lambda$ the linear space of $\mathbb{R}-$valued step functions on $[0, T]$, that is, $\psi \in \Lambda$ if

$$\psi(t) = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i-1})}(t),$$

where $t \in [0,T], x_i \in \mathbb{R}$ and $0 = s_1 < s_2 < \cdots < s_n = T$. For $\psi \in \Lambda$ we define its Wiener integral with respect to $\beta^H$ by

$$\int_0^T \psi(\sigma) d\beta^H(\sigma) = \sum_{i=1}^{n-1} x_i(\beta^H(s_{i+1}) - \beta^H(s_i)).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\Lambda$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

Then, the mapping

$$\psi = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i+1})} \mapsto \int_0^T \psi(\sigma) d\beta^H(\sigma)$$

is an isometry between $\Lambda$ and the linear space span $\{\beta^H(t), t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{\beta^H(t), t \in [0,T]\}$ (see [29]). The image of an element $\psi \in \mathcal{H}$ by this isometry is called the Wiener integral of $\psi$ with respect to $\beta^H$. Our next goal is to give an explicit expression for this integral. To this end, consider the kernel

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du,$$

where $c_H = \left( \frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})} \right)^{1/2}$, with $B(\cdot,\cdot)$ denoting the Beta function, and $t \leq s$.

It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K_H^* : \Lambda \longrightarrow L^2([0,T])$ given by

$$(K_H^* \Phi)(s) = \int_s^t \Phi(t) \frac{\partial K_H}{\partial t}(t,s) dt.$$

Then

$$(K_H^* 1_{[0,t]})(s) = K_H(t,s) 1_{[0,t]}(s),$$

and $K_H^*$ is an isometry between $\Lambda$ and $L^2([0,T])$ that can be extended to $\Lambda$ (see [2]). Considering $W = \{W(t), t \in [0,T]\}$ defined by

$$W(t) = \beta^H((K_H^*)^{-1} 1_{[0,t]}),$$

it turns out that $W$ is a Wiener process and $\beta^H$ has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s) dW(s).$$

In addition, for any $\Phi \in \Lambda$,

$$\int_0^T \Phi(s) \beta^H(s) dW(s) = \int_0^T (K_H^* \Phi)(t) dW(t)$$

if and only if $K_H^* \Phi \in L^2([0,T])$.

Also denoting

$$L^2_{\mathcal{H}}([0,T]) = \{\Phi \in \Lambda, K_H^* \Phi \in L^2([0,T])\},$$

since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{2.1}$$

see [26]. Moreover, the following useful result holds:

**Lemma 2.2.** *([27]) For $\Phi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T \int_0^T |\Phi(r)\|\Phi(u)\|r-u|^{2H-2} drdu \leq c_H \|\Phi\|^2_{L^{1/H}([0,T])}.$$

Next, we consider a fractional Brownian motion with values in a Hilbert space and give the definition of the corresponding stochastic integral.

Let $Q \in L(\mathcal{K}, \mathcal{H})$ be a non-negative self$-$adjoint operator. Denote by $L^0_Q(\mathcal{K}, \mathcal{H})$ the space of all $\xi \in L(\mathcal{K}, \mathcal{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|^2_{L^0_Q(\mathcal{K},\mathcal{H})} = tr(\xi Q \xi^*).$$

Then $\xi$ is called a $Q$-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$.

Let $\{\beta^H_n(t)\}_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions that are mutually independent on $(\Omega, \mathcal{F}, P)$. The following series

$$\sum_{n=1}^{\infty} \beta^H_n(t) e_n, \quad t \geq 0,$$

where $\{e_n\}_{n\in N}$ is a complete orthonormal basis in $\mathcal{K}$, does not necessarily converge in the space $\mathcal{K}$. Thus, we consider a $\mathcal{K}-$valued stochastic process $B^H_Q(t)$ given formally by the following series:

$$B^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t) Q^{\frac{1}{2}} e_n, \quad t \geq 0,$$

which is well-defined as a $\mathcal{K}$-valued $Q$-cylindrical fractional Brownian motion.

Let $\varphi : [0,T] \mapsto L^Q_0(\mathcal{K}, \mathcal{H})$ such that

$$\sum_{n=1}^{\infty} \|K^*_H(\varphi Q^{\frac{1}{2}} e_n)\|_{L^{1/H}([0,T];\mathcal{H})} < \infty. \tag{2.2}$$

**Definition 2.3.** *Let $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K}, \mathcal{H})$ satisfy (2.2). Then, its stochastic integral with respect to the fractional Brownian motion $B^H_Q$ is defined, for $t \geq 0$, as*

$$\int_0^t \varphi(s) dB^H_Q(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t (K^*_H(\varphi Q^{1/2} e_n))(s) dW(s).$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{1/2} e_n\|_{L^{1/H}([0,T];\mathcal{H})} < \infty, \tag{2.3}$$

then in particular (2.2) holds, which follows immediately from (2.1).

**Lemma 2.4.** *([11]) If $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K}, \mathcal{H})$ satisfies*

$$\int_0^T \|\varphi(s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} ds < \infty,$$

*then the series in (2.3) is well defined as a $\mathcal{H}$-valued random variable and we have*

$$E\Big|\int_0^t \varphi(s)dB_Q^H(s)\Big|^2 \leq 2Ht^{2H-1}\int_0^t \|\varphi(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})}ds.$$

Assume that $S(t)$ is an analytic semigroup with infinitesimal generator $A$ such that $0 \in \rho(A)$ (the resolvent set of $A$). Then, it is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \leq 1$ as a closed linear invertible operator with its domain $D((-A)^\alpha)$ being dense in $\mathcal{H}$. We denote by $H_\alpha$ the Banach space $D((-A)^\alpha)$ endowed with the norm $\|y\|_\alpha = \|(-A)^\alpha y\|$, which is equivalent to the graph norm of $(-A)^\alpha$. In the sequel, $\mathcal{H}_\alpha$ represents the space $D((-A)^\alpha)$ with the norm $\|\cdot\|_\alpha$. Then, we have the following well-known properties that appear in ([30]).

**Lemma 2.5.** **(i):** *If $0 < \beta < \alpha \leq 1$, then $\mathcal{H}_\alpha \subset \mathcal{H}_\beta$ and the embedding is compact whenever the resolvent operator of $A$ is compact.*

**(ii):** *For each $0 < \alpha \leq 1$, there exists a positive constant $C_\alpha$ such that $\|(-A)^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}e^{-\lambda t},\ t > 0,\ \lambda > 0.$*

**Lemma 2.6.** *([18]) Let $v(\cdot)$,$w(\cdot)$ : $[0,T] \longrightarrow [0,\infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta\int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}ds,\ t \in J,$$

*then*

$$v(t) \leq e^{\theta^n\Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)}\sum_{j=0}^{n-1}\left(\frac{\theta T^\alpha}{\alpha}\right)^j w(t),$$

*for every $t \in [0,T]$ and every $n \in \mathbb{N}$ such that $n\alpha > 1$, where $\Gamma(\cdot)$ is the Gamma function.*

**Definition 2.7.** *The map $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ is said to be $L^2$-Carathéodory if*

**(i):** *$t \mapsto f(t,v)$ is measurable for each $v \in \mathcal{D}_{\mathcal{F}_0}$;*

**(ii):** *$v \mapsto F(t,v)$ is continuous for almost all $t \in J$;*

**(iii):** *for each $q > 0$, there exists $\alpha_q \in L^1(J,\mathbb{R}^+)$ such that*

$$E|f(t,v)|^2 \leq \alpha_q(t),\ \textit{for all } \|v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq q \textit{ and for a.e. } t \in J.$$

## 3. Existence result

Now we first define the concept of mild solution to our problem.

**Definition 3.1.** *Given $\phi \in \mathcal{D}_{\mathcal{F}_0}$, a $\mathcal{H}-$valued stochastic process $\{y(t), t \in (-\infty,T]\}$ is called a mild solution of the problem (1.1)-(1.3) if $y(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $y(t) = \phi(t)$ on $(-\infty,0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1,2,\cdots,m$, the restriction of $y(\cdot,\cdot)$ to $[0,T) - \{t_1,t_2,\cdots,t_m\}$ is continuous, and for each $s \in [0,t)$, the function $AS(t-s)g(s,x_s)$ is integrable, and $y$ satisfies the integral equation*

$$\begin{aligned}
y(t) \quad = \quad & S(t)[\phi(0) - g(s,\phi)] + g(t,y_t) + \int_0^t AS(t-s)g(s,y_s)ds \\
& + \int_0^t S(t-s)f(s,y_s)ds + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) \qquad (3.1)\\
& + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)),\ t \in J.
\end{aligned}$$

Notice that this concept of solution can be considered as more general than the classical concept of solution to equation (1.1)-(1.3). A continuous solution of (3.1) is called a mild solution of (1.1)-(1.3).

Our main result in this section is based on the following fixed point theorem due to Burton and Kirk [10].

**Theorem 3.2.** *Let $X$ be a Banach space, and $\Phi_1, \Phi_2 : X \to X$ be two operators satisfying:*

(1) $\Phi_1$ *is a contraction, and*
(2) $\Phi_2$ *is completely continuous*

*Then, either the operator equation $y = \Phi_1(y) + \Phi_2(y)$ possesses a solution, or the set* $\Xi = \left\{ y \in X : \lambda\Phi_1(\frac{y}{\lambda}) + \lambda\Phi_2(y) = y, \text{ for some } \lambda \in (0,1) \right\}$ *is unbounded.*

We are now in a position to state and prove our local existence result for the problem (1.1)-(1.3). First we will list the following hypotheses which will be imposed in our main theorem.

- (H1) $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t),\ t \geq 0$ and there exists a constant $M$ such that $\{\|S(t)\|^2 \leq M\}$ for all $t \geq 0$ and $\|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}$, for all $t > 0$.
- (H2) There exist constants $0 < \beta < 1$, $L_g \geq 0$ and a bounded continuous function $\zeta : J \longrightarrow \mathbb{R}^+$ such that $g$ is $\mathcal{H}_\beta$-valued, $(-A)^\beta g$ is continuous, and
  **(i):** $E|(-A)^\beta g(t,y)|^2 \leq \zeta(t)\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}},\ t \in J,\ y \in \mathcal{D}_{\mathcal{F}_0}$
  **(ii):** $E|(-A)^\beta g(t,y_1) - (-A)^\beta g(t,y_2)|^2 \leq L_g\|y_1 - y_2\|^2_{\mathcal{D}_{\mathcal{F}_0}},\ t \in J,$

  $$y_1 \text{ and } y_2 \in \mathcal{D}_{\mathcal{F}_0} \text{ with } L_0 = 4\widetilde{K}^2 L_g\left(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\right) < 1.$$
- (H3) For all $y \in \mathcal{H}$, there exist constants $d_k > 0$, $k = 1, \ldots, m, \ldots$ for each $|I_k(y)| \leq d_k$, and $\displaystyle\sum_{k=0}^{\infty} d_k < \infty$.
- (H4) $f$ is a $L^2$-Caratheodory map and for every $t \in [0,T]$ the function $t \to f(t, y_t),\ y_t \in \mathcal{D}_{\mathcal{F}_0}$, is mesurable.
- (H5) The function $\sigma : J \longrightarrow L_Q(\mathcal{K}, \mathcal{H})$ satisfies

  $$\int_0^T \|\sigma(s)\|^2_{L_Q} ds < \infty.$$
- (H6) For the initial value $\phi \in \mathcal{D}_{\mathcal{F}_0}$, there exists a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty),\ \psi(0) = 0$ and $p \in L^1(J, \mathbb{R}_+)$ such that $E|f(t,y)|^2 \leq p(t)\psi(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}})$, for a.e. $t \in J$ and $y \in \mathcal{D}_{\mathcal{F}_0}$
  with
  $$\int_{\eta K_0}^{\infty} \frac{du}{\psi(u)} > \eta K_2 \int_0^T p(s)ds,$$
  where $K_0 = \frac{4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 4\widetilde{K}^2 F}{1-24\widetilde{K}^2(-A)^{-\beta}\zeta^*}$, $K_2 = \frac{24T\widetilde{K}^2 M}{1-24\widetilde{K}^2(-A)^{-\beta}\zeta^*}$, and
  $$\eta = e^{K_1^n(\Gamma(2\beta-1))^n T^{n(2\beta-1)}/\Gamma(n(2\beta-1))} \sum_{j=0}^{n-1}\left(\frac{K_1 T^{2\beta-1}}{2\beta-1}\right).$$

**Theorem 3.3.** *Assume that hypotheses (H1)-(H6) hold. If* $12\widetilde{K}^2(-A)^{-\beta}\theta_1 < 1$, *then, problem (1.1)-(1.3) possesses at least one mild solution on* $(-\infty, T]$.

*Proof.* Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $\Phi : \mathcal{D}_{\mathcal{F}_T} \to \mathcal{D}_{\mathcal{F}_T}$ defined by

$$\Phi(y)(t) = \begin{cases} \phi(t), \ \text{if } t \in (-\infty, 0], \\ S(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \displaystyle\int_0^t AS(t-s)g(s, y_s)ds \\ + \displaystyle\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \quad \text{if } t \in J. \end{cases}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in J; \end{cases}$$

then $\widehat{\phi} \in \mathcal{D}_{\mathcal{F}_T}$.
Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t \le T$. It is evident that $z$ satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} z(t) = \quad & -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ & + \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in J. \end{aligned}$$

Set $\mathcal{D}'_{\mathcal{F}_T} = \{y \in \mathcal{D}_{\mathcal{F}_T}, \quad \text{such that} \quad y_0 = 0 \in \mathcal{D}_{\mathcal{F}_0}\}$; for any $y \in \mathcal{D}_{\mathcal{F}_0}$ we have

$$\|y\|_{\mathcal{F}_T} = \|y_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t\in J}(E\|y(t)\|^2)^{\frac{1}{2}} = \sup_{t\in J}(E\|y(t)\|^2)^{\frac{1}{2}},$$

Then $(\mathcal{D}'_{\mathcal{F}_T}, \|\cdot\|_{\mathcal{F}_T})$ is a Banach space.

Let $\mathcal{B}_q = \left\{ y \in \mathcal{D}'_{\mathcal{F}_T}, \quad \|y\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \le q, \ q \ge 0 \right\}$. Clear that the set $\mathcal{B}_q$ is a bounded closed convex set in $\mathcal{D}'_{\mathcal{F}_T}$ for each $q \ge 0$ and for each $y \in \mathcal{B}_q$. we have

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \le \quad & 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ & \le 4(\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \widetilde{K}^2(q + ME|\phi(0)|^2)) \\ & = q'. \end{aligned}$$

Let the operator $\widehat{\Phi} : \mathcal{D}'_{\mathcal{F}_T} \to \mathcal{D}'_{\mathcal{F}_T}$ be defined by

$$\widehat{\Phi}(z) = \begin{cases} 0, \ \text{if } t \in (-\infty, 0], \\ -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \displaystyle\int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ + \displaystyle\int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), \qquad t \in J. \end{cases}$$

Now, consider the two operators $\widehat{\Phi}_1, \widehat{\Phi}_2$ defined by:

$$\widehat{\Phi}_1(z)(t) = \begin{cases} 0, & \text{if} \quad t \in (-\infty, 0], \\ -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \displaystyle\int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) & \\ + \displaystyle\int_0^t S(t-s)g(s)dB_Q^H(s), & \text{if} \quad t \in [0, T], \end{cases}$$

and

$$\widehat{\Phi}_2(z)(t) = \begin{cases} 0, & \text{if} \quad t \in (-\infty, 0], \\ \displaystyle\int_0^t S(t-s)f(s, z_s) + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), & \text{if} \quad t \in [0, T]. \end{cases}$$

Clear that

$$\widehat{\Phi}_1 + \widehat{\Phi}_2 = \widehat{\Phi}.$$

Then the problem of finding a solution of (1.1)-(1.3) is reduced to finding a solution of the operator equation $y(t) = \widehat{\Phi}_1(y)(t) + \widehat{\Phi}_2(y)(t)$, $t \in (-\infty, T]$. We shall show that the operators $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ satisfy all the conditions of Theorem 3.2. The proof will be given in several steps.

**Step 1**: $\Phi_1$ is a contraction.

Let $v, u \in \mathcal{D}'_{\mathcal{F}_T}$. Then, for $t \in J$,

$$\begin{aligned} E|\widehat{\Phi}_1(v)(t) - \widehat{\Phi}_1(u)(t)|^2 &\leq 2E|g(t, v_t + \widehat{\phi}_t) - g(t, u_t + \widehat{\phi}_t)|^2 \\ &\quad + 2E\left|\int_0^t AS(t-s)(g(s, v_s + \widehat{\phi}_s) - g(s, u_s + \widehat{\phi}_s))\right|^2 \\ &\leq 2L_g\|(-A)^{-\beta}\|^2\|v(t) - u(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\quad + 2T\int_0^t \frac{C_{1-\beta}^2}{t - s^{2(1-\beta)}} L_g\|v(s) - u(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}} ds, \\ &\leq 2L_g\left(\|(-A)^{-\beta}\|^2 + \tfrac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\right) \times \left[2\widetilde{K}^2 \sup_{0\leq s\leq T} E|v(s) - u(s)|^2\right. \\ &\quad \left. + 2\widetilde{N}\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}}\right], \\ &\leq L_0 \sup_{0\leq s\leq T} E|v(s) - u(s)|^2. \end{aligned}$$

Since $\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$, $\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$. Taking the supremum over $t$, we obtain

$$\|\widehat{\Phi}_1(v) - \widehat{\Phi}_1(u)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq L_0\|v - u\|^2_{\mathcal{D}'_{\mathcal{F}_T}},$$

where $L_0 = 4\widetilde{K}^2 L_g\left(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\right) < 1$. Thus $\widehat{\Phi}_1$ is a contraction.

Next, we prove that the operator $\widehat{\Phi}_2$ is completely continuous.
**Step 2**: $\widehat{\Phi}_2$ is continuous.

Let $z^n$ be a sequence such that $z^n \to z$ in $\mathcal{D}'_{\mathcal{F}_T}$. Then, for $t \in J$, and thanks to $(H1), (H3)$ and $(H4)$, $I_k$ , $k = 1, 2, \cdots, m$, is continuous.

By the dominated convergence theorem, we have

$$\begin{aligned}
&E|\widehat{\Phi}_2(z^n)(t)-\widehat{\Phi}_2(z)(t)|^2\\
&\quad\le 2E\Big|\int_0^t S(t-s)(f(s,(z_s^n+\widehat{\phi}_s))-f(s,z_s+\widehat{\phi}_s))ds\Big|^2\\
&\quad+2E\Big|\sum_{0\le t_k\le t}|S(t-t_k)|^2|I_k(z^n_{t_k^-}+\widehat{\phi}(t_k^-))-I_k(z_{t_k^-}+\widehat{\phi}(t_k^-))\Big|^2\\
&\quad\le 2MT\int_0^t E\Big|(f(s,(z_s^n+\widehat{\phi}_s))-f(s,z_s+\widehat{\phi}_s))ds\Big|^2\\
&\quad+2ME\Big|I_k(z^n(t_k^-)+\widehat{\phi}(t_k^-))-I_k(z(t_k^-)+\widehat{\phi}(t_k^-))\Big|^2\to 0 \text{ as } n\to+\infty.
\end{aligned}$$

Thus, $\Phi_2$ is continuous.

**Step 3**: $\widehat{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{D}'_{\mathcal{F}_T}$.

Indeed, it is enough to show that for any $q>0$, there exists a positive constant $l$ such that for each $z\in\mathcal{B}_q=\{z\in\mathcal{D}'_{\mathcal{F}_T}:\|z\|^2_{\mathcal{D}'_{\mathcal{F}_T}}\le q\}$, one has $\|\widehat{\Phi}_2(z)\|^2_{\mathcal{D}'_{\mathcal{F}_T}}\le l$. Let $z\in\mathcal{B}_q$; then for each $t\in[0,T]$, we have

$$\begin{aligned}
|\widehat{\Phi}_2z(t)|^2 &\le \left|\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s)ds+\sum_{0\le t_k\le t}S(t-t_k)I_k(z^n_{t_k^-}+\widehat{\phi}(t_k^-))\right|^2\\
&\le 2M\Big|\int_0^t f(s,z_s+\widehat{\phi}_s)ds\Big|^2+2M\sum_{0\le t_k\le t}\Big|I_k(z^n(t_k^-)+\widehat{\phi}(t_k^-))\Big|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
E|\widehat{\Phi}_2z(t)|^2 &\le 2TM\int_0^t p(s)\psi(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}'_{\mathcal{F}_T}})ds\\
&\quad+2M\sum_{0\le t_k\le t}E|I_k(z(t_k^-)+\widehat{\phi}(t_k^-))|^2\\
&\le 2MT\psi(q')\int_0^T p(s)ds+2M\Big(\sum_{k=1}^m d_k\Big)^2.
\end{aligned}$$

Then we have

$$E|\widehat{\Phi}_2z(t)|^2\le 2MT\psi(q')\|p\|_{L^1}+2M\Big(\sum_{k=1}^m d_k\Big)^2=l.$$

**Step 4**: $\widehat{\Phi}_2$ maps bounded sets into equicontinuous sets in $\mathcal{D}'_{\mathcal{F}_T}$.

Let $0 < \epsilon \le \tau_1 < \tau_2 \in J$, $\tau_1, \tau_2 \neq t_i$, $i = 1, \cdots, m$, and $\mathcal{B}_q$ be a bounded set of $\mathcal{D}'_{\mathcal{F}_T}$ as in Step 3. Let $z \in \mathcal{B}_q$; then we have

$$\begin{aligned}
E|(\widehat{\Phi}_2 y)(\tau_2) - (\widehat{\Phi}_2 y)(\tau_1)|^2 \ &\le \ 6T\int_0^{\tau_1-\epsilon} |S(\tau_2 - s) - S(\tau_1 - s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&\quad +6T\int_{\tau_1-\epsilon}^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&\quad +6T\int_{\tau_1}^{\tau_2} |S(\tau_2 - s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&\quad +4\sum_{0<t_k<\tau_1} |S(\tau_2 - t_k) - S(\tau_1 - t_k)|^2 E|I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-))|^2 \\
&\quad +4\sum_{\tau_1 \le t_k < \tau_2} |S(\tau_2 - t_k)|^2 |I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-))|^2 \\
&\le \ 6T\int_0^{\tau_1-\epsilon} |S(\tau_2 - s) - S(\tau_1 - s)|^2 \alpha_{q'}(s) ds \\
&\quad +6T\int_{\tau_1-\epsilon}^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)|^2 \alpha_{q'}(s) ds \\
&\quad +6TM\int_{\tau_1}^{\tau_2} \alpha_{q'}(s) ds \\
&\quad +4\sum_{0<t_k<\tau_1} |S(\tau_2 - t_k) - S(\tau_1 - t_k) d_k|^2 \\
&\quad +4M\Big(\sum_{\tau_1 \le t_k < \tau_2} d_k\Big)^2.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and $\epsilon$ sufficiently small, since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [30]. This proves the equicontinuity. Here, we consider the case $0 < \tau_1 < \tau_2 \le T$, since the cases $\tau_1 < \tau_2 \le 0$ and $\tau_1 \le 0 \le \tau_2 \le T$ are easier to handle.

**Step 5**: $(\widehat{\Phi}_2 \mathcal{B}_q)(t)$ is precompact in $\mathcal{H}$

As a consequence of Steps 2 to 4, together with the Arzelá-Ascoli theorem, it suffices to show that $\widehat{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in $\mathcal{H}$.

Let $0 < t < T$ be fixed and let $\epsilon$ be a real number satisfying $0 < \epsilon < t$. For $y \in \mathcal{B}_q$, we define

$$(\widehat{\Phi}_{2\epsilon} z)(t) = S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon) f(s, z_s + \widehat{\phi}_s) ds + S(\epsilon)\sum_{0<t_k<t-\epsilon} S(t-t_k-\epsilon) I_k(z(t_k^-) + \widehat{\phi}(t_k^-)).$$

Since $S(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\widehat{\Phi}_{2\epsilon}(z)(t) : \ z \in \mathcal{B}_q\}$$

is precompact in $\mathcal{H}$ for every $\epsilon$ and $0 < \epsilon < t$. Moreover, for every $z \in \mathcal{B}_q$ we have

$$\begin{aligned} E|(\widehat{\Phi}_2 y)(t)-(\widehat{\Phi}_{2\epsilon}(y)(t)|^2 &\leq 4T\int_{t-\epsilon}^t |S(t,s)|^2E|f(s,z_s+\widehat{\phi}_s)|^2ds \\ &\quad +4\sum_{t-\epsilon<t_k<t}|S(t-t_k)|^2E|I_k(z(t_k^-)+\widehat{\phi}(t_k^-))|^2 \\ &\leq 4TM\int_{t-\epsilon}^t \alpha_{q'}(s)ds+4M\Big(\sum_{0<t_k<t-\epsilon} d_k\Big)^2. \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $Y_\epsilon(t)=\{\widehat{\Phi}_{2\epsilon}(z)(t):\ z\in \mathcal{B}_q\}$. Hence the set $Y(t)=\{\widehat{\Phi}_2(z)(t):\ y\in\mathcal{B}_q\}$ is precompact in $\mathcal{H}$, and therefore, the operator $\widehat{\Phi}_2:\mathcal{D}^{'}_{\mathcal{F}_T}\to\mathcal{D}^{'}_{\mathcal{F}_T}$ is completely continuous.

**Step 5** : A priori bounds.

Now it remains to show that the set

$$\Xi=\{z\in\mathcal{D}^{'}_{\mathcal{F}_T}: z=\lambda\widehat{\Phi}_2(z)+\lambda\widehat{\Phi}_1\left(\frac{z}{\lambda}\right), \text{for some } 0<\lambda<1\}$$

is bounded. For each $t\in J$

$$\begin{aligned} z(t) &= -S(t)g(0,\phi)+g(t,z_t+\widehat{\phi}_t)+\int_0^t AS(t-s)g(s,z_s+\widehat{\phi}_s)ds \\ &\quad +\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s)ds+\int_0^t S(t-s)\sigma(s)dB_Q^H(s) \\ &\quad +\sum_{0<t_k<t} S(t-t_k)I_k(z(t_k)+\widehat{\phi}(t_k)). \end{aligned}$$

This implies, for each $t\in J$,

$$\begin{aligned} E|z(t)|^2 &\leq 6(-A)^{-\beta}M\zeta(t)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+6(-A)^{-\beta}\zeta(t)\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\quad +6T\int_0^t\frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\zeta(s)\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds+6MT\int_0^t p(s)\psi(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds \\ &\quad +12MHt^{2H-1}\int_0^t\|\sigma(s)\|^2_{L_Q^0}ds+6M\Big(\sum_{k=1}^m d_k\Big)^2 \\ &\leq F+6(-A)^{-\beta}\zeta(t)\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\quad +6T\int_0^t\frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\zeta(s)\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds \\ &\quad +6MT\int_0^t p(s)\psi(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds, \end{aligned}$$

where

$$F=6(-A)^{-\beta}M\zeta^*\|\phi\|_{\mathcal{D}_{\mathcal{F}_0}}+12MHT^{2H-1}\int_0^T\|\sigma(s)\|^2_{L_Q^0}ds+6M\Big(\sum_{k=1}^m d_k\Big)^2,$$

and

$$\zeta^*=\sup_{t\in J}|\zeta(t)|.$$

But

$$\begin{aligned} \|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} &\leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\leq 4\widetilde{K}^2\sup_{s\in[0,T]}E|z(s)|^2+4\widetilde{K}^2ME|\widehat{\phi}(0)|^2+4\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\leq 4\widetilde{K}^2\sup_{s\in[0,T]}E|z(s)|^2+4\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \end{aligned}$$

where

$$\alpha^2 = \max\{\widetilde{K}^2, \widetilde{N}^2\}.$$

If we set $w(t)$ the right hand side of the above inequality we have that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq w(t),$$

and therefore (3.2) becomes

$$\begin{aligned} E|z(t)|^2 \leq & F + 6(-A)^{-\beta}\zeta(t)w(t) + 6T\int_0^t \frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}}\zeta(s)w(s)ds \\ & + 6MT\int_0^t p(s)\psi(w(s))ds. \end{aligned} \tag{3.2}$$

Using (3.2) in the definition of $w$, we have that

$$\begin{aligned} w(t) \leq \quad & 4\widetilde{K}^2\left(F + 6(-A)^{-\beta}\zeta^* w(t) + 6T\int_0^t \frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}}\zeta^* w(s)ds\right. \\ & \left. +6MT\int_0^t p(s)\psi(w(s))ds\right) + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{aligned} \tag{3.3}$$

Thus, we obtain

$$w(t) \leq K_0 + K_1\int_0^t \frac{w(s)}{(t-s)^{2(1-\beta)}}ds + K_2\int_0^t p(s)\psi(w(s))ds. \tag{3.4}$$

where

$$K_0 = \frac{4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 4\widetilde{K}^2 F}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*},$$

$$K_1 = \frac{24T\widetilde{K}^2 C^2_{1-\beta}\zeta^*}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*},$$

and

$$K_2 = \frac{24T\widetilde{K}^2 M}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*}.$$

By Lemma 2.6, we get

$$w(t) \leq \eta\left(K_0 + K_2\int_0^t p(s)\psi(w(s))ds\right), \tag{3.5}$$

where

$$\eta = e^{C_1^n(\Gamma(2\beta-1))^n T^{n(2\beta-1)}/\Gamma(n(2\beta-1))}\sum_{j=0}^{n-1}\left(\frac{K_2 T^{2\beta-1}}{2\beta-1}\right).$$

Let us denote the right-hand side of the inequality (3.5) by $v(t)$. Then we have

$$v(0) = \eta K_0,\ w(t) \leq v(t),\ t \in J,$$

and

$$v'(t) = \eta K_2 p(t)\psi(w(t)),\ t \in J.$$

Using the increasing character of $\psi$, we obtain

$$v'(t) \leq \eta K_2 p(t)\psi(v(t)),\ \text{for } a.e.\ t \in J.$$

This implies, for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \eta K_2 \int_0^T p(s)ds \Rightarrow \Gamma(v(t)) \leq \eta K_2 \|p\|_{L^1},$$

where $\Gamma$ is nondegreasing function defined by

$$\Gamma(z) = \int_{\eta K_0}^{z} \frac{du}{\psi(u)}.$$

Hence

$$v(t) \leq \Gamma^{-1}\left(\eta K_2 \|p\|_{L^1}\right) := K \Rightarrow w(t) \leq v(t) \leq K,\ t \in J.$$

From equation (4.5), we obtain that

$$\begin{aligned} E|z(t)|^2 \leq & F + 6(-A)^{-\beta}\zeta^* K + 6T\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\zeta^* K \\ & + 6MT\int_0^T p(s)\psi(K)ds = L, \end{aligned} \tag{3.6}$$

$$\tag{3.7}$$

Thus

$$\|z\|^2_{\mathcal{D}_{\mathcal{F}'_T}} \leq L.$$

As a consequence of Theorem 3.2, we deduce that $\widehat{\Phi}$ has a fixed point, since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, T]$. Then $y$ is a fixed point of the operator $\Phi$ which is a mild solution of the problem (1.1)-(1.3). □

## 4. Global existence and uniqueness result

In this section we establish a result for the global existence of mild solutions ($T = \infty$) for our semilinear equations with infinite delay. In order to achieve this end, we need to impose some stronger assumptions, but will obtain the mild solutions defined in $\mathbb{R}$, something that will be necessary for the study of attractivity of solutions in the next section.

We will need to introduce the following hypotheses which are assumed thereafter:

- $(H1^{'})$The semigroup $S(t)$ satisfies the additional condition:

$$\exists \lambda > 0 \text{ and } \exists M > 0 \text{ such that } \|S(t)\| \leq Me^{-\lambda t}.$$

- $(H2^{'})$ There exist constants, $L_f$ and a function $p \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ such that:

  **(i):** $E|f(t,y) - f(t,x)|^2 \leq L_f \|y - x\|^2_{\mathcal{D}_{\mathcal{F}_0}},\ t \in J = [0, \infty)$;

  **(ii):** $E|f(t,y)|^2 \leq p(t)(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 1)$ for a.e. $t \in [0, \infty)$ and each $y \in \mathcal{D}_{\mathcal{F}_0}$;

  **(iii):** For every $t \in [0, \infty)$ the function $t \to f(t, y_t),\ y_t \in \mathcal{D}_{\mathcal{F}_0}$ is mesurable.

- $(H3^{'})$ The function $\sigma : J = [0, \infty) \longrightarrow L_Q^0(\mathcal{K}, \mathcal{H})$ satisfies

$$\int_0^\infty e^{2\gamma s}\|\sigma(s)\|^2_{L_Q^0} ds < \infty,\ \gamma > 0.$$

**Theorem 4.1.** *Assume that* $f(t,0) = g(t,0) = 0$, $\forall t \geq 0$, $k \in \mathbb{N}$. *Assume that hypotheses* $(H2)(ii)$, $(H3)$ *and* $(H1^{'})$-$(H3^{'})$ *hold. If*

$$L_1 = 6\widetilde{K}^2\left(L_g|(-A)^{-\beta}|^2 + M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + M^2 L_f \lambda^{-2}\right) < 1,$$

*then, there exists unique mild solution to (1.1)-(1.3) defined on* $(-\infty,\infty)$.

*Proof.* We shall consider the space

$$\begin{aligned}\mathcal{D}_{\mathcal{F}_\infty} = \big\{ & y\colon\ (-\infty,\infty)\times\Omega \to \mathcal{H},\ y_k \in C(J_k,\mathcal{H}) \text{ for } k = 1,\dots,\ y_0 \in \mathcal{D}_{\mathcal{F}_0}, \\ & \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ \ k = 1,2,\cdots, \\ & \text{and } \sup_{t\in[0,\infty)} E(|y(t)|^2) < \infty\big\},\end{aligned}$$

where $y_k$ denotes the restriction of $y$ to $J_k = (t_{k-1}, t_k]$, $k = 1,2,\cdots$; $\lim_{k\to\infty} t_k = \infty$ and $J_0 = (-\infty, 0]$. Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$.

Consider the set $Z^0_{\mathcal{F}_\infty} = \{y \in \mathcal{D}_{\mathcal{F}_\infty} : \sup_{t\in J} E\|y\|^2 < \infty\}$ endowed with the norm

$$\|y\|_{Z^0_{\mathcal{F}_\infty}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0\leq s\leq\infty} (E\|y(s)\|^2)^{\frac{1}{2}}.$$

We transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $\Psi : Z^0_{\mathcal{F}_\infty} \to Z^0_{\mathcal{F}_\infty}$ defined by

$$\Psi(y)(t) = \begin{cases} \phi(t), \ \text{if } t \in (-\infty,0], \\ S(t)[\phi(0) - g(0,\phi)] + g(t,y_t) + \displaystyle\int_0^t AS(t-s)g(s,y_s)ds \\ + \displaystyle\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \ \text{if } t \in [0,\infty). \end{cases}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ S(t)\phi(0), & t \in [0,\infty). \end{cases}$$

Then $\widehat{\phi} \in \mathcal{D}_{\mathcal{F}_\infty}$. Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t < \infty$. It is evident that $z$ satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} z(t) = \ & -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ & + \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in J. \end{aligned}$$

Set

$$\begin{aligned} Z^1_{\mathcal{F}_\infty} = \big\{ & z\colon\ (-\infty,\infty)\times\Omega \to \mathcal{H},\ z_k \in C(J_k,\mathcal{H}) \text{ for } k = 1,\dots m,\ z \in Z^0_{\mathcal{F}_\infty} \\ & \text{and } z_0 = 0, \text{ and there exist } z(t_k^-) \text{ and } z(t_k^+) \text{ with } z(t_k) = z(t_k^-),\ \ k \geq 1, \\ & \text{and } \sup_{t\in[0,\infty)} E(|z(t)|^2) < \infty\big\}. \end{aligned}$$

For any $z \in Z^1_{\mathcal{F}_\infty}$, we have

$$\|z\|_{Z^1_{\mathcal{F}_\infty}} = \|z_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t\in J}(E|z(t)|^2)^{\frac{1}{2}} = \sup_{t\in J}(E|z(t)|^2)^{\frac{1}{2}}$$

Thus, $(Z^1_{\mathcal{F}_\infty}, \|.\|_{Z^1_{\mathcal{F}_\infty}})$ is a Banach space.

Let the operator $\widehat{\Psi} : Z^1_{\mathcal{F}_\infty} \to Z^1_{\mathcal{F}_\infty}$ be defined by

$$\widehat{\Psi}(z) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \displaystyle\int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ + \displaystyle\int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB^H_Q(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in [0, +\infty). \end{cases}$$

The problem of finding a solution of problem (1.1)-(1.3) is reduced to finding a solution of the operator equation $\widehat{\Psi}(z)(t) = z(t)$, $t \in [0, \infty)$.

Consider the set

$$\mathcal{B} = \{z \in Z^1_{\mathcal{F}_\infty} : \exists a > 0,\ M^* = M^*(\phi, a) \text{ such that } E|z(t)|^2 \le M^* e^{-at},\ t \ge 0\};$$

then, $\mathcal{B} \subset Z^1_{\mathcal{F}_\infty}$ is closed.

Now, we will show that by using the Banach fixed point theorem, the operator $\widehat{\Psi}$ has a fixed point.

**Step 1**: We first verify $\widehat{\Psi}(\mathcal{B}) \subset \mathcal{B}$. We denote by $M_i^*, i = 1, 2, \cdots$ finite positive constants depending on $\phi$ and $a$.

For any $z \in \mathcal{B}$, we have

$$\begin{aligned} \widehat{\Psi}(z)(t) =& - S(t)g(0, \phi(0)) + g(t, z_t + \widehat{\phi}_t) \\ &+ \int_0^t AS(t-s)g(s, z_t + \widehat{\phi}_t)ds + \int_0^t S(t-s)f(s, z_t + \widehat{\phi}_t)ds \\ &+ \int_0^t S(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k) + \widehat{\phi}(t_k)) \\ =&: \sum_{1\le i\le 6} \eta_i(t). \end{aligned}$$

By assumption $(H1^{'})$, we have

$$E|\eta_1(t)|^2 \le M^2 E|g(0, \phi(0))|^2 e^{-\lambda t} \le M_1^* e^{-\lambda t}. \tag{4.1}$$

To estimate $\eta_i(t)$, $i = 2, \cdots, 5$, we observe that for $z \in \mathcal{D}^{'}_{\mathcal{F}_\infty}$,
the following useful estimate holds

$$\begin{aligned} E\|z(t)\|^2 \le& 2\widetilde{K}^2 M^* e^{-at} + 2\widetilde{N}\|z_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} e^{-at} \\ \le& \ 2\widetilde{K}^2 M^* e^{-at}, \end{aligned}$$

where $\|z_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$.

By assumption $(H2)(ii)$ we have

$$\begin{aligned}\mathbb{E}|\eta_2(h)|^2 \leq & |(-A)^{-\beta}|^2 E|(-A)^{\beta} g(t, z_t + \widehat{\phi}_t) - (-A)^{\beta} g(t, 0)|^2 \\ \leq & \ |(-A)^{-\beta}|^2 L_g E|z(t)|^2 \\ \leq & \ 2\widetilde{K}^2 M^* \|(-A)^{-\beta}\|^2 e^{-at}.\end{aligned}$$

Then we have

$$E|\eta_2(t)|^2 \leq M_2^* e^{-at}. \tag{4.2}$$

Using Lemma 2.5, Hölder's inequality and assumption (H3) we obtain that

$$\begin{aligned}E|\eta_3(t)|^2 \leq & E\Big|\int_0^t AS(t-s)g(t, z_s + \widehat{\phi}_s)\Big|^2 \\ \leq & \int_0^t \Big|(-A)^{1-\beta} S(t-s)\Big| ds \int_0^t \Big|(-A)^{1-\beta} S(t-s)\Big| E|(-A)^{\beta} g(s, z_s + \widehat{\phi}_s)|^2 ds \\ \leq & M_{1-\beta}^2 L_g \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E|z(s)|^2 ds. \\ \leq & 2M_{1-\beta}^2 L_g \frac{\Gamma(\beta)}{\lambda^{\beta}} \widetilde{K}^2 M^* e^{-at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\ \leq & 2M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{\beta}(\lambda-a)^{\beta}} \widetilde{K}^2 M^* e^{-at}.\end{aligned}$$

We therefore have

$$E|\eta_3(t)|^2 \leq M_3^* e^{-at} \tag{4.3}$$

Similarly, by assumption $(H2^{'})$,

$$\begin{aligned}\mathbb{E}|\eta_4(t)|^2 \leq & E\Big|\int_0^t S(t-s) f(t, z_s + \widehat{\phi}_s) ds\Big|^2 \\ \leq & \ M^2 L_f \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} E|z(s)|^2 ds \\ \leq & \ 2M^2 L_f \lambda^{-1} \widetilde{K}^2 M^* \int_0^t e^{-\lambda(t-s)} e^{-as} ds \\ \leq & \ 2M^2 L_f \lambda^{-1} (\lambda - a)^{-1} \widetilde{K}^2 M^* e^{-at} \\ \leq & \ M_4^* e^{-at}\end{aligned}$$

Therefore,

$$E\|\eta_4(t)\|^2 \leq M_4^* e^{-at}. \tag{4.4}$$

Now, for the term $\eta_5(t)$, we have

$$\eta_5(t) \leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|_{L_Q}^2 ds. \tag{4.5}$$

From this inequality we can ensure that

$$E|\eta_5(t)|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda_1 t} \int_0^t e^{2\lambda_2 s} \|\sigma(s)\|_{L_Q}^2 ds, \tag{4.6}$$

where $\lambda_1 = \lambda \wedge \lambda_2$. Indeed, if $\lambda < \lambda_2$, then $\lambda_1 = \lambda$ and we have

$$E|\eta_5(t)|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\lambda s} \|\sigma(s)\|_{L_Q}^2 ds$$

$$\leq 2M^2Ht^{2H-1}e^{-2\lambda_1 t}\int_0^t e^{2\lambda_2 s}\|\sigma(s)\|^2_{L_Q}ds.$$

If $\lambda_2 < \lambda$, then $\lambda_1 = \lambda_2$ and we have

$$\begin{aligned}E|\eta_5(t)|^2 &\leq 2M^2Ht^{2H-1}e^{-2\lambda_2 t}\int_0^t e^{-2(\lambda-\lambda_2)(t-s)}e^{2\lambda_2 S}\|\sigma(s)\|^2_{L_Q}ds\\ &\leq 2M^2Ht^{2H-1}e^{-2\lambda_1 t}\int_0^\infty e^{2\lambda_2 s}\|\sigma(s)\|^2_{L_Q}ds.\end{aligned}$$

Since $\sup_{t\geq 0}(t^{2H-1}e^{-\lambda_1 t}) < \infty$ , this, together with (4.6), gives us

$$E|\eta_5(t)|^2 \leq M_5^* e^{-\lambda_2 t}.$$

From $(H3)$ and Hölder's inequality, we obtain the following estimate for $\eta_6(t)$

$$\begin{aligned}E|\eta_6(t)|^2 \;\;&\leq\;\; E\Big|\sum_{0<t_k<t}S(t-t_k)I_k(z(t_k)+\widehat{\phi}(t_k))\Big|^2\\ &\quad +E\Big(\Big|\sum_{0<t_k<t}S(t-t_k)\Big|\Big|I_k(z(t_k)+\widehat{\phi}(t_k))\Big|\Big)^2\\ &\leq\;\; M^2e^{-at}\sum_{k=1}^\infty d_k\sum_{k=1}^\infty d_ke^{-2\lambda t_k} \leq M_6^*e^{-at}.\end{aligned}$$

We therefore have

$$E\|\eta_4(t)\|^2 \leq M_6^* e^{-at}. \tag{4.7}$$

Combining (4.1)-(4.7) we see that there exist $\overline{M}^* > 0$ and $\overline{a} > 0$ such that

$$E\|\widehat{\Psi}(t)\|^2 \leq \overline{M}^* e^{-\overline{a}t},\ t\geq 0$$

Hence, we can conclude that $\widehat{\Psi}(\mathcal{B}) \subset \mathcal{B}$.

**Step 2**: Now, we prove that $\widehat{\Psi}$ is a contracting mapping in $\mathcal{B}$.

For every $z_1, z_2 \in \mathcal{B}$ and $t \in [0,\infty)$, we obtain

$$\begin{aligned}&E|\widehat{\Psi}(z_1)(t)-\widehat{\Psi}(z_2)(t)|^2\\ &\quad\leq 3E|g(t,z_{1t}+\widehat{\phi}_t)-g(t,z_{2t}+\widehat{\phi}_t)|^2\\ &\qquad+3E\Big|\int_0^t AS(t-s)(g(s,z_{1s}+\widehat{\phi}_s)-g(s,z_{2s}+\widehat{\phi}_s))\Big|^2ds\\ &\qquad+3E\Big|\int_0^t S(t-s)(f(s,z_{1s}+\widehat{\phi}_s)-f(s,z_{2s}+\widehat{\phi}_s))\Big|^2ds\\ &\quad\leq 3L_g\|(-A)^{-\beta}\|^2E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}\\ &\qquad+3M_{1-\beta}^2L_g\int_0^t(t-s)^{\beta-1}e^{-\lambda(t-s)}ds\int_0^t(t-s)^{\beta-1}e^{-\lambda(t-s)}E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds,\\ &\qquad+3M^2L_f\int_0^t e^{-\lambda(t-s)}ds\int_0^t e^{-\lambda(t-s)}E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds\\ &\quad\leq 3L_g\|(-A)^{-\beta}\|^2E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}\\ &\qquad+3M_{1-\beta}^2L_g\frac{\Gamma^2(\beta)}{\lambda^{2\beta}}E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}\\ &\qquad+3M^2L_f\lambda^{-2}E\|z_1(t)-z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(3L_g\|(-A)^{-\beta}\|^2 + 3M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + 3M^2 L_f \lambda^{-2}\right) \\
&\quad \times \left[2\widetilde{K}^2 \sup_{t\geq 0} E|z_1(t)-z_2(t)|^2 + 2\widetilde{N}^2\|z_{10}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}^2\|z_{20}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\right], \\
&\leq L_1 \sup_{t\geq 0} E|z_1(t)-z_2(t)|^2.
\end{aligned}
$$

Hence, $\widehat{\Psi}$ is a contraction mapping on $\mathcal{B}$ and therefore $\widehat{\Psi}$ has a unique fixed point, since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, \infty)$. Then $y$ is a fixed point of the operator $\Psi$ which is a mild solution of the problem (1.1)-(1.3). This completes the proof. □

## 5. Attractivity of solutions

In this section we study the local attractivity of solutions of the problem (1.1)-(1.3)

**Definition 5.1.** *([13]) We say that solutions of (1.1)-(1.3) are locally attractive if there exists a closed ball $\bar{\mathcal{B}}(z^*,\rho)$ in the space $Z^1_{\mathcal{F}_\infty}$ for some $z^* \in Z^0_{\mathcal{F}_\infty}$ such that for arbitrary solutions $z$ and $\tilde{z}$ of (1.1)-(1.3) belonging to $\bar{\mathcal{B}}(z^*,\rho)$ we have*

$$\lim_{t\longrightarrow+\infty} E|z(t)-\tilde{z}(t)|^2 = 0.$$

Under the assumptions of Section 3 and 4 , let $z^*$ be a solution of (1.1)-(1.3) and $\bar{B}(z^*,\rho)$ the closed ball in $\mathcal{D}_{\mathcal{F}_T'}$ with $\rho$ satisfying

$$\rho \geq \frac{6M^2\lambda^{-1}\|p\|_{L^1}}{1 - 12L_g\left(\|(-A)^{-\beta}\|^2 + \frac{C_{1-\beta}^2\Gamma^2(\beta)}{\lambda^{2\beta}}\right) \times \widetilde{K}^2 - 24M^2\lambda^{-1}\widetilde{K}^2\|p\|_{L^1}}.$$

Moreover, we assume that

$$\lim_{t\longrightarrow\infty}\zeta(t) = 0,\ \lim_{t\longrightarrow\infty}\int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s) = 0 \text{ and } \lim_{t\longrightarrow\infty}\int_0^t e^{-\lambda(t-s)}p(s)(s) = 0. \tag{5.1}$$

Then, for $z \in \bar{B}(z^*,\rho)$ by $(H1')$, $(H2')$ and $(H2)$ we have

$$
\begin{aligned}
&E|z(t)-z^*(t)|^2 \\
&= E|\widehat{\Psi}_1(z)(t) - \widehat{\Psi}_1(z^*)(t)|^2 \\
&\leq 3E|g(t, z_t+\widehat{\phi}_t) - g(t, z_t^*+\widehat{\phi}_t^*)|^2 \\
&\quad +3\int_0^t AS(t-s)(g(s,z_s+\widehat{\phi}_s) - g(s,z_s^*+\widehat{\phi}_s^*))\Big|^2 \\
&\quad +3E\Big|\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s) - f(s,z_s^*+\widehat{\phi}_s^*)|^2 ds \\
&\leq 3L_g\|(-A)^{-\beta}\|^2\|(z(t)+\widehat{\phi}(t)) - (z^*(t)+\widehat{\phi}^*(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\quad +3\frac{\Gamma(\beta)}{\lambda^\beta}\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{\beta-1}}e^{-2\lambda(t-s)}L_g\|(z(t)+\widehat{\phi}(t)) - (z^*(t)+\widehat{\phi}^*(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds, \\
&\quad +3\lambda^{-1}M^2\int_0^t e^{-\lambda(t-s)}p(s)\left[\|z+\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|z^*+\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\right]
\end{aligned}
$$

$$\leq 12L_g\left(\|(-A)^{-\beta}\|^2+\frac{C_{1-\beta}^2\Gamma^2(\beta)}{\lambda^{2\beta}}\right)\times \widetilde{K}^2\rho$$
$$+6M^2\lambda^{-1}(4\widetilde{K}^2\rho+1)\|p\|_{L^1}\leq \rho,$$

Therefore, we have $\Phi(\bar{B}(z^*,\rho))\subset \bar{B}(z^*,\rho)$.

So, for each solution of problem (1.1)-(1.3) $z\in \bar{B}(z^*,\rho)$ and $t\in J=[0,\infty)$, we have

$$\begin{aligned}
E|z(t)-z^*(t)|^2 &= E|\widehat{\Phi}(z)(t)-\widehat{\Phi}(z^*)(t)|^2\\
&\leq 3E|g(t,z_t+\widehat{\phi}_t)-g(t,z_t^*+\widehat{\phi}_t^*)|^2\\
&\quad +3E\left|\int_0^t AS(t-s)(g(s,z_s+\widehat{\phi}_s)-g(s,z_s^*+\widehat{\phi}_s^*))ds\right|^2\\
&\quad +3E\left|\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s^*)-f(s,z_s^*+\widehat{\phi}_s^*)ds\right|^2\\
&\leq 6\|(-A)^{-\beta}\|^2\zeta(t)(\|z(t)+\widehat{\phi}(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*(t)+\widehat{\phi}^*(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}})\\
&\quad +6C_{1-\beta}^2\tfrac{\Gamma(\beta)}{\lambda^\beta}\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s)(\|z(s)+\widehat{\phi}(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*(s))+\widehat{\phi}^*(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds,\\
&\quad +6\lambda^{-1}M^2\int_0^t e^{-\lambda(t-s)}p(s)\Big[\|z+\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*+\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}+2\Big]\\
&\leq 12\|(-A)^{-\beta}\|^2(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho)\zeta(t)\\
&\quad +12C_{1-\beta}^2\tfrac{\Gamma(\beta)}{\lambda^\beta}(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho))\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s)ds\\
&\quad +12\lambda^{-1}M^2(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho+2)\int_0^t e^{-\lambda(t-s)}p(s)ds,
\end{aligned}$$

where

$$\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}=\max\{\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}},\|\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}\}.$$

Hence, from (5.1), we conclude that

$$\lim_{t\longrightarrow\infty}E|z(t)-\widetilde{z}(t)|^2=0.$$

Consequently, the solutions of problem (1.1)-(1.3) are locally attractive.

## 6. An example

Consider the following stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} d[u(t,\xi)+G(t,u(t-h,\xi))]=\dfrac{\partial^2}{\partial\xi^2}u(t,\xi)+F(t,u(t-h,\xi))\\ \qquad\qquad +\sigma(t)\dfrac{dB_Q^H}{dt},\quad t\geq 0,\quad t\neq t_k,\quad 0\leq\xi\leq\pi,\\ u(t_k^+,\xi)-u(t_k^-,\xi)=\alpha_k u(t_k^-,\xi),\qquad k=1,\cdots,m,\\ u(t,0)=u(t,\pi)=0,\qquad t\geq 0,\\ u(t,\xi)=\phi(t,\xi),\qquad -\infty\leq t\leq 0, 0\leq\xi\leq\pi,\end{cases}\tag{6.1}$$

where $\alpha_k>0$, $B_Q^H$ denotes a fractional Brownian motion and $G,F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Let

$$y(t)(\xi)=u(t,\xi)\ t\in J=[0,T],\ \xi\in[0,\pi],$$

$$I_k(y(t_k)) = \alpha_k u(t_k^-, \xi),\ \xi \in [0,\pi],\ k = 1, \cdots, m,$$
$$g(t,\phi)(\xi) = G(t, \phi(-h,\xi)),\ \theta \in (-\infty, 0],\ \xi \in [0,\pi],$$
$$f(t,\phi)(\xi) = F(t, \phi(-h,\xi)),\ \theta \in (-\infty, 0],\ \xi \in [0,\pi],$$
$$\phi(\theta)(\xi) = \phi(\theta,\xi),\ \theta \in (-\infty, 0],\ \xi \in [0,\pi].$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. We define the operator $A$ by $Au = u^{''}$, with domain

$$D(A) = \{u \in \mathcal{H}, u^{''} \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}.$$

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} e^{-n^2 t}\langle z, e_n\rangle e_n,\ z \in \mathcal{H},$$

and $A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$, which is given by $S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n$, $u \in \mathcal{H}$, where $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of $A$. Due to the fact that the semigroup $\{S(t)\}$ is analytic and compact, there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$ for all $t \in J$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\gamma_n^H(t)e_n,$$

where $H \in (1/2, 1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions that are mutually independent. Assume now that

**(i):** There exist positive number, $d_k,\ k = 1, \cdots, m$ such that

$$|I_k(\xi)| \leq d_k,$$

for any $\xi \in \mathbb{R}$

**(ii):** The function $g : [0,T] \times \mathcal{H} \longrightarrow \mathcal{H}$ defined by $g(t,u)(.) = G(t, u(.))$ is continuous and we impose suitable conditions on $G$ to satisfy assumption $(H2)$ .

**(iii):** Assume that there exists an integrable function $\eta : [0,T] \longrightarrow \mathbb{R}^+$ such that

$$E|F(t, u(\omega)|^2 \leq \eta(t)\psi(E|u(\omega)|^2)$$

for any $t \in [0,T]$ and any random variable $u(\cdot) \in L^2(\Omega)$, where $\psi : [0,\infty) \longrightarrow (0,\infty)$ is continuous and nondecreasing with

$$\int_1^{\infty} \frac{ds}{\psi(s)} = +\infty.$$

**(iv):** The function $g : [0,T] \longrightarrow L^2_Q(\mathcal{K},\mathcal{H})$ is bounded, that is, there exists a positive constant $L$ such that

$$\int_0^T \|\sigma(s)\|^2_{L^2_Q} < L.$$

The problem (6.1) can be written in the abstract form

$$\begin{cases} d[y(t)+g(t,y_t)] = [Ay(t)+f(t,y_t)]dt + \sigma(t)dB^H_Q(t), & t \in J := [0,T], \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \ k = 1,\dots,m, \\ y(t) = \phi(t), \quad \text{for a.e. } t \in (-\infty,0]. \end{cases}$$

Thanks to those assumptions, it is straightforward to check that all the conditions of Theorem 3.3 hold. Hence, we can conclude that the problem (6.1) has at least one mild solution on $(-\infty,T]$.

In the case of $t \in J = [0,+\infty)$ we observe that

**($i'$):** $\|S(t)\| \le e^{-\pi^2 t}$, and $\|(-A)^{\frac{3}{4}}\| = 1$;

**($ii'$):** The function $f : [0,\infty) \times \mathcal{H} \longrightarrow \mathcal{H}$ defined by $f(t,u)(.) = F(t,u(.))$ is continuous and it is easy to impose suitable conditions on $F$ to make assumption $(H2')$ hold;

**($iii'$):** The function $g : [0,T] \longrightarrow L^2_Q(\mathcal{K},\mathcal{H})$ is bounded, that is, there exists a positive constant $L$ such that

$$\int_0^\infty e^{\gamma s}\|\sigma(s)\|^2_{L^0_Q} < L;$$

**($iv'$):** There exist positive number, $d_k, \ k = 1,\cdots,m,\cdots$ such that

$$|I_k(\xi)| \le d_k \text{ and } \sum_{k=1}^\infty d_k < \infty$$

for any $\xi \in \mathbb{R}$.

Thus, the problem (6.1) can be written in the abstract form

$$\begin{cases} d[y(t)+g(t,y_t)] = [Ay(t)+f(t,y_t)]dt + \sigma(t)dB^H_Q(t), & t \in J := [0,\infty); \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \ k = 1,\dots,m,\dots, \\ y(t) = \phi(t), \quad \text{for a.e. } t \in (-\infty,0]. \end{cases}$$

Thanks to these assumptions, it is straightforward to check that $(H1')$-$(H3')$, (H2) and (H3) hold. The assumptions in Theorem 4.1 are fulfilled, and conclude that system (6.1) has a unique mild solution on $(-\infty,\infty)$, which implies that the mild solution of (6.1) is locally attractive.

## 7. Concluding remarks

In this paper, by using the fixed point theory we investigated the problem (1.1)-(1.3) under various assumptions on the right hand-side and we have obtained a number of new results regarding the existence of mild solution and the local attractivity of mild solution. The main assumptions on the right hand-side are the Carathéodory or Lipschitz conditions, and for jump functions we require they be continuous bounded or satisfy a Lipschitz condition.

In the case where the jump functions $I_k \equiv 0$ the existence and exponential stability of mild solutions for (1.1) and (1.3) was studied by Boufoussi and Hajji [9], Caraballo *et al.* [11], Caraballo and Diop [12] and Revathi *et al* [33] . Also, it is interesting to consider the case when the jump functions $I_k$ are multivalued and coupled system of impulsive stochastic differential equations and inclusions. Such a case is motivated by some models from control theory [1, 7]. In the current paper, we have focused on the existence and attractivity of mild solutions for impulsive neutral stochastic differential equations with infinite delay.

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