Strongly Convergent Approximations to Fixed Points of Total Asymptotically Nonexpansive Mappings

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Abstract

In this work we prove a new strong convergence result of the regularized successive approximation method given by

$$y_{n+1} = q_n z_0 + (1 - q_n) T^n y_n, \quad n = 1, 2, ...,$$

where

$$\lim_{n\to\infty} q_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} q_n = \infty,$$

for T a total asymptotically nonexpansive mapping, i.e., T is such that

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)} \phi(\|x - y\|) + k_n^{(2)},$$

where k_n^1 and k_n^2 are real null convergent sequences and $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and such that $\phi(0) = 0$ and $\lim_{t\to\infty} \frac{\phi(t)}{t} \leq C$ for a certain constant $C > 0$.

Among other features, our results essentially generalize existing results on strong convergence for T nonexpansive and asymptotically nonexpansive. The convergence and stability analysis is given for both self- and nonself-mappings.

Running title: Strong convergence to fixed points

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1 Introduction

Iterative procedures for nonlinear operators have been largely studied by many authors in the last decades. One of the first results of this nature was obtained by Browder [5] for nonexpansive self-mappings defined on Hilbert spaces. Here Browder studied the iterative method:

$$x_\omega = \omega z_0 + (1-\omega)Tx_\omega. \tag{1.1}$$

for Ω a closed and convex subset of H, $z_0 \in \Omega$ an arbitrary point and $T : \Omega \to \Omega$ a nonexpansive mapping with nonempty fixed point set $\mathcal{N}(T) := \{x \in \Omega : Tx = x\}$.

In [5], Browder proved that $\lim\limits_{\omega\to 0} x_\omega$ exists and is a fixed point of T. This result was extended by Reich [17] to the case when X is a uniformly smooth Banach space. Furthermore, he showed that the fixed point set of T is a sunny nonexpansive retract of Ω.

The recursive formula (explicit scheme)

$$y_1 \in \Omega, \quad y_{n+1} = q_n z_0 + (1-q_n)Ty_n, \quad n = 1, 2, ..., \tag{1.2}$$

was introduced by Halpern [12] who discussed its convergence in the framework of Hilbert spaces. Later it has been investigated in [12, 18, 19, 20] with different additional properties on the sequence $\{q_n\}$, the operator T and the space X.

Browder's and Halpern's iterative procedures have motivated different schemes to find fixed points of asymptotically nonexpansive mappings (see Remark 1.2 for definition). In this way, T.C. Lim and H.K. Xu [15] studied the algorithm for T asymptotically nonexpansive which generates the sequence (implicit scheme)

$$x_n = q_n z_0 + (1-q_n)T^n x_n. \tag{1.3}$$

They showed that the sequence $\{x_n\}$ converges strongly to a fixed point of T in the framework of a uniformly smooth Banach space, under suitable conditions on the coefficients. Very recently, in [6], the strong convergence of the explicit scheme given by

$$y_1 \in \Omega \qquad y_{n+1} = q_n z_0 + (1-q_n)T^n y_n, \quad n = 1, 2, ..., \tag{1.4}$$

where $z_0 \in \Omega$, and

$$\lim_{n\to\infty} q_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} q_n = \infty, \tag{1.5}$$

has been studied in uniformly smooth spaces. It is worthwhile to point out that the convergence of the implicit scheme given by (1.3) is an important tool in order to prove the strong convergence of explicit schemes as (1.2).

In this paper we will consider the class of the total asymptotically nonexpansive mappings which have been introduced very recently in [2].

Definition 1.1 (cf. [2]) *A mapping $T : \Omega \to \Omega$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $k_n^{(1)}$, $k_n^{(2)} \to 0$ as $n \to \infty$, and a continuous function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that*

$$\|T^n x - T^n y\| \le \|x-y\| + k_n^{(1)}\phi(\|x-y\|) + k_n^{(2)}. \tag{1.6}$$

Remark 1.2 *If $\phi(\lambda) \equiv 0$ then (1.6) takes the form*

$$\|T^n x - T^n y\| \le \|x - y\| + k_n^{(2)}.$$

Hence, if Ω is a bounded set and T^N is continuous for some integer $N \ge 1$ the mapping T is of asymptotically nonexpansive type. If $\phi(\lambda) = \lambda$ then we can write

$$\|T^n x - T^n y\| \le (1 + k_n^{(1)})\|x - y\| + k_n^{(2)}.$$

In addition, if $k_n^{(2)} = 0$ for all $n \ge 1$ then we obtain the definition of asymptotically nonexpansive mapping:

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad k_n \to 1.$$

If $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$ for all $n \ge 1$ then we obtain the class of nonexpansive mappings:

$$\|Tx - Ty\| \le \|x - y\|.$$

If $k_1^{(2)} = 0$ then it follows from (1.6) that T is uniformly continuous, however, it can be uniformly continuous even if $k_1^{(2)} \ne 0$.

To construct the strong convergent approximations to solutions of the equation

$$Tx = x \tag{1.7}$$

with a total asymptotically nonexpansive mapping T, we apply the iterative scheme given by (1.4).

For nonexpansive operators T, the algorithm (1.4) is written down in the following form:

$$y_1 \in \Omega, \quad y_{n+1} = q_n z_0 + (1 - q_n) T y_n, \quad n = 1, 2, ..., \tag{1.8}$$

where $\lim\limits_{n\to\infty} q_n = 0$.

We show next that (1.8) is the regularized successive approximation method for (1.7). As it is known, equation (1.7) is equivalent to

$$Ax = 0 \tag{1.9}$$

with the accretive operator $A = I - T : \Omega \to \Omega$. That is, in this case

$$\langle Ax - Ay, J(x - y)\rangle \ge 0, \tag{1.10}$$

where J stands for the normalized duality map. If x^* is a solution of (1.7) then $Ax^* = 0$. In the sequel, we assume that the fixed point set $\mathcal{N}(T)$ of T is not empty. We emphasize that the problem (1.7) belongs to the class of ill-posed problems (for more on ill-posted problems see [4]). Strongly convergent approximations to x^* can be obtained only by using some regularization procedure.

Let ω be a parameter such that $0 < \omega < 1$ and $\omega \to 1$. Obviously, if x^* is a solution of (1.9) then it is solution of the equation

$$\omega Ax = 0 \tag{1.11}$$

for any fixed $\omega > 0$. Using the general theory (see for example [4], Section 2.7), construct for (1.11) the operator regularization method with regularization parameter $\alpha = 1-\omega \to 0$, namely,

$$\omega Ax + (1-\omega)(x - z_0) = 0, \tag{1.12}$$

where $z_0 \in \Omega$. It is easy to see that (1.12) is equivalent to

$$x = (1-\omega)z_0 + \omega Tx. \tag{1.13}$$

Denote

$$T_\omega x = (1-\omega)z_0 + \omega Tx.$$

Since Ω is convex and closed, we have that $T_\omega : \Omega \to \Omega$, and (1.13) can be rewritten as

$$x = T_\omega x. \tag{1.14}$$

Consequently, by Banach Contraction Principle, equation (1.12) has a unique solution x_ω and the successive approximation method

$$x_1 \in \Omega \qquad x_{n+1} = (1-\omega)z_0 + \omega Tx_n$$

converges strongly to x_ω. Let X be uniformly smooth and $\omega_k \to 1$ as $k \to \infty$. Consider now the regularized equation

$$\omega_k Ax + (1-\omega_k)(x - z_0) = 0 \tag{1.15}$$

with k fixed and denote by x_k its unique solution. Then there exists $\bar{x}^* \in \mathcal{N}(T)$ such that $x_k \to \bar{x}^*$ as $k \to \infty$. Moreover (see [4]), $\bar{x}^*$ satisfies the inequality

$$\langle \bar{x}^* - z_0, J(\bar{x}^* - x^*)\rangle \geq 0 \quad \forall x^* \in \mathcal{N}(T).$$

It can be shown in the same way that (1.8) with T^n in place of T is the regularized successive approximation method for the equation (1.7) with total asymptotically nonexpansive mapping. Indeed, if total asymptotically nonexpansive mappings are considered in place of nonexpansive mappings, then

$$\langle A^n x - A^n y, J(x-y)\rangle \geq -k_n^{(1)}\phi(\|x-y\|)\|x-y\| - k_n^{(2)}\|x-y\|, \tag{1.16}$$

where $A^n = I - T^n$. It is clear that the analysis of strong convergence is more difficult in this situation, moreover, very little is known about the structure of the solution set. In particular the same holds for asymptotically nonexpansive mappings for which (1.16) is

$$\langle A^n x - A^n y, J(x-y)\rangle \geq -k_n^{(1)}\|x-y\|^2. \tag{1.17}$$

The main result of this paper, Theorem 3.1 in Section 3, states a strong convergence result for the iterative scheme (1.4) in reflexive Banach spaces with a weakly continuous duality map on unbounded domains. Notice that it is an open question wether a reflexive Banach space admitting a weakly sequentially continuous duality mapping is uniformly smooth. An implicit scheme convergence result is also proved. This result is used to guarantee the existence of sunny nonexpansive retractions.

In Section 4 we study the same iterative scheme for total asymptotically nonexpansive nonself-mapping. Finally, in Section 5, our last section we investigate the stability problem for iterative schemes with respect to perturbations of constraint sets for nonexpansive nonself-mappings.

2 Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$, let X^* be its dual space with the norm $\|\cdot\|_*$ and, as usual, denote the duality pairing of X and X^* by $\langle\varphi, x\rangle$, where $x \in X$ and $\varphi \in X^*$ (in other words, $\langle\varphi, x\rangle$ is the value of φ at x).

It is said that X is uniformly smooth if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = 1$ and $\|y\| \le \delta$, the inequality

$$2^{-1}(\|x+y\| + \|x-y\|) - 1 \le \varepsilon\|y\|$$

holds. The function

$$\rho_X(\tau) = \sup\{2^{-1}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = \tau\}$$

is called the modulus of smoothness of the space X.

This function is increasing and approaches to zero as $\tau \to 0$. Denote $h_X(\tau) = \dfrac{\rho_X(\tau)}{\tau}$. Observe that the space X is uniformly smooth if and only if $\lim\limits_{\tau\to 0} h_X(\tau) = 0$.

Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $\psi(t) \to \infty$ as $t \to \infty$ and $\psi(0) = 0$. The generalized duality mapping $J_\psi : X \to 2^{X^*}$ associated to a gauge function ψ is defined as

$$J_\psi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \psi(\|x\|)\|x\|,\ \|x^*\| = \psi(\|x\|)\}, \qquad x \in X.$$

In the case that $\psi(t) = t$ then $J_\psi = J$ which is the normalized duality map.

We say that a Banach space X has a weakly continuous duality map ([5]) if there exists a gauge function ψ for which the generalized duality map J_ψ is single-valued and weak-to-weak* sequentially continuous.

It is well-known that J_ψ is the subdifferential, in the sense of convex analysis, of the convex function

$$\Phi(t) = \int_0^t \psi(\tau)\, d\tau, \ \text{ for } \ \tau \ge 0,$$

and that J_ψ is single-valued if and only if X is smooth. We will need the following subdifferential inequality which is known to hold in smooth spaces:

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\psi(x+y)\rangle$$

for any $x, y \in X$.

Next we introduce some definitions and auxiliary results that will be needed in the sequel.

Definition 2.1 *Let X be a Banach space and C a nonempty closed convex subset of X. An operator $T : C \to X$ is demiclosed (at y) if $T(x) = y$ whenever $\{x_n\} \subseteq C$ is a sequence weakly convergent to x and $T(x_n) \to y$ as $n \to \infty$.*

Definition 2.2 *A Banach space X satisfies the Opial's condition if for each sequence $\{x_n\}$ in X, the relation $x_n \rightharpoonup x$ implies that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $x \neq y$.

Definition 2.3 *A Banach space X satisfies the Generalized Gossez-Lami Dozo property (GGLD-property) if*

$$\liminf_{n\to\infty} \|x_n\| < \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_m - x_n\|$$

whenever $\{x_n\}$ is a weak null sequence which is not norm convergent.

The following demiclosedness principle can be found in [11].

Theorem 2.4 *Let X be a Banach space with GGLD-property and Opial's condition. Let C be a weakly compact convex subset of X and $T : C \to C$ a uniformly continuous mapping of asymptotically nonexpansive type. Then $I - T$ is demiclosed at zero.*

The following result is well-known (see [13] and [14]).

Proposition 2.5 *If in a reflexive Banach space X the duality mapping J is weakly continuous then X satisfies GGLD-property and Opial's condition.*

The next corollary follows as a consequence of this proposition and a careful reading of the original proof of Theorem 2.4.

Corollary 2.6 *Let X be a reflexive Banach space with a weakly continuous duality mapping J. Let C be a closed convex subset of X and $T : C \to C$ a uniformly continuous mapping and total asymptotically nonexpansive with bounded orbits. Then $I - T$ is demiclosed at zero.*

We will also use the concept of a sunny nonexpansive retraction [10] and, in particular, its characterization by means of the duality map in a smooth Banach space.

Definition 2.7 *Let C be a non-empty subset of a Banach space X and D a subset of C. A mapping $Q : C \to D$ is said to be*

(i) a retraction onto D if $Q^2 = Q$;

(ii) a nonexpansive retraction if it also satisfies the inequality

$$\|Qx - Qy\| \leq \|x - y\|, \quad \text{for all } x, y \in C;$$

(iii) a sunny retraction if for all $x \in C$ and for all $0 \leq t < \infty$,

$$Q(Qx + t(x - Qx)) = Qx, \quad \text{whenever } Qx + t(x - Qx) \in C.$$

Proposition 2.8 *Assume that C is a non-empty closed convex subset of a smooth Banach space X and D is a subset of C. Then a nonexpansive mapping $Q: C \to D$ is a sunny retraction if and only if for all $x \in C$ and for all $\xi \in D$,*

$$\langle x - Qx, J(\xi - Qx)\rangle \le 0.$$

In particular, there is at most one sunny nonexpansive retraction on D.

Remark 2.9 *Proposition 2.5 and Corollary 2.6 remain still valid if the normalized duality mapping J is replaced by the duality mapping J_ψ with the gauge function $\psi(t)$. Moreover, we can use J_ψ to characterize sunny nonexpansive retractions in a smooth Banach space given by Proposition 2.8.*

Let G_1 and G_2 be nonempty closed subsets of X. The Hausdorff distance between G_1 and G_2 is defined by the following formula:

$$\mathcal{H}(G_1, G_2) = \max\{\sup_{z_1 \in G_1} \inf_{z_2 \in G_2} \|z_1 - z_2\|, \ \sup_{z_1 \in G_2} \inf_{z_2 \in G_1} \|z_1 - z_2\|\}.$$

We need the following lemma [3] in order to prove the main result of Section 5.

Lemma 2.10 *If X is a uniformly smooth Banach space, Ω_1 and Ω_2 are closed convex subsets of X such that the Hausdorff distance $\mathcal{H}(\Omega_1, \Omega_2) \le \sigma$ and Q_{Ω_1} and Q_{Ω_2} are the (unique) sunny nonexpansive retractions onto the subsets Ω_1 and Ω_2, respectively, then*

$$\|Q_{\Omega_1} x - Q_{\Omega_2} x\|^2 \le 16R(2r + d) h_X(16LR^{-1}\sigma), \tag{2.1}$$

where $r = \|x\|$, $d = \max\{d_1, d_2\}$, $R = 2(2r + d) + \sigma$ and $1 < L < 1.7$ is the Figiel constant [1, 2, 9]. Here $d_i =$ dist (θ, Ω_i), $i = 1, 2$, and θ is the origin of the space X.

We will often apply the following lemma on numerical recurrent inequalities.

Lemma 2.11 *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be nonnegative, $\{\alpha_n\}$ be positive real numbers such that*

$$\lambda_{n+1} \le \lambda_n - \alpha_n \lambda_n + \gamma_n, \quad \forall\, n \ge 1.$$

Let for all $n > 1$

$$\frac{\gamma_n}{\alpha_n} \le c_1 \quad \text{and} \quad \alpha_n \le \alpha. \tag{2.2}$$

Then $\lambda_n \le max\{\lambda_1, K_*\}$, where $K_* = (1 + \alpha)c_1$. In addition, if

$$\sum_1^\infty \alpha_n = \infty \quad \text{and} \quad \frac{\gamma_n}{\alpha_n} \to 0$$

then $\lambda_n \to 0$ as $n \to \infty$.

3 Convergence Analysis of Successive Approximation Method

The goal of this section is to prove strong convergence of the regularized successive approximation method (1.4). Let us consider the explicit scheme (1.4) given by

$$y_0 \in \Omega, \quad y_{n+1} = q_n z_0 + (1-q_n)T^n y_n, \quad n = 1, 2, ...,$$

with

$$\lim_{n\to\infty} q_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} q_n = \infty. \tag{3.1}$$

Theorem 3.1 *Let Ω be a nonempty closed and convex subset of a smooth reflexive Banach space X with a weakly sequentially continuous duality map J_ψ, $T : \Omega \to \Omega$ a uniformly continuous mapping which is total asymptotically nonexpansive with nonempty fixed point set $\mathcal{N}(T)$. Let $\mathcal{N}(T)$ be such that there exists a sunny nonexpansive retraction $Q : \Omega \to \mathcal{N}(T)$.*

Let $z_0 \in \Omega$ and $\{q_n\} \subset (0, 1]$ a sequence satisfying (3.1). Let the sequence $\{y_n\}$ be generated by (1.4). Assume that

$$\lim_{n\to\infty} \frac{k_n^{(1)} + k_n^{(2)}}{q_n} = 0, \tag{3.2}$$

and that there exist positive constants M_0 and M_1 such that $\phi(\lambda) \le M_0\lambda$ for $\lambda \ge M_1$. Suppose that $\lim_n \|y_n - Ty_n\| = 0$, then $\{y_n\}$ converges strongly to the fixed point $\bar{x}^* = Qz_0$ of T.

Proof. Firstly we observe that the sequence $\{y_n\} \subset \Omega$ because Ω is convex. Take $x^* \in \mathcal{N}(T)$. It follows from (1.4) that

$$\|y_{n+1} - x^*\| \le q_n\|z_0 - x^*\| + (1-q_n)\|T^n y_n - T^n x^*\|$$

$$\le q_n\|z_0 - x^*\| + (1-q_n)\Big(\|y_n - x^*\| + k_n^{(1)}\phi(\|y_n - x^*\|) + k_n^{(2)}\Big).$$

Denoting $\lambda_n = \|y_n - x^*\|$ we have

$$\lambda_{n+1} \le (1-q_n)\lambda_n + q_n\|z_0 - x^*\| + (1-q_n)\Big(k_n^{(1)}\phi(\lambda_n) + k_n^{(2)}\Big). \tag{3.3}$$

Since ϕ is continuous it attains its maximum M on $[0, M_1]$. Then it is easy to verify that for all $\lambda \in [0, \infty)$

$$\phi(\lambda) \le M + M_0\lambda.$$

The inequality (3.3) is rewritten as

$$\lambda_{n+1} \le \lambda_n - \Big(q_n - (1-q_n)k_n^{(1)}M_0\Big)\lambda_n + \gamma_n,$$

where

$$\gamma_n = (1-q_n)\Big(k_n^{(1)}M + k_n^{(2)}\Big) + q_n\|z_0 - x^*\|.$$

Without loss of generality, in view of (3.2), we assume that there exist constants $\alpha \in (0,1)$ and $M_2 > 0$ such that for all $n \geq 1$

$$\frac{k_n^{(1)}}{q_n} \leq \frac{M_0(1-\alpha)}{1-q_n}, \tag{3.4}$$

and

$$\frac{\gamma_n}{q_n} \leq \alpha M_2.$$

Then

$$\lambda_{n+1} \leq \lambda_n - \alpha q_n \lambda_n + \gamma_n.$$

By Lemma 2.11, we conclude that

$$\lambda_n \leq \max\{\lambda_1, (1+\alpha)M_2\}.$$

Thus, the sequence $\{y_n - x^*\}$ is bounded, which, clearly, implies that $\{y_n\}$ is a bounded sequence.

Applying the subdifferential inequality to

$$y_{n+1} - Qz_0 = (1-q_n)(T^n y_n - Qz_0) + q_n(z_0 - Qz_0)$$

we deduce that

$$\Phi(\|y_{n+1} - Qz_0\|) \leq \Phi((1-q_n)\|T^n y_n - Qz_0\|) + q_n\langle z_0 - Qz_0, J_\psi(y_{n+1} - Qz_0)\rangle.$$

Since T is total asymptotically nonexpansive and $Qz_0 \in \mathcal{N}(T)$, we have

$$\|T^n y_n - Qz_0\| \leq \|y_n - Qz_0\| + \nu_n,$$

where $\nu_n = Mk_n^{(1)} + M_0 k_n^{(1)}\|y_n - Qz_0\| + k_n^{(2)}$ is bounded and vanishes as $n \to \infty$. Now, since Φ is a convex and nondecreasing, for n large enough we have

$$\begin{aligned}\Phi(\|T^n y_n - Qz_0\|) &\leq (1-\nu_n)\Phi(\|y_n - Qz_0\|) + \nu_n \Phi(\|y_n - Qz_0\| + 1) \\ &\leq \Phi(\|y_n - Qz_0\|) + \nu_n M_3\end{aligned}$$

for M_3 a suitable constant. Consequently

$$\begin{aligned}\Phi(\|y_{n+1} - Qz_0\|) \leq (1-q_n)\Phi(\|y_n - Qz_0\|) + (1-q_n)\nu_n M_3 + \\ + q_n\langle z_0 - Qz_0, J_\psi(y_{n+1} - Qz_0)\rangle.\end{aligned} \tag{3.5}$$

We claim that

$$\limsup_{n\to\infty}\langle z_0 - Qz_0, J_\psi(y_n - Qz_0)\rangle \leq 0.$$

Indeed, since the sequence $\{y_n\}$ is bounded and the space X reflexive there exists a subsequence $\{y_{n_k}\}$ which is weakly convergent in Ω. Let $\bar{y}$ be its weak limit. We can fix this subsequence so that

$$\limsup_{n\to\infty}\langle z_0 - Qz_0, J_\psi(y_n - Qz_0)\rangle = \lim_{k\to\infty}\langle z_0 - Qz_0, J_\psi(y_{n_k} - Qz_0)\rangle.$$

But we know that $y_{n_k} - Ty_{n_k} \to 0$ as $k \to \infty$, so, from the demiclosedness principle, we have that $\bar{y}$ is a fixed point of T. From the weak continuity of J_ψ and the characterization of sunny nonexpansive retraction (Proposition 2.8), our claim follows in the following way

$$\limsup_{n\to\infty} \langle z_0 - Qz_0, J_\psi(y_n - Qz_0)\rangle = \langle z_0 - Qz_0, J_\psi(\bar{y} - Qz_0)\rangle \leq 0.$$

Let us consider now (3.5), which we rewrite as follows

$$\lambda_{n+1} \leq \lambda_n - q_n\lambda_n + \gamma'_n$$

where $\lambda_n = \Phi(\|y_n - Qz_0\|)$ and

$$\gamma'_n = q_n\langle z_0 - Qz_0, J_\psi(y_{n+1} - Qz_0)\rangle + (1 - q_n)\nu_n M_3.$$

If we make $\alpha_n = q_n$ and $\gamma_n = \max\{0, \gamma'_n\}$, we can apply Lemma 2.11 to deduce that $\lambda_n \to 0$ as $n \to \infty$. Therefore $\{y_n\}$ converges strongly to Qz_0 and the proof is complete. ■

Next we study different situations that guarantee the fulfillment of some of the conditions imposed in Theorem 3.1. Observe first that if $\{y_n\}$ has a limit then

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.6}$$

Lemma 3.2 *Condition (3.6) is sufficient to guarantee* $\lim_n \|y_n - Ty_n\| = 0$ *in the previous theorem.*

Proof. Following the proof of Theorem 3.1, the sequence $\{y_n - x^*\}$ is bounded, say $\|y_n - x^*\| \leq C_1$. It is clear that

$$\|y_n\| \leq \|y_n - x^*\| + \|x^*\| \leq C_1 + \|x^*\| = C.$$

Observe that if $C_1 \leq M_1$ then $\phi(\|y_n - x^*\|) \leq M$. At the same time, if $C_1 \geq M_1$ then $\phi(\|y_n - x^*\|) \leq M_0\|y_n - x^*\| \leq M_0C_1$. Therefore,

$$\phi(\|y_n - x^*\|) \leq \max\{M, M_0C_1\} = \overline{M}.$$

In addition,

$$\begin{aligned}\|T^n y_n\| &\leq \|T^n y_n - T^n x^*\| + \|x^*\| \\ &\leq \|y_n - x^*\| + k_n^{(1)}\phi(\|y_n - x^*\|) + k_n^{(2)} + \|x^*\|.\end{aligned} \tag{3.7}$$

This means that the sequence $\{T^n y_n\}$ is bounded too. Then algorithm (1.4) yields the following limit equality:

$$\lim_{n\to\infty} (y_{n+1} - T^n y_n) = \lim_{n\to\infty} \Big(q_n(z_0 - T^n y_n)\Big) = 0. \tag{3.8}$$

Since

$$\|y_{n+1} - T^n y_{n+1}\| \leq \|y_{n+1} - T^n y_n\| + \|T^n y_n - T^n y_{n+1}\|,$$

we deduce that

$$\|y_{n+1} - T^n y_{n+1}\| \le \|y_{n+1} - T^n y_n\| + \|y_n - y_{n+1}\| + k_n^{(1)}\phi(\|y_n - y_{n+1}\|) + k_n^{(2)}.$$

In addition, from (3.8) and the fact that, by hypothesis, $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty}(y_n - T^{n-1}y_n) = 0. \tag{3.9}$$

Now,

$$\|y_n - Ty_n\| \le \|y_n - T^n y_n\| + \|T^n y_n - Ty_n\|. \tag{3.10}$$

In view of (3.8),

$$\|T^n y_n - y_n\| \le \|T^n y_n - y_{n+1}\| + \|y_{n+1} - y_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Since T is uniformly continuous, there exists a continuous increasing function $\omega : \mathbf{R} \to \mathbf{R}$ with $\omega(0) = 0$ satisfying the relations

$$\|T^n y_n - Ty_n\| = \|T(T^{n-1}y_n) - Ty_n\| \le \omega(\|T^{n-1}y_n - y_n\|).$$

By (3.9), it is easy to see that

$$\|T^n y_n - Ty_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Now (3.10) and (3.11) complete the proof. ∎

The next two propositions give conditions for the fulfillment of condition (3.6).

Proposition 3.3 *Let Ω be a closed convex subset of a Banach space X, $T : \Omega \to \Omega$ a total asymptotically nonexpansive mapping with nonempty fixed point set $\mathcal{N}(T)$. Take z_0 in Ω, $x^* \in \mathcal{N}(T)$ and $\{q_n\}$ a decreasing sequence in $(0,1)$ satisfying (3.1). Let the sequence $\{y_n\}$ be generated by (1.4). If the sequence $\{T^n y_n\}$ is bounded and*

$$\lim_{n\to\infty}\|T^n y_n - T^{n-1}y_{n-1}\| = 0$$

then (3.6) holds.

Proof. We have from (1.4)

$$y_{n+1} - y_n = (q_n - q_{n-1})z_0 + (1 - q_n)T^n y_n - (1 - q_{n-1})T^{n-1}y_{n-1}.$$

Then

$$\|y_{n+1} - y_n\| \le |q_n - q_{n-1}|(\|z_0\| + \|T^n y_n\|) + (1 - q_n)\|T^n y_n - T^{n-1}y_{n-1}\|.$$

The assumptions of the Proposition imply the claim. ∎

Proposition 3.4 *Let Ω be a closed convex subset of a Banach space X, $T : \Omega \to \Omega$ a total asymptotically nonexpansive mapping with a nonempty fixed point set $\mathcal{N}(T)$. Take z_0 some point in Ω, $x^* \in \mathcal{N}(T)$ and $\{q_n\}$ a decreasing sequence in $(0, 1)$ satisfying (3.1). Let the sequence $\{y_n\}$ be generated by (1.4). If (3.2) holds, there exist positive constants M_0 and M_1 such that $\phi(\lambda) \leq M_0\lambda$ for $\lambda \geq M_1$, and, if additionally, we assume that $\lim \frac{q_{n-1}}{q_n}$ exists and*

$$\lim_{n\to\infty} \frac{\|T^n y_{n-1} - T^{n-1} y_{n-1}\|}{q_n} = 0 \tag{3.12}$$

then (3.6) holds.

Proof. It is not difficult to state the following difference:

$$y_{n+1} - y_n = (1 - q_n)(T^n y_n - T^n y_{n-1}) + (q_n - q_{n-1})(z_0 - x^*)$$
$$+ (q_{n-1} - q_n)(T^{n-1} y_{n-1} - T^n x^*) + (1 - q_n)(T^n y_{n-1} - T^{n-1} y_{n-1}).$$

We have

$$\|T^n y_n - T^n y_{n-1}\| \leq \|y_n - y_{n-1}\| + k_n^{(1)} M + k_n^{(1)} M_0 \|y_n - y_{n-1}\| + k_n^{(2)}.$$

Further,

$$\|(q_n - q_{n-1})(z_0 - x^*) + (q_{n-1} - q_n)(T^{n-1} y_{n-1} - T^{n-1} x^*)\|$$
$$\leq |q_n - q_{n-1}|(\|z_0 - x^*\| + \|y_{n-1} - x^*\| + k_{n-1}^{(1)} M + k_{n-1}^{(1)} M_0 \|y_{n-1} - x^*\| + k_{n-1}^{(2)}).$$

Let

$$\mu_n = \|z_0 - x^*\| + \|y_n - x^*\| + k_n^{(1)} M + k_n^{(1)} M_0 \|y_n - x^*\| + k_n^{(2)}.$$

Since, by Theorem 3.1, $\{y_n\}$ is bounded, there exists a constant $M_4 > 0$ such that $\mu_n \leq M_4$ for all $n \geq 1$. Then, since in (3.4) we can chose α so close to 1 as needed,

$$\|y_{n+1} - y_n\| \leq \|y_n - y_{n-1}\| - kq_n \|y_n - y_{n-1}\| + (1 - q_n)(k_n^{(1)} M + k_n^{(2)})$$
$$+ |q_n - q_{n-1}| M_4 + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|$$

for a certain positive constant k. Denoting $\lambda_n = \|y_n - y_{n-1}\|$ one gets

$$\lambda_{n+1} \leq \lambda_n - kq_n\lambda_n + (1 - q_n)(k_n^{(1)} M + k_n^{(2)}) + |q_n - q_{n-1}| M_4 + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|.$$

Since $\lim \frac{q_{n-1}}{q_n}$ exists and $\sum_1^\infty q_n = \infty$ we conclude that $\lim \frac{q_{n-1}}{q_n} = 1$ and so

$$\lim \frac{|q_n - q_{n-1}|}{q_n} = 0. \tag{3.13}$$

Now (3.2), (3.12), (3.13) and Lemma 2.11 complete the proof. ■

The second part of this section is devoted to the study of the convergence of the implicit scheme (1.3) for asymptotically nonexpansive mappings and the existence of sunny nonexpansive retractions as those in Theorem 3.1.

Theorem 3.5 *Let X be a smooth reflexive Banach space which has a weakly sequentially continuous duality map J_ψ associated to a gauge function ψ, let Ω be a nonempty closed and convex subset of X, and $T : \Omega \to \Omega$ an asymptotically nonexpansive mapping with nonempty fixed point set $\mathcal{N}(T)$. Let $z_0 \in \Omega$ and $\{q_n\} \subseteq (0, 1]$ be a sequence such that $\lim_n q_n = 0$ and $\lim_n \dfrac{k_n^{(1)}}{q_n} = 0$. Then,*

(i) for $n \in \mathbf{N}$ large enough, there is a unique $x_n \in \Omega$ such that

$$x_n = q_n z_0 + (1 - q_n)T^n x_n. \tag{3.14}$$

If additionally we suppose that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then

(ii) the sequence $\{x_n\}$ strongly converges to a fixed point of T and $\mathcal{N}(T)$ is a sunny nonexpansive retract of Ω.

Proof. It is not hard to check that for n large enough the mapping $T_n x = q_n z_0 + (1 - q_n)T^n x$ is a contraction. Therefore the Banach Contraction Principle implies that the sequence $\{x_n\}$ is well-defined. Next we show that this sequence is bounded. Let $x^* \in \mathcal{N}(T)$, then

$$\|x_n - x^*\| = \|q_n(z_0 - x^*) + (1 - q_n)(T^n x_n - x^*)\| \le q_n\|z_0 - x^*\| + (1 - q_n)\|T^n x_n - x^*\|$$

$$\le q_n\|z_0 - x^*\| + (1 - q_n)(1 + k_n^{(1)})\|x_n - x^*\|.$$

Henceforth

$$\|x_n - x^*\| \le \frac{q_n}{q_n - (1 - q_n)k_n^{(1)}}\|z_0 - x^*\|.$$

The boundedness of $\{x_n\}$ follows from the condition $\lim_{n\to\infty} \dfrac{k_n^{(1)}}{q_n} = 0$. Since X is reflexive, there exists a weakly convergent subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$. Let $\bar{y} \in \Omega$ its weak limit, then the demiclosedness principle implies that $\bar{y} = T\bar{y}$. Further we will see that $\{x_{n_k}\}$ strongly converges to $\bar{y}$. Indeed,

$$x_{n_k} - \bar{y} = q_{n_k}(z_0 - \bar{y}) + (1 - q_{n_k})(T^{n_k} x_{n_k} - \bar{y}),$$

from the subdifferential inequality, we obtain

$$\Phi(\|x_{n_k} - \bar{y}\|) \le \Phi((1 - q_{n_k})\|T^{n_k} x_{n_k} - \bar{y}\|) + q_{n_k}\langle z_0 - \bar{y}, J_\psi(x_{n_k} - \bar{y})\rangle.$$

On the other hand,

$$\Phi((1 - q_{n_k})\|T^{n_k} x_{n_k} - \bar{y}\|) \le \Phi((1 - q_{n_k})(1 + k_{n_k}^{(1)})\|x_{n_k} - \bar{y}\|).$$

Since $(1 - q_{n_k})(1 + k_{n_k}^{(1)}) < 1$ for n_k large enough, we also have from the convexity of the function Φ,

$$\Phi((1 - q_{n_k})\|T^{n_k} x_{n_k} - \bar{y}\|) \le (1 - q_{n_k})(1 + k_{n_k}^{(1)})\Phi(\|x_{n_k} - \bar{y}\|).$$

Hence

$$(q_{n_k} + q_{n_k} k_{n_k}^{(1)} - k_{n_k}^{(1)})\Phi((\|x_{n_k} - \bar{y}\|) \le q_{n_k} \langle z_0 - \bar{y}, J_\psi(x_{n_k} - \bar{y})\rangle. \tag{3.15}$$

Now, since $x_{n_k} \rightharpoonup \bar{y}$ and J_ψ is w-w* continuous,

$$\langle z_0 - \bar{y}, J_\psi(x_{n_k} - \bar{y})\rangle \to 0 \quad \text{as} \quad k \to \infty.$$

It is easy to check that $\dfrac{q_{n_k}}{q_{n_k} + q_{n_k} k_{n_k}^{(1)} - k_{n_k}^{(1)}} \to 1$ as $k \to \infty$, so taking limit in (3.15),

$$\lim_{k\to\infty} \Phi(\|x_{n_k} - \bar{y}\|) \le 0,$$

the continuity of Φ finally implies that $x_{n_k} \to \bar{y}$.

Next we show that $\{x_n\}$ is convergent. Let $x_{n_k} \to z$ and $x_{n_p} \to z'$, we will see that $z = z'$. Let $x^* \in \mathcal{N}(T)$, then

$$\begin{aligned}\langle x_n - T^n x_n, J_\psi(x_n - x^*)\rangle &= \langle x_n - x^*, J_\psi(x_n - x^*)\rangle + \langle x^* - T^n x_n, J_\psi(x_n - x^*)\rangle \ge \\ &\ge \|x_n - x^*\|\psi(\|x_n - x^*\|) - \|x^* - T^n x_n\|\|J_\psi(x_n - x^*)\|_* \\ &\ge \|x_n - x^*\|\psi(\|x_n - x^*\|) - (1 + k_n^{(1)})\|x_n - x^*\|\psi(\|x_n - x^*\|) \\ &= -k_n^{(1)}\|x_n - x^*\|\psi(\|x_n - x^*\|).\end{aligned}$$

Since $\|x_n - x^*\|$ is bounded and

$$x_n - T^n x_n = \frac{q_n}{1 - q_n}(z_0 - x_n),$$

we have that

$$\langle z_0 - x_n, J_\psi(x_n - x^*)\rangle \ge 0,$$

where K is a positive number. Then, since z and z' are both fixed points of T, one gets

$$\limsup_{p\to\infty}\langle x_{n_p} - z_0, J_\psi(x_{n_p} - z)\rangle \le 0 \ \text{ and } \ \limsup_{k\to\infty}\langle x_{n_k} - z_0, J_\psi(x_{n_k} - z')\rangle \le 0.$$

Now, since J_ψ is w-w* continuous, we may conclude that

$$\langle z - z', J_\psi(z - z')\rangle \le 0$$

and so $z = z'$.

We will finish the proof if we show that the mapping Q, defined as $Qz_0 := \lim_n x_n$, is a sunny nonexpansive retraction from Ω onto $\mathcal{N}(T)$. Indeed, we know that if $x^* \in \mathcal{N}(T)$ then

$$\langle x_n - z_0, J_\psi(x_n - x^*)\rangle \le \frac{(1 - q_n)k_n^{(1)}}{q_n}\|x_n - x^*\|\phi(\|x_n - x^*\|).$$

Taking limit it follows that

$$\langle Qz_0 - z_0, J_\psi(Qz_0 - x^*)\rangle \le 0,$$

which states, by Proposition 2.8, that Q is as required. ∎

Remark 3.6 *If Ω is bounded in the previous theorem then $\mathcal{N}(T) \neq \emptyset$ [15].*

Remark 3.7 *In [6] similar results to Theorems 3.1 and 3.5 are stated, however techniques are different from those used in this section. Moreover we do not require the domain Ω to be bounded.*

Remark 3.8 *The existence of the sunny nonexpansive retraction in Theorem 3.1 is not guaranteed in general. Theorem 3.5 implies however that such a retraction exists if the mapping is asymptotically nonexpansive. Moreover if Ω is supposed to be bounded and $\mathcal{N}(T) \neq \emptyset$, then there exists a nonexpansive retraction from Ω onto $\mathcal{N}(T)$ (see [8]).*

Remark 3.9 *It is worthwhile to note that when T is a nonexpansive mapping and T^n is replaced by T, Theorem 3.5 coincides with a result in a recent paper by H. K. Xu in [21]. The convergence of the explicit scheme in Theorem 3.1 seems to be new even for the case of nonexpansive mappings.*

4 Successive Approximation Method for Nonself-mappings

We consider now the case of total asymptotically nonexpansive nonself-mappings $T : \Omega \to X$. In place of (1.4), we investigate the iterative process in the form

$$y_1 \in \Omega \qquad y_{n+1} = q_n z_0 + (1 - q_n)(Q_\Omega T)^n y_n, \quad n = 1, 2, ..., \tag{4.1}$$

where Q_Ω is a nonexpansive retraction of X onto Ω. A modification on the definition of total asymptotically nonexpansive mapping is, however, needed.

Definition 4.1 (cf. [2, 7]) *An operator $T : \Omega \to X$ is said to be total asymptotically nonexpansive if there exist a nonexpansive retraction $Q_\Omega : X \to \Omega$, nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $k_n^{(1)}$, $k_n^{(2)} \to 0$ as $n \to \infty$, and a continuous functions $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that*

$$\|T(Q_\Omega T)^{n-1} Q_\Omega x - T(Q_\Omega T)^{n-1} Q_\Omega y\| \leq \|x - y\| + k_n^{(1)} \phi(\|x - y\|) + k_n^{(2)}. \tag{4.2}$$

Theorem 4.2 *Let Ω be a closed convex subset of a smooth reflexive Banach space X with a weakly sequentially continuous duality map J_ψ, $T : \Omega \to X$ a total asymptotically nonexpansive mapping with nonempty fixed point set $\mathcal{N}(T)$. Under the conditions of Theorem 3.1 replacing $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ by $\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0$, the sequence $\{y_n\}$ generated by (4.1) strongly converges to the fixed point $\bar{x}^* = Qz_0$ given that there exists a sunny nonexpansive retraction $Q : \Omega \to \mathcal{N}(Q_\Omega T)$.*

Proof. First of all, we note that $\mathcal{N}(Q_\Omega T)$ is non-empty because $\mathcal{N}(T) \subseteq \mathcal{N}(Q_\Omega T)$. We show next that $\{y_n\} \subseteq \Omega$ is bounded. Take $x^* \in \mathcal{N}(T)$. In view of the inequalities

$$\|y_{n+1} - x^*\| \le q_n\|z_0 - x^*\| + (1-q_n)\|(Q_\Omega T)^n y_n - (Q_\Omega T)^n x^*\|$$

$$\le q_n\|z_0 - x^*\| + (1-q_n)\|Q_\Omega T(Q_\Omega T)^{n-1} y_n - Q_\Omega T(Q_\Omega T)^{n-1} x^*\|$$

$$\le q_n\|z_0 - x^*\| + (1-q_n)\|T(Q_\Omega T)^{n-1} y_n - T(Q_\Omega T)^{n-1} x^*\|$$

$$\le q_n\|z_0 - x^*\| + (1-q_n)\Big(\|y_n - x^*\| + k_n^{(1)}\phi(\|y_n - x^*\|) + k_n^{(2)}\Big),$$

we conclude, by analogy with Theorem 3.1, that there exist positive constants C, C_1 and $\bar{M}$ such that $\|y_n\| \le C$, $\|y_n - x^*\| \le C_1$ and $\phi(\|y_n - x^*\|) \le \bar{M}$. Since Q_Ω is a nonexpansive mapping, it is not difficult to verify that there exists a constant $\bar{C} > 0$ such that $\|(Q_\Omega T)^n y_n\| \le \bar{C}$.

It follows from (4.1) that

$$\lim_{n\to\infty}\Big(y_{n+1} - (Q_\Omega T)^n y_n\Big) = 0. \tag{4.3}$$

On the other hand, we can show that

$$\lim_{n\to\infty}\Big(y_{n+1} - (Q_\Omega T)^n y_{n+1}\Big) = 0. \tag{4.4}$$

Indeed,

$$\|y_{n+1} - (Q_\Omega T)^n y_{n+1}\| \quad\le\quad \|y_{n+1} - (Q_\Omega T)^n y_n\| + \|(Q_\Omega T)^n y_n - (Q_\Omega T)^n y_{n+1}\|.$$

Due to the total asymptotical nonexpansiveness of T, we obtain

$$\|(Q_\Omega T)^n y_n - (Q_\Omega T)^n y_{n+1}\| \le \|T(Q_\Omega T)^{n-1} y_n - T(Q_\Omega T)^{n-1} y_{n+1}\|$$

$$\le \|y_n - y_{n+1}\| + k_n^{(1)}\phi(\|y_n - y_{n+1}\|) + k_n^{(2)}.$$

Now the boundedness of $\{(Q_\Omega T)^n y_n\}$, the fact that $\|y_n - y_{n+1}\| \to 0$ and (4.3) prove (4.4).

Further, we have

$$\|y_n - Q_\Omega T y_n\| \le \|y_n - y_{n+1}\| + \|y_{n+1} - (Q_\Omega T)^n y_n\| + \|(Q_\Omega T)^n y_n - Q_\Omega T y_n\|.$$

From the uniform continuity of T, we estimate the last term in the form:

$$\|Q_\Omega T(Q_\Omega T)^{n-1} y_n - Q_\Omega T y_n\| \le \|T(Q_\Omega T)^{n-1} y_n - T y_n\| \le \omega(\|(Q_\Omega T)^{n-1} y_n - y_n\|).$$

By (4.3) and (4.4), one gets

$$\lim(y_n - Q_\Omega T y_n) = 0.$$

Notice that $Q_\Omega T$ is a mapping as required in Theorem 3.1 and then we can apply Theorem 3.1 to the sequence $\{y_n\}$. Indeed, we write

$$y_{n+1} - Qz_0 = (1 - q_n)((Q_\Omega T)^n y_n - Qz_0) + q_n(z_0 - Qz_0).$$

we use the subdifferential inequality as in Theorem 3.1, and the rest of the proof follows the pattern with the only difference that we obtain that $\{y_n\}$ strongly converges to $Qz_0 = \bar{x}^*$ which is a fixed point of $Q_\Omega T$. ■

5 Stability Analysis for Nonexpansive Nonself-mappings

Next we study the stability problem for iterative processes with respect to perturbations of constraint sets. Specifically, we consider a process which involves nonexpansive nonself-mappings in the following form:

$$y_1 \in \Omega_1, \quad y_{n+1} = q_n z_0 + (1 - q_n) Q_{\Omega_{n+1}} T y_n, \quad n = 1, 2, ..., \tag{5.1}$$

where $Q_{\Omega_n} : X \to \Omega_n$ is a sunny nonexpansive retraction, and the proximity between the original set Ω and Ω_n with $n = 1, 2, ...$ is given by the Hausdorff distance:

$$\mathcal{H}(\Omega_n, \Omega) \leq \sigma_n. \tag{5.2}$$

Let $G = \cap_n \Omega_n$ and $\bar{G} = \Omega \cap G \neq \emptyset$.

Theorem 5.1 *Let X be a uniformly smooth Banach space which has a weakly sequentially continuous duality map J_ψ. Assume that $D \subset X$ is a closed convex set, $\Omega \subset D$ and $\Omega_n \subset D$, $n = 1, 2, ...$ are closed convex subsets of X with property (5.2). Let $T : D \to X$ be a nonexpansive mapping with fixed point set $\mathcal{N}(T)$ such that $\mathcal{N}(T) \cap \Omega \neq \emptyset$. Take z_0 some point in $\bar{G}$, $x^* \in \mathcal{N}(T) \cap \Omega$, $\{q_n\}$ a sequence in $(0, 1)$ satisfying (3.1) and a non-increasing sequence $\sigma_n \leq \sigma$, such that $\sigma_n \to 0$ as $n \to \infty$. Let the sequence $\{y_n\}$ be generated by (5.1). We suppose that the sequence $\{Ty_n\}$ is bounded,*

$$\lim_{n\to\infty} \frac{|q_n - q_{n-1}|}{q_n} = 0 \tag{5.3}$$

and

$$\lim_{n\to\infty} \frac{\sqrt{h_X(\sigma_n)}}{q_n} = 0. \tag{5.4}$$

Then $\{y_n\}$ converges strongly to the fixed point $\bar{x}^* = Qz_0$ of $Q_\Omega T$, where $Q : \Omega \to \mathcal{N}(Q_\Omega T)$ is the unique sunny nonexpansive retraction onto $\mathcal{N}(Q_\Omega T)$.

Proof. We show first that $\{y_n\}$ is bounded. It is not difficult to see that

$$\|y_{n+1} - x^*\| = \|q_n z_0 + (1 - q_n) Q_\Omega T y_n - Q_\Omega T x^* + (1 - q_n)(Q_{\Omega_{n+1}} T y_n - Q_\Omega T y_n)\|$$

$$\leq q_n \|z_0 - x^*\| + (1 - q_n)\|Q_\Omega T y_n - Q_\Omega T x^*\| + (1 - q_n)\|Q_{\Omega_{n+1}} T y_n - Q_\Omega T y_n\|.$$

Since $\{Ty_n\}$ is bounded and due to Lemma 2.10, there exist positive constants M_5 and M_6 such that

$$\|Q_{\Omega_{n+1}}Ty_n - Q_\Omega Ty_n\| \le M_5\sqrt{h_X(M_6\sigma_n)}.$$

This implies

$$\|y_{n+1} - x^*\| \le q_n\|z_0 - x^*\| + (1-q_n)\|y_n - x^*\| + M_5\sqrt{h_X(M_6\sigma_n)}.$$

Denoting $\lambda_n = \|y_n - x^*\|$ we obtain

$$\lambda_{n+1} \le (1-q_n)\lambda_n + q_n\|z_0 - x^*\| + M_5\sqrt{h_X(M_6\sigma_n)}.$$

From Lemma 2.11 it follows that $\{y_n - x^*\}$ is bounded. Let $\|y_n - x^*\| \le C_1$ for all n.

Next we evaluate the following difference:

$$\begin{aligned} y_{n+1} - y_n = (1-q_n)(Q_{\Omega_{n+1}}Ty_n - Q_{\Omega_n}Ty_{n-1}) + (q_n - q_{n-1})(z_0 - x^*) \\ +(q_{n-1} - q_n)(Q_{\Omega_n}Ty_{n-1} - Q_\Omega Tx^*) \\ = (1-q_n)(Q_{\Omega_{n+1}}Ty_n - Q_{\Omega_{n+1}}Ty_{n-1}) + (1-q_n)(Q_{\Omega_{n+1}}Ty_{n-1} - Q_{\Omega_n}Ty_{n-1}) \\ +(q_n - q_{n-1})(z_0 - x^*) + (q_{n-1} - q_n)(Q_{\Omega_n}Ty_{n-1} - Q_{\Omega_n}Tx^*) \\ +(q_{n-1} - q_n)(Q_{\Omega_n}Tx^* - Q_\Omega Tx^*). \end{aligned}$$

Using the following estimates

$$\|Q_{\Omega_{n+1}}Ty_n - Q_{\Omega_{n+1}}Ty_{n-1}\| \le \|y_n - y_{n-1}\|,$$

$$\|Q_{\Omega_n}Ty_{n-1} - Q_{\Omega_n}Tx^*\| \le \|y_{n-1} - x^*\|,$$

$$\|Q_{\Omega_n}Tx^* - Q_\Omega Tx^*\| \le M_7\sqrt{h_X(M_8\sigma_n)},$$

and

$$\|Q_{\Omega_{n+1}}Ty_{n-1} - Q_{\Omega_n}Ty_{n-1}\| \le M_9\sqrt{h_X(M_{10}\sigma_n)},$$

for suitable constants $M_7, ..., M_{10}$ (see Lemma 2.10), we obtain

$$\begin{aligned} \|y_{n+1} - y_n\| \le (1-q_n)\|y_n - y_{n-1}\| + (1-q_n)M_9\sqrt{h_X(M_{10}\sigma_n)} \\ +|q_n - q_{n-1}|\Big(\|z_0 - x^*\| + C_1) + M_7\sqrt{h_X(M_8\sigma_n)}\Big). \end{aligned}$$

Denoting $\lambda_n = \|y_n - y_{n-1}\|$ one has

$$\lambda_{n+1} \le (1-q_n)\lambda_n + \gamma_n,$$

where

$$\begin{aligned} \gamma_n = |q_n - q_{n-1}|(\|z_0 - x^*\| + C_1) + |q_n - q_{n-1}|M_7\sqrt{h_X(M_8\sigma_n)} \\ + (1-q_n)M_9\sqrt{h_X(M_{10}\sigma_n)}. \end{aligned}$$

Now, by (5.3), (5.4) and Lemma 2.11, we conclude that $\|y_n - y_{n-1}\| \to 0$.

Next we show that

$$\lim_{n\to\infty} \|y_n - Q_\Omega T y_n\| = 0. \tag{5.5}$$

Indeed,

$$\begin{aligned} \|y_n - Q_\Omega T y_n\| &\leq \|y_n - Q_{\Omega_n} T y_{n-1}\| + \|Q_{\Omega_n} T y_{n-1} - Q_\Omega T y_{n-1}\| \\ &\quad + \|Q_\Omega T y_{n-1} - Q_\Omega T y_n\| \\ &\leq \|y_n - Q_{\Omega_n} T y_{n-1}\| + M_7 \sqrt{h_X(M_8 \sigma_n)} + \|y_{n-1} - y_n\|, \end{aligned}$$

which states our claim since $\|y_n - y_{n-1}\| \to 0$ and $\sqrt{h_X(M_8\sigma_n)} \to 0$ as $n \to \infty$, and

$$\|y_n - Q_{\Omega_n} T y_{n-1}\| = q_n (z_0 - Q_{\Omega_n T y_{n-1}})$$

which, from the boundedness of $\{Q_{\Omega_n T y_{n-1}}\}$, also tends to 0.

Now we write

$$y_{n+1} - Qz_0 = (1 - q_n)(Q_{\Omega_{n+1}} T y_n - Qz_0) + q_n (z_0 - Qz_0),$$

and apply the subdifferential inequality to J_ψ as in Theorem 3.1 to deduce that

$$\phi(\|y_{n+1} - Qz_0\|) \leq \phi((1 - q_n)\|Q_{\Omega_{n+1}} T y_n - Qz_0\|) + q_n \langle z_0 - Qz_0, J_\psi(y_{n+1} - Qz_0)\rangle.$$

Since $Qz_0 \in \mathcal{N}(Q_\Omega T) \subseteq \Omega$,

$$\begin{aligned} \|Q_{\Omega_{n+1}} T y_n - Qz_0\| &\leq \|Q_{\Omega_{n+1}} T y_n - Q_{\Omega_{n+1}} T Q z_0\| + \|Q_{\Omega_{n+1}} T Q z_0 - Q_\Omega T Q z_0\| \\ &\leq \|y_n - Qz_0\| + \nu_n, \end{aligned}$$

where $\nu_n = M_7 \sqrt{h_X(M_8 \sigma_{n+1})}$ is bounded and vanishes as $n \to \infty$. From (5.4) we can follow the same reasoning as in Theorem 3.1 to obtain (for n large enough)

$$\begin{aligned} \phi(\|y_{n+1} - Qz_0\|) \leq (1 - q_n)\phi(\|y_n - Qz_0\|) + (1 - q_n)\nu_n M_{11} + \\ + q_n \langle z_0 - Qz_0, J_\psi(y_{n+1} - Qz_0)\rangle. \end{aligned} \tag{5.6}$$

for M_{11} a suitable constant. We claim that

$$\limsup_{n\to\infty} \langle z_0 - Qz_0, J_\psi(y_n - Qz_0)\rangle \leq 0. \tag{5.7}$$

Since $\{y_n\}$ is bounded there exists a subsequence $\{y_{n_k}\}$, $y_{n_k} \in \Omega_{n_k}$ for each k, which weakly converges to some point $\bar{y}$ and

$$\limsup_{n\to\infty} \langle z_0 - Qz_0, J_\psi(y_n - Qz_0)\rangle = \lim_{k\to\infty} \langle z_0 - Qz_0, J_\psi(y_{n_k} - Qz_0)\rangle.$$

Now we apply the following result:

Lemma 5.2 *[16] If the set Ω is convex and closed set in reflexive Banach space X and the sequence of sets $\Omega_n \subseteq X$ satisfy the limit relation $\mathcal{H}(\Omega_n, \Omega) \to 0$ as $n \to \infty$ then every weak limit point u of any sequence $\{u_n\}$, $u_n \in \Omega_n$, belongs to the subset Ω.*

To deduce that $\bar{y} \in \Omega$. The rest of the proof follows the pattern of Theorem 3.1 once we prove that $\bar{y}$ is a fixed point of $Q_\Omega T$. To prove this we make use of Opial's condition used in the following form:

$$\liminf_{l\to\infty} \|y_{n_l} - \bar{y}\| < \liminf_{l\to\infty} \|y_{n_l} - Q_\Omega T\bar{y}\|$$
$$\leq \liminf_{l\to\infty} (\|y_{n_l} - Q_\Omega T y_{n_l}\| + \|Q_\Omega T y_{n_l} - Q_\Omega T\bar{y}\|)$$
$$\leq \liminf_{l\to\infty} \|Q_\Omega T y_{n_l} - Q_\Omega T\bar{y}\| \leq \liminf_{l\to\infty} \|y_{n_l} - \bar{y}\|. \quad \blacksquare$$

Remark 5.3 *Notice that in this theorem we do not obtain convergence to a fixed point of T. Convergence to a fixed point of T can be obtained if we impose certain boundary conditions on T as, for instance, that $T(\partial\Omega) \subseteq \Omega$. It is not hard to see that in this case $\mathcal{N}(Q_\Omega T) = \mathcal{N}(T)$.*

Remark 5.4 *The sunny nonexpansive retraction onto $\mathcal{N}(Q_\Omega T)$ always exists in this case since $Q_\Omega T$ is a nonexpansive self-mapping [8].*

Remark 5.5 *If the condition $\bar{G} = \Omega \cap G \neq \emptyset$ does not hold we prove the previous theorem in the following way. Instead of Ω_n we consider the collection of sets $\Omega'_n = \overline{co}(\Omega \cup \Omega_n)$, where $\overline{co}(A)$ stands for the closed convex closure of a set A. It is easy to see that $\mathcal{H}(\Omega, \Omega'_n) \to 0$ as $n \to \infty$. Now it suffices to follow the same proof.*

Remark 5.6 *The sequence $\{Ty_n\}$ is bounded if, for instance, $\{y_n\}$ is bounded. This obviously holds if, for instance, Ω is bounded.*

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