On a Modification of the Jacobi Linear Functional: Asymptotic Properties and Zeros of the Corresponding Orthogonal Polynomials

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Abstract. The paper deals with orthogonal polynomials in the case where the orthogonality condition is related to semiclassical functionals. The polynomials that we discuss are a generalization of Jacobi polynomials and Jacobi-type polynomials. More precisely, we study some algebraic properties as well as the asymptotic behaviour of polynomials orthogonal with respect to the linear functional $\mathcal{U}$

$$\mathcal{U} = \mathcal{J}_{\alpha,\beta} + A_1\delta(x-1) + B_1\delta(x+1) - A_2\delta'(x-1) - B_2\delta'(x+1),$$

where $\mathcal{J}_{\alpha,\beta}$ is the Jacobi linear functional, i.e.

$$\langle \mathcal{J}_{\alpha,\beta}, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x, \quad \alpha,\beta > -1,\ p \in \mathbb{P},$$

and $\mathbb{P}$ is the linear space of polynomials with complex coefficients. The asymptotic properties are analyzed in $(-1,1)$ (inner asymptotics) and $\mathbb{C} \setminus [-1,1]$ (outer asymptotics) with respect to the behaviour of Jacobi polynomials. In a second step, we use the above results in order to obtain the location of zeros of such orthogonal polynomials. Notice that the linear functional $\mathcal{U}$ is a generalization of one studied by T. H. Koornwinder when $A_2 = B_2 = 0$. From the point of view of rational approximation, the corresponding Markov function is a perturbation of the Jacobi–Markov function by a rational function with two double poles at ± 1. The denominators of the $[n-1/n]$ Padé approximants are our orthogonal polynomials.

Mathematics Subject Classifications (2000): 33C45, 42C05.

Key words: semiclassical orthogonal polynomials, asymptotics, zeros.

1. Introduction

In this work we will study a generalization of the Jacobi polynomials introduced in Koornwinder (1984). Such polynomials are orthogonal with respect to the Jacobi measure *'perturbed'* by the addition of two delta Dirac measures as well as their derivatives at the points $x = \pm 1$.

Such a kind of modification of a positive linear functional appear when an extension of the Gauss–Lobatto quadrature formulas is considered. In fact, in Bernardi and Maday (1991) such quadrature formulas are used in a spectral method for solving a one-dimensional fourth-order differential problem. Here, the boundary conditions are values of the solution and its first derivative in the ends of the interval $(-1, 1)$.

The aforementioned modifications were firstly studied in Krall (1940) when he considered the polynomial solution of certain fourth-order linear differential equations. There Krall obtained, apart the classical orthogonal polynomials (Hermite, Jacobi, Laguerre and Bessel), three new families of orthogonal polynomials with respect to positive measures with an absolutely continuous part plus some mass points. More precisely, the so-called classical-type orthogonal polynomials appear. Another approach to this subject was presented in Krall (1981).

The analysis of the asymptotic properties of polynomials orthogonal with respect to a perturbation of a measure via the addition of mass points was introduced by Nevai (1979). In particular, he proved how the location of the mass points with respect to the support of the measure has an influence in the asymptotic behaviour of perturbed polynomials.

The algebraic properties for such polynomials have attracted the interest of many researchers. A general approach when a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one delta Dirac measure was started by Chihara (1985) in the positive definite case and Marcellán and Maroni (1992) for quasi-definite linear functionals. From the point of view of differential equations, see Marcellán and Ronveaux (1989). For two point masses there exist very few examples in the literature (see Draïdi, 1990; Koekoek, 1990; Koornwinder, 1984; Kwon and Park, 1997) but the difficulties increase as shown in Draïdi and Maroni (1988).

Special emphasis was placed on the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel) within the framework of the so-called semiclassical orthogonal polynomials. Notice that every positive linear functional induces an inner product in a natural way. However, in general, if we consider a linear functional, no inner product can be defined. Nevertheless, the concept of orthogonality with respect to the linear functional has a sense (see Chihara, 1978; Section 2 in Chapter 1). In Koornwinder (1984), the Jacobi case with two masses at points $x = \pm 1$ was considered. The hypergeometric representation of the resulting polynomials as well as the existence of a second-order differential equation that such polynomials satisfy have been established. Also the particular cases of the Krall-type polynomials (A. M. Krall, 1981; H. L. Krall, 1940) have been obtained from this general case as special cases or limit cases. The Laguerre case was considered in detail in Koekoek and Koekoek (1991), Koekoek (1988, 1990).

The perturbation of a linear functional via the addition of the derivatives of a delta Dirac measure was started in Belmehdi and Marcellán (1992). In particular, necessary and sufficient conditions for the existence of a sequence of polynomials orthogonal with respect to such a linear functional are obtained. Furthermore, an extensive study for the new orthogonal polynomials was performed when the initial functional is semiclassical. This problem can be considered as a limit case of two masses located in two close points. The study of such a kind of modifications of a linear functional has known an increasing interest during the past years since their applications in approximation theory (see Gonchar (1975) for the bounded case and López (1989) for the unbounded one).

First, in Álvarez-Nodarse and Marcellán (1995, 1996), the perturbation of the Laguerre linear functional when we add the linear functional $M_0\delta(x) + M_1\delta'(x)$ is analyzed. More precisely they studied the behavior of the polynomials and their zeros as well as the hypergeometric character of them.

More recently, Arvesú *et al.* (1998) analyzed a generalization of the Bessel polynomials, which appears when one perturbs the Bessel linear functional by the addition of the linear functional $M_0\delta(x) + M_1\delta'(x)$. In particular, the hypergeometric character of these polynomials and the behavior of their zeros were studied. In this paper, we will deal with the Jacobi case.

1.1. SUMMARY AND STRUCTURE OF THE PAPER

In spite of the long history (more than one hundred years ago) of studying of orthogonal polynomials (OP) with respect to semiclassical functionals (see Laguerre, 1885) the theory of semiclassical OP does not enjoy the same level of development and completeness as the theory of classical OP, even, considering the recent powerful modern tools developed to work out with them (see Draïdi, 1990; Draïdi and Maroni, 1988; Kwon and Park, 1997; Marcellán and Maroni, 1992; Maroni, 1991). The present paper intends to cover this void, and introduces a constructive approach to the study of a special class of semiclassical OP. Also, we show that the semiclassical Jacobi-type OP (see Section 3 below) inherits (up to some adaptations) many of the remarkable properties which satisfy classical orthogonal polynomials: exact expression for the coefficients of the hypergeometric series, recurrence relations, etc.

The plan of the paper is the following. Section 2 summarizes the basic notions and tools to work out with the classical and semiclassical OP, with particular attention to Jacobi polynomials. In Section 3 we give necessary and sufficient conditions in order to guarantee the existence of the semiclassical orthogonal polynomial sequence. A symmetry property in the same sense as in Koornwinder (1984) is considered. Also, the order of the class for the semiclassical linear functional $\mathcal{U}$ (see (19) below) is established, together with the corresponding distributional equation. In Section 4 a general formula for the semiclassical Jacobi-type orthogonal polynomials in terms of the classical ones and their first and second derivatives is given. This fact allows us to study the asymptotic properties which are useful to investigate in Section 5 the location, and asymptotic distribution of zeros of such polynomials. Finally, Section 6 is devoted to conclusions and open problems.

2. Preliminaries and Notations

Here we present the basic notions, definitions, and notations of the paper. Also, we have enclosed some formulas for the Jacobi polynomials which are useful in the analysis of polynomials orthogonal with respect to the linear functional (19), see Section 3 below. Later on, we summarize the basic tools to work out with semiclassical orthogonal polynomials.

2.1. CLASSICAL ORTHOGONAL POLYNOMIALS: JACOBI POLYNOMIALS

In this paper we will always be considering monic orthogonal polynomials from the linear space $\mathbb{P}$ of polynomials with complex coefficients. $\mathbb{P}_n$ stands for the subset of polynomials of degree not greater than n.

Most of the properties which are used to characterize the classical orthogonal polynomials (OP) in a number of ways (see Chihara, 1978; Szegő, 1975) follow from the fact that the weight functions ρ involved in the orthogonality condition

$$\int_\Gamma P_n(x)x^k \rho(x)\,\mathrm{d}x = 0, \quad 0 \leqslant k < n-1, \tag{1}$$

satisfy the Pearson differential equation

$$[\sigma(x)\rho(x)]' + \tau(x)\rho(x) = 0, \tag{2}$$

with σ, a polynomial of degree at most 2, and τ, a polynomial of degree exactly 1. The position of the singularities of the first-order differential equation (2) leads to four different possibilities of classical weights (see Table I).

One aspect in the theory of classical orthogonal polynomials is that the solutions of the differential equation (2) together with the condition for the path of integration Γ

$$\sigma(x)\rho(x)x^k|_\Gamma = 0, \quad k = 0, 1, \ldots, \tag{3}$$

determine an integral moment functional

$$\langle \mathcal{U}, x^k \rangle = \int_\Gamma x^k \rho(x)\,\mathrm{d}x. \tag{4}$$

Table I. Classical weight functions.

Name	$\sigma(x)$	$\rho(x)$
Jacobi	$x(a-x)$	$x^\beta (a-x)^\alpha$
Laguerre	x	$x^\alpha \mathrm{e}^{\beta x}$
Hermite	const	$\mathrm{e}^{\beta x^2}$
Bessel	x^2	$x^\alpha \mathrm{e}^{\gamma/x}$

Later, we will return to these types of classical moment functionals to study their generalization.

Before proceeding to sketch some remarkable properties as well as some formulas of the Jacobi polynomials, let us specify that we will use the Jacobi polynomials, which are monic polynomials of degree n orthogonal to all lower degree polynomials with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ on $[-1, 1]$, where $\alpha, \beta > -1$, i.e., they are orthogonal with respect to the linear functional $\mathcal{J}_{\alpha,\beta}$ on the linear space $\mathbb{P}$ of polynomials with real coefficients defined by

$$\langle \mathcal{J}_{\alpha,\beta}, P\rangle = \int_{-1}^{1} P(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x, \quad \alpha, \beta > -1,\ P \in \mathbb{P}, \tag{5}$$

and we will denote these monic polynomials by $P_n^{\alpha,\beta}(x)$. Thus, the orthogonality relation is

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x = \delta_{n,m}\|P_n^{\alpha,\beta}\|^2, \tag{6}$$

where

$$\|P_n^{\alpha,\beta}\|^2 = \frac{2^{\alpha+\beta+2n+1} n!\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2},$$

and $(n)_k$ with $k = 1, 2, \ldots$ is the Pochhammer symbol or shifted factorial defined by

$$(n)_0 := 1, \qquad (n)_k := n(n+1)(n+2)\cdots(n+k-1) = \frac{\Gamma(n+k)}{\Gamma(n)}.$$

The change of variable $x \mapsto (2x/a) - 1$ gives Jacobi polynomials on $[0, a]$ for the weight function $\rho(x) = x^{\beta}(a-x)^{\alpha}$ with singularities at 0 and a (see Table I).

Now, we will list several properties of the Jacobi polynomials, most of which can be found in the literature of special functions, see for instance the classical monograph *Orthogonal Polynomials* by Szegő (1975, Chapter 5).

The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are the polynomial solution of the second-order linear differential equation of hypergeometric type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \tag{7}$$

where

$$\sigma(x) = (1-x^2), \qquad \beta - \alpha - (\alpha+\beta+2)x,$$
$$\lambda_n = n(n+\alpha+\beta+1),$$

respectively.

The notation $y^{(k)}(x)$ and $D^k y(x)$, $k \in \mathbb{N}$ are used along the paper to denote the kth derivative. $(y)^{(k)}(x_0)$ indicates that the kth derivative of y is evaluated at the point x_0.

The Jacobi polynomials verify the differentiation formula

$$D^\nu P_n^{\alpha,\beta}(x) = \frac{n!}{(n-\nu)!} P_{n-\nu}^{\alpha+\nu,\beta+\nu}(x), \quad \nu = 0, 1, \ldots, \tag{8}$$

where $n = 1, 2, \ldots$. Furthermore, the following symmetry property holds

$$P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x). \tag{9}$$

Among other properties there is one very simple and useful for computing aims. It is the exact expression, in terms of the coefficients of the polynomials σ and τ only, for the coefficients of the so-called *structure relation*

$$(1-x^2)P_n'(x) = \tilde{\alpha}_n P_{n+1}(x) + \tilde{\beta}_n P_n(x) + \tilde{\gamma}_n P_{n-1}(x), \quad n \geqslant 0, \tag{10}$$

where $P_n(x) = P_n^{\alpha,\beta}(x)$,

$$\begin{aligned} \tilde{\alpha}_n &= -n, \\ \tilde{\beta}_n &= \frac{2n(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \tilde{\gamma}_n &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{11}$$

and the *three-term recurrence relation*,

$$xP_n(x) = P_{n+1}(x) + \beta_n^{\alpha,\beta} P_n(x) + \gamma_n^{\alpha,\beta} P_{n-1}(x), \quad n \geqslant 0, \tag{12}$$

where $P_n(x) = P_n^{\alpha,\beta}(x)$,

$$\begin{aligned} \beta_n^{\alpha,\beta} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \gamma_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{13}$$

providing that $P_{-1} = 0$.

The Jacobi polynomials have the following representation as hypergeometric series

$$P_n^{\alpha,\beta}(x) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right), \tag{14}$$

where

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}.$$

A consequence of this representation is

$$P_n^{\alpha,\beta}(1) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}, \qquad P_n^{\alpha,\beta}(-1) = \frac{(-1)^n 2^n(\beta+1)_n}{(n+\alpha+\beta+1)_n}. \tag{15}$$

Throughout this work we will use

$$K_n^{\alpha,\beta(p,q)}(x,y) = \sum_{m=0}^{n} \frac{(P_m^{\alpha,\beta})^{(p)}(x)(P_m^{\alpha,\beta})^{(q)}(y)}{\|P_m^{\alpha,\beta}\|^2} = \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_n^{\alpha,\beta(0,0)}(x,y), \tag{16}$$

in order to denote the kernels of the Jacobi polynomials, as well as their derivatives with respect to x and y, respectively. For $p = q = 0$ and $n = 1, 2, \ldots$ the well-known Christoffel–Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x)P_m^{\alpha,\beta}(y)}{\|P_m^{\alpha,\beta}\|^2} = \frac{P_n^{\alpha,\beta}(x)P_{n-1}^{\alpha,\beta}(y) - P_{n-1}^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)}{(x-y)\|P_{n-1}^{\alpha,\beta}\|^2}, \tag{17}$$

holds.

2.2. SEMICLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS s: CLASSIFICATION

The notion of classical orthogonal polynomials associated with classical weights (see Table I) can be generalized in a very natural way by omitting the restriction on the degrees of the polynomials σ and τ (they are polynomials of degree at most 2 and exactly 1, respectively) in Equation (2). So, the moment functional (4) defined by (2) and (3) is called semiclassical of class s, being

$$s = \min\{\max\{\deg\sigma - 2, \deg\tau - 1\}, \text{ such that } D(\sigma\mathcal{U}) = \tau\mathcal{U}\}.$$

For $s = 0$ one gets the ordinary classical case, and $s > 0$ corresponds to the semiclassical case of class s.

Now, we will list some known results concerning semiclassical linear functionals.

Let $\mathcal{U}$ be a linear functional on the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $S(\mathcal{U})(z)$ be its Stieltjes function defined by

$$S(\mathcal{U})(z) = -\sum_{n\geqslant 0} \frac{\mathcal{U}_n}{z^{n+1}},$$

where $\mathcal{U}_n = \langle \mathcal{U}, x^n \rangle$, $n \geqslant 0$, are the moments of $\mathcal{U}$ and $\langle \cdot, \cdot \rangle$ means the duality bracket. By a convention, we will suppose that $\mathcal{U}_0 = 1$.

Let $\mathbb{P}'$ be the algebraic dual space of $\mathbb{P}$ and $\mathbb{D}$ the linear space generated by $\{D^n\delta\}_{n\geqslant 0}$, where $D^n\delta$ means the nth derivative of the Dirac delta in the origin.

We consider the isomorphism $\mathcal{I}\colon \mathbb{D} \to \mathbb{P}$ given as follows (see Maroni, 1991): For

$$\mathcal{U} = \sum_{n\geqslant 0} \mathcal{U}_n \frac{(-1)^n}{n!} D^n\delta, \quad \mathcal{I}(\mathcal{U})(z) = \sum_{n\geqslant 0} \mathcal{U}_n z^n.$$

Then,

$$S(\mathcal{U})(z) = -z^{-1}\mathcal{I}(\mathcal{U})(z^{-1}).$$

Let introduce $\langle p\mathcal{U}, q\rangle = \langle \mathcal{U}, pq\rangle$ for every polynomial $q(z)$, and define

$$(\mathcal{U}p)(z) = \sum_{m=0}^{n}\left(\sum_{j=m}^{n} a_j \mathcal{U}_{j-m}\right) z^m, \quad p(z) = \sum_{j=0}^{n} a_j z^j,$$

and

$$(\theta_0\, p)(z) = \frac{p(z) - p(0)}{z}.$$

Hence, $S(p\mathcal{U})(z) = p(z)S(\mathcal{U})(z) + (\mathcal{U}\theta_0\, p)(z)$, for a polynomial $p(z)$.

We define the functional $x^{-1}\mathcal{U}$ and the product of two linear functionals in the following way

$$\langle x^{-1}\mathcal{U}, p\rangle = \langle \mathcal{U}, \theta_0 p\rangle, \qquad \langle \mathcal{U}\mathcal{V}, p\rangle = \langle \mathcal{U}, \mathcal{V}p\rangle.$$

Then it is straightforward to prove that

(i) $x(x^{-1}\mathcal{U}) = \mathcal{U}$.
(ii) $x^{-1}(x\mathcal{U}) = \mathcal{U} - \mathcal{U}_0\delta$.
(iii) $x^{-2}(x^2\mathcal{U}) = x^{-1}(x^{-1}\mathcal{U}) = \mathcal{U} - \mathcal{U}_0\delta + \mathcal{U}_1 D\delta$.

DEFINITION 1. A linear functional $\mathcal{U}$ is said to be a quasi-definite or regular (see Chihara, 1978) functional if there exists a sequence of monic orthogonal polynomials (MOPS), $\{P_n\}_{n\geqslant 0}$ with respect to $\mathcal{U}$, i.e., it satisfies

(i) $P_n(x) = x^n +$ lower degree terms.
(ii) $\langle \mathcal{U}, P_n P_m\rangle = k_n\delta_{nm}, k_n \neq 0, n = 0, 1, 2, \ldots$.

A MOPS $\{P_n\}_{n\geqslant 0}$ with respect to a quasi-definite linear functional satisfies the following three-term recurrence relation

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geqslant 0,$$
$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0,$$

with $\gamma_n \neq 0, n \geqslant 0$ and $\gamma_0 = 1 = \langle \mathcal{U}, P_0^2\rangle$.

PROPOSITION 1 (Chihara, 1978). *A linear functional $\mathcal{U}$ is quasi-definite if and only if* $\det[\mathcal{U}_{i+j}]_{i,j=0}^{n} \neq 0$, *for all* $n \geqslant 0$.

Other results concerning the algebra of linear functionals are (see (Bouakkaz and Maroni, 1991; Marcellán and Prianes, 1996; Maroni, 1991) for a more comprehensive approach):

LEMMA 1. *For* $p, q \in \mathbb{P}$ *and for* $\mathcal{U}, \mathcal{V} \in \mathbb{P}'$*, we have*

(i) $x^{-1}(p\mathcal{U}) + \langle \mathcal{U}, \theta_0 p \rangle \delta = p(x^{-1}\mathcal{U})$,
(ii) $q(\mathcal{U}\theta_0\, p) - \mathcal{U}\theta_0(q\, p) = -\theta_0[(p\mathcal{U})q]$,
(iii) $\theta_0(\mathcal{U}p) = \mathcal{U}(\theta_0 p)$,
(iv) $\mathcal{U}(pq) = (p\mathcal{U})q + xq(\mathcal{U}\theta_0 p)$,
(v) $p(\mathcal{U}\mathcal{V}) = (p\mathcal{V})\mathcal{U} + x(\mathcal{V}\theta_0 p)\mathcal{U}$.

In terms of the Stieltjes functions,

LEMMA 2. *For* $p \in P$ *and for* $\mathcal{U}, \mathcal{V} \in \mathbb{P}'$*, we have*

(i) $S'(\mathcal{U})(z) = S(D\mathcal{U})(z)$,
(ii) $S(\mathcal{U}\mathcal{V})(z) = -zS(\mathcal{U})(z)S(\mathcal{V})(z)$,
(iii) $S(x^{-1}\mathcal{U})(z) = z^{-1}S(\mathcal{U})(z)$,
(iv) $z^{-1}(\mathcal{U}\theta_0 p)(z) = -z^{-2}S(\langle \mathcal{U}, \theta_0 p\rangle, \delta)(z^{-1}) + (\mathcal{U}\theta_0^2 p)(z)$.

Finally,

THEOREM 1. *Let* $\mathcal{U}$ *be a semiclassical linear functional, and define the set*

$$\widetilde{X}_\sigma = \{\tilde{x} \in \mathbb{C} : \sigma(\tilde{x}) = 0\}.$$

Then, the order of the class of $\mathcal{U}$ *is given by*

$$s = \max\{\deg \sigma - 2, \deg \tau - 1\},$$

if and only if one of the following statements holds

(i) *Either* $\forall \tilde{x} \in \widetilde{X}_\sigma$ *one has* $\tau(\tilde{x}) - \sigma'(\tilde{x}) \neq 0$.
(ii) *Or if* $\tilde{x} \in \widetilde{X}_\sigma$ *satisfies* $\tau(\tilde{x}) - \sigma'(\tilde{x}) = 0$ *then* $\langle \mathcal{U}, \tilde{\tau} + \tilde{\sigma}' \rangle \neq 0$*, where* $\tilde{\sigma}(x)$ *and* $\tilde{\tau}(x)$ *are two polynomials, such that*

$$\sigma(x) = (x - \tilde{x})\tilde{\sigma}(x), \qquad \tau(x) - \sigma'(x) = (x - \tilde{x})\tilde{\tau}(x).$$

2.3. CONNECTION WITH APPROXIMATION THEORY

The study of the modification of a measure via the addition of the derivatives of a delta Dirac measure is intimately related with approximation theory (see Gonchar (1975) for the bounded case and López (1989) for the unbounded one). In fact, the denominators $q_n(x)$ of the main diagonal sequence for Padé approximants of Stieltjes-type meromorphic functions

$$\int \frac{d\mu(x)}{z - x} + \sum_{j=1}^{m} \sum_{i=0}^{N_j} A_{i,j} \frac{i!}{(z - c_j)^{i+1}}, \quad A_{N_j,j} \neq 0,$$

satisfy the orthogonal relations

$$\int p(x)q_n(x)\,\mathrm{d}\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}\,(p(z)q_n(z))^{(i)}\Big|_{z=c_j} = 0, \tag{18}$$

where $p(x)$ is a polynomial of degree at most $n-1$.

3. The Definition and Orthogonal Relation

Consider the linear functional $\mathcal{U}$ on $\mathbb{P}$, defined as

$$\langle \mathcal{U}, P\rangle = \langle \mathcal{J}_{\alpha,\beta}, P\rangle + A_1P(1) + B_1P(-1) + A_2P'(1) + B_2P'(-1), \tag{19}$$

where $\mathcal{J}_{\alpha,\beta}$ is the Jacobi linear functional (5), and A_1, B_1, A_2 and B_2 are, in general, complex numbers.

DEFINITION 2. If there exists a polynomial $\widetilde{P}_n(x)$ such that it satisfies the following conditions:

$$\begin{aligned} &\text{(i)} \quad \deg \widetilde{P}_n(x) \leqslant n, \\ &\text{(ii)} \quad \langle \mathcal{U}, \widetilde{P}_n(x)x^k\rangle = 0, \quad 0 \leqslant k < n, \end{aligned} \tag{20}$$

then, it is called a semiclassical Jacobi-type orthogonal polynomial.

The *existence* of $\widetilde{P}_n(x) = \sum_{k=0}^{n} a_{k,n}x^k$ ($a_{n,n} = 1$) is not always guaranteed, in spite of its n unknown coefficients are 'determined' by using the orthogonality condition, i.e., (20) gives a system of n linear algebraic homogeneous equations, nevertheless the matrix of coefficients for such a system depends on n, A_1, B_1, A_2 and B_2, then it may have a trivial solution. However, the *uniqueness* is always guaranteed.

Because we are not very skilled in doing heavy algebraic computation, we cannot do anything different from it (or we do not know another way to avoid it) when studying the algebraic properties of the semiclassical Jacobi-type orthogonal polynomials. So, in this direction and for later development of the paper it will be essential to be able to express the semiclassical Jacobi-type OP as a simple combination of the Jacobi polynomials and their first and second derivatives. The first step for this purpose is to obtain an explicit expression for the Jacobi kernels. For this aim it will be convenient to have an expression of the Jacobi polynomial $P_{n-1}(x)$ in the form of a simple relation containing $P_n(x)$ and $P_n'(x)$. Indeed:

LEMMA 3. *The following expression*

$$\begin{aligned} \hat{\gamma}_n^{\alpha,\beta} D\, P_{n-1}^{\alpha,\beta}(x) = [x(n-2) - (\tilde{\beta}_n + n\beta_n^{\alpha,\beta})]D\, P_n^{\alpha,\beta}(x) + \\ + nP_n^{\alpha,\beta}(x) + (1-x^2)D^2P_n^{\alpha,\beta}(x), \end{aligned} \tag{21}$$

where

$$D = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \hat{\gamma}_n^{\alpha,\beta} = (2n + \alpha + \beta + 1)\gamma_n^{\alpha,\beta},$$

and $\tilde{\beta}_n$, $\beta_n^{\alpha,\beta}$, $\gamma_n^{\alpha,\beta}$ *are given in* (11) *and* (13)*, respectively, holds.*

Proof. If one multiplies both sides of the three-term recurrence relation (12) by $\tilde{\alpha}_n$ (see (10)) and subtracts it from (10) one cancels the term $P_{n+1}(x)$. Then, we have

$$\hat{\gamma}_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x) = [n(x - \beta_n^{\alpha,\beta}) - \tilde{\beta}_n] P_n^{\alpha,\beta}(x) + \\ + (1 - x^2) D\, P_n^{\alpha,\beta}(x). \tag{22}$$

Hence, taking derivative in the above formula we obtain (21). □

The next proposition deals with Jacobi kernels, probably its proof results rather tedious, but explicit knowledge of the Jacobi kernels will be useful in the later development of the paper. All the work done here will rather simplify the subsequent working out with semiclassical Jacobi-type OP. Now we will show how, for certain particular cases of Jacobi kernels (see formula (16) and taking into account that p and q can be 0 or 1, indistinctly) it is possible to write them as a simple relation which combines the Jacobi polynomials and their derivatives.

PROPOSITION 2. *Let the couple of integers* (p, q) *be* $(0, 0)$, $(0, 1)$ *and* $(1, 1)$. *Then, the Jacobi kernels* (16) *evaluated in the points* $x = 1$ *and* $y = \pm 1$ *are the following:*

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(1, 1) &= \frac{n(n + \beta)}{(\alpha + 1)} \kappa_n^{\alpha,\beta} P_n^{\alpha,\beta}(1), \\ K_{n-1}^{\alpha,\beta(0,0)}(1, -1) &= -n \kappa_n^{\alpha,\beta} P_n^{\alpha,\beta}(-1), \end{aligned} \tag{23}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(1, 1) &= \frac{(n - 1)(n + \beta)}{(\alpha + 2)} \kappa_n^{\alpha,\beta} (P_n^{\alpha,\beta})'(1), \\ K_{n-1}^{\alpha,\beta(0,1)}(1, -1) &= -(n - 1) \kappa_n^{\alpha,\beta} (P_n^{\alpha,\beta})'(-1), \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{K_{n-1}^{\alpha,\beta(1,1)}(1, 1)}{K_{n-1}^{\alpha,\beta(0,1)}(1, 1)} &= \frac{(\alpha + 2)(\lambda_n - n) - (\alpha + 1)(\alpha + \beta + 2)}{2(\alpha + 1)(\alpha + 3)}, \\ \frac{K_{n-1}^{\alpha,\beta(1,1)}(1, -1)}{K_{n-1}^{\alpha,\beta(0,1)}(1, -1)} &= \frac{\lambda_n - (n + \alpha + \beta + 2)}{2(\alpha + 1)}, \end{aligned} \tag{25}$$

where $\kappa_n^{\alpha,\beta} = P_n^{\alpha,\beta}(1) \| P_{n-1}^{\alpha,\beta} \|^{-2} / \hat{\gamma}_n^{\alpha,\beta}$.

For the order of derivatives indicated in Proposition 2, the remaining three couples of Jacobi kernels (16), at the points $x = -1$ and $y = \pm 1$ can be found from (23)–(25) and

$$\begin{aligned} K_n^{\alpha,\beta(0,0)}(x, y) &= K_n^{\beta,\alpha(0,0)}(-x, -y), \\ K_n^{\alpha,\beta(0,1)}(x, y) &= -K_n^{\beta,\alpha(0,1)}(-x, -y), \\ K_n^{\alpha,\beta(1,1)}(x, y) &= K_n^{\beta,\alpha(1,1)}(-x, -y), \end{aligned} \tag{26}$$

which follows from (16), (17), and (9).

Proof. Here we prove (23)–(25), in such a way, that the proof be useful to the aim of expressing the semiclassical Jacobi-type OP as $x^n + \tilde{b}_n x^{n-1} +$ lower degree terms (see formula (44) below).

Let us start with the kernel $K_{n-1}^{\alpha,\beta(0,0)}(x, 1)$. Evaluating (17) in $y = 1$, and replacing the terms $P_{n-1}^{\alpha,\beta}(1)$ and $P_{n-1}^{\alpha,\beta}(x)$ according to (22), we have

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(x, 1) &= \frac{1}{x-1} \frac{P_n^{\alpha,\beta}(x) P_{n-1}^{\alpha,\beta}(1) - P_{n-1}^{\alpha,\beta}(x) P_n^{\alpha,\beta}(1)}{\| P_{n-1}^{\alpha,\beta} \|^2} \\ &= \kappa_n^{\alpha,\beta}[(1+x) D\, P_n^{\alpha,\beta}(x) - n P_n^{\alpha,\beta}(x)]. \end{aligned} \tag{27}$$

Now, we continue with $K_{n-1}^{\alpha,\beta(0,1)}(x, 1)$. By taking first derivative with respect to y in both sides of (17), one yields

$$\begin{aligned} &K_{n-1}^{\alpha,\beta(0,1)}(x, y) \\ &\quad = \sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x) D P_m^{\alpha,\beta}(y)}{\| P_m^{\alpha,\beta} \|^2} \\ &\quad = \frac{K_{n-1}^{\alpha,\beta(0,0)}(x, y)}{(x-y)} + \frac{P_n^{\alpha,\beta}(x) D\, P_{n-1}^{\alpha,\beta}(y) - P_{n-1}^{\alpha,\beta}(x) D\, P_n^{\alpha,\beta}(y)}{(x-y) \| P_{n-1}^{\alpha,\beta} \|^2}. \end{aligned} \tag{28}$$

Evaluating (28) at $y = 1$ we get

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(x, 1) &= \frac{K_{n-1}^{\alpha,\beta(0,0)}(x, 1)}{x-1} + \frac{P_n^{\alpha,\beta}(x) (P_{n-1}^{\alpha,\beta})'(1)}{\| P_{n-1}^{\alpha,\beta} \|^2 (x-1)} - \\ &\quad - \frac{(P_n^{\alpha,\beta})'(1) P_{n-1}^{\alpha,\beta}(x)}{\| P_{n-1}^{\alpha,\beta} \|^2 (x-1)}. \end{aligned} \tag{29}$$

By using Lemma 3, i.e., the expression (21) evaluated at $x = 1$, one can substitute $u(P_{n-1}^{\alpha,\beta})'(1)$ according to (21), then using (22) one expresses $P_{n-1}^{\alpha,\beta}(x)$ in terms of $D P_n^{\alpha,\beta}(x)$ and $P_n^{\alpha,\beta}(x)$. So, taking into account (27) the expression (29) becomes

$$K_{n-1}^{\alpha,\beta(0,1)}(x, 1)$$

$$= \frac{(P_n^{\alpha,\beta})'(1)[(1+x)D\,P_n^{\alpha,\beta}(x) - nP_n^{\alpha,\beta}(x)]}{\|P_{n-1}^{\alpha,\beta}\|^2\hat{\gamma}_n^{\alpha,\beta}} +$$

$$+ \frac{[(1+x)P_n^{\alpha,\beta}(1)D\,P_n^{\alpha,\beta}(x) - 2(P_n^{\alpha,\beta})'(1)P_n^{\alpha,\beta}(x)]}{\|P_{n-1}^{\alpha,\beta}\|^2\hat{\gamma}_n^{\alpha,\beta}(x-1)}. \tag{30}$$

Now, based on the second-order linear differential Equation (7) one can transform the rational part, in the right-hand side of (30), into a polynomial of degree $n-1$. In fact, using

$$(P_n^{\alpha,\beta})'(1) = \frac{\lambda_n}{2(\alpha+1)}P_n^{\alpha,\beta}(1), \quad n \geqslant 1,$$

which also follows from (7), we get

$$\begin{aligned}(\alpha+1)&K_{n-1}^{\alpha,\beta(0,1)}(x,1)\\ &= 2^{-1}\lambda_n K_{n-1}^{\alpha,\beta(0,0)}(x,1) +\\ &\quad + \kappa_n^{\alpha,\beta}\frac{[(x+1)(\alpha+1)D\,P_n^{\alpha,\beta}(x) - \lambda_n P_n^{\alpha,\beta}(x)]}{(x-1)}.\end{aligned} \tag{31}$$

This yields

$$\begin{aligned}(\alpha+1)&K_{n-1}^{\alpha,\beta(0,1)}(x,1)\\ &= 2^{-1}\lambda_n K_{n-1}^{\alpha,\beta(0,0)}(x,1) - \kappa_n^{\alpha,\beta}[(\beta+1)D\,P_n^{\alpha,\beta}(x) +\\ &\quad + (x+1)D^2P_n^{\alpha,\beta}(x)].\end{aligned} \tag{32}$$

Handling as above, starting from (17) evaluated at $y=-1$, it is easy to obtain similar expressions for the kernels $K_{n-1}^{\alpha,\beta(0,0)}(x,-1)$ and $K_{n-1}^{\alpha,\beta(0,1)}(x,-1)$. Indeed,

$$\begin{aligned}&K_{n-1}^{\alpha,\beta(0,0)}(x,-1) = (-1)^n\kappa_n^{\beta,\alpha}[(x-1)D\,P_n^{\alpha,\beta}(x) - nP_n^{\alpha,\beta}(x)],\\ &(\beta+1)K_{n-1}^{\alpha,\beta(0,1)}(x,-1)\\ &\quad = -2^{-1}\lambda_n K_{n-1}^{\alpha,\beta(0,0)}(x,-1) +\\ &\qquad + (-1)^{n+1}\kappa_n^{\beta,\alpha}[(\alpha+1)D\,P_n^{\alpha,\beta}(x) + (x-1)D^2P_n^{\alpha,\beta}(x)].\end{aligned} \tag{33}$$

Finally, we obtain the kernel $K_{n-1}^{\alpha,\beta(1,1)}(x,1)$ and $K_{n-1}^{\alpha,\beta(1,1)}(x,-1)$ in terms of the polynomials $P_n^{\alpha,\beta}(x)$ and their first, second, and third derivatives. From (32)–(33) it is straightforward to compute them, i.e.,

$$\begin{aligned}&\frac{2(\alpha+1)}{\kappa_n^{\alpha,\beta}}K_{n-1}^{\alpha,\beta(1,1)}(x,1)\\ &\quad = [\lambda_n(x+1) - 2(\beta+2)]D^2P_n^{\alpha,\beta}(x) -\\ &\qquad - [\lambda_n(n-1)DP_n^{\alpha,\beta}(x) + 2(x+1)D^3P_n^{\alpha,\beta}(x)],\end{aligned} \tag{34}$$

$$\frac{2(\beta+1)}{(-1)^n \kappa_n^{\beta,\alpha}} K_{n-1}^{\alpha,\beta(1,1)}(x,-1)$$
$$= [\lambda_n(1-x) - 2(\alpha+2)] D^2 P_n^{\alpha,\beta}(x) +$$
$$+ \lambda_n(n-1) D\, P_n^{\alpha,\beta}(x) + 2(x-1) D^3 P_n^{\alpha,\beta}(x). \tag{35}$$

As simple consequences of (27) and (32)–(35), the proposition holds. □

The asymptotic formulas for the Jacobi polynomials and their kernels (23)–(25) will help us to establish a condition on the existence of the semiclassical Jacobi-type orthogonal polynomials.

From the asymptotic formulas of the Jacobi kernels (see formulas (36)–(38) below) we will conclude that for n large enough the polynomial $\widetilde{P}_n(x)$ exists.

COROLLARY 1. *The following asymptotic formulas and expressions for the Jacobi kernels*

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(1,1) &\sim \frac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)2^{\alpha+\beta+1}}, \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) &\sim \frac{(-1)^{n+1} n^{\alpha+\beta+1}}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}, \end{aligned} \tag{36}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(1,1) &= (\alpha+1) K_{n-1}^{\alpha+1,\beta(0,0)}(1,1), \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) &= -2(\alpha+1) K_{n-1}^{\alpha+1,\beta+1(0,0)}(1,-1), \end{aligned} \tag{37}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(1,1)}(1,1) &= (\alpha+2) K_{n-1}^{\alpha+1,\beta(0,1)}(1,1), \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) &= 2(\beta+2) K_{n-1}^{\alpha+1,\beta+1(0,1)}(1,-1), \end{aligned} \tag{38}$$

hold.

The notation $x_n \sim y_n$ means that x_n behaves as y_n when $n \to \infty$, more precisely, $\lim_{n\to\infty} x_n/y_n = 1$.

To obtain the other kernels, as well as their estimates, we can use the symmetry properties (26) and (36)–(38).

Proof. Using the asymptotic formula for the Gamma function (see Olver (1974, formula 8.16, p. 88) and (Szegő (1975))

$$\Gamma(ax+b) \sim \sqrt{2\pi}\, \mathrm{e}^{-ax} (ax)^{ax+b-\frac{1}{2}}, \quad x \gg 1,\ a, b, x \in \mathbb{R}, \tag{39}$$

and taking into account (6), (8) and (15), we find the following asymptotic formulas for $k \in \mathbb{N}$

$$(P_n^{\alpha,\beta})^{(k)}(1) \sim \frac{\sqrt{\pi}\, n^{\alpha+2k+\frac{1}{2}}}{\Gamma(\alpha+k+1)2^{n+\alpha+\beta+k}}, \qquad \|P_{n-1}^{\alpha,\beta}\|^2 \sim \frac{\pi}{2^{2n+\alpha+\beta-2}}. \tag{40}$$

Hence, based on (40) and Proposition 2 (formulas (23)–(25)) the corollary holds. □

PROPOSITION 3. *The orthogonal polynomial sequence* $\{\widetilde{P}_k\}_{k\geqslant 0}$ *with respect to the moment functional* $\mathcal{U}$ *exists if and only if the determinant of the matrix* $\mathcal{K} = \mathcal{K}(n)$ (*see* (46) *below*) *is different from zero for every* $n \geqslant 0$.

Proof. Assume the sequence $\{\widetilde{P}_k\}_{k\geqslant 0}$ exists. Hence, one can write the Fourier expansion of the semiclassical Jacobi-type orthogonal polynomials in terms of the Jacobi polynomials

$$\widetilde{P}_n(x) := P_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = P_n^{\alpha,\beta}(x) + \sum_{k=0}^{n-1} a_{n,k} P_k^{\alpha,\beta}(x). \tag{41}$$

The coefficients $a_{n,k}$ can be found using the orthogonality of the polynomials $\widetilde{P}_n(x)$ with respect to $\mathcal{U}$, i.e.,

$$\langle \mathcal{U}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x)\rangle = 0, \quad 0 \leqslant k < n.$$

Putting (41) in (19) we get:

$$\begin{aligned}
&\langle \mathcal{U}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x)\rangle \\
&\quad = \langle \mathcal{J}_{\alpha,\beta}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x)\rangle + A_1 \widetilde{P}_n(1) P_k^{\alpha,\beta}(1) + B_1 \widetilde{P}_n(-1) P_k^{\alpha,\beta}(-1) + \\
&\qquad + A_2 \widetilde{P}_n(x) P_k^{\alpha,\beta}(x))'\big|_{x=1} + B_2(\widetilde{P}_n(x) P_k^{\alpha,\beta}(x))'\big|_{x=-1}.
\end{aligned} \tag{42}$$

If we use the decomposition (41), and taking into account the orthogonality of the Jacobi polynomials with respect to the linear functional $\mathcal{J}_{\alpha,\beta}$, we find the following expression for the coefficients $a_{n,k}$

$$\begin{aligned}
a_{n,k} = &-\frac{A_1 \widetilde{P}_n(1) P_k^{\alpha,\beta}(1) + B_1(\widetilde{P}_n)'(-1) P_k^{\alpha,\beta}(-1)}{\| P_k^{\alpha,\beta}\|^2} - \\
&-\frac{A_2[(\widetilde{P}_n)'(1) P_k^{\alpha,\beta}(1) + \widetilde{P}_n(1)(P_k^{\alpha,\beta})'(1)]}{\| P_k^{\alpha,\beta}\|^2} - \\
&-\frac{B_2[(\widetilde{P}_n)'(-1) P_k^{\alpha,\beta}(-1) + \widetilde{P}_n(-1)(P_k^{\alpha,\beta})'(-1)]}{\| P_k^{\alpha,\beta}\|^2}.
\end{aligned} \tag{43}$$

Thus, (41) becomes

$$\begin{aligned}
\widetilde{P}_n(x) = P_n^{\alpha,\beta}(x) &- A_1 \widetilde{P}_n(1) K_{n-1}^{\alpha,\beta(0,0)}(x,1) - \\
&- B_1 \widetilde{P}_n(-1) K_{n-1}^{\alpha,\beta(0,0)}(x,-1) - A_2(\widetilde{P}_n)'(1) K_{n-1}^{\alpha,\beta(0,0)}(x,1) - \\
&- B_2(\widetilde{P}_n)'(-1) K_{n-1}^{\alpha,\beta(0,0)}(x,-1) - A_2 \widetilde{P}_n(1) K_{n-1}^{\alpha,\beta(0,1)}(x,1) - \\
&- B_2 \widetilde{P}_n(-1) K_{n-1}^{\alpha,\beta(0,1)}(x,-1).
\end{aligned} \tag{44}$$

This is in accordance with the fact that $\widetilde{P}_n(x)$ cannot vanish for every $n \geqslant 0$.

Now we show in a very simple form that $\det \mathcal{K} \neq 0$, and by the way we write $\widetilde{P}_n(1)$, $\widetilde{P}_n(-1)$, $(\widetilde{P}_n)'(1)$ and $(\widetilde{P}_n)'(-1)$ in terms of Jacobi polynomials, Jacobi kernels and masses A_1, A_2, B_1 and B_2. For both aims one takes derivatives in (44) and evaluate the resulting equation, as well as (44), at $x = 1$ and $x = -1$. This leads us to a linear system of equations

$$\mathcal{K} \cdot \widetilde{\mathcal{P}}_n = \mathcal{P}_n, \tag{45}$$

being

$$\mathcal{K} := \mathcal{I}_4 + \mathcal{K}_4(n), \tag{46}$$

where $\mathcal{I}_4$ is the identity matrix and $\mathcal{K}_4(n) = (\mathcal{M}_1|\mathcal{M}_2|\mathcal{M}_3|\mathcal{M}_4)$ with the following column vectors

$$\mathcal{M}_1 = A_1 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \end{pmatrix} + A_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) \end{pmatrix},$$

$$\mathcal{M}_2 = B_1 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,0)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \end{pmatrix} + B_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) \\ K_{n-1}^{\alpha,\beta(1,1)}(-1,-1) \end{pmatrix},$$

$$\mathcal{M}_3 = A_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \end{pmatrix}, \qquad \mathcal{M}_4 = B_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,0)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \end{pmatrix}.$$

$\widetilde{\mathcal{P}}_n$ and $\mathcal{P}_n$ are the column vectors

$$\widetilde{\mathcal{P}}_n = \begin{pmatrix} \widetilde{P}_n(1) \\ \widetilde{P}_n(-1) \\ (\widetilde{P}_n)'(1) \\ (\widetilde{P}_n)'(-1) \end{pmatrix}, \qquad \mathcal{P}_n = \begin{pmatrix} P_n^{\alpha,\beta}(1) \\ P_n^{\alpha,\beta}(-1) \\ (P_n^{\alpha,\beta})'(1) \\ (P_n^{\alpha,\beta})'(-1) \end{pmatrix},$$

respectively.

Then, by the Cramer's rule, the system (45) has a unique solution if and only if the determinant of $\mathcal{K}$ is different from zero.

Moreover, if $\mathcal{K}_j(\mathcal{P}_n)$ denotes the matrix obtained substituting the j column in $\mathcal{K}$ by $\mathcal{P}_n$. Then,

$$\begin{aligned} \widetilde{P}_n(1) &= \frac{\det \mathcal{K}_1(\mathcal{P}_n)}{\det \mathcal{K}}, & \widetilde{P}_n(-1) &= \frac{\det \mathcal{K}_2(\mathcal{P}_n)}{\det \mathcal{K}}, \\ (\widetilde{P}_n)'(1) &= \frac{\det \mathcal{K}_3(\mathcal{P}_n)}{\det \mathcal{K}}, & (\widetilde{P}_n)'(-1) &= \frac{\det \mathcal{K}_4(\mathcal{P}_n)}{\det \mathcal{K}}. \end{aligned} \tag{47}$$

Conversely, assume that $\det \mathcal{K}$ does not vanish for every $n \geqslant 0$ and define $\widetilde{P}_n(x)$ by means of expressions (44) and (47). Then, $\{\widetilde{P}_k\}_{k\geqslant 0}$ is a monic OPS with respect to $\mathcal{U}$. □

Observe that $\det \mathcal{K}$ depends on n, α, β, A_1, B_1, A_2, and B_2. So, the family of manifolds $F_n(\alpha, \beta, A_1, B_1, A_2, B_2) = \det \mathcal{K} = 0$ determines a set of parameters for which the existence of semiclassical Jacobi-type orthogonal polynomials is not guaranteed. How to cover this lack? What kind of conditions are needed to work out with semiclassical Jacobi-type orthogonal polynomials? The following corollary helps us to establish a certain existence condition for the nth OP $\widetilde{P}_n(x)$.

COROLLARY 2. *Let us define*

$$c(\alpha) = \frac{A_2 B_2}{2^{4(\alpha+1)}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(\alpha+4)}, \tag{48}$$

and A_1, B_1, A_2 *and* B_2 *be different from zero in the expression* (19). *Then, for* n *large enough, the existence of* $\widetilde{P}_n(x)$ *is always guaranteed. Furthermore,*

$$\lim_{n\to\infty} \frac{\det \mathcal{K}}{c(\alpha)c(\beta)n^{16+4\alpha+4\beta}} = 1, \quad \alpha, \beta > -1,$$

where $c(\beta)$ *is obtained from* (48) *replacing* α *by* β $(\alpha \mapsto \beta)$.

Proof. Substituting the asymptotic formulas (36)–(38) in (46), and doing a cumbersome calculation we find for n large enough a rather lengthy expression for $\det \mathcal{K}$ (any symbolic computer algebra package like *Mathematica* can help to calculate it). So, we will not write it here, and provide only the order of n in the power series decomposition of $\mathcal{K}$. More precisely, it is

$$\det \mathcal{K} \sim c(\alpha)c(\beta)n^{16+4\alpha+4\beta}.$$

Hence, the corollary holds. □

The assumption 'n large enough' guarantees the existence of the semiclassical Jacobi-type orthogonal polynomials for every nonzero value of the masses A_1, B_1, A_2 and B_2. So, we work out with these kind of polynomials, without any problem, forgetting about a possible 'pathological' election of the masses A_1, B_1, A_2 and B_2.

PROPOSITION 4. *The following symmetry properties for the semiclassical Jacobi-type orthogonal polynomials and their first derivatives hold*

$$P_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-x) = (-1)^n P_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2}(x), \tag{49}$$

$$(P_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-x) = (-1)^{n+1} (P_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2})'(x). \tag{50}$$

Proof. Using the definition of the functional $\mathcal{U}$ (see (19)) the proof is straightforward. □

3.1. ORDER OF THE CLASS AND DIFFERENTIAL DISTRIBUTIONAL EQUATION FOR $\mathcal{U}$

Here we will determine the order of the class for the Jacobi-type moment functional (19), as well as the differential distributional equation that such a functional satisfies.

Let us rewrite (19) in the form:

$$\mathcal{U} = \mathcal{J}_{\alpha,\beta} + A_1\delta(x-1) + B_1\delta(x+1) - A_2\delta'(x-1) - B_2\delta'(x+1). \quad (51)$$

PROPOSITION 5. *The moment linear functional $\mathcal{U}$ verifies the differential distributional equation*

$$D[(1-x^2)^3\mathcal{U}] = [\beta - \alpha - (\alpha+\beta+6)x](1-x^2)^2\mathcal{U}, \quad (52)$$

being $\mathcal{U}$ a semiclassical functional of class $s = 4$.

Proof. The product of $(1-x^2)^2$ by the functional $\mathcal{U}$ leads to

$$(1-x^2)^2\mathcal{U} = (1-x^2)^2\mathcal{J}_{\alpha,\beta}. \quad (53)$$

Now, before taking derivatives in (53), it is convenient to remember the distributional equation for the Jacobi moment functional, i.e.,

$$D[(1-x^2)\mathcal{J}_{\alpha,\beta}] = [\beta - \alpha - (\alpha+\beta+2)x]\mathcal{J}_{\alpha,\beta}.$$

Thus, one has

$$\begin{aligned} &D[(1-x^2)^2\mathcal{U}] \\ &\quad = D[(1-x^2)^2\mathcal{J}_{\alpha,\beta}] \\ &\quad = -2x(1-x^2)\mathcal{J}_{\alpha,\beta} + (1-x^2)D[(1-x^2)\mathcal{J}_{\alpha,\beta}] \\ &\quad = -2x(1-x^2)\mathcal{J}_{\alpha,\beta} + (1-x^2)[\beta-\alpha-(\alpha+\beta+2)x]\mathcal{J}_{\alpha,\beta} \\ &\quad = (1-x^2)[\beta-\alpha-(\alpha+\beta+4)x]\mathcal{J}_{\alpha,\beta}. \end{aligned}$$

If we multiply the above expression by $(1-x^2)$, then

$$\begin{aligned} (1-x^2)D[(1-x^2)^2\mathcal{U}] &= [\beta-\alpha-(\alpha+\beta+4)x](1-x^2)^2\mathcal{J}_{\alpha,\beta} \\ &= [\beta-\alpha-(\alpha+\beta+4)x](1-x^2)^2\mathcal{U}, \end{aligned}$$

from which (52) holds.

To determine the order of the class it is enough to apply Theorem 1. Thus, $s = 4$. □

4. Hypergeometric Character

PROPOSITION 6. *The semiclassical Jacobi-type orthogonal polynomial* $\widetilde{P}_n(x)$ *can be represented, up to a multiplicative constant, by a generalized hypergeometric series. More precisely,*

$$\widetilde{P}_n(x) = \gamma \frac{2^{n-3}(\alpha+3)_{n-3}}{(n+\alpha+\beta+1)_n} \times$$
$$\times \ {}_6F_5\left(\begin{matrix} -n, n+\alpha+\beta+1, \beta_0+1, \beta_1+1, \beta_2+1, \beta_3+1 \\ \alpha+3, \beta_0, \beta_1, \beta_2, \beta_3 \end{matrix} \middle| \frac{1-x}{2} \right), \quad (54)$$

where γ, β_0, β_1, β_2 *and* β_3 *are constants depending on* n, α, β, *and the masses* A_1, B_1, A_2, *and* B_2.

Proof. Let us define the polynomials in k,

$$a_1(k) = 8A_n(n+\alpha)(k+a_1)(k+a_2), \quad a_1 = \alpha+1,\ a_2 = a_1+1,$$
$$a_2(k) = -4B_n(n+\alpha)(k+b_1)(k+b_2)(k+b_3),$$
$$b_1 = -n, \quad b_2 = -b_1+a_1+\beta,\ b_3 = a_2,$$
$$a_3(k) = \frac{8nC_n(n+\alpha)(n+\alpha+1)}{(n+b_2)(n+b_2+1)}(k+c_1)(k+c_2)(k+c_3),$$
$$c_1 = a_2, \quad c_2 = b_2,\ c_3 = b_2+1,$$
$$a_4(k) = \frac{2D_n(n+b_2-2)(n+b_2-1)}{(n-1)}(k+d_1)(k+d_2)(k+d_3),$$
$$d_1 = b_1,\ d_2 = b_1+1,\ d_3 = a_2,$$
$$a_5(k) = 2nE_n(n+\alpha)(k+e_1)(k+e_2)(k+e_3)(k+e_4),$$
$$e_1 = b_1,\ e_2 = d_2,\ e_3 = b_2,\ e_4 = b_2+1,$$
$$a_6(k) = -\frac{4nF_n(n+\alpha)(n-1)b_2}{(n+b_2)(n+b_2+1)}(k+f_1)(k+f_2)(k+f_3)(k+f_4),$$
$$f_1 = b_1,\ f_2 = b_2,\ f_3 = b_2+1,\ f_4 = b_2+2,$$
$$a_7(k) = \frac{G_n(n+b_2-2)(1-n-b_2)}{(n-2)}(k+g_1)(k+g_2)(k+g_3)(k+g_4),$$
$$g_1 = b_1,\ g_2 = b_1+1,\ g_3 = b_1+2,\ g_4 = b_2.$$

Substituting (14) in (64) one finds

$$\widetilde{P}_n(x) = \frac{2^{n-3}(\alpha+3)_{n-3}}{(n+\alpha+\beta+1)_n} \sum_{k=0}^{\infty} \left[\sum_{i=1}^{7} a_i(k) \right] \times$$
$$\times \frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\alpha+3)_k} \left(\frac{1-x}{2} \right)^k. \quad (55)$$

Taking into account that the expression inside the quadratic brackets is a polynomial in k of degree 4, and denoting it by

$$p_4(k) = \left[\sum_{i=1}^{7} a_i(k)\right] := \frac{p_4^{(4)}(k)}{4!}(k+\beta_0)(k+\beta_1)(k+\beta_2)(k+\beta_3),$$

one can write

$$\begin{aligned}\widetilde{P}_n(x) = p_4^{(4)}(k)\frac{2^{n-3}(\alpha+3)_{n-3}}{4!\,(n+\alpha+\beta+1)_n}\sum_{k=0}^{\infty}\Bigg[\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+3)_k}\times \\ \times\frac{(k+\beta_0)(k+\beta_1)(k+\beta_2)(k+\beta_3)}{k!}\left(\frac{1-x}{2}\right)^k\Bigg],\end{aligned} \tag{56}$$

being $p_4^{(4)}(x)/4!$ the leading coefficient of $p_4(k)$

$$\begin{aligned}\frac{p_4^{(4)}(k)}{4!} = 2nE_n(n+\alpha) - \frac{4nF_n(n-1)(n+\alpha)(n+a_1)}{(n+b_2)(n+b_2+1)} - \\ - \frac{G_n(n+b_2-1)(n+b_2-2)}{(n-2)}.\end{aligned} \tag{57}$$

Since

$$(k+\beta_i) = \frac{\beta_i(\beta_i+1)_k}{(\beta_i)_k},$$

where $-\beta_i$ with $i = 0, 1, 2, 3$ are the zeros of $p_4(k)$ depending on $n, \alpha, \beta, A_1, A_2, B_1, B_2$, the expression in (56) becomes

$$\begin{aligned}\widetilde{P}_n(x) = \gamma(n)\frac{2^{n-3}(\alpha+3)_{n-3}}{(n+\alpha+\beta+1)_n}\sum_{k=0}^{\infty}\Bigg[\frac{(-n)_k(n+\alpha+\beta+1)_k}{k!\,(\alpha+3)_k}\times \\ \times\frac{(1+\beta_0)_k(1+\beta_1)_k(1+\beta_2)_k(1+\beta_3)_k}{(\beta_0)_k(\beta_1)_k\,(\beta_2)_k(\beta_3)_k}\left(\frac{1-x}{2}\right)^k\Bigg],\end{aligned} \tag{58}$$

where

$$\begin{aligned}\gamma(n) := 8A_n(n+\alpha)a_1a_2 + 4nB_n(n+\alpha)a_2b_2 + \\ + \frac{8nC_n(n+\alpha)(n+a_1)a_2b_2(b_2+1)}{(n+b_2)(n+b_2+1)} + \\ + 2nD_na_2 + 2n(n-1)E_n(n+\alpha)b_2(b_2+1) + \\ + \frac{4n(n-1)F_n(n+\alpha)(n+a_1)b_2(b_2+1)(b_2+2)}{(n+b_2)(n+b_2+1)} + \\ + n(n-1)G_nb_2(n+b_2-2)(n+b_2-1),\end{aligned} \tag{59}$$

which is nothing other than the hypergeometric representation (54). □

The zeros of $p_4(k)$ are in general complex numbers. If for some $i = 0, 1, 2, 3$, the value of β_i is a positive integer we need to take the analytic continuation of the hypergeometric series (54).

5. Some Asymptotic Formulas

In this section we will study some asymptotic formulas for the semiclassical Jacobi-type orthogonal polynomials. More precisely, the relative asymptotics $\widetilde{P}_n(x)/P_n^{\alpha,\beta}(x)$ outside the interval $[-1, 1]$ and the difference between the new polynomials and the classical ones inside $[-1, 1]$.

For such a purpose will be useful to have an explicit representation of the semi-classical Jacobi-type orthogonal polynomials in terms of the classical ones. To do that we rewrite Equation (44) in the form

$$\begin{aligned}\widetilde{P}_n(x) = (1 + n\zeta_n + n\eta_n)P_n^{\alpha,\beta}(x) + \\ + [\zeta_n(1-x) - \eta_n(1+x) + (\beta+1)\chi_n + (\alpha+1)\omega_n]DP_n^{\alpha,\beta}(x) + \\ + [\chi_n(1+x) - \omega_n(1-x)]D^2P_n^{\alpha,\beta}(x),\end{aligned} \tag{60}$$

where

$$\begin{aligned}\zeta_n &= \left[B_1 - \frac{\lambda_n B_2}{2(\beta+1)}\right]C_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2} - B_2 D_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2}, \\ \eta_n &= \left[A_1 + \frac{\lambda_n A_2}{2(\alpha+1)}\right]C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} + A_2 D_n^{\alpha,\beta,A_1,B_1,A_2,B_2},\end{aligned} \tag{61}$$

$$\begin{aligned}(\alpha+1)\chi_n &= A_2 C_n^{\alpha,\beta,A_1,B_1,A_2,B_2}, \\ (\beta+1)\omega_n &= B_2 C_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2},\end{aligned} \tag{62}$$

and

$$\begin{aligned}C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \kappa_n^{\alpha,\beta}\widetilde{P}_n(1), \\ D_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \kappa_n^{\alpha,\beta}(\widetilde{P}_n)'(1).\end{aligned} \tag{63}$$

Notice that ζ_n, η_n, χ_n, and ω_n depend on $n, \alpha, \beta, A_1, B_1, A_2$, and B_2.

Now, using (8)–(12) we can rewrite (60) as follows

$$\begin{aligned}\widetilde{P}_n(x) = A_n P_n^{\alpha,\beta}(x) + n[B_n P_{n-1}^{\alpha+1,\beta+1}(x) + C_n P_n^{\alpha+1,\beta+1}(x)] + \\ + n[D_n P_{n-2}^{\alpha+1,\beta+1}(x) + (n-1)E_n P_{n-2}^{\alpha+2,\beta+2}(x)] + \\ + n(n-1)[F_n P_{n-1}^{\alpha+2,\beta+2}(x) + G_n P_{n-3}^{\alpha+2,\beta+2}(x)],\end{aligned} \tag{64}$$

where

$$
\begin{aligned}
B_n &= \zeta_n - \eta_n + C_n \beta_{n-1}^{\alpha+1,\beta+1} + (\beta+1)\chi_n + (\alpha+1)\omega_n, \\
A_n &= 1 - nC_n, \qquad C_n = -(\zeta_n + \eta_n), \qquad D_n = C_n \gamma_{n-1}^{\alpha+1,\beta+1}, \\
E_n &= \chi_n - \omega_n + F_n \beta_{n-2}^{\alpha+2,\beta+2}, \qquad F_n = \chi_n + \omega_n, \qquad G_n = F_n \gamma_{n-2}^{\alpha+2,\beta+2}.
\end{aligned} \tag{65}
$$

Remark 1. The semiclassical Jacobi-type orthogonal polynomials satisfy a second-order linear differential equation (SODE). To deduce it one can rewrite the representation formula (60) in terms of the polynomials and their first derivatives, and using the fact that the Jacobi polynomials satisfy a SODE.

THEOREM 2. *For n large enough the semiclassical Jacobi-type OP satisfy the following outer and inner asymptotics, respectively,*

$$
\begin{aligned}
\frac{\widetilde{P}_n(z)}{P_n^{\alpha,\beta}(z)} \;=\; & 1 + \frac{2(\beta+2)}{n}\left[1 - \sqrt{\frac{z-1}{z+1}}\right] + \\
& + \frac{2(\alpha+2)}{n}\left[1 - \sqrt{\frac{z+1}{z-1}}\right] + \mathrm{o}\left(\frac{1}{n}\right),
\end{aligned} \tag{66}
$$

where $z \in \mathbb{C}\backslash[-1,1]$*,*

$$
\begin{aligned}
& 2^{n+\alpha+\beta}[\widetilde{P}_n(x) - P_n^{\alpha,\beta}(x)] \\
& \quad \sim \frac{[\sin(\frac{\theta}{2})]^{-\alpha-\frac{3}{2}}[\cos(\frac{\theta}{2})]^{-\beta-\frac{3}{2}}}{n} \times \\
& \quad \times \Big\{(\alpha+\beta+4)\sin\theta\cos\big[n\theta + \tfrac{1}{2}(\alpha+\beta+1)\theta - -\tfrac{1}{2}(\alpha+\tfrac{1}{2})\pi\big] - \\
& \quad - 2(\alpha+2)\cos\big[n\theta + \tfrac{1}{2}(\alpha+\beta+3)\theta - -\tfrac{1}{2}(\alpha+\tfrac{3}{2})\pi\big]\Big\} + \mathrm{O}\left(\frac{1}{n^2}\right),
\end{aligned} \tag{67}
$$

$x \in (-1,1)$.

Proof. The existence of $\widetilde{P}_n(x)$ for n large enough is guaranteed for any choice of nonzero masses A_1, B_1, A_2 and B_2. Now using the symmetry property (9) and the asymptotic formulas (36) and (40), we can compute the asymptotic behavior of the semiclassical Jacobi-type orthogonal polynomials, as well as their first derivatives at the points ± 1, i.e.,

$$
\begin{aligned}
& \widetilde{P}_n(1) \sim \frac{\sqrt{\pi}\,\Gamma(\alpha+4)}{2^{n-2}A_2 n^{\frac{7}{2}+\alpha}}, \qquad \widetilde{P}_n(-1) \sim (-1)^{n+1}\frac{\sqrt{\pi}\,\Gamma(\beta+4)}{2^{n-2}B_2 n^{\frac{7}{2}+\beta}}, \\
& (\widetilde{P}_n)'(1) \sim -\frac{\sqrt{\pi}\,\Gamma(\alpha+3)}{2^{n-1}A_2 n^{\frac{3}{2}+\alpha}}, \qquad (\widetilde{P}_n)'(-1) \sim \frac{(-1)^{n+1}\sqrt{\pi}\,\Gamma(\beta+3)}{2^{n-1}B_2 n^{\frac{3}{2}+\beta}}.
\end{aligned} \tag{68}
$$

From (40)–(68) we can give the estimates for the constants defined by (61)–(63)

$$
\begin{aligned}
&C^{\alpha,\beta,A_1,B_1,A_2,B_2} \sim 2\frac{\Gamma(\alpha+4)}{A_2\Gamma(\alpha+1)n^4}, \\
&D^{\alpha,\beta,A_1,B_1,A_2,B_2} \sim -\frac{\Gamma(\alpha+3)}{A_2\Gamma(\alpha+1)n^2}, \\
&\zeta_n \sim 2\frac{(\beta+2)}{n^2}, \qquad \eta_n \sim 2\frac{(\alpha+2)}{n^2}, \qquad \chi_n \sim 2\frac{(\alpha+2)(\alpha+3)}{n^4}, \\
&\omega_n \sim -2\frac{(\beta+2)(\beta+3)}{n^4}.
\end{aligned} \tag{69}
$$

Finally, from (60), taking derivatives twice and using (40) and (69), we find

$$
(\widetilde{P}_n)''(1) \sim -\frac{\sqrt{\pi}n^{\alpha+\frac{9}{2}}(\alpha+2)(\alpha+5)}{\Gamma(\alpha+5)2^{n+\alpha+\beta+2}}. \tag{70}
$$

To obtain the relative asymptotics $\widetilde{P}_n(z)/P_n^{\alpha,\beta}(z)$, outside the interval $[-1,1]$ we need to do some handling. First, we multiply (60) by $\sigma(z)$, and using the SODE (7) we find the following equivalent representation formula

$$
\sigma(z)\widetilde{P}_n(z) = a(z;n)P_n^{\alpha,\beta}(z) + b(z;n)DP_n^{\alpha,\beta}(z), \tag{71}
$$

where $a(z;n), b(z;n)$ are polynomials of uniformly bounded degree in z with coefficients depending on n given by

$$
\begin{aligned}
a(z;n) = {}& (1+n\zeta_n+n\eta_n)\sigma(z) - \lambda_n[\chi_n(1+z) - \omega_n(1-z)], \\
b(z;n) = {}& [\zeta_n(1-z) - \eta_n(1+z) + (\beta+1)\chi_n + (\alpha+1)\omega_n]\sigma(z) - \\
& - \tau(z)[\chi_n(1+z) - \omega_n(1-z)].
\end{aligned} \tag{72}
$$

Second, we will rewrite (71) in the form

$$
\widetilde{P}_n(z) = \tilde{a}(z;n)P_n^{\alpha,\beta}(z) + \tilde{b}(z;n)DP_n^{\alpha,\beta}(z), \tag{73}
$$

where

$$
\begin{aligned}
\tilde{a}(z;n) &= (1+n\zeta_n+n\eta_n) - \lambda_n\left[\frac{\chi_n}{(1-z)} - \frac{\omega_n}{(1+z)}\right] \\
&\sim 1 + 2\frac{(\alpha+\beta+4)}{n} - \\
&\quad - \frac{2}{n^2}\left[\frac{(\alpha+2)(\alpha+3)}{(1-z)} + \frac{(\beta+2)(\beta+3)}{(1+z)}\right],
\end{aligned} \tag{74}
$$

$$
\begin{aligned}
\tilde{b}(z;n) &= [\zeta_n(1-z) - \eta_n(1+z) + (\beta+1)\chi_n + (\alpha+1)\omega_n] - \\
&\quad - \tau(x)\left[\frac{\chi_n}{(1-z)} - \frac{\omega_n}{(1+z)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sim \frac{2}{n^2}[(\beta+2)(1-z)-(\alpha+2)(1+z)]+\\
&\quad+\frac{2}{n^4}[(\beta+1)(\alpha+2)(\alpha+3)-(\alpha+1)(\beta+2)(\beta+3)]-\\
&\quad-2\frac{[\beta-\alpha-(\alpha+\beta+2)z]}{n^4}\times\\
&\quad\times\left[\frac{(\alpha+2)(\alpha+3)}{(1-z)}+\frac{(\beta+2)(\beta+3)}{(1+z)}\right].
\end{aligned}
\tag{75}
$$

Then, from (73) and using (74)–(76), as well as,

$$
\frac{1}{n}\frac{D\,P_n^{\alpha,\beta}(z)}{P_n^{\alpha,\beta}(z)}=\frac{1}{\sqrt{z^2-1}}+\mathrm{o}(1),
\tag{76}
$$

we obtain the following estimate for the ratio (66).

In order to obtain the asymptotic behavior of the difference between the new polynomials and the classical ones, when z belongs to $[-1,1]$, we use the Darboux formula for the asymptotics of the Jacobi polynomials on the interval $\theta \in [\varepsilon, \pi-\varepsilon]$, $0<\varepsilon \ll 1$ (Szegő, 1975, Equation 8.21.10, p. 196)

$$
\begin{aligned}
a_n P_n^{\alpha,\beta}(\cos\theta) &= \frac{(\sin\frac{\theta}{2})^{-\alpha-\frac{1}{2}}(\cos\frac{\theta}{2})^{-\beta-\frac{1}{2}}}{\sqrt{n\pi}}\times\\
&\quad\times\cos\bigl[n\theta+\tfrac{1}{2}(\alpha+\beta+1)\theta-\tfrac{1}{2}(\alpha+\tfrac{1}{2})\pi\bigr]+\mathrm{O}\left(\frac{1}{n^{\frac{3}{2}}}\right),
\end{aligned}
\tag{77}
$$

with $a_n=\frac{(n+\alpha+\beta+1)_n}{2^n\,n!}\sim\frac{2^{n+\alpha+\beta}}{\sqrt{n\pi}}$.

The expression (64), as well as the following asymptotic estimates for the coefficients

$$
\begin{aligned}
&A_n\sim 1+2\frac{(\alpha+\beta+4)}{n}, \qquad B_n\sim 2\frac{(\beta-\alpha)}{n^2},\\
&C_n\sim -2\frac{(\alpha+\beta+4)}{n^2}, \qquad D_n\sim -\frac{(\alpha+\beta+4)}{2n^2},\\
&E_n=\mathrm{O}\left(\frac{1}{n^6}\right), \qquad F_n=\mathrm{O}\left(\frac{1}{n^4}\right), \qquad G_n=\mathrm{O}\left(\frac{1}{n^4}\right),
\end{aligned}
\tag{78}
$$

follow from (69).

Then, using (64) and (77)–(78) we deduce (67). □

6. Zeros

Here we will study the properties of the zeros of the semiclassical Jacobi-type orthogonal polynomials, for nonzero values of the masses, and will present some results concerning their asymptotic behaviour.

THEOREM 3. *Suppose* $n \gg N$ $(N \in \mathbb{N})$. *Then, the semiclassical Jacobi-type orthogonal polynomial* $\widetilde{P}_n(x)$ *has at least* $n-4$ *different real zeros in* $(-1, 1)$.

Proof. Since, for n large enough

$$\langle \mathcal{U}, \widetilde{P}_n(x)\,(1-x^2)^2\rangle = \langle \mathcal{J}_{\alpha,\beta}, \widetilde{P}_n(x)\,(1-x^2)^2\rangle = 0,$$

the semiclassical orthogonal polynomial $\widetilde{P}_n(x)$ changes its sign in the interval $(-1, 1)$. Let $x_1, x_2, \ldots, x_k$ be the different real zeros of odd multiplicity of $\widetilde{P}_n(x)$ in $(-1, 1)$. Hence, if $q(x) = (x-x_1)(x-x_2)\ldots(x-x_k)$, then the product $\widetilde{P}_n(x)q(x)$ does not change its sign in $(-1, 1)$.

Now define

$$h(x) = (1-x^2)^2 q(x).$$

Thus

$$\langle \mathcal{U}, \widetilde{P}_n(x)\,h(x)\rangle = \langle \mathcal{J}_{\alpha,\beta}, \widetilde{P}_n(x)\,h(x)\rangle > 0,$$

so $\deg h(x) \geqslant n$, i.e., $k \geqslant n-4$. □

COROLLARY 3. *Suppose that all the masses involved in the linear functional* $\mathcal{U}$ (*see* (19)) *are real, and* A_2, B_2 *have the same sign. Then, for* n *large enough the semiclassical Jacobi-type orthogonal polynomial has exactly* $n-3$ *different real zeros belonging to the interval* $(-1, 1)$.

Proof. Let us consider the case of even n and $A_2, B_2 > 0$ (the procedure to prove the other cases is completely analog to the developed here).

Since $\widetilde{P}_n(1) > 0$ and $\widetilde{P}_n'(1) < 0$ (see (68)), then for some positive $x > 1$, the polynomial $\widetilde{P}_n(x)$ has a minimum. This implies that on the right of $x = 1$ it has only two zeros (see the argument below for the point -1), which can be complex conjugates, real and simple or with multiplicity 2. Again from (68), since $\widetilde{P}_n(-1)$ and $\widetilde{P}_n'(-1)$ are negative the polynomial $\widetilde{P}_n(x)$ is a convex upward function for $x < -1$ and has a simple real zero; otherwise the number of zeros off $[-1, 1]$ would be greater than 4, which yields a contradiction. □

COROLLARY 4. *Let* A_1, A_2, B_1, B_2 *be real numbers, and* $A_2 > 0$, $B_2 < 0$. *Then, for* n *large enough the semiclassical Jacobi-type orthogonal polynomial has exactly* 4 *zeros off* $[-1, 1]$. *Two of them are located on the right of* $x = 1$, *and the other two are on the left of* $x = -1$. *Thus, in* $(-1, 1)$ *are* $n-4$ *real and simple zeros.*

Proof. Its enough to take into account the formulas (68) and analyze two cases: First, when n is even; second, when n is odd. So, counting the number of possible zeros outside the interval $(-1, 1)$ the corollary holds. □

By the two previous corollaries we can deduce the existence of complex conjugate zeros outside the interval $(-1, 1)$. Then it is interesting to give some bounds for them, in order to have some control about their asymptotic behaviour. The following proposition deals with Corollaries 3 and 4, and the masses are subjected to the restrictions imposed there.

PROPOSITION 7. *Let ϵ and δ be real and complex numbers, respectively, where $0 < \epsilon < 2$, and $\Re\delta > 0$. Let $z_{1,2}$ denotes z_1, z_2, indistinctly, and $z_1 = \bar{z}_2$ (the same for $z'_{1,2}$). Then, for n large enough, the bounds for the real and imaginary part of complex conjugate zeros $z_1 = \bar{z}_2$ and $z'_1 = \bar{z}'_2$, located on the right-hand side of the point $x = 1$, and on the left-hand side of $x = -1$, respectively, are the following:*

$$\begin{aligned} &1 < \Re(z_{1,2}) < 1 + \frac{\Re(\delta)}{n^{\frac{\epsilon}{2}}}, \\ &0 \leqslant \Im(z_{1,2}) < \widetilde{P}_n(1), \quad n \gg N, \quad N \in \mathbb{N}. \\ &-1 - \frac{\Re(\delta)}{n^{\frac{\epsilon}{2}}} < \Re(z'_{1,2}) < -1, \\ &0 \leqslant \Im(z'_{1,2}) < \widetilde{P}_n(-1). \end{aligned} \tag{79}$$

The above proposition guarantees that the complex conjugate zeros off the orthogonality interval $(-1, 1)$, tend to the end points.

Proof. Let us consider the situation when two zeros belong to the domain $\mathbb{C} \setminus \{z : -\infty < \Re z \leqslant 1\}$ (analogously for $\mathbb{C} \setminus \{z : -1 \leqslant \Re z < \infty\}$). Now, observe that for n large enough, from (40) and (68), the semiclassical polynomial $\widetilde{P}_n(x)$ in $x = 1$ takes a positive value tending to zero, while its first derivative is a negative value tending to zero, whereas the classical Jacobi polynomials and its first derivative $P_n^{\alpha,\beta}(1)$ and $(P_n^{\alpha,\beta})'(1)$, respectively, tend to $+\infty$.

Let $\{\tilde{z}_k\}_{k=1}^{\infty}$ be the sequence

$$\tilde{z}_k = 1 + \frac{\delta}{k^{2-\epsilon}},$$

where $0 < \epsilon < 2$, and $\delta \in \mathbb{C}$ is any complex constant number such that $\Re\delta > 0$. Notice that, such a sequence converges to 1 when k tends to ∞.

Theorem 2 gives the outer asymptotics for $z \in \mathbb{C}\setminus[-1, 1]$, so for n large enough one can evaluate (66) in the point $\tilde{z}_n$. Thus,

$$\frac{\widetilde{P}_n(\tilde{z}_n)}{P_n^{\alpha,\beta}(\tilde{z}_n)} = 1 - \frac{2(\alpha + 2)}{n^{\frac{\epsilon}{2}}}\left(\frac{2}{\delta}\right)^{\frac{1}{2}} + O\left(\frac{1}{n}\right). \tag{80}$$

So, (80) shows that for $n \gg N$ ($N \in \mathbb{N}$) the polynomials $P_n^{\alpha,\beta}(z)$ and $\widetilde{P}_n(z)$ have the same asymptotic behaviour in $\tilde{z}_n$. Now, taking into account the aforementioned fact on the asymptotic behavior of classical and semiclassical polynomials, respectively, in the point $x = 1$, the complex conjugate zeros of the semiclassical Jacobi-type orthogonal polynomials are inside the disk of radius $|\tilde{z}_n|$ centered on $z = 1$. Then, $|z_1|, |z_2| < |\tilde{z}_n|$, $n \gg N$ ($N \in \mathbb{N}$). Moreover, for the imaginary part of z_1, z_2 one can establish a sharp bound due to the fact that $\widetilde{P}_n(x)$ in $x = 1$ is positive, its first derivative is negative, and $\widetilde{P}_n(x) \to \infty$, when $x \to \infty$. Hence, the result holds. □

The above proposition is also valid when the two zeros off the interval $[-1, 1]$ are real and simple or with multiplicity 2 (of course, omitting the analysis of imaginary part).

PROPOSITION 8. *Suppose that $A_1, A_2, B_1, B_2 \in \mathbb{R}$. Then, for n large enough, if $A_2 < 0$ and $B_2 > 0$ the semiclassical polynomial $\widetilde{P}_n(x)$, orthogonal with respect to the linear functional $\mathcal{U}$ has exactly $n-2$ zeros belonging to the set $(-1, 1)$, and the two remainder zeros are outside the interval $(-1, 1)$ being one positive and the other one negative.*

Proof. Let $x_1, x_2, \ldots, x_k$ be the different real zeros of odd multiplicity of $\widetilde{P}_n(x)$ on the interval $(-1, 1)$ and

$$q(x) = (x - x_1)(x - x_2) \ldots (x - x_k).$$

Obviously, the product $\widetilde{P}_n(x)q(x)$ does not change its sign in $(-1, 1)$. Now define a new polynomial $h(x)$ as

$$h(x) = (1 - x^2)q(x) = (1 - x^2)(x - x_1)(x - x_2) \ldots (x - x_k),$$

since, for n large enough, the signs of

$$\int_{-1}^{1} (1 - x^2)q(x)\widetilde{P}_n(x)\rho(x)\,\mathrm{d}x \quad \text{and} \quad 2[-A_2\widetilde{P}_n(1)q(1) + B_2\widetilde{P}_n(-1)q(-1)]$$

are the same, then

$$\langle \mathcal{U}, \widetilde{P}_n(x)\,h(x)\rangle \neq 0, \tag{81}$$

which yields $\deg h(x) \geqslant n$, i.e. $k \geqslant n - 2$.

To prove that $\widetilde{P}_n(x)$ has one simple negative zero and one simple positive zero outside $[-1, 1]$ we use the fact that for n large enough $\widetilde{P}_n(1) < 0$, $\widetilde{P}_n'(1) > 0$ (formula (68)) and the fact that the polynomial $\widetilde{P}_n(x)$ is a continuous convex upward function for $x > 1$, then in some positive value $x > 1$ it changes its sign. Using the same argument when n is even, and equivalently when n is odd, we can prove (based on the expressions (68)) that the polynomial $\widetilde{P}_n(x)$ has one simple real negative zero on the left-hand side of $x = -1$. This implies that $k = n - 2$, hence the proposition holds. □

When $A_2 < 0$, $B_2 > 0$ by Proposition 8 we have only 2 zeros outside the interval $(-1, 1)$. So, we denote the zero on the left-hand side of -1 by $x_{n,1} < -1$ (analogously for the other case $x_{n,n} > 1$). Now we proceed to study, in more detail, the speed of convergence of these zeros ($x_{n,1}$ and $x_{n,n}$) to the end points of the interval $(-1, 1)$.

COROLLARY 5. *For n large enough* $1 < x_{n,n} < 1 + 2(\alpha+3)/n^2$ *and* $-1 > x_{n,1} > -1 - 2(\beta+3)/n^2$.

Proof. Let us start with the zero $x_{n,n}$. Using the formulas (68) one can define a linear function

$$f(x) = (\widetilde{P}_n)'(1)(x-1) + \widetilde{P}_n(1), \tag{82}$$

whose zero (denoted by x_0) bounds by the right the zero $x_{n,n}$ of $\widetilde{P}_n(x)$, i.e., $1 < x_{n,n} < x_0$. Hence, from (82) one has

$$1 < x_{n,n} < x_0 = 1 + \frac{2(\alpha+3)}{n^2} + \mathrm{o}(n^{-2}).$$

A similar procedure leads to prove the bound of $x_{n,1}$, i.e., one gets a linear function which passes by the point $(-1, \widetilde{P}_n(-1))$, such that $f'(x) = (\widetilde{P}_n)'(-1)$. So, the corollary holds. □

Remark 2. If n tends to infinity, all the zeros of the semiclassical Jacobi-type orthogonal polynomial $\widetilde{P}_n(x)$ off $[-1, 1]$ tend to ± 1.

Concerning the distribution of zeros for the semiclassical Jacobi-type orthogonal polynomials inside $[-1, 1]$ the next result shows that it is an arcsin distribution.

THEOREM 4. *Let* ν_n *be the discrete unit measure defined on the Borel sets in* $\mathbb{C}$ *having mass* $1/n$ *at each zero of* $\widetilde{P}_n(x)$. *Then*

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}}, \tag{83}$$

in the weak star topology.

Proof. From (64) and (78), we get

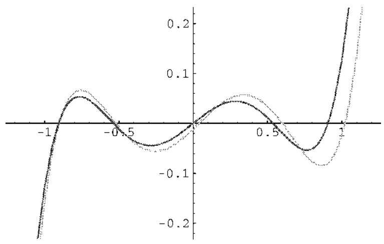
$$\begin{aligned} &\|\widetilde{P}_n(x)\|_{[-1,1]} \\ &\quad \leqslant |A_n|\|P_n^{\alpha,\beta}(x)\|_{[-1,1]} + n|B_n|\|P_{n-1}^{\alpha+1,\beta+1}(x)\|_{[-1,1]} + \\ &\quad + n|C_n|\|P_n^{\alpha+1,+1}(x)\|_{[-1,1]} + n|D_n|\|P_{n-2}^{\alpha+1,\beta+1}(x)\|_{[-1,1]} + \\ &\quad + n(n-1)\big[|E_n|\|P_{n-2}^{\alpha+2,+2}(x)\|_{[-1,1]} + |F_n|\|P_{n-1}^{\alpha+2,\beta+2}(x)\|_{[-1,1]}\big] + \\ &\quad + n(n-1)|G_n|\|P_{n-3}^{\alpha+2,\beta+2}(x)\|_{[-1,1]}, \end{aligned} \tag{84}$$

where $\|\cdot\|_{[-1,1]}$ denotes the sup-norm in the interval $[-1, 1]$. Because of $\overline{\lim}_{n\to\infty}\|P_n^{\alpha,\beta}(x)\|_{[-1,1]}^{1/n} = 1/2$ (see Szegő, 1975), we deduce

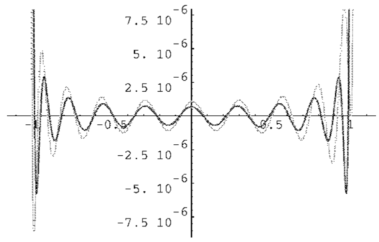
$$\overline{\lim_{n\to\infty}}\|\widetilde{P}_n(x)\|_{[-1,1]}^{1/n} \leqslant \tfrac{1}{2}. \tag{85}$$

Thus, from Theorem 2.1 in Blatt *et al.* (1988)

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}}. \qquad \square \tag{86}$$

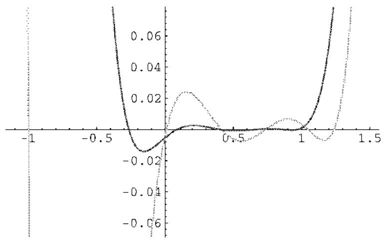


Comparison between classical (bold line) and semiclassical polynomials for $n = 5$, $A_1 = -0.5$, $B_1 = 0$, $A_2 = 0$, $B_2 = 0$, $\alpha = \beta = 0$, and $x \in [-1.2, 1.2]$.

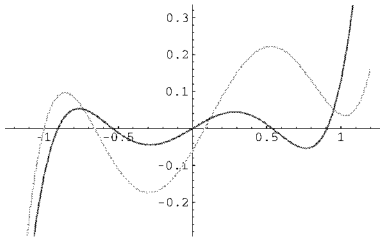


Comparison between classical (bold line) and semiclassical polynomials for $n = 20$, $A_1 = 0$, $B_1 = 0$, $A_2 = -4$, $B_2 = 2$, $\alpha = \beta = 1$, and $x \in [-1.1, 1.1]$.

Figure 1. Numerical tests performed by using the symbolic computer algebra package *Mathematica* to compare the behaviour between the Jacobi orthogonal polynomials and semiclassical Jacobi-type orthogonal polynomials as well as their number of zeros inside the orthogonality interval. Notice, that owing to the small value of the degree n, the parameters A_i, B_i, $i = 1, 2$, α and β are given in such a way that the existence of semiclassical orthogonal polynomials is guaranteed. Observe that, even for small n, the zeros of semiclassical Jacobi-type orthogonal polynomials go out the interval of orthogonality.



Comparison between classical (bold line) and semiclassical polynomials for $n = 6$, $A_1 = B_1 = 0$, $A_2 = 10$, $B_2 = -10^3$, $\alpha = 0$, $\beta = 10$, and $x \in [-1.1, 1.5]$.



Comparison between classical (bold line) and semiclassical polynomials for $n = 5$, $A_1 = B_1 = 0$, $A_2 = -0.5$, $B_2 = -10^6$, $\alpha = \beta = 0$, and $x \in [-1.2, 1.2]$.

Figure 2. Numerical tests performed by using the symbolic computer algebra package *Mathematica* to compare the behaviour between the Jacobi orthogonal polynomials and semiclassical Jacobi-type orthogonal polynomials as well as their number of zeros inside the orthogonality interval. Notice, that owing to the small value of the degree n, the parameters A_i, B_i, $i = 1, 2$, α and β are given in such a way that the existence of semiclassical orthogonal polynomials is guaranteed. Observe that, even for small n, the zeros of semiclassical Jacobi-type orthogonal polynomials go out the interval of orthogonality.

In Figures 1 and 2 we show some numerical examples when the zeros are located outside the interval $[-1, 1]$.

7. Conclusions and Open Problems

The notion of semiclassical OP seems to be very appropriate to generalize the very classical OP of Jacobi, Laguerre, and Hermite. So, the example analyzed in this paper (semiclassical Jacobi-type OP with respect to a moment functional of class $s = 4$) is, probably, a good starting point, since other semiclassical cases like Laguerre-type, Hermite-type (see Álvarez-Nodarse and Marcellán, 1995; Koekoek and Koekoek, 1991; Koekoek, 1988), and, of course, the very classical OP should be connected by limit transitions to the semiclassical Jacobi-type OP as the starting family. The present paper helps us to understand the analysis of semiclassical OP (very close to the classical ones), and then one has a good basis for studying other extensions, for which there are quite a few possibilities for research.

The semiclassical Jacobi-type OP inherit some of the most remarkable properties of classical Jacobi polynomials. Then, the powerful tools which have been created for the treatment of classical OP may be used (up to a some adaptation) for an analysis of semiclassical OP. Obviously, such a mathematical aspect gives the opportunity for the theory of semiclassical OP to develop as far as the theory of classical OP.

The semiclassical OP are easily classified by using the order of the class of the semiclassical moment functional from which they are coming from (see the distributional equation (52)). Thus, some open problems can be raised:

(1) The semiclassical Jacobi-type OP show new interesting phenomena: They can be represented by means of generalized hypergeometric series (${}_6F_5$). Can any semiclassical 'classical-type' OP be represented as ${}_{s+p}F_{s+q}$ hypergeometric functions? (where p and q correspond to the hypergeometric representation of classical OP, and s is the order of the new class).

(2) The classical OP of Jacobi, Laguerre, and Hermite satisfy a linear second-order differential equation. There is a linear differential equation of order

$$\mathcal{O} = 2(s+1) + 2(\alpha + \tilde{r} - r),$$

for the semiclassical symmetric 'classical-type' OP, being $\alpha > -1$, s the order of the class of the corresponding semiclassical functional $\mathcal{U}$, $\tilde{r}$ the total number of masses in $\mathcal{U}$, and r the number of different masses?

A deeper problem is to characterize all the semiclassical OP satisfying a $\mathcal{O}$-order linear differential equation, extending the Bochner's result for classical OP.

We suggest the reader to consult the excellent survey (W. N. Everitt, K. H. Kwon, L. L. Littlejohn, and R. Wellman 2001) as well as the reference contained therein, for the latest known results concerning the classification of differential equations.

Acknowledgements

The first author (J.A.) was partially supported by Dirección General de Investigación del Ministerio de Ciencia y Tecnología of Spain under grants BFM2000-0029 and BFM2000-0206-C04-01. The research of the authors (F.M. and R.A.N.) was partially supported by Dirección General de Investigación del Ministerio de Ciencia y Tecnología of Spain under grants BFM2000-0206-C04-01 and BFM2000 -0206-C04-02, respectively. In addition, the first author (J.A.) thanks Dirección General de Investigación de la Comunidad Autónoma de Madrid for its financial support.

References

Álvarez-Nodarse, R. and Marcellán, F. (1995) A generalization of the classical Laguerre polynomials, *Rend. Circ. Mat. Palermo, Serie II* **44**, 315–329.

Álvarez-Nodarse, R. and Marcellán, F. (1996) A generalization of the classical Laguerre polynomials: Asymptotic properties and zeros, *Appl. Anal.* **62**, 349–366.

Arvesú, J. (1999) *Analytic and Algebraic Properties of Polynomials with Several Models of Orthogonality: q-discretes, Sobolev-type and Semiclassicals*, Carlos III University of Madrid.

Arvesú, J., Álvarez-Nodarse, R., Marcellán, F. and Kwon, K. H. (1998) Some extension of the Bessel-type orthogonal polynomials, *Integral Trans. Special Funct.* **7**(3–4), 191–214.

Belmehdi, S. L. and Marcellán, F. (1992) Orthogonal polynomials associated to some modifications of a linear functional, *Appl. Anal.* **46**(1), 1–24.

Bernardi, C. and Maday, Y. (1991) Some spectral approximation of one-dimensional fourth order problems, In: P. Nevai and A. Pinkus (eds), *Progress in Approximation Theory*, Academic Press, San Diego, pp. 43–116.

Blatt, H. P., Saff, E. B. and Simkani, M. (1988) Jentzsch–Szegő type theorems for the zeros of best approximants, *J. London Math. Soc.* **38**, 307–316.

Bouakkaz, H. and Maroni, P. (1991) Polynômes orthogonaux de Laguerre–Hahn, In: C. Brezinski *et al.* (eds), *Ann. Comput. Appl. Math.* 9, J. C. Baltzer AG, Basel, pp. 189–194.

Chihara, T. S. (1978) *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York.

Chihara, T. S. (1985) Orthogonal polynomials and measures with end point masses, *Rocky Mountain J. Math.* **15**(3), 705–719.

Draïdi, N. (1990) Sur l'adjonction de deux masses de Dirac à une forme linéaire régulière quelconque, Thèse Doctorat de l'Université Pierre et Marie Curie, Paris.

Draïdi, N. and Maroni, P. (1988) Sur l'adjonction de deux masses de Dirac à une forme régulière quelconque, In: A. Cachafeiro and E. Godoy (eds), *Polinomios ortogonales y sus aplicaciones*, Universidad de Santiago, Vigo, Spain, pp. 83–90.

Everitt, W. N., Kwon, K. H., Littlejohn, L. L. and Wellman, R. (2001) Orthogonal polynomial solutions of linear ordinary differential equations, *J. Comput. Appl. Math.* **135**, 85–109.

Gonchar, A. A. (1975) On the convergence of Padé approximants for some classes of meromorphic functions, *Mat. Sb.* **97**(139), 607–629; English transl. in *Math. USSR Sb.* **26**, 555–575.

Koekoek, J. and Koekoek, R. (1991) On a differential equation for Koornwinder's generalized Laguerre polynomials, *Proc. Amer. Math. Soc.* **112**, 1045–1054.

Koekoek, R. (1988) Koornwinder's Laguerre polynomials, *Delft Progress Report.* **12**, 393–404.

Koekoek, R. (1990) Generalizations of the classical Laguerre polynomials and some q-analogues, PhD Thesis, Delft University of Technology.

Koornwinder, T. H. (1984) Orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, *Canad. Math. Bull.* **27**(2), 205–214.

Krall, A. M. (1981) Orthogonal polynomials satisfying fourth order differential equations, *Proc. Roy. Soc. Edinburgh* **87**, 271–288.

Krall, H. L. (1940) On orthogonal polynomials satisfying a certain fourth order differential equation, *The Pennsylvania State College Bull.* **6**, 1–24.

Kwon, K. H. and Park, S. B. (1997) Two points masses perturbation of regular moment functionals, *Indag. Math.* (*N.S.*) **8**(1), 79–93.

Laguerre, E. (1885) Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coéfficients sont rationnels, *J. Math. Pures Appl.* **4**(1), 135–165; *Œuvres* 2, Chelsea, New York, 1972, pp. 685–711.

López, G. (1989) Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotic for orthogonal polynomials, *Math. USSR Sb.* **64**, 207–227.

Marcellán, F. and Maroni, P. (1992) Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique, *Ann. Mat. Pura Appl.* **IV**(CLXII), 1–22.

Marcellán, F. and Prianes, E. (1996) Orthogonal polynomials and Stieltjes functions: The Laguerre–Hahn case, *Rend. Mat.* **16**, 117–141.

Marcellán, F. and Ronveaux, A. (1989) Differential equations for classical type orthogonal polynomials, *Canad. Math. Bull.* **32**(4), 404–411.

Maroni, P. (1991) Une théorie algébrique des polynômes orthogonaux: Applications aux polynômes orthogonaux semiclassiques, *Ann. Comput. Appl. Math.* **9**, 98–130.

Nevai, P. (1979) Orthogonal polynomials, *Mem. Amer. Math. Soc.* **213**.

Olver, F. W. J. (1974) *Asymptotics and Special Functions*, Academic Press Inc., New York.

Szegő, G. (1975) *Orthogonal Polynomials*, 4th edn, Amer. Math. Soc. Colloq. Publ. 23, Providence, RI.