arXiv:1403.5093v1 [math.FA] 20 Mar 2014

INVARIANT SUBSPACES AND DEDDENS ALGEBRAS

MIGUEL LACRUZ

Abstract. It is shown that if the Deddens algebra $\mathcal{D}_T$ associated with a quasinilpotent operator T on a complex Banach space is closed and localizing then T has a nontrivial closed hyperinvariant subspace.

We shall represent by $\mathcal{B}(E)$ the algebra of all bounded linear operators on a complex Banach space E. Recall that the commutant of an operator $T \in \mathcal{B}(E)$ is the subalgebra $\{T\}' \subseteq \mathcal{B}(E)$ of all operators that commute with T. A subspace $F \subseteq E$ is said to be invariant under an operator $T \in \mathcal{B}(E)$ provided that $TF \subseteq F$. A subspace $F \subseteq E$ is said to be invariant under a subalgebra $\mathcal{R} \subseteq \mathcal{B}(E)$ provided that F is invariant under every $R \in \mathcal{R}$. A subspace $F \subseteq E$ is said to be hyperinvariant under an operator $T \in \mathcal{B}(E)$ provided that F is invariant under the commutant $\{T\}'$. A subalgebra $\mathcal{R} \subseteq \mathcal{B}(E)$ is said to be transitive provided that the only closed subspaces invariant under $\mathcal{R}$ are the trivial ones, that is, $F = \{0\}$ and $F = E$. As it turns out, this is equivalent to saying that for every $x \in E\backslash\{0\}$, the subspace $\{Rx \colon R \in \mathcal{R}\}$ is dense in E.

Recall that an operator $T \in \mathcal{B}(E)$ is said to be quasinilpotent provided that $\sigma(T) = \{0\}$. According with the spectral radius formula, T is quasinilpotent if and only if

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0. \tag{1}$$

Let $T \in \mathcal{B}(E)$ and consider the *Deddens algebra* $\mathcal{D}_T$ associated with T, that is, the family of those operators $X \in \mathcal{B}(E)$ for which there is a constant $M > 0$ such that for every $n \in \mathbb{N}$ and for every $f \in E$,

$$\|T^n X f\| \leq M \|T^n f\|. \tag{2}$$

When T is invertible this is equivalent to saying that

$$\sup_{n\in\mathbb{N}} \|T^n X T^{-n}\| < \infty. \tag{3}$$

It is easy to see that $\mathcal{D}_T$ is indeed a unital subalgebra of $\mathcal{B}(E)$ with the nice property that $\{T\}' \subseteq \mathcal{D}_T$. Also, $\mathcal{D}_T = \mathcal{B}(E)$ in case T is an isometry. These algebras are named after Deddens because he first introduced them in the 1970s in the context of nest algebras [4]. The description of Deddens algebras associated with some special classes of operators has been recently obtained by Petrovic [14, 15].

Let $T \in \mathcal{B}(E)$. A complex scalar λ is said to be an *extended eigenvalue* for T provided that there exists a nonzero operator $X \in \mathcal{B}(E)$ such that $TX = \lambda XT$. Such an operator is called an *extended eigenoperator* for T corresponding to the extended eigenvalue λ. These notions became popular back in the 1970s when searching for invariant subspaces. Recently, the concepts of extended eigenvalue and extended eigenoperator have received a considerable amount of attention, both in the context of invariant subspaces [8, 9] and in the study of extended eigenvalues and extended eigenoperators for some special classes of operators [1, 2, 5, 12, 13, 14].

Date: March 21, 2014.

2010 *Mathematics Subject Classification.* Primary 47A15; Secondary 47L10 .

Key words and phrases. Deddens algebra; Extended eigenvalue; Invariant subspace; Localizing algebra.

This research was partially supported by Ministerio de Ministerio de Economía y Competitividad under grant MTM 2012-30748, and by Junta de Andalucía under grant FQM-3737.

Let $\mathcal{E}_T(\lambda)$ denote the set of extended eigenoperators of T associated with an extended eigenvalue λ and let $\mathcal{E}_T$ denote the union of the sets $\mathcal{E}_T(\lambda)$ when λ runs through all the extended eigenvalues for T with $|\lambda| \leq 1$. It is easy to see that $\{T\}' \subseteq \mathcal{E}_T \subseteq \mathcal{D}_T$. Both inclusions may be proper; for instance, Petrovic [14] showed that if W is an injective unilateral shift on a Hilbert space then boths inclusions $\{W\}' \subset \mathcal{E}_W$ and $\mathcal{E}_W \subset \mathcal{D}_W$ are proper.

A subspace $\mathcal{X} \subseteq \mathcal{B}(E)$ is said to be *localizing* provided that there is a closed ball $B \subseteq E$ such that $0 \notin B$ and such that for every sequence of vectors (f_n) in B there is a subsequence (f_{n_j}) and a sequence of operators (X_j) in $\mathcal{X}$ such that $\|X_j\| \leq 1$ and such that the sequence $(X_j f_{n_j})$ converges in norm to some nonzero vector. This notion was introduced by Lomonosov, Radjavi, and Troitsky [11] as a side condition to build invariant subspaces. A typical example of a localizing algebra is a subalgebra $\mathcal{R} \subseteq \mathcal{B}(E)$ such that the closure in the weak operator topology of the unit ball of $\mathcal{R}$ contains a nonzero compact operator.

Rodríguez-Piazza and the author studied some properties of localizing algebras in a recent paper [8]. They also obtained a result on the existence of invariant subspaces that extends and unifies previous results of Scott Brown [3] and Kim, Moore and Pearcy [6], on the one hand, and Lomonosov, Radjavi and Troitsky [11], on the other hand. The result goes as follows.

Theorem 1. *Let $T \in \mathcal{B}(E)$ be a nonzero operator, let $\lambda \in \mathbb{C}$ be an extended eigenvalue of T such that the subspace $\mathcal{E}_T(\lambda)$ of all associated extended eigenoperators is localizing and suppose that either*

(1) $|\lambda| \neq 1$, *or*
(2) $|\lambda| = 1$ *and T is quasinilpotent.*

Then T has a nontrivial closed hyperinvariant subspace.

The aim of this note is to provide an extension of part (2) in Theorem 1 by replacing the assumption that the subspace $\mathcal{E}_T(\lambda)$ be localizing with the assumption that the Deddens algebra $\mathcal{D}_T$ be closed and localizing. Our main result can be stated as follows.

Theorem 2. *Let $T \in \mathcal{B}(E)$ be a nonzero quasinilpotent operator. If the Deddens algebra $\mathcal{D}_T$ is closed and localizing then T has a nontrivial closed hyperinvariant subspace.*

Notice that, under the assumption that $\mathcal{D}_T$ be closed, part (2) of Theorem 1 is a consequence of Theorem 2 since $\mathcal{E}_T(\lambda) \subseteq \mathcal{D}_T$, and that Theorem 2 is strictly more general than part (2) of Theorem 1, because the inclusion $\mathcal{E}_T \subseteq \mathcal{D}_T$ is proper in general. Let us start with a general result about Deddens algebras before we proceed with the proof of Theorem 2. This result characterizes when the Deddens algebra $\mathcal{D}_T$ is closed in the operator norm. The corresponding result for spectral radius algebras was obtained by Lambert and Petrovic [7].

Lemma 3. *Let $T \in \mathcal{B}(E)$. The following conditions are equivalent:*

(1) *The Deddens algebra $\mathcal{D}_T$ is closed in the operator norm topology.*
(2) *There is a constant $M > 0$ such that for every $X \in \mathcal{D}_T$, for all $n \in \mathbb{N}$ and for all $f \in E$ we have*

$$\|T^n X f\| \leq M\|X\| \cdot \|T^n f\| \tag{4}$$

Proof of Lemma 3. Suppose $\mathcal{D}_T$ is closed and consider for every $k \in \mathbb{N}$ the closed set $\mathcal{F}_k$ of those operators $X \in \mathcal{D}_T$ that satisfy the inequality $\|T^n X f\| \leq k\|T^n f\|$ for all $n \in \mathbb{N}$ and for all $f \in E$. Then we have

$$\mathcal{D}_T = \bigcup_{k \in \mathbb{N}} \mathcal{F}_k.$$

It follows from Baire's theorem that there is some $k_0 \in \mathbb{N}$ such that $\mathcal{F}_{k_0}$ has nonempty interior, that is, there is some $X_0 \in \mathcal{F}_{k_0}$ and there is some $\varepsilon > 0$ such that $\{X \in \mathcal{D}_T \colon \|X - X_0\| \leq \varepsilon\} \subseteq \mathcal{F}_{k_0}$. Let $Y \in \mathcal{D}_T$

such that $\|Y\| \leq 1$ and let $X = X_0 + \varepsilon Y$. Then we have $\varepsilon\|T^n Y f\| \leq \|T^n X_0 f\| + \|T^n X f\| \leq 2k_0\|T^n f\|$. Finally, we conclude that for every $X \in \mathcal{D}_T$, for all $n \in \mathbb{N}$ and for all $f \in \mathcal{D}_T$ we have

$$\|T^n X f\| \leq \frac{2k_0}{\varepsilon}\|X\| \cdot \|T^n f\|.$$

The converse is trivial because if such a constant $M > 0$ exists then

$$\mathcal{D}_T = \bigcap_{n\in\mathbb{N}} \bigcap_{f\in E} \{X \in \mathcal{B}(E)\colon \|T^n X f\| \leq M\|X\| \cdot \|T^n f\|\}.$$

so that $\mathcal{D}_T$ is closed since it is the intersection of a family of closed sets. □

An easy proof of the nontrivial part of Lemma 3 can be obtained from the uniform boundedness principle in the special case that T is an invertible operator. The proof goes as follows.

Proof of Lemma 3 when T is invertible. Consider the operator $\Phi\colon \mathcal{D}_T \to \mathcal{D}_T$ defined by the expression $\Phi(X) = TXT^{-1}$ for all $X \in \mathcal{D}_T$. Notice that $\Phi^n(X) = T^n X T^{-n}$ for all $n \in \mathbb{N}$, so that $\sup_n \|\Phi^n(X)\| < \infty$. Now it follows from the uniform boundedness principle that $\sup_n \|\Phi_n\| < \infty$. This means that there is a constant $M > 0$ such that $\|T^n X T^{-n}\| \leq M\|X\|$ for all $n \in \mathbb{N}$ and for all $X \in \mathcal{D}_T$. Therefore $\|T^n X T^{-n} g\| \leq M\|X\| \cdot \|g\|$ for all $g \in E$, and taking $g = T^n f$ we get $\|T^n X f\| \leq M\|X\| \cdot \|T^n f\|$. □

The key for the proof of Theorem 2 is a lemma that we have extracted from the proof of Theorem 2.3 in the paper of Lomonosov, Radjavi and Troitsky [11]. This lemma can be stated as follows.

Lemma 4. *Let $T \in \mathcal{B}(E)$ be a nonzero operator such that $\{T\}'$ is a transitive algebra, let $\mathcal{R} \subseteq \mathcal{B}(E)$ be a localizing algebra such that $\{T\}' \subseteq \mathcal{R}$, and let $B \subseteq E$ be a closed ball that makes $\mathcal{R}$ a localizing algebra. There is a constant $c > 0$ such that for every $f \in B$ there is an $X \in \mathcal{R}$ such that $TXf \in B$ and $\|X\| \leq c$.*

Proof of Lemma 4. Assume the commutant $\{T\}'$ is a transitive algebra. Since the closed subspace $\ker T$ is invariant under $\{T\}'$ and since $T \neq 0$, we must have $\ker T = \{0\}$, so that T is injective. We ought to show that there is some constant $c > 0$ such that for every $f \in B$ there is an $X \in \mathcal{R}$ such that $\|X\| \leq c$ and $TXf \in B$. We proceed by contradiction. Otherwise, for every $n \in \mathbb{N}$ there is an $f_n \in B$ such that the condition $X \in \mathcal{R}$ and $TXf_n \in B$ implies $\|X\| > n$. Since $\mathcal{R}$ is localizing, there is a subsequence (f_{n_j}) and there is a sequence (X_j) in $\mathcal{R}$ with $\|X_j\| \leq 1$, and such that $(X_j f_{n_j})$ converges in norm to some nonzero vector $f \in E$. Therefore, $(TX_j f_{n_j})$ converges in norm to Tf. Since T is injective, $Tf \neq 0$, and since $\{T\}'$ is transitive, there is an $R \in \{T\}'$ such that $RTf \in \operatorname{int} B$. Hence, there is some $j_0 \geq 1$ such that $RTX_j f_{n_j} \in B$ for all $j \geq j_0$. Since $RT = TR$, we have $TRX_j f_{n_j} \in B$ for all $j \geq j_0$. Since $RX_j \in \mathcal{R}$, the choice of the sequence (f_n) implies $\|RX_j\| > n_j$ for all $j \geq j_0$. Finally, this leads to a contradiction, because $\|RX_j\| \leq \|R\|$ for all $j \geq 1$. This completes the proof of Lemma 4. □

The technique for the proof of Theorem 2 is an iterative procedure that is reminiscent of an argument at the end of the proof in Hilden's simplification for the striking theorem of Lomonosov [10] that a nonzero compact operator on a complex Banach space has a nontrivial hyperinvariant subspace. We recommend the book of Rudin [16] for an exposition of this argument.

Proof of Theorem 2. Start with any vector $f_0 \in B$ and use Lemma 4 to choose an operator $X_1 \in \mathcal{D}_T$ such that $\|X_1\| \leq c$ and such that $TX_1 f_0 \in B$. Now use again Lemma 4 to choose another operator $X_2 \in \mathcal{D}_T$ such that $\|X_2\| \leq c$ and $TX_2TX_1 f_0 \in B$. Continue this ping pong game to obtain a sequence of vectors $f_n \in B$ and a sequence of operators $X_n \in \mathcal{D}_T$ such that $\|X_n\| \leq c$ and such that $f_n = TX_n \cdots TX_1 f_0$. Now apply Lemma 3 to find a constant $M > 0$ such that $\|T^n X f\| \leq M\|X\| \cdot \|T^n f\|$ for every $X \in \mathcal{D}_T$, for all $n \in \mathbb{N}$ and for all $f \in H$. Notice that $\|f_1\| = \|TX_1 f_0\| \leq cM\|Tf_0\|$. Also, notice that

$$\|f_2\| = \|TX_2TX_1 f_0\| \leq cM\|T^2 X_1 f_0\| \leq (cM)^2\|T^2 f_0\|,$$

and in general $\|f_n\| \leq (cM)^n \|T^n f_0\|$. Let $d = \min\{\|x\|: x \in B\}$. It is plain that $d > 0$ because $0 \notin B$. Then, for all $n \in \mathbb{N}$ we have $0 < d \leq \|f_n\| \leq (cM)^n \|T^n f_0\|$, and this gives information on the spectral radius of T, namely,

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \geq \frac{1}{cM} > 0.$$

We arrived at a contradiction because T is quasinilpotent. This completes the proof of Theorem 2. □

References

[1] A. Biswas, A. Lambert, and S. Petrovic, *Extended eigenvalues and the Volterra operator,* Glasg. Math. J. **44** (2002), 521–534, MR 1956558.

[2] A. Biswas and S. Petrovic, *On extended eigenvalues of operators,* Integr. Equ. Oper. Theory, **55** (2006), 233–248, MR 2234256.

[3] S. Brown, *Connections between an operator and a compact operator that yield hyperinvariant subspaces,* J. Operator Theory **1** (1979), 1–21. MR 526293.

[4] J. Deddens, *Another description of nest algebras,* Hilbert space operators, Lecture Notes in Mathematics **693** (Springer, Berlin, 1978) 77–86, MR 0526534.

[5] P. S. Bourdon and J. H. Shapiro, *Intertwining relations and extended eigenvalues for analytic Toeplitz operators,* Illinois J. Math. **52** (2008). 1007–1030, MR 2546021.

[6] H. W. Kim, R. Moore and C. M. Pearcy, *A variation of Lomonosov's theorem,* J. Operator Theory **2** (1979), 131–140. MR 553868.

[7] A. Lambert and S. Petrovic, *Beyond hyperinvariance for compact operators,* J. Funct. Anal. **219** (2005) 93–108, MR 2108360.

[8] M. Lacruz and L. Rodríguez-Piazza, *Localizing algebras and invariant subspaces,* to appear in J. Operator Theory.

[9] A. Lambert, *Hyperinvariant subspaces and extended eigenvalues,* New York J. Math. **10** (2004) 83–88, MR 2052366.

[10] V. I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator,* Funkcional. Anal. i Priloen. 7 (1973), 55–56, MR 0548852.

[11] V.I. Lomonosov, H. Radjavi and V.G. Troitsky, *Sesquitransitive and localizing operator algebras,* Integral Equations Operator Theory **60** (2008), 405–418, MR 2392834.

[12] V. Lauric, *The set of extended eigenvalues of a weighted Toeplitz operator,* Acta Sci. Math. (Szeged) **72** (2006), 691–700, MR 2289761.

[13] S. Petrovic, *On the extended eigenvalues of some Volterra operators,* Integr. Equ. Operator Theory **57** (2007), 593–598, MR 2313287.

[14] S. Petrovic, *Spectral radius algebras, Deddens algebras, and weighted shifts,* Bull. Lond. Math. Soc. **43** (2011), 513–522, MR 2820141.

[15] S. Petrovic, *Deddens algebras and shift,* Complex Anal. Oper. Theory **5** (2011), 253–259, MR 2773065.

[16] W. Rudin, *Functional Analysis,* McGraw-Hill, New York 1991, MR 1157815.

M. Lacruz, Facultad de Matemáticas, Universidad de Sevilla, Avenida Reina Mercedes, 41012 Seville (Spain)
E-mail address: `lacruz@us.es`