ON REGULARIZATION IN SUPERREFLEXIVE BANACH SPACES BY INFIMAL CONVOLUTION FORMULAS

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Abstract. We present here a new method for approximating functions defined on superreflexive Banach spaces by differentiable functions with α-Hölder derivatives (for some $0 < \alpha \leq 1$). The smooth approximation is given by means of an explicit formula enjoying good properties from the minimization point of view. For instance, for any function f which is bounded below and uniformly continuous on bounded sets this formula gives a sequence of Δ-convex $\mathcal{C}^{1,\alpha}$ functions converging uniformly on bounded sets to f and preserving the infimum and the set of minimizers of f. The techniques we develop are based on the use of *extended inf-convolution* formulas and convexity properties such as the preservation of smoothness for the convex envelope of certain differentiable functions.

0. Introduction and Preliminaries

This paper introduces an explicit regularization procedure for functions defined on superreflexive Banach spaces. For any bounded below l.s.c. (resp. uniformly continuous on bounded sets) function f on a superreflexive Banach space X we give by means of a "standard" formula a sequence of $\mathcal{C}^{1,\alpha}$-smooth functions converging pointwise (resp. uniformly on bounded sets) to f (where $0 < \alpha \leq 1$ only depends on X). Under some additional conditions, the convergence of the sequence of approximate functions is uniform on the whole space X. Moreover, the approximate functions preserve the infimum and the set of minimizers of f. We remark that these features altogether cannot be easily obtained from regularization methods like the *smooth partitions of the unity* techniques (for a detailed study of this topic we refer to Chapter VIII.3 of [DGZ], the references therein and [Fr]) or other results that only ensure the existence of smooth approximates (for instance, see [DFH]).

In Hilbert spaces, our work is closely linked with the *Lasry-Lions approximation method* (introduced in [LL] and subsequently studied by several authors, such as [AA]) and its more general version given by T. Strömberg in [St$_2$]. Actually, we improve the results of [St$_2$] in the superreflexive case by providing the best uniformly smooth approximation possible for this setting. Nonetheless, we want to remark that the approximate functions explained herein cannot be reduced to those of Strömberg (or Lasry-Lions approximates in Hilbert spaces); we refer to the remark after **Proposition 8** for a more precise explanation. Our approach for smooth

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regularization in non-Hilbert spaces comes from two main facts: the density of the linear span of the convex functions (studied in [C]) and the smoothness of the convex envelope of a "somehow" smooth function. In this direction, we also present more general versions of certain results in [GR] for infinite dimensional Banach spaces.

This paper is organized in the following way. Our main result of this paper, **Theorem 1**, and several corollaries are explained in SECTION 1. The proof of **Theorem 1** is showed in SECTION 4 with the tools provided by sections 2 and 3. SECTION 2 deals with the existence of approximates for a given function f using some results on extended inf-convolution formulas. SECTION 3 develops a procedure for regularizating certain Δ-convex approximates. This procedure is based on the smoothness of the convex envelope of certain "somehow" smooth functions.

NOTATION: In what follows, X denotes a Banach space and $\|\cdot\|$ an equivalent norm on X. By B_X we mean the unit closed ball of X under the norm $\|\cdot\|$ and by $B_X(r)$ the closed ball of radius $r > 0$. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *proper* if $f \not\equiv +\infty$ and $\mathfrak{S}_{\mathfrak{Inf}}(f)$ is the (possibly empty) set $\{x \in X : f(x) = \inf f\}$. We will deal with the pointwise, compact, uniform on bounded sets and uniform on X convergence in the set of lower semi-continuous (in short, l.s.c.) functions on X, abbreviated respectively by τ_p, τ_K, τ_b and τ_u.

A function defined on X is called Δ-*convex* if it can be expressed as the difference of two continuous convex functions. The convex envelope $\operatorname{co} f$ of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is defined as the greatest proper convex l.s.c. function below f (if there exists a convex minorant of f). The explicit value of the convex envelope of f at a point $x \in X$ is given by the formula

$$(\operatorname{co} f)(x) = \inf_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} \lambda_i f(x_i) : x = \sum_{i=1}^{n} \lambda_i x_i,\ \sum_{i=1}^{n} \lambda_i = 1,\ \left(x_i, \lambda_i\right)_{i=1}^{n} \subset (X \times \mathbb{R}_+) \right\}. \quad (1)$$

Unless stated otherwise, differentiability will be understood in the *Fréchet sense*. The following notation is used throughout this work. By $\mathcal{C}^{1,u}(X)$ (respectively $\mathcal{C}^{1,u}_{\mathcal{B}}(X)$) we understand the set of differentiable functions defined on X with uniformly continuous (resp. uniformly continuous on bounded sets) derivative. Similarly, $\mathcal{C}^{1,\alpha}(X)$ (resp. $\mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$) stands for the class of functions on X having α-Hölder continuous (resp. α-Hölder continuous on bounded sets) derivative ($0 < \alpha \le 1$).

1. THE MAIN RESULT

We begin by stating the main result of this work.

Theorem 1. *Let $p > 1$, X be a Banach space and $\|\cdot\|$ be an equivalent norm on X which is locally uniformly convex and uniformly smooth. For any proper lower semi-continuous bounded below function $f : X \to \mathbb{R} \cup \{+\infty\}$, consider the sequence of Δ-convex functions given by the formula*

$$\Delta^p_n f := \operatorname{co} g^p_n - 2^{p-1} n \|\cdot\|^p \quad (n \in \mathbb{N}),$$

where g^p_n at a point $x \in X$ is defined as

$$g^p_n(x) := \inf_{y\in X} \left\{ f(y) + 2^{p-1} n\|x\|^p + 2^{p-1} n\|y\|^p - n\|x + y\|^p \right\} + 2^{p-1} n\|x\|^p.$$

Then the following assertions are satisfied:

(i) *For all n, $\inf f \leq \Delta_n^p f \leq f$ and $\mathfrak{S}_{\mathfrak{Inf}}(\Delta_n^p f) = \mathfrak{S}_{\mathfrak{Inf}}(f)$.*

(ii) *$(\Delta_n^p f)_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathcal{B}}^{1,u}(X)$ and $(\Delta_n^p f)_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathcal{B}}^{1,\alpha}(X)$ provided that the modulus of smoothness of the norm $\|\cdot\|$ is of power type $1+\alpha$; actually, we have that $(\Delta_n^{1+\alpha} f)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,\alpha}(X)$.*

(iii) *$\Delta_n^p f \xrightarrow[n]{\tau_p} f$ pointwise and $\Delta_n^p f \xrightarrow[n]{\tau_K} f$ if $f : X \to \mathbb{R}$ is continuous.*

If moreover the norm $\|\cdot\|$ is uniformly convex then

(iv) *$\Delta_n^p f \xrightarrow[n]{\tau_b} f$ whenever f is uniformly continuous on bounded sets.*

(v) *$\Delta_n^p f \xrightarrow[n]{\tau_u} f$ provided that f is uniformly continuous on X (not necessarily bounded below) and the modulus of convexity of the norm $\|\cdot\|$ is of power type p ($p \geq 2$).*

Remark. It is well-known that the existence of a uniformly smooth norm $\|\cdot\|$ on a Banach space X implies the superreflexivity of X (and reciprocally, the articles [E] of P. Enflo and [Pi] of G. Pisier tell us that any superreflexive Banach space admits an equivalent uniformly smooth norm). Similarly, we want to point out that the conclusions of **Theorem 1** cannot be expected outside the superreflexive setting.

First, the τ_b-density of the set of Δ-convex functions defined on X in the set of functions on X that are uniformly continuous on bounded sets is equivalent to the superreflexivity of the Banach space X(as it was proved in [C]). On the other hand, the existence of $\mathcal{C}^{1,\alpha}$ bump functions (for some $0 < \alpha \leq 1$) on X implies the existence of an equivalent norm $\|\cdot\|$ on X with modulus of smoothness of power type $1+\alpha$ (see Theorem V.3.1. of [DGZ]).

Remark. The optimal application of **Theorem 1** is achieved when we consider a Hilbertian norm $\|\cdot\|$. In this case, taking $p = 2$ in **Theorem 1** we obtain similar approximation results as those given by the Lasry-Lions approximation method (see [LL]). Nevertheless, the different sequences of approximates are not the same even in this setting (see remark after **Proposition 8**).

We proceed to state some corollaries to **Theorem 1**. They are related with certain results known on a superreflexive Banach spaces from the existence of *smooth partitions of the unity* (see Theorem VIII.3.2 in [DGZ]). Their proof is easily obtained appealing to **Theorem 1** and Pisier's renorming Theorem (the original proof can be found in [P]; we refer to [L] for a simpler and more geometrical proof).

The first corollary improves Corollary 1 of [St$_2$] for superreflexive Banach spaces.

Corollary 2. *Let X be a superreflexive Banach space. Then there exists some $0 < \alpha \leq 1$ such that any non-empty closed set F of X is the set of zeros of a Δ-convex $\mathcal{C}^{1,\alpha}$-differentiable function on X. Moreover, F is the limit for the Hausdorff distance of a sequence of sets $\mathfrak{S}_n = \{x \in X : f_n(x) < \sigma_n \in \mathbb{R}\}$ ($n \in \mathbb{N}$) where the functions $(f_n)_n$ are Δ-convex and in $\mathcal{C}_{\mathcal{B}}^{1,\alpha}(X)$.*

Proof of Corollary 2. For a superreflexive Banach space X, Pisier's renorming Theorem ensures the existence of an equivalent norm $\|\cdot\|$ on X with modulus of smoothness of power type q ($1 < q \leq 2$). Given a closed set F in X, consider the proper function d defined at a point $x \in X$ as $d(x) := \operatorname{dist}(x, F) = \inf_{y\in F} \|x - y\|$ (d is proper because F is not empty). By **Theorem 1**(i)–(ii) we have that the function $\Delta_1^q d$ is Δ-convex, $\mathcal{C}^{1,q-1}$-differentiable and satisfies that $\mathfrak{S}_{\mathfrak{Inf}}(\Delta_1^q(d)) = \mathfrak{S}_{\mathfrak{Inf}}(d) = F$.

Moreover, using Asplund averaging technique (see Proposition IV.5.2 of [DGZ]), we can assume that the modulus of convexity of the norm $\|\cdot\|$ is in addition of power type p (for some $p \geq 2$). Since d is Lipschitz continuous on X, from **Theorem 1**(iv) it follows for every n that $F \subseteq \{\Delta_n^p d(x) < \frac{1}{n} : x \in X\} := \mathfrak{S}_n$, where $\Delta_n^p d$ is a Δ-convex $\mathcal{C}_{\mathcal{B}}^{1,q-1}$-differentiable function and $(\mathfrak{S}_n)_n$ converges to F for the Hausdorff distance. □

The next corollary gives a slightly stronger version of some others approximation results obtained by using partition of the unity techniques (for instance, see Theorem 1 of [NS]).

Corollary 3. *For any superreflexive Banach space X there is $0 < \alpha \leq 1$ so that for every uniformly continuous on bounded sets (resp. uniformly continuous) function on X one has the following: f is the uniform limit on any fixed bounded set B of X (resp. on X) of a sequence of Δ-convex $\mathcal{C}^{1,\alpha}$-differentiable (resp. $\mathcal{C}_{\mathcal{B}}^{1,\alpha}$-differentiable) functions having the same infimum and set of minimizers on B as f.*

Proof of the Corollary 3. Appealing again to Pisier's renorming Theorem for superreflexive Banach spaces, we can suppose that there is an equivalent norm $\|\cdot\|$ on X with modulus of smoothness of power type q ($1 < q \leq 2$). Fix some bounded set B of X and define $\tilde{f} := \max\{f, \inf_B f\}$. Since f is uniformly continuous on B, we have that $\inf_B f > -\infty$. Therefore, $\tilde{f}$ is uniformly continuous on bounded sets and bounded below. Note that trivially $\tilde{f}(x) = f(x)$ for all $x \in B$ and then the infimum and set of minimizers on B of f and $\tilde{f}$ are the same. Hence, **Theorem 1**(ii) and (vi) tell us that the sequence $(\Delta_n^q \tilde{f})_n$ satisfies the required conditions of the claim for $\alpha = q - 1$. If f is uniform continuous on X, the proof of Corollary 3 follows the same lines, using the existence on X of an equivalent norm $\|\cdot\|$ with non-trivial moduli of convexity and smoothness and **Theorem 1**(v). □

The last corollary is an extension of Remark (viii) in [LL]. It deals with the property of extending and regularizing functions defined on subsets of superreflexive Banach spaces to the whole space.

Corollary 4. *Let X be a superreflexive Banach space. The following holds true for some $0 < \alpha \leq 1$ depending only on X:*

Let S be a subset of X and $f : S \to \mathbb{R}$ be a function that is uniformly continuous on bounded sets of S. Then for every $r > 0$ and $\varepsilon > 0$ there exists a Δ-convex function $F_{r,\varepsilon} : X \to \mathbb{R}$ satisfying the following conditions:

(i) $\inf_S f = \inf_X F_{r,\varepsilon}$ *and* $\mathfrak{S}_{\mathfrak{Inf}}(f) = \mathfrak{S}_{\mathfrak{Inf}}(F_{r,\varepsilon})$,

(ii) $F_{r,\varepsilon} \in \mathcal{C}^{1,\alpha}(X)$, *for some* $0 < \alpha \leq 1$, *and*

(iii) $f(x) - \varepsilon \leq F_{r,\varepsilon}(x) \leq f(x)$ *for every* $x \in S \cap B_X(r)$.

Proof of the Corollary 4. By the same argument as above, let $\|\cdot\|$ be an equivalent norm on X with modulus of smoothness $1 + \alpha$ (for some $0 < \alpha \leq 1$). Consider the following simple extension of f:

$$F(x) := \begin{cases} f(x) & \text{for } x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $\mathfrak{S}_{\mathfrak{Inf}}(F) = \mathfrak{S}_{\mathfrak{Inf}}(f) \subset S$. It is not hard to see using **Proposition 8**(i) and the proof of **Proposition 6**(v) that the sequence $(\Delta_n^{1+\alpha} F)_{n \in \mathbb{N}}$, which satisfies (i) and (ii) of **Theorem 1**, also converges uniformly on bounded sets of S to f. □

The proof of **Theorem 1** will be done in a general scheme involving two main steps. First, we explain an *extended inf-convolution* formula that gives us a standard way to approximate functions on X. Then, we develop some convexity techniques in order to get smooth Δ-convex functions between the functions given by the extended inf-convolution formula.

2. The extended inf-convolution

In this section we explain the convergence results we need in the proof of **Theorem 1**. First, we introduce the definition of *extended inf-convolution*. This definition generalizes the classical one of *inf-convolution* (see [St$_1$] for a general survey of the subject) and will be an important tool in our work.

Definition. For any application $K : X \times X \to \mathbb{R} \cup \{+\infty\}$ and any function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define the *extended inf-convolution* of f by K as the function

$$\big(f \,\Box\, K\big)(x) := \inf_{y \in X} \Big\{ f(y) + K(x,y) \Big\}, \quad x \in X.$$

K will be called the *kernel* of the extended inf-convolution.

Example. If for $g : X \to \mathbb{R} \cup \{+\infty\}$ we consider the kernel $K_g(x,y) := g(x-y)$, then the extended inf-convolution $\big(f \,\Box\, K_g\big)$ is nothing else but the classical inf-convolution $\big(f \,\Box\, g\big)$.

Before the statement of the main result of this section, we need to define some natural properties of kernels.

Definition. A kernel K is *pointwise separating* if for every $x_0 \in X$ and every $\delta > 0$ there exists $C_{x_0,\delta} > 0$ such that $K(x_0, y) \geq C_{x_0,\delta}$ whenever $\|x_0 - y\| \geq \delta$.

A kernel K is called *uniformly separating on bounded sets* if for all $r > 0$ and $\delta > 0$ there exists $C_{r,\delta} > 0$ so that $K(x,y) \geq C_{r,\delta}$ provided $\|x\| \leq r$ and $\|x - y\| \geq \delta$.

A kernel K is *uniformly separating* if for every $\delta > 0$ there is some $\beta_\delta > 0$ in such a way that $K(x,y) \geq \beta_\delta \|x - y\|$ whenever $\|x - y\| \geq \delta$.

Definition. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a kernel K, we define the following sequences of functions:

$$I_{K,n} f := \big(f \,\Box\, nK\big) \quad \text{and} \quad S_{K,n} f := -\big(-f \,\Box\, nK\big) \quad (n \in \mathbb{N}).$$

Remark. For any Hilbert norm $\|\cdot\|$ consider the kernel $K_L(x,y) = \|x - y\|^2$. Then, with our notation the sequence $\Big(S_{K_L,m}\big(I_{K_L,n} f\big)\Big)_{m>n}$ denotes the *Lasry-Lions approximates* of f related to the norm $\|\cdot\|$.

Remark. Note that the Lasry-Lions approximates commutes with translations in the same way as the classical inf-convolution also does. This is a consequence of the following property of the kernel: $K_L(x - a, y) = K_L(x, y + a)$ (for all x, y and a). However, the problem of regularizing (not necessarily convex) functions in a non-Hilbert space leads naturally to more general kernels which do not yield translation-invariant approximates.

The next facts are easy to check.

Facts 5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function.*

1 *For $x \in X$,*

$$I_{K,n}f = \inf_{y\in X} \Big\{ f(y) + nK(x,y) \Big\},$$
$$S_{K,n}f(x) = -I_{K,n}(-f)(x) = \sup_{y\in X} \Big\{ f(y) - nK(x,y) \Big\}.$$

2 *Let C be a constant. Then $I_{K,n}(f+C) = I_{K,n}f + C$, for any n.*

3 *Suppose that the kernel K is positive (i.e., $K(x,y) \geq 0$ for all $x,y \in X$) then*

(i) *$\big(I_{K,n}f\big)_{n\in\mathbb{N}}$ is an increasing sequence of functions bounded below by $\inf f$.*

(ii) *If $f \leq g$, then $I_{K,n}f \leq I_{K,n}g$ for any n.*

(iii) *$I_{K,m}\big(I_{K,n}f\big) \leq I_{K,m}\big(I_{K,m}f\big)$, for any $m > n$.*

We now proceed to state and prove a technical proposition which is the main result of this section.

Proposition 6. *Let $K : X \times X \to \mathbb{R}$ a kernel satisfying the following conditions:*

(1) *K is positive and $K(x,x) = 0$ for all $x \in X$,*

(2) *K is symmetric (i.e., $K(x,y) = K(y,x)$ for all $x,y \in X$),*

(3) *$K(x,y) \xrightarrow[y\to\infty]{} +\infty$ uniformly on bounded sets,*

(4) *K is uniformly continuous (resp. Lipschitz continuous) on bounded sets and*

(5) *K is pointwise separating.*

Then for every proper l.s.c. bounded below function $f : X \to \mathbb{R} \cup \{+\infty\}$ the following statements hold:

(i) *$I_{K,n}f \leq S_{K,n}\big(I_{K,n}f\big) \leq f$.*

(ii) *$\inf I_{K,n}f = \inf f$ and $\mathfrak{S}_{\mathfrak{Inf}}(I_{K,n}f) = \mathfrak{S}_{\mathfrak{Inf}}(f)$.*

(iii) *$I_{K,n}f$ is uniformly continuous (resp. Lipschitz continuous) on bounded sets.*

(iv) *$I_{K,n}\big(I_{K,n}f\big) \xrightarrow[n\to\infty]{\tau_p} f$ and $I_{K,n}\big(I_{K,n}f\big) \xrightarrow[n\to\infty]{\tau_K} f$ when f is continuous.*

If in addition K is uniformly separating on bounded sets then

(v) *$I_{K,n}\big(I_{K,n}f\big) \xrightarrow[n\to\infty]{\tau_b} f$ whenever f is uniformly continuous on bounded sets.*

Finally, when K is uniformly separating one has

(vi) *$I_{K,n}\big(I_{K,n}f\big) \xrightarrow[n\to\infty]{\tau_u} f$ provided f is uniformly continuous on X (not necessarily bounded below).*

Remark. The sequence of functions $I_{K,n}\big(I_{K,n}f\big)$ plays an important auxiliary rôle in this work; namely, it provides a lower bound for the sequence $(\Delta_{K,n}f)_{n\in\mathbb{N}}$ in **Proposition 8**(i).

Proof of the Proposition 6.
(i) Since $K(x,x) = 0$ we get that $I_{K,n}f \leq f$ (take $y = x$ in the infimal definition of $I_{K,n}f$ at any point $x \in X$). Therefore we deduce that

$$S_{K,n}(I_{K,n})f = -I_{K,n}(-I_{K,n}f) \geq I_{K,n}f.$$

To see the other inequality, notice that from **Fact 5-1** we obtain for $x \in X$ the expression

$$S_{K,n}(I_{K,n}f)(x) = \sup_{y\in X} \inf_{z\in X} \Big\{ f(z) + n\big(K(y,z) - K(x,y)\big) \Big\}. \tag{2}$$

For some fixed x, if we take $z = x$ in **(2)** we conclude from the symmetry of K that $S_{K,n}(I_{K,n}f)(x) \le f(x)$.
(ii) From **(i)** and **Fact 5-1**(i) we have $\inf I_{K,n}f = \inf f$ and $\mathfrak{S}_{\mathfrak{Inf}}(f) \subseteq \mathfrak{S}_{\mathfrak{Inf}}(I_{K,n}f)$. Consider any minimum $x_0 \in X$ of $I_{K,n}f$. Then, there exists a sequence $(y_k)_{k\in\mathbb{N}} \subset X$ so that

$$\inf f = I_{K,n}f(x_0) \le f(y_k) + nK(x_0, y_k) \xrightarrow[k\to\infty]{} \inf f. \tag{3}$$

Hence, since K is positive it follows from **(3)** that

$$\lim_{k\to\infty} f(y_k) = \inf f \text{ and } \lim_{k\to\infty} K(x_0, y_k) = 0. \tag{4}$$

But K is pointwise separating, so the second part of **(4)** implies that $y_k \longrightarrow x_0$. Using the lower-semicontinuity of f and the first part of **(4)** we conclude that

$$\inf f \le f(x_0) \le \lim_{k\to\infty} f(y_k) = \inf f.$$

and this proves assertion (ii).

Before proceeding with the rest of the proof, we set up the following useful definition:

$$\Omega_n(x) := \{y \in X : f(y) + nK(x,y) \le I_{K,n}f(x) + 1\} \quad (x \in X, n \in \mathbb{N}) \tag{5}$$

With these notations, we remark that for $n \in N$ and $x \in X$

$$I_{K,n}f(x) = \inf_{y\in\Omega_n(x)} \{f(y) + nK(x,y)\} \ge \inf_{\Omega_n(x)} f \tag{6}$$

(the last inequality coming from the positivity of K).

It is clear from **(6)** that the behaviour of $I_{K,n}f$ is directly linked with the size of the sets $\{\Omega_n(x)\}_{x\in X}$. We shall see that the growth condition (3) ensures that the sets $\Omega_n(x)$ are not arbitrarily big when x runs on bounded sets of X. More precisely, we claim the following.

Claim 6.1. *For any $r > 0$, the set $\Omega_r := \bigcup_{n\in\mathbb{N}} \bigcup_{\|x\|\le r} \Omega_n(x)$ is bounded.*

The proof of this claim is based on the next simple fact.

Fact 6.2. *For any $r > 0$,* $\sup\left\{\frac{I_{K,n}f(x)}{n} : x \in B_X(r),\, n \in \mathbb{N}\right\} := M_r < +\infty.$

Proof of the Fact 6.2. Since f is proper, take y_0 such that $f(y_0) \le \inf f + 1 < +\infty$. Then by definition of $I_{K,n}f$ it follows that for any $x \in X$

$$\frac{I_{K,n}f(x)}{n} \le \frac{f(y_0)}{n} + K(x, y_0) \le \inf f + 1 + \sup\{K(x, y_0) : x \in B_X(r)\},$$

and this expression is bounded above on bounded sets because K is uniformly continuous (or Lipschitz continuous) on bounded sets. The proof of **Fact 6.2** is finished.

Proof of the Claim 6.1. For $r_0 > 0$, let $M_{r_0} > 0$ be the upper bound defined in **Fact 6.2**. Thus, for any $x \in B_X(r_0)$ and $n \in \mathbb{N}$ if $y \in \Omega_n(x)$ it follows from the definition of $\Omega_n(x)$, given in **(5)**, that

$$K(x,y) \le \frac{1}{n}\Big(I_{K,n}f(x) + 1 - f(y)\Big) \le M_{r_0} + 1 - \inf f. \tag{7}$$

But the growth condition on K given by (3) implies that the set of y satisfying **(7)** is uniformly bounded for $x \in B_X(r_0)$. The proof of **Claim 6.1** is done.

We can now continue with the proof of Proposition 6.
(iii) Suppose the kernel K is Lipschitz continuous on bounded sets (the proof for the uniformly continuous case is practically the same). For $r_0 > 0$ take $x, x' \in B_X(r_0)$ and let L_{K,r_0} be the Lipschitz constant of K on $B_X(r_0) \times \Omega_{r_0}$ (Ω_{r_0} being bounded by **Claim 6.1**). Using the equality of **(6)** we can construct a sequence $(y_k)_{k\in\mathbb{N}} \subset \Omega_{r_0}$ in such a way that for every $k \in \mathbb{N}$ one has $f(y_k) + nK(x', y_k) \le I_{K,n}f(x') + \frac{1}{k}$. Therefore, we obtain

$$\begin{aligned} I_{K,n}f(x') - I_{K,n}f(x) &\le f(y_k) + nK(x', y_k) - f(y_k) - nK(x, y_k) + \frac{1}{k} \\ &\le nL_{K,r_0}\|x' - x\| + \frac{1}{k} \xrightarrow[k\to\infty]{} nL_{K,r_0}\|x' - x\|. \end{aligned}$$

This concludes the proof of (iii).

We first prove (iv), (v) and (vi) for $(I_{K,n}f)_n$ instead of $\big(I_{K,n}(I_{K,n}f)\big)_n$. We will complete the proof afterwards.
(iv') Fix $x_0 \in X$. If $\lim_{n\to\infty} I_{K,n}f(x_0) = \sup_n I_{K,n}f(x_0) = +\infty$ then by **(i)** one has $f(x_0) = +\infty$ and the result holds. Thus, suppose that $I_{x_0} := \lim_n I_{K,n}f(x_0) < +\infty$. By the infimal definition of $I_{K,n}f$ at x_0, we can choose a sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that

$$I_{K,n}f(x_0) \le f(y_n) + nK(x_0, y_n) \le I_{K,n}f(x_0) + \frac{1}{n} \xrightarrow[n\to\infty]{} I_{x_0} \tag{8}$$

Hence, from **(8)** it follows for $n \in \mathbb{N}$ that

$$K(x_0, y_n) \le \frac{1}{n}\big(I_{K,n}f(x_0) - f(y_n)\big) + \frac{1}{n^2} \le \frac{1}{n}\big(I_{x_0} - \inf f\big) + \frac{1}{n^2} \xrightarrow[n\to\infty]{} 0. \tag{9}$$

But K is pointwise separating, so we have from **(9)** that $(y_n)_n$ is norm converging to x_0. Using the lower-semicontinuity of f, the positivity of K in **(8)** and **(i)**, we get that

$$f(x_0) \le \liminf_{n\to\infty} f(y_n) \le I_{x_0} \le f(x_0).$$

If f is continuous, since by **Fact 5-3**(i) and **(iii)** $(I_{K,n}f)_n$ is an increasing sequence of continuous functions, Dini's Theorem tell us that the pointwise convergence of $(I_{K,n}f)_n$ to f is actually uniform on compact sets.
(v') Let f be an uniformly continuous function on bounded sets and O_{r_0} be the oscillation of f on the set $B_X(r_0) \cup \Omega_{r_0}$, for some fixed $r_0 > 0$. Then, for any $n \in \mathbb{N}$, $x \in B_X(r_0)$ and $y \in \Omega_n(x)$ after the first inequality of **(7)** and **(i)** we have that

$$K(x, y) \le \frac{1}{n}\big(I_{K,n}f(x) + 1 - f(y)\big) \le \frac{1}{n}\big(f(x) - f(y) + 1\big) \le \frac{1}{n}(O_{r_0} + 1) \xrightarrow[n\to\infty]{} 0. \tag{10}$$

Suppose that K is uniformly separating on bounded sets . Then, a direct consequence of **(10)** is that $\lim_n \operatorname{diam}(\Omega_n(x)) = 0$ uniformly on $B_X(r_0)$. Therefore, it follows from **(i)**, **(6)** and the uniform continuity of f on $B_X(r_0)$ that

$$f(x) \ge \lim_{n\to\infty} I_{K,n}f(x) \ge \lim_{n\to\infty} \inf_{\Omega_n(x)} f \xrightarrow[n\to\infty]{} f(x) \tag{11}$$

uniformly on $x \in B_X(r_0)$.
(vi') Suppose that f is uniformly continuous on X. Then f satisfy the following fact (whose simple proof is left as an exercise to the reader):

$$\text{there exists } \alpha > 0 \text{ such that } f(x) - f(y) \le \max\{1, \alpha\|x-y\|\} \text{ for all } x, y \in X. \quad (12)$$

Then, in the same way as in **(10)** before, using this time **(12)**, we deduce that for $n \in \mathbb{N}$, $x \in X$ and any $y \in \Omega_n(x)$

$$K(x,y) \le \frac{1}{n}(f(x) - f(y) + 1) \le \max\left\{\frac{1}{n}, \frac{\alpha}{n}\|x-y\|\right\} + \frac{1}{n}. \quad (13)$$

For $1 > \delta > 0$, since K is uniformly separating there is some $\beta_\delta > 0$ so that from **(13)** we deduce for $x \in X$ and $y \in \Omega_n(x)$ that

$$\|x-y\| \le \max\left\{\frac{1}{n\beta_\delta}, \frac{\alpha}{n\beta_\delta}\|x-y\|\right\} + \frac{1}{n\beta_\delta} \text{ whenever } \|x-y\| > \delta. \quad (14)$$

Hence, taking n big so that $\max\left\{\frac{2}{n\beta_\delta}, \frac{2\alpha}{n\beta_\delta}\right\} \le \delta < 1$, **(14)** shows for every $x \in X$ that $\operatorname{diam}(\Omega_n(x)) \le 2\delta$.

That is, we have shown that $\operatorname{diam}(\Omega_n(x)) \to 0$ uniformly on $x \in X$. Therefore, as f is uniformly continuous on X we can repeat the same reasonings of **(11)** to conclude that $(I_{K,n}f)_n$ converges to f uniformly on X.

(iv) and (v) are straightforward corollaries of (iv') and (v') if we remark the following.

Suppose that for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that $f - \frac{\varepsilon}{2} \le I_{K,n_0}f$ on some set S (S being a singleton, or a compact set or a bounded set of X). By **Fact 5-3**(i) and **(iii)**, we can then apply (iv') (or (v')) to the bounded below, uniformly continuous function $I_{K,n_0}f$ to obtain $m > n_0$ such that $I_{K,n_0}f - \frac{\varepsilon}{2} \le I_{K,m}(I_{K,n_0}f)$ on the same S. Thus, by **Fact 5-3**(iii) and **(i)** it follows that

$$f - \varepsilon \le I_{K,n_0}f - \frac{\varepsilon}{2} \le I_{K,m}(I_{K,n_0}f) \le I_{K,m}(I_{K,m}f) \le f \text{ on } S.$$

(vi) is also easily deduced from (vi') through the following argument.

If $f - \varepsilon \le I_{K,n}f \le f$, for some $\varepsilon > 0$ and $n \in \mathbb{N}$, then applying **Facts 5-2** and **5-3**(ii) we get that

$$f - 2\varepsilon \le I_{K,n}f - \varepsilon = I_{K,n}(f - \varepsilon) \le I_{K,n}(I_{K,n}f) \le I_{K,n}f \le f. \quad \square$$

Remark. With the above techniques it is not difficult to check that $\big(I_{K,n}(I_{K,n}f)\big)_n$ converges to f for the *epigraphical distance* (see [AW] for the definition). We refer to the proof of Lemma 3(v) in [St$_2$] for details.

3. Convexity techniques and smoothness results

In this section we shall show a procedure to obtain smooth functions from the operators $I_{K,n}(\cdot)$ and $S_{K,n}(\cdot)$. We will need to impose some additional conditions of convexity and smoothness on the kernel K to achieve the smooth regularization. The interesting feature of these convexity arguments is the preservation of the approximating properties obtained in the previous section.

The main tool we shall use to get smooth regularization is explained in the next theorem. It deals with the smooth properties inherited by the convex envelop of a "somehow" smooth function.

Theorem 7. *Let $c : X \to \mathbb{R}$ be a differentiable function, and $d : X \to \mathbb{R}$ be a convex function. Denote by h their difference $h := c - d$ and assume that $\operatorname{co} h$ makes sense. Then the following statements are fulfiled.*

(i) *If $c \in \mathcal{C}^{1,u}(X)$ (resp. $c \in \mathcal{C}^{1,\alpha}(X)$, for some $0 < \alpha \leq 1$) then $\operatorname{co} h \in \mathcal{C}^{1,u}(X)$ (resp. $\operatorname{co} h \in \mathcal{C}^{1,\alpha}(X)$).*

(ii) *If $c \in \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ (resp. $c \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$, for some $0 < \alpha \leq 1$) and h is uniformly continuous on bounded sets and strongly coercive (i.e., $\lim_{x\to\infty} \frac{h(x)}{\|x\|} = +\infty$) then $\operatorname{co} h \in \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ (resp. $\operatorname{co} h \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$).*

Remark. A proof for the finite dimensional version of **Theorem 7**(ii) with $d \equiv 0$ can be found in [GR]. Our more general proof does not require local compactness and relies upon ideas of the work [Fa]. The fact that the convex envelope of a smooth function c "perturbed" by a non-smooth concave function $-d$ is still smooth will be crucial later (namely, when we check the smoothness of the sequence $(\Delta_{K,n} f)_{n\in\mathbb{N}}$ in **Proposition 8**).

Notice that the uniform continuity hypothesis on the derivative of c cannot be weakened in the infinite dimensional setting. There are bounded below $\mathcal{C}^\infty$-differentiable functions on ℓ_2 whose convex envelope is not even Gâteaux differentiable (see Example II.5.6(a) in [DGZ]).

Proof of the Theorem 7. Denote by $\nu := \operatorname{co} h = \operatorname{co}(c - d)$.

(i) Suppose that $c \in \mathcal{C}^{1,\alpha}(X)$ (the proof for the other case is similar). Since ν is convex, a necessary and sufficient condition for $\nu \in \mathcal{C}^{1,\alpha}(X)$ is that for every x, $y \in X$ one has

$$\nu(x+y) + \nu(x-y) - 2\nu(x) \leq L\|y\|^{1+\alpha}, \text{ for some } L > 0. \tag{15}$$

(see Lemma V.3.5 of [DGZ]). We shall check this condition for ν.

For $\varepsilon > 0$ and $x \in X$, by the expression of the convex envelope of a function given in **(1)**, we can choose $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n > 0$ so that

$$\sum_{i=1}^n \lambda_i = 1, \ \sum_{i=1}^n \lambda_i x_i = x \text{ and } \sum_{i=1}^n \lambda_i h(x_i) \leq \nu(x) + \frac{\varepsilon}{2}. \tag{16}$$

Note that from the two first parts of **(16)** we also have

$$x \pm y = \left(\sum_{i=1}^n \lambda_i x_i\right) \pm \left(\sum_{i=1}^n \lambda_i y\right) = \sum_{i=1}^n \lambda_i (x_i \pm y). \tag{17}$$

Thus, it follows from **(1)** and **(17)** that

$$\nu(x \pm y) \leq \sum_{i=1}^n \lambda_i h(x_i \pm y). \tag{18}$$

Let $L > 0$ be the α-Hölder continuity constant of the derivative of c. Putting together the last part of **(16)**, **(18)** and using the convexity of d, we get

$$\nu(x+y)+\nu(x-y)-2\nu(x) \leq \sum_{i=1}^n \lambda_i h(x_i+y) + \sum_{i=1}^n \lambda_i h(x_i-y) - 2\sum_{i=1}^n \lambda_i h(x_i) + \varepsilon =$$

$$\sum_{i=1}^n \lambda_i \Big(c(x_i+y)+c(x_i-y)-2c(x_i)\Big) + 2\sum_{i=1}^n \lambda_i \left(d(x_i) - \frac{d(x_i+y)+d(x_i-y)}{2}\right) + \varepsilon \leq$$

$$\sum_{i=1}^n \lambda_i \Big(c(x_i+y)-c(x_i)+c(x_i-y)-c(x_i)\Big) + \varepsilon \leq \sum_{i=1}^n \lambda_i 2^\alpha L\|y\|^{1+\alpha} + \varepsilon = 2^\alpha L\|y\|^{1+\alpha} + \varepsilon,$$

because $c \in \mathcal{C}^{1,\alpha}(X)$ (and therefore satisfy **(15)** for $L' = 2^\alpha L$). As ε is arbitrary, the condition **(15)** holds for ν.
(ii) can be proved reproducing the same lines as before, bearing in mind that the lack of uniformity for the derivative of c on X can be replaced by the next "localization" property of the convex envelope of a strongly coercive function h.

Claim 7.1. *Let $h : X \to \mathbb{R}$ be a function which is uniformly continuous on bounded sets and strongly coercive. Then for every $r > 0$ there exists $\rho_r > 0$ so that for all $\|x\| \le r$ one has*

$$\operatorname{co} h(x) = \inf\Big\{ \sum_{i=1}^n \lambda_i h(x_i) : (x_i)_{i=1}^n \subset B_X(\rho_r),\ \lambda_i > 0,\ \sum_{i=1}^n \lambda_i = 1,\ x = \sum_{i=1}^n \lambda_i x_i \Big\}.$$

Proof of the Claim 7.1. First, note that under the hypothesis of **Claim 7.1**, h is bounded below, so that $\operatorname{co} h$ makes sense. Fix $r_0 > 0$ and let m_{r_0} be the infimum of h on X and M_{r_0} be the supremum of h on $B_X(r_0+1)$ ($M_{r_0} < +\infty$, because of the uniform continuity of f on $B_X(r_0+1)$). Consider the following family of hyperplanes:

$$\mathcal{H}_{r_0} := \bigcup_{\substack{x \in B_X(r_0) \\ v \in B_X,\, v^* \in B_{X^*}}} \Big\{ H_{x,v}(z) = m_{r_0} + \big(h(x+v) - m_{r_0}\big) v^*(z-x) : v^*(v) = 1 \Big\}. \tag{19}$$

Notice that for $\|x\| \le r_0$ and $v \in B_X$ we have the following

$$H_{x,v}(x) = m_{r_0} \le \operatorname{co} h(x) \text{ and } H_{x,v}(x+v) = h(x+v) \le M_{r_0}. \tag{20}$$

Since h is strongly coercive, we get from **(19)** that

$$\sup_{H \in \mathcal{H}_{r_0}} H(z) \le m_{r_0} + (M_{r_0} - m_{r_0})(\|z\| + r_0) < h(z) \text{ provided } \|z\| > \rho_{r_0}, \tag{21}$$

for some $\rho_{r_0} > 0$. Let us show that ρ_{r_0} satisfy the conclusion of the claim.
The strategy is to replace any convex combination that appears in the definition of the convex envelope **(1)** by another smaller convex combination with "uniformly bounded vertices". This idea is formally explained in the next fact.

Fact 7.2. *For $\|x\| \le r_0$, consider any finite convex combination ($x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n > 0$ and $\sum_{i=1}^n \lambda_i = 1$) such that $\sum_{i=1}^n \lambda_i x_i = x$. If $\|x_{i_0}\| > \rho_{r_0}$, for some $1 \le i_0 \le n$, then there exists $x'_{i_0} \in B_X(\rho_{r_0})$ and $\lambda'_1, \ldots, \lambda'_n > 0$ so that $\sum_{i=1}^n \lambda'_i = 1$,*

$$\sum_{\substack{i=1 \\ i \ne i_0}}^n \lambda'_i x_i + \lambda'_{i_0} x'_{i_0} = x \text{ and } \sum_{\substack{i=1 \\ i \ne i_0}}^n \lambda'_i h(x_i) + \lambda'_{i_0} h(x'_{i_0}) \le \sum_{i=1}^n \lambda_i h(x_i).$$

Proof of the Fact 7.2. For simplicity, take $i_0 = 1$. Since $\|x_1\| > \rho_{r_0}$, it follows from **(21)** that $H_{x,v_{x_1}}(x_1) < h(x_1)$, where we take $v_{x_1} := \frac{x_1 - x}{\|x_1 - x\|}$. But by the first part of **(20)** we also have $H_{x,v_{x_1}}(x) = m_{r_0} \le \sum_{i=1}^n \lambda_i h(x_i)$. Hence, the segment

$I_{x,x_1} := \Big[\big(x, \sum_{i=1}^n \lambda_i h(x_i)\big), \big(x_1, h(x_1)\big)\Big] \subset X \times \mathbb{R}$ belongs to the upper half-space define by $H_{x,v_{x_1}}$. Therefore, the equality in the second part of **(20)** implies that

$$I_{x,x_1} \cap \big\{(x+v_{x_1}, t) : t \in \mathbb{R}\big\} = \big\{(x+v_{x_1}, s)\big\} \text{ where } h(x+v_{x_1}) \le s. \tag{22}$$

If we define $x' := \sum_{i>1}^n \frac{\lambda_i}{1-\lambda_1} x_i$ (so that we have $x = \lambda_1 x_1 + (1-\lambda_1)x'$), using barycentric coordinates on the segment $\Big[\big(x', \sum_{i>1}^n \frac{\lambda_i}{1-\lambda_1} h(x_i)\big), \big(x_1, h(x_1)\big)\Big]$ we can compute some $\mu \ge 0$ in such a way that

$$\begin{aligned}\left(x, \sum_{i=1}^n \lambda_i h(x_i)\right) = \mu\left(x', \sum_{i>1}^n \frac{\lambda_i}{1-\lambda_1} h(x_i)\right) + (1-\mu)(x+v_{x_1}, s) = \\ \left(\sum_{i>1}^n \frac{\mu\lambda_i}{1-\lambda_1} x_i, \sum_{i>1}^n \frac{\mu\lambda_i}{1-\lambda_1} h(x_i)\right) + (1-\mu)(x+v_{x_1}, s).\end{aligned} \tag{23}$$

But **(22)** and **(23)** together give that $x = \sum_{i>1}^n \frac{\mu\lambda_i}{1-\lambda_1} x_i + (1-\mu)(x+v_{x_1})$ and

$$\sum_{i>1}^n \frac{\mu\lambda_i}{1-\lambda_1} h(x_i) + (1-\mu)h(x+v_{x_1}) \le \sum_{i>1}^n \frac{\mu\lambda_i}{1-\lambda_1} h(x_i) + (1-\mu)s = \sum_{i=1}^n \lambda_i h(x_i).$$

This concludes the proof of **Fact 7.2**.

Then the proof of **Claim 7.1** is done and, therefore, **Theorem 7** is proved. □

Remark. **Claim 7.1** is false for functions h failing the strong coerciveness condition $\liminf_{x\to\infty} \frac{h(x)}{\|x\|} = +\infty$. For instance, consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \sqrt{|x|}$.

Theorem 7 can be applied as an useful tool to regularize functions on infinite dimensional Banach spaces. Our next proposition, which provides the smoothness assertions we need for proving **Theorem 1**, is a good example of this feature. We keep the notation used in SECTION 2.

Proposition 8. *Let $K : X \times X \to \mathbb{R}$ be a kernel satisfying the following conditions:*

(1) *K is positive and $K(x,x) = 0$ for all $x \in X$,*
(2) *K is symmetric,*
(3) *$K(x,y) \xrightarrow[y\to\infty]{} +\infty$ uniformly on bounded sets,*
(4) *K is uniformly continuous on bounded sets and*
(5) *$K(x,y) = c_K(x) - d_K(x,y)$ where $d_K(\cdot, y)$ is a lower semi-continuous convex function, for all $y \in X$.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semi-continuous function and consider the sequence of Δ-convex functions $\Delta_{K,n} f$ defined as follows

$$\Delta_{K,n} f := \mathrm{co}\,(I_{K,n} f + n c_K) - n c_K \quad (n \in \mathbb{N}).$$

Then the following assertions hold.

(i) *$I_{K,n}\big(I_{K,n} f\big) \le \Delta_{K,n} f \le f$.*
(ii) *If $c_K \in \mathcal{C}^{1,u}(X)$ (resp. $c_K \in \mathcal{C}^{1,\alpha}(X)$, for some $0 < \alpha \le 1$), then one has $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,u}(X)$ (resp. $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,\alpha}(X)$).*

(iii) *If* $c_K \in \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ *(resp.* $c_K \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$*, for some* $0 < \alpha \leq 1$*) and* c_K *is strongly coercive then* $(\Delta_{K,n}f)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ *(resp.* $(\Delta_{K,n}f)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$*) provided* f *is bounded below.*

Remark. For every pair of function f, g on X, denote by $f\blacktriangle g := \operatorname{co}(f+g) - g$. Suppose that $\|\cdot\|$ is a Hilbert norm and consider the kernel $K_L(x-y) := \|x-y\|^2$. The Lasry-Lions approximates of a function f by the norm $\|\cdot\|$ satisfy the following relation for $m > n$ (see Proposition 2(i) of [St$_2$])

$$\Big(S_{K_L,m}\big(I_{K_L,n}f\big)\Big) = I_{K_L,m-n}\big(f\blacktriangle nc\big).$$

Compare with the expression given by **Proposition 8**

$$\Delta_{K_L,n}f = \big(I_{K_L,n}f\big)\blacktriangle nc.$$

Proof of the Proposition 8. For any function $g : X \to \mathbb{R} \cup \{+\infty\}$, denote by

$$D_n g(x) := \sup\{g(y) + n d_K(x,y) : y \in X\}.$$

Since $d_K(\cdot, y)$ is a l.s.c. convex function, we have that $D_n g$ is l.s.c. and convex too. Note that by **Fact 5-1** we also have the following decomposition:

$$S_{K,n}g = \sup_{y\in X}\Big\{g(y) - n\big(c_K(x) - d_K(x,y)\big)\Big\} = D_n g(x) - nc_K(x). \tag{24}$$

On the other hand, (1) and (2) ensure that (i) of **Proposition 6** holds true. Hence, for all $n \in \mathbb{N}$ from **(24)** we get that

$$I_{K,n}(I_{K,n}f) \leq S_{K,n}\big(I_{K,n}(I_{K,n}f)\big) = D_n\big(I_{K,n}I_{K,n}f\big) - nc_K \leq I_{K,n}f \leq f \tag{25}$$

Now, we make the next simple but crucial observation.

Fact 8.1. *Let* c, d *and* e *be three functions such that* $d - c \leq e$ *and suppose that* d *is l.s.c. and convex. Then we have that* $d - c \leq \operatorname{co}(e+c) - c \leq e$.

Proof of the Fact 8.1. It suffices to note that the convexity of d implies the equivalent inequality $d \leq \operatorname{co}(e+c) \leq e + c$.

Applying **Fact 8.1** to the inequality **(25)** we obtain that

$$I_{K,n}I_{K,n}f \leq D_n\big(I_{K,n}I_{K,n}f\big) - nc_K \leq \operatorname{co}(I_{K,n}f + nc_K) - nc_K = \Delta_{K,n}f \leq I_{K,n}f.$$

At this point, another important remark turns up. By definition of $I_{K,n}f$ at any point $x \in X$ one has

$$(I_{K,n}f + nc_K)(x) = 2nc_K(x) + \inf_{y\in X}\big\{f(y) - d_K(x,y)\big\} = 2nc_K(x) - D_n(-f)(x), \tag{26}$$

where $D_n(-f)$ is a convex function.

Therefore, if $c_K \in \mathcal{C}^{1,u}(X)$ (or $c_K \in \mathcal{C}^{1,\alpha}(X)$) **Theorem 7**(i) can be applied to $\operatorname{co}(I_{K,n}f(x) + nc_K(x))$ because of **(26)**. This shows **Proposition 8**(ii).

The proof for the case of $c_K \in \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ (or $c_K \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$) can be done in a similar way from **Theorem 7**(ii). Notice that the function $I_{K,n}f + nc_K$ is uniformly continuous on bounded sets since K satisfies (1)–(4) and therefore **Proposition 6**(iii) holds. On the other hand, the strong coerciveness of c_K implies for a bounded below function f that

$$\frac{I_{K,n}f(x) + nc_K(x)}{\|x\|} \geq \frac{\inf f}{\|x\|} + n\frac{c_K(x)}{\|x\|} \xrightarrow[x\to\infty]{} +\infty. \ \square$$

4. The proof of the main result

With the tools shown in Section 2 and Section 3, we are now ready to prove our main result.

Proof of the Theorem 1. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a bounded below l.s.c. function. For $p > 1$, we define the following kernel $K_p : X \times X \to \mathbb{R}$ as

$$K_p(x, y) := 2^{p-1}\|x\|^p + 2^{p-1}\|y\|^p - \|x + y\|^p. \tag{27}$$

Let us first check the following two basic properties of K_p.
(1) Clearly, $K_p(x, x) = 0$ (for all $x \in X$). Also, K_p is positive since one has that

$$\|x + y\|^p \le \big(\|x\| + \|y\|\big)^p \le 2^{p-1}\big(\|x\|^p + \|y\|^p\big).$$

(2) K_p is obviously symmetric.

If we set $c_{K_p}(x) := 2^{p-1}\|x\|^p$ and $d_{K_p}(x, y) := \|x + y\|^p - 2^{p-1}\|y\|^p$ $(x,\, y \in X)$, we have $K_p(\cdot, y) = c_{K_p}(\cdot) - d_{K_p}(\cdot, y)$.

With this notation and the definition of g^p_n given in **Theorem 1**, one has

$$(I_{K_p,n}f + nc_{K_p}) = g^p_n \quad \text{for every } n \in \mathbb{N}. \tag{28}$$

Hence, using again the notation of **Theorem 1** and **Proposition 8**(i), it follows that

$$I_{K_p,n}\big(I_{K_p,n}f\big) \le \Delta_{K_p,n}f = \operatorname{co} g^p_n - 2^{p-1}n\|\cdot\|^p = \Delta^p_n f \le f. \tag{29}$$

Therefore, by **(29)** the statements (i), (iii), (iv) and (v) of **Theorem 1** hold true if we check that K_p satisfies the assumptions of **Proposition 6**.

We proceed to show the following growth property of K_p, that trivially implies the condition (3) of **Proposition 6**.

Claim 1.1. *For any $p > 1$ there exists $\gamma_p > 0$ and $\eta_p > 1$ so that $K_p(x, y) \ge \gamma_p\|y\|^p$ whenever $\|y\| \ge \eta_p\|x\|$.*

Proof of Claim 1.1. Take $\eta > 1$ and $x, y \in X$ such that $\eta\|x\| \le \|y\|$. After the computation

$$K_p(x, y) \ge \|y\|^p\left(2^{p-1}\left|\frac{\|x\|}{\|y\|}\right|^p + 2^{p-1} - \left\|\frac{y}{\|y\|} + \frac{x}{\|y\|}\right\|^p\right) \ge$$
$$\|y\|^p\left(2^{p-1} - \left|1 + \frac{\|x\|}{\|y\|}\right|^p\right) \ge \|y\|^p\left(2^{p-1} - (1 + \frac{1}{\eta})^p\right).$$

The claim is proved by choosing $\eta_p > 1$ such that $\gamma_p := \big(2^{p-1} - (1 + \frac{1}{\eta_p})^p\big) > 0$. □

It is clear that K_p is Lipschitz continuous on bounded sets. The next claim takes care of the *separating* properties of K_p.

Claim 1.2. *Suppose that the norm $\|\cdot\|$ is l.u.c. at $x_0 \in X$ (resp. UC) then for every $\varepsilon > 0$ there exists $C_{\varepsilon,x_0} > 0$ such that $K_p(x_0, y) \ge C_{\varepsilon,x_0}\|x_0 - y\|^p$ whenever $\|x_0 - y\| \ge \varepsilon$ (resp. for all $r > 0$ and $\varepsilon > 0$ there exists $C_{\varepsilon,r} > 0$ such that $K_p(x,y) \ge C_{\varepsilon,r}\|x - y\|^p$ provided $\|x - y\| \ge \varepsilon$ and $\|x\| \le r$).*

Proof of the Claim 1.2. We only prove the claim under the uniform convexity assumption. The proof for the l.u.c. case is completely similar. We proceed by contradiction.

Suppose that the claim is false. Then by definition of K_p (see **(27)**), there are two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X so that $(x_n)_n$ is bounded, $\|x_n - y_n\| \geq \varepsilon_0 > 0$ for all $n \in \mathbb{N}$ and

$$K_p(x_n, y_n) = 2^{p-1}\|x_n\|^p + 2^{p-1}\|y_n\|^p - \|x_n + y_n\|^p \leq \frac{1}{n}\|x_n - y_n\|^p. \tag{30}$$

Moreover, without loss of generality we can suppose that $\|y_n\| \geq \|x_n\| > 0$ for all n. We then consider $0 < \beta_n = \frac{\|x_n\|}{\|y_n\|} \leq 1$. From **(30)** it follows

$$0 \leq 2^{p-1}\big({\beta_n}^p + 1\big) - \big(\beta_n + 1\big)^p \leq \frac{1}{n}\big(\beta_n + 1\big)^p \xrightarrow[n\to\infty]{} 0.$$

Hence,

$$2^{p-1}\frac{{\beta_n}^p + 1}{(\beta_n + 1)^p} \xrightarrow[n\to\infty]{} 1. \tag{31}$$

From **(31)** it follows that, $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\frac{\|x_n\|}{\|y_n\|} = 1$. Then, since the sequence $(x_n)_n$ is bounded, so is $(y_n)_n$ and therefore we have that $\lim_{n\to\infty}\big(\|x_n\| - \|y_n\|\big) = 0$. But using **(30)** again, we obtain that the bounded sequences $(x_n)_n$ and $(y_n)_n$ verify

$$\lim_{n\to\infty}\Big(\|x_n\| - \|\frac{x_n + y_n}{2}\|\Big) = \lim_{n\to\infty}\big(\|x_n\| - \|y_n\|\big) = 0.$$

Nonetheless, by hypothesis we have that $\|x_n - y_n\| \geq \varepsilon_0 > 0$ for all n. That is a contradiction with the uniform convexity of the norm $\|\cdot\|$. □

Another important fact is that K_p is uniformly separating when the modulus of convexity of the norm $\|\cdot\|$ is of power type p. This is a consequence from results of [H]. Indeed, for any pair x, $y \in X$ we have the following stronger inequality

$$K_p(x, y) = 2^{p-1}\|x\|^p + 2^{p-1}\|y\|^p - \|x + y\|^p \geq C_{\|\cdot\|}\|x - y\|^p,$$

for some $0 < C_{\|\cdot\|} \leq 1$ (for instance, see [C] Lemma 3.1).

Hence, using **Proposition 6** together with the inequality **(29)** we deduce (i), (iii), (iv) and (v) of **Theorem 1**. It remains to prove the assertion (ii), for which we use **Proposition 8**.

More precisely, we observe that in the decomposition **(28)** $d_K(\cdot, y)$ is a convex function for every $y \in X$. Moreover, for $p > 1$ is easy to verify that c_{K_p} is strongly coercive; that is,

$$\frac{c_{K_p}(x)}{\|x\|} = 2^{p-1}\|x\|^{p-1} \xrightarrow[x\to\infty]{} +\infty.$$

Therefore, by **Proposition 8**(iii) the regularity of $\Delta_n^p f = \Delta_{K_p,n} f$ can be deduced from the regularity of $c_{K_p} = 2^{p-1}\|\cdot\|^p$.

Recall now that for any norm $\|\cdot\|$ on X, the fact of being US (resp. with modulus of smoothness of power type $1+\alpha$) is equivalent to $\|\cdot\| \in \mathcal{C}^{1,u}(X)$ (resp. $\|\cdot\| \in \mathcal{C}^{1,\alpha}(X)$). Therefore, $c_{K_p} = 2^{p-1}\|\cdot\|^p \in \mathcal{C}^{1,u}_{\mathcal{B}}(X)$ (or $c_{K_p} \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$) whenever the norm $\|\cdot\|$ is US (or with modulus of smoothness of power type $1+\alpha$).

In the last case of (ii), for a norm $\|\cdot\|$ with modulus of smoothness of power type $1+\alpha$ (or equivalently $\|\cdot\| \in \mathcal{C}^{1,\alpha}(X)$), we can achieve a smoother behaviour of the sequence $(\Delta_n^p f)$ by choosing the proper value of p: $(\Delta_n^{1+\alpha} f)_n \subset \mathcal{C}^{1,\alpha}(X)$. This is a corollary of **Proposition 8**(ii) and the next lemma.

Lemma 1.3. *If* $\|\cdot\| \in \mathcal{C}^{1,\alpha}(X)$ *then* $\|\cdot\|^{1+\alpha} \in \mathcal{C}^{1,\alpha}(X)$.

Proof of the Lemma 1.3. This fact relies strongly in the convexity and homogeneity of a norm. Since it is clear that $\|\cdot\|^{1+\alpha} \in \mathcal{C}^{1,\alpha}_{\mathcal{B}}(X)$, let $C > 0$ be the α-Hölder continuity constant of the derivative of the norm $\|\cdot\|$ in B_X. We shall show that the condition **(15)** holds true for $\|\cdot\|^{1+\alpha}$. Take any x, $y \in X$ and denote by ω the maximum of $\|x\|$ and $\|y\|$. The lemma is proved by the next computation.

$$\begin{aligned}\|x+y\|^{1+\alpha} + &\|x-y\|^{1+\alpha} - 2\|x\|^{1+\alpha} = \\ &\omega^{1+\alpha}\Big(\Big\|\frac{x}{\omega}+\frac{y}{\omega}\Big\|^{1+\alpha} - \Big\|\frac{x}{\omega}\Big\|^{1+\alpha} + \Big\|\frac{x}{\omega}-\frac{y}{\omega}\Big\|^{1+\alpha} - \Big\|\frac{x}{\omega}\Big\|^{1+\alpha}\Big) \leq \\ &\omega^{1+\alpha}2^{\alpha}C\Big\|\frac{y}{\omega}\Big\|^{1+\alpha} = 2^{\alpha}C\|y\|^{1+\alpha}.\end{aligned}$$

By the above, this concludes the proof of **Theorem 1**. □

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