FIXED POINTS OF NONEXPANSIVE MAPPINGS IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. Let K be a compact metrizable space and $C(K)$ the Banach space of all real continuous functions defined on K with the maximum norm. It is known that $C(K)$ fails to have the weak fixed point property for nonexpansive mappings (w-FPP) when K contains a perfect set. However the space $C(\omega^n+1)$, where $n\in\mathbb{N}$ and ω is the first infinite ordinal number, enjoys the w-FPP and so $C(K)$ also satisfies this property if $K^{(\omega)}=\emptyset$. It is unknown if $C(K)$ has the w-FPP when K is a scattered set such that $K^{(\omega)}\neq\emptyset$. In this paper we prove that certain subspaces of $C(K)$, with $K^{(\omega)}\neq\emptyset$, satisfy the w-FPP. To prove this result we introduce the notion of ω-almost weak orthogonality and we prove that an ω-almost weakly orthogonal closed subspace of $C(K)$ enjoys the w-FPP. We show an example of an ω-almost weakly orthogonal subspace of $C(\omega^\omega+1)$ which is not contained in $C(\omega^n+1)$ for any $n\in\mathbb{N}$

Introduction

Let K be a compact metrizable space and $C(K)$ the Banach space of all real continuous functions defined on K with the maximum norm. It is well known (see [13], [15]) that many topological properties of K are strongly related to geometrical properties of $C(K)$. In this paper we are specially concerned with a geometrical property: The weak fixed point property for nonexpansive mappings. A Banach space X is said to have the weak fixed point property for nonexpansive mappings (w-FPP) if every nonexpansive mapping T defined from a nonempty convex weakly compact subset M of X into itself has a fixed point.

Whether or not every Banach space has the w-FPP was an open question for some years. In 1981, Alspach [1] solved this problem by proving that the Lebesgue space $L^1([0,1])$ fails to have the w-FPP. Despite the fact that no explicit example is known in any other Banach space, Alspach's example provides the failure of the w-FPP for any space containing isometrically $L^1([0,1])$. In particular, $C([0,1])$ which is universal for separable Banach spaces fails to have the w-FPP. In fact, it is known [13, Main Theorem] that $C(K)$ contains isometrically $L^1([0,1])$ if and only if K is a compact set which is not scattered (that is, K contains a perfect non-void subset). Thus, $C(K)$ fails to have the w-FPP if K is not scattered. On the other hand, it is known [5] that the space $C(\omega^n+1)$, where $n\in\mathbb{N}$ and ω is the first infinite ordinal number (for ordinal numbers we follow the notation in [15]) enjoys the w-FPP and

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so $C(K)$ also satisfies this property if $K^{(\omega)} = \emptyset$ where $K^{(\alpha)}$ denotes the α-derived set of K, α being any ordinal number. It is unknown if $C(K)$ has the w-FPP when K is a scattered set such that $K^{(\omega)} \neq \emptyset$. In Section 2, we will prove that a class of subspaces of $C(K)$ does satisfy the w-FPP. To do that, we introduce the notion of ω-almost weak orthogonality and we prove that any ω-almost weakly orthogonal closed subspace of $C(K)$ enjoys the w-FPP. This notion is a wide extension of the concept of weakly orthogonal Banach lattice, defined by Borwein and Sims [3]. We will prove that the class of metrizable compact sets K such that $C(K)$ is weakly orthogonal, is very strict. Actually, this class only contains those compact sets with finitely many accumulations points. However, we will show an example of an ω-almost weakly orthogonal subspace X of $C(\omega^\omega+1)$ which is not contained in any space $C(\omega^n+1)$ for $n \in \mathbb{N}$, i.e. for any topological compact space K, such that X can be lattice isomorphically embedded in $C(K)$, we have $K^{(\omega)} \neq \emptyset$. Furthermore, we prove that $C(\omega^n+1)$ is ω-almost weakly orthogonal, which means that our result is a strict extension of Corollary 3 in [5].

1. Preliminaries

In this section we introduce some known results related to the weak fixed point property, which will be used throughout this paper. For more details the reader may consult, for instance [7, 11]. We also recall some classical topological and metric results concerning spaces of continuous functions.

Let X be a Banach space and let M be a nonempty convex weakly compact subset of X. Let $T : M \to M$ be a nonexpansive map (i.e. $\|Tx-Ty\| \leq \|x-y\|$ for every $x, y \in M$) which is fixed point free. Using Zorn's Lemma we can find a subset C of M which is convex, weakly compact, $\mathrm{diam}(C) > 0$, $T(C) \subset C$ and minimal in the following sense: There is no nonempty convex weakly compact proper subset of C which is invariant under T.

On the other hand, it is well known that we can obtain a sequence (x_n) of approximated fixed points for T, that is, $\lim_n \|x_n - Tx_n\| = 0$.

Goebel-Karlovitz' Lemma [6], [10].

Let C be a convex weakly compact subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. Assume that C is minimal for T and let (x_n) be an approximated fixed point sequence. Then

$$\lim_n \|x_n - x\| = diam(C)$$

for every $x \in C$.

We recall some well known topological results.

Definition. *Let M be a topological space and A a subset of M. The set A is said to be perfect if it is closed and has no isolated points, i.e. A is equal to the set of its own accumulation points. The space M is said to be scattered if it contains no perfect non-void subset.*

Cantor-Bendixson Theorem [15,page 148]. *Let A be a topological space. Then there exists an ordinal number α such that $A^{(\alpha+1)} = A^{(\alpha)}$. Moreover $A^{(\alpha)} = \emptyset$ if and only if A is scattered.*

Mazurkiewicz-Sierpiński Theorem. *Every scattered first-countable compact topological space is homeomorphic to a countable compact ordinal.*

In fact, the Mazurkiewicz-Sierpiński Theorem proves that K is homeomorphic to $\omega^{\alpha-1}m+1$ if α is the smallest ordinal such that $K^{(\alpha)}=\emptyset$ and m is the (finite) number of elements in $K^{(\alpha-1)}$. As a consequence $C(K)$ is isometric and order isomorphic to $C(\omega^{\alpha-1}m+1)$.

The following extension theorem will be used in this paper:

Borsuk-Dugundji Theorem [15, page 365]. *Let L be a closed nonempty subset of a metric space K. Then there exists a linear extension $\Lambda : C(L)\to C(K)$ such that $\|\Lambda\|=1$.*

2.Fixed point results

Definition 2.1. *Let X be a subspace of a Banach lattice. We say that X is w-weakly orthogonal if for every weakly null sequence $(x_n)\subset X$ there exists some $p\in\mathbb{N}$ such that*

$$\liminf_{n_p\to\infty}...\liminf_{n_1\to\infty}\||x_{n_p}|\wedge...\wedge|x_{n_1}|\|=0.$$

We say that X is w-almost weakly orthogonal if for every weakly null sequence $(x_n)\subset B_X$ where B_X is the closed unit ball in X, there exists some $p\in\mathbb{N}$ such that

$$\liminf_{n_p\to\infty}...\liminf_{n_1\to\infty}\||x_{n_p}|\wedge...\wedge|x_{n_1}|\|<1/p.$$

REMARKS:

(1) It is clear that the w-weakly orthogonality implies the w-almost weakly orthogonality. We will prove in Theorem 3.3. that, in general, these notions are different.

(2) Recall that a Banach lattice is said to be weakly orthogonal if for every weakly null sequence (x_n) the equality $\liminf_n\liminf_m\||x_n|\wedge|x_m|\|=0$ holds. This concept was used by B. Sims and J. Borwein to prove that every weakly orthogonal Banach lattice with Riesz angle less than 2 has the w-FPP. As a consequence, the authors deduce in [3] that the space of all real convergent sequences c (which can be isometrically indentified with the space of continuous functions on the one point compactification of the integer numbers) has the w-FPP. However, we will later check that the class of metric compact sets for which we can deduce the w-FPP in $C(K)$ by means of Borwein-Sims' result is very strict.

In order to prove the main theorem of this section we need the following lemma:

Lemma 2.2. *Let C be a convex weakly compact set with $diam(C)=1$, such that $0\in C$ and C is minimal for a nonexpansive mapping $T:C\to C$. Let $(x_n(k))_{n\in\mathbb{N}}$ be an a.f.p.s. for each $k=1,...,p$. Then for every $\epsilon>0$ there exists a sequence $(w_n)_{n\in\mathbb{N}}\subset C$ such that $\|w_n\|>1-\epsilon$ for every $n\in\mathbb{N}$ and $\limsup_n\|w_n-x_n(k)\|\le\frac{p-1}{p}$ for every $k=1,...,p$.*

Proof. We use a similar argument as in the proof of Theorem 1 in [8]. Fix $\epsilon>0$ and by Goebel-Karlovitz' Lemma there exists some $\delta>0$ such that $\|x\|>1-\epsilon$

if $x \in C$ and $\|Tx - x\| < \delta$. Choose $\gamma < \min\{1, \delta\}$ and for every $n \in \mathbb{N}$ define $S_n : C \to C$ given by

$$S_n(x) = (1-\gamma)Tx + \gamma \frac{x_n(1) + \cdots + x_n(p)}{p}.$$

It is clear that S_n is a contraction which has a (unique) fixed point w_n. It is easy to check that $\|Tw_n - w_n\| \leq \gamma < \delta$ so $\|w_n\| > 1 - \epsilon$ for every $n \in \mathbb{N}$. Moreover, for every $k \in \{1, ..., p\}$ and $n \in \mathbb{N}$ we have

$$\|w_n - x_n(k)\| = \left\|(1-\gamma)Tw_n + \gamma \frac{x_n(1) + \cdots x_n(k)}{p} - x_n(k)\right\| \leq$$

$$(1-\gamma)\|Tw_n - Tx_n(k)\| + (1-\gamma)\|Tx_n(k) - x_n(k)\| + \gamma \|\frac{x_1(1) + \cdots x_n(p)}{p} - x_n(k)\| \leq$$

$$(1-\gamma)\|w_n - x_n(k)\| + (1-\gamma)\|Tx_n(k) - x_n(k)\| + \gamma \frac{p-1}{p}.$$

Thus $\|w_n - x_n(k)\| \leq \frac{1-\gamma}{\gamma}\|Tx_n(k) - x_n(k)\| + \frac{p-1}{p}$. Taking limit when n goes to infinity we obtain that $\limsup_n \|w_n - x_n(k)\| \leq \frac{p-1}{p}$.

Theorem 2.3. *Let X be an w-almost weakly orthogonal closed subspace of $C(K)$ where K is a compact space. Then X has the w-FPP.*

Proof. By contradiction we assume that X fails to have the w-FPP. Thus we can find a convex weakly compact set C of X with $\mathrm{diam}(C) = 1$ and such that C is minimal invariant for a nonexpansive mapping T. Let (x_n) be an approximated fixed point sequence that, by translation, we can consider that is weakly null. Since $0 \in C$ and diam $(C) = 1$ we know that (x_n) is in B_X. Since X is ω-almost weakly orthogonal, there exists some $p \in \mathbb{N}$ (depending on (x_n)) and $c < 1/p$ such that

$$\liminf_{n_p \to \infty} ... \liminf_{n_1 \to \infty} \||x_{n_p}| \wedge ... \wedge |x_{n_1}|\| < c.$$

Fixed $n \in \mathbb{N}$ we can find $x_{n_1}(1), x_{n_1}(2), ..., x_{n_1}(p) \in \{x_n : n \in \mathbb{N}\}$ with $\||x_{n_1}(1)| \wedge ... \wedge |x_{n_1}(p)|\| \leq c + \frac{1}{n}$. Thus, by an induction argument, we can construct subsequences $(x_{n_s}(1))_{s \in \mathbb{N}}$, $(x_{n_s}(2))_{s \in \mathbb{N}}$, ..., $(x_{n_s}(p))_{s \in \mathbb{N}}$ of (x_n), which are also approximated fixed point sequences, and satisfy

$$\lim_{s \to \infty} \||x_{n_s}(1)| \wedge ... \wedge |x_{n_s}(p)|\| \leq c.$$

Taking $\epsilon \in (0, \frac{1}{p} - c)$ we choose a sequence $(w_s)_{s \in \mathbb{N}}$ given by Lemma 2.2.

It is not difficult to check that for every $s \in \mathbb{N}$ we have

$$|w_s| \leq |w_s - x_{n_s}(1)| \vee ... \vee |w_s - x_{n_s}(p)| + |x_{n_s}(1)| \wedge ... \wedge |x_{n_s}(p)|$$

which implies, using the triangular inequality and that X is a space of continuous functions, that

$$\|w_s\| \leq \|w_s - x_{n_s}(1)\| \vee ... \vee \|w_s - x_{n_s}(p)\| + \||x_{n_s}(1)| \wedge ... \wedge |x_{n_s}(p)|\|$$

Taking limit as s goes to infinity we obtain

$$\lim_s \|w_s\| \leq \lim_s \|w_s - x_{n_s}(1)\| \vee ... \vee \lim_s \|w_s - x_{n_s}(p)\| + \lim_{s\to\infty} \||x_{n_s}(1)| \wedge ... \wedge |x_{n_s}(p)|\| \leq$$

$$\leq \frac{p-1}{p} + c < 1 - \epsilon,$$

which contradicts the fact that $\|w_n\| > 1 - \epsilon$ for every $n \in \mathbb{N}$.

3. Some ω-almost weak orthogonal spaces

We will look for properties assuring that a subspace of $C(K)$ is ω-almost weakly orthogonal. We start giving a characterization of the spaces $C(K)$ which are ω-weak orthogonal.

Theorem 3.1. *Let K be a compact metrizable space. Then, the following conditions are all equivalent:*
(1) $C(K)$ is ω- weakly orthogonal
(2) $C(K)$ is ω-almost weakly orthogonal
(3) $K^{(\omega)} = \emptyset$.

Proof. We first prove that (3) ⇒ (1). Assume that $K^{(\omega)} = \emptyset$ and $\{f_n\}$ is a weakly null sequence in $C(K)$. We claim that

$$\liminf_{n_p\to\infty} ... \liminf_{n_1\to\infty} \||f_{n_p}| \wedge ... \wedge |f_{n_1}|\| = 0$$

if $K^{(p)} = \emptyset$. If $K^{(\omega)} = \emptyset$ we know (by compactness) that there exists $p \in \mathbb{N}$ such that $K^{(p)} = \emptyset$ and the result follows. We use an induction argument on p. It is clear that the claim holds if $p = 1$. Assume that the claim holds for $f_n \in C(L)$ where L is any compact set such that $L^{(p-1)} = \emptyset$ and let K be a compact set with $K^{(p)} = \emptyset$. Take (f_n) a weakly null sequence in $C(K)$.

Since $K^{(p-1)}$ is a finite set, we can write $K^{(p-1)} = \{t_1, ..., t_m\}$. Fix a positive integer n_p and choose open neighborhoods V_i of t_i, $i = 1, ..., m$ such that $|f_{n_p}(t) - f_{n_p}(t_i)| < \frac{1}{n_p}$ if $t \in V_i$. Set $L = K \setminus \cup_{i=1}^m V_i$, which is a compact set with $L^{(p-1)} \subset K^{(p-1)} \cap L = \emptyset$. Consider the weakly null sequence $(g_n) \subset C(L)$ defined by $g_n(t) = f_n(t)$ for every $t \in L$. Therefore, according to the induction hypotheses we know that

$$\liminf_{n_{p-1}\to\infty} ... \liminf_{n_1\to\infty} \||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\| = 0.$$

Let $t \in K$. If $t \in L$ we have

$$|f_{n_p}| \wedge |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \leq |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \leq \||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\|.$$

If $t \in K \setminus L = \cup_{i=1}^m V_i$ we also have

$$|f_{n_p}| \wedge |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \leq |f_{n_p}|(t) \leq \max_{i=1,...,m} |f_{n_p}(t_i)| + \frac{1}{n_p}.$$

Taking supremum we have

$$\||f_{n_p}| \wedge ... \wedge |f_{n_1}|\| \leq \max\left\{\||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\|, \max_{i=1,...,m} |f_{n_p}(t_i)| + \frac{1}{n_p}\right\}$$

Finally, taking limits we obtain

$$\liminf_{n_p\to\infty} ... \liminf_{n_1\to\infty} \||f_{n_p}| \wedge ... \wedge |f_{n_2}|\| \leq \lim_{n_p\to\infty} \left(\max_{i=1,\dots,m} |f_{n_p}(t_i)| + \frac{1}{n_p} \right) = 0$$

Next, we prove that (2) $\Rightarrow$ (3). Assume that $C(K)$ is an ω-almost weakly orthogonal Banach lattice. Then K is a scattered set. Indeed, otherwise we obtain a contradiction because $C(K)$ contains $L^1([0,1])$ and has the w-FPP according to Theorem 2.3. So, assume by contradiction that K is scattered and $K^{(\omega)} \neq \emptyset$. In this case, we can assume that $\omega^\omega + 1$ is a closed subset of K. We use the sequence (f_n) constructed in [14] for the space $C(Q)$ where Q is a compact subset of $\omega^\omega + 1$. Indeed, this sequence is a weakly null $\{0,1\}$-valued sequence which satisfies that for any finite sets of integers $\{m_1 < m_2 < ... < m_{m_1+1}\}$ there exists $t \in Q$ such that

$$f_{m_1}(t) = ... = f_{m_{m_i+1}}(t) = 1.$$

Thus, for any $p \in \mathbb{N}$ we have

$$\liminf_{n_1\to\infty} ... \liminf_{n_p\to\infty} \||f_{n_1}| \wedge ... \wedge |f_{n_p}|\| = 1$$

which shows that $C(Q)$ is not ω-almost weakly orthogonal.Using the Borsuk-Dugundji Theorem we obtain that $C(K)$ is not ω-almost weakly orthogonal. Finally, since (1) obviously implies (2) we conclude the proof.

REMARKS. (1) The metrizability assumption of K can be replaced in Theorem 3.1 by a much weaker notion. Indeed, we say that a compact space K has the property (D) if each point $t \in K$ has a neighborhood basis consisting of a decreasing (possibly transfinite) sequence $\{U_\alpha\}_{\alpha<\tau}$ of closed and open sets with the additional property that $(\cap_{\alpha<\beta} U_\alpha) \setminus U_\beta$ contains at most one point for each limit ordinal $\beta < \tau$. Notice that every first countable regular compact space and every ordinal satisfy the property (D). If K is a compact scattered space with the property (D) then K is homeomorphic to the ordinal $\omega^{\alpha-1}m+1$ (where α is the smallest ordinal such that $K^{(\alpha)} = \emptyset$ and m is the finite number of elements in $K^{(\alpha-1)}$) [12, page 34]. Therefore, if K is a compact scattered space with the property (D) and satisfies $K^\omega \neq \emptyset$, we can assume that $\omega^\omega + 1$ is contained in K and the proof of Theorem 3.1 equally holds. Therefore, Theorem 3.1 can be applied, for instance, when K is any compact ordinal number bigger than the first uncountable ordinal ω_1 to prove that $C(K)$ is not ω-almost weakly orthogonal.

(2) It is easy to check that the first part of the proof of Theorem 3.1 (for the special case $p = 2$) proves that $C(K)$ is weakly orthogonal when $K^{(2)} = \emptyset$. We will prove after Theorem 3.3 that this is a characterization of weakly orthogonality for spaces $C(K)$.

Using Theorem 2.3 and Theorem 3.1 we easily derive a result which contains Corollary 3 in [5].

Corollary 3.2. *Let K be a compact set with $K^{(w)} = \emptyset$. Then $C(K)$ has the w-FPP.*

REMARK.
When K is an infinite metric compact space, it is known (see [2]) that $K^{(w)} = \emptyset$ if and only if $C(K)$ is isomorphic to c_0. Thus, we can state the above corollary as follows: *If $C(K)$ is isomorphic to c_0, then $C(K)$ has the w-FPP.* This result is, in some sense, surprising, because an isomorphic property implies the existence of fixed points for nonexpansive mappings which is, clearly, an isometric property. (Recall [4] that $L_1[0,1]$, which fails to have the w-FPP, can be renormed in such a way that the new space has normal structure (which implies the w-FPP) and this new norm is as close (in the Banach-Mazur distance) to the original norm as wanted). Moreover, it was known [3] that any Banach space X isomorphic to c_0 such that the Banach-Mazur distance between X and c_0 is less than 2, has the w-FPP. However, Corollary 3.2 assures the w-FPP for a class of spaces which are isomorphic to c_0 where Banach-Mazur distance is arbitrarily large. Indeed, if $K^{(p)} \neq \emptyset$ and $K^{(p+1)} = \emptyset$, then the Banach-Mazur distance $d(c_0, C(K))$ is greater than p (see [2, Remark 1]).

In the following theorem we construct a space X which shows that the notions of ω-almost weak orthogonality and ω-weak orthogonality are different. Moreover X is a subspace of $C(\omega^\omega + 1)$ which is not contained in any $C(\omega^n + 1)$ for $n \in \mathbb{N}$. This fact let us assure that Theorem 2.3 is a strict improvement of the results in [5] for subspaces of $C(K)$.

Theorem 3.3. *There exists a subspace of $C(\omega^\omega + 1)$ which is ω-almost weakly orthogonal and it is not order isomorphically contained in any space $C(\omega^n + 1)$ for $n \in \mathbb{N}$.*

Proof. Denote $A_p = [\omega^{p-1} + 1, \omega^p] = (\omega^{p-1}, \omega^p + 1)$ which is a clopen subset of $\omega^\omega + 1$. To simplify the notation, we shall write $\langle m_1, ..., m_k\rangle$ to denote the ordinal $\omega^{p-1} m_1 + ... + \omega^{p-k} m_k$. Consider the subset B_p of A_p defined by

$$B_p = \{\alpha = \langle m_1, ..., m_k\rangle : k = 1, 2, ..., p\,,\ 1 < m_1 < m_2 < ... < m_{k-1} < m_k\} \cup \{\omega^p\}.$$

We claim that B_p is a closed subset of A_p. Indeed, assume that $t = \lim_{s\to\infty} t_s$ where $t_s = \langle m_1(s), ..., m_{k(s)}(s)\rangle \in B_p$. There is a subsequence, denoted again t_s such that for any $i = 1, ..., p$ we have either $\lim_s m_i(s) = \infty$ or $m_i(s)$ is a constant, say m_i. If for every $i = 1, ..., p$ we have the second alternative, the result is clear. Otherwise, assume that $j = \min\{i : \lim_s m_i(s) = \infty\}$. Thus, $t = \langle m_1, ..., m_{j-1} + 1\rangle \in B_p$ if $j > 1$, or $t = \omega^p$ if $j = 1$. Hence B_p is a closed subset of A_p and so it is a compact metrizable space.

For any positive integer $n > 1$, we define a sequence $\{h_n\}$ in $C(B_p)$ in the following way: $h_n(\langle m_1, ..., , m_k\rangle) = 1$ if $n \in \{m_1, ..., m_{k-1}, m_k - 1\}$ and $h_n(t) = 0$ otherwise. We claim that h_n is a continuous function. It suffices to prove that $B_{n,p} = \{\langle m_1, ..., m_k\rangle \in B_p : n \in \{m_1, ..., m_{k-1}, m_k - 1\}\}$ is an open and closed subset of B_p. To prove that $B_{n,p}$ is a closed subset of B_p, assume that $t_s = \langle m_1(s), ..., m_{k(s)}(s)\rangle$ is a sequence in $B_{n,p}$ (i.e. $m_1(s) < m_2(s) < ... < m_{k(s)}(s)$ and $n \in \{m_1(s), ..., m_{k(s)-1}, m_{k(s)}(s) - 1\}$) convergent to $t = \langle m_1, ..., m_k\rangle \in B_p$. Without loss of generality, we can assume that there exists $j > 1$ such that $m_i(s) =$

m_i for any s, $i = 1, ...j-1$ and $m_j(s) \to_s \infty$. Thus $n < m_i(s)$ for any $i \geq j$ and s large enough which implies that n belongs to $\{m_1, ..., m_{j-1}\}$ and $t = \langle m_1, ..., m_{j-1} + 1\rangle$ belongs to $B_{n,p}$. On the other hand, to prove that $B_{n,p}$ is an open subset of B_p, assume that $t_s \to_s t = \langle m_1, ..., m_k\rangle \in B_{n,p}$, where $t_s \in B_p$. For s large enough we have $t_s = t$ or $t_s = \langle m_1, ..., m_{k-1}, m_k - 1, m_{k+1}(s), ...\rangle \in B_{n,p}$.

It is easy to check that the sequence $\{h_n\}$ is weakly null. Furthermore for any $t \in B_p$ we have that card $\{n \in \mathbb{N} : h_n(t) \neq 0\} \leq p$ and for any choice of distinct positive integers $n_1, ..., n_p$ greater than 1, there exists $t \in B_p$ such that $\||h_{n_1}| \wedge ... \wedge |h_{n_p}|\| = 1$. Denote

$$K = \cup_{p=1}^{\infty} B_p \cup \{\omega^\omega\}.$$

We claim that K is a closed subset of $\omega^\omega+1$ and so it is a metrizable compact space. Indeed, assume that (t_n) is a sequence in K convergent to $t \in \omega^\omega+1$. If there exists $k \in \mathbb{N}$ such that $t_n \in \cup_{p=1}^k B_p$ for every $n \in \mathbb{N}$ then $t \in K$ because $\cup_{p=1}^k B_p$ is a closed subset of $\omega^\omega+1$. Otherwise, for any $k \in \mathbb{N}$ there exists n_k with $t_{n_k} \notin \cup_{p=1}^k B_p$ which implies that $t = \omega^\omega$. Define $h_n^{(p)} : K \to \{0, 1\}$ by $h_n^{(p)}(t) = h_n(t)$ if $t \in B_p$ and $h_n^{(p)}(t) = 0$ otherwise. Since B_p is a clopen subset of K, we know that $h_n^{(p)}$ is a weakly null sequence in $C(K)$. We define

$$f_n = h_n^{(1)} + \sum_{p=2}^{n} \frac{1}{4p} h_n^{(p)}$$

which is also a weakly null sequence in $C(K)$. Let X be the closed space generated by (f_n). Then, X is a subspace of $C(K)$ which is not ω-weakly orthogonal because for any $p \in \mathbb{N}$ we have

$$\liminf_{n_1} ... \liminf_{n_p} \||f_{n_1}| \wedge ... \wedge |f_{n_p}|\|_{C(K)}$$

$$\geq \frac{1}{4p} \liminf_{n_1} ... \liminf_{n_p} \||h_{n_1}^{(p)}| \wedge ... \wedge |h_{n_p}^{(p)}|\|_{C(K)}$$

$$= \frac{1}{4p} \liminf_{n_1} ... \liminf_{n_p} \||h_{n_1}| \wedge ... \wedge |h_{n_p}\|_{C(B_p)} = \frac{1}{4p}.$$

Thus X is not ω-weakly orthogonal and by Theorem 3.1, X is not order isomorphic to any subspace of $C(\omega^n + 1)$. However X is ω-almost weakly orthogonal. Indeed, let f be a mapping in span (f_n); i.e. $f = \lambda_2 f_2 + ... + \lambda_n f_n$. For $a \in B_p$ we have

$$f(a) = \lambda_{n_1} f_{n_1}(a) + ... + \lambda_{n_q} f_{n_q}(a)$$

for some $1 < n_1 < ... < n_q$, $q \leq p$ because card $(\{n \in \mathbb{N} : f_n(a) \neq 0\}) \leq p$. Thus

$$|f(a)| \leq (|\lambda_{n_1}| + ... + |\lambda_{n_q}|) \frac{1}{4p} \leq \frac{1}{4} \max\{|\lambda_{n_i}| : i = 1, ..., q\}$$

Hence for some $i \in \{1, ..., q\}$ we have $|\lambda_{n_i}| \geq 4|f(a)|$. Since there exists $a_1 \in B_1$ $(a_1 = n_i + 1)$ satisfying $f_{n_i}(a_1) = 1$ and $f_{n_j}(a_1) = 0$ if $j \neq i$, we have $|f(a_1)| = |\lambda_{n_i}| \geq 4|f(a)|$. Thus

$$\|f_{|B_p}\| \leq \frac{1}{4}\|f\|.$$

Assume that (g_n) is a weakly null sequence in B_X. By approximation, we can assume that (g_n) is in span (f_n). Since $C(B_1)$ is order isometrically contained in $C(\omega+1)$ and this space is weakly orthogonal we know that

$$\liminf_{n_1\to\infty}\liminf_{n_2\to\infty}\||g_{n_1|B_1}|\wedge|g_{n_2|B_1}|\|=0.$$

On the other hand

$$\|g_{n_i|K\setminus B_1}\|\leq\frac{1}{4}\quad i=1,2.$$

Thus

$$\liminf_{n_1\to\infty}\liminf_{n_2\to\infty}\||g_{n_1}|\wedge|g_{n_2}|\|\leq\frac{1}{4}<\frac{1}{2}$$

and X is ω-almost weakly orthogonal.

Since any compact metrizable set K such that $K^{(2)}\neq\emptyset$ contains homeomorphically ω^2+1 and so B_2, the proof of Theorem 3.3 and Borsuk-Dugundji Theorem let assure that $C(K)$ is not weakly orthogonal if $K^{(2)}\neq\emptyset$. Thus, using the remark after Theorem 3.1 we can state the following result showing that the class of compact metrizable spaces K such that $C(K)$ is weakly orthogonal is very strict.

Theorem 3.4. *Let K be a metrizable compact space. Then the following properties are equivalent:*
(1) $C(K)$ is a weakly orthogonal Banach lattice.
(2) $K^{(2)}=\emptyset$

REMARK. In [9] the notion of convex orthogonality is defined and used to prove that the space c has the w-FPP. We recall that a Banach space E is said to be orthogonally convex if for every weakly null sequence $\{x_n\}$ in E with $D(x_n)>0$ there exists a positive number λ such that $A_\lambda(x_n)<D(x_n)$ where

$$D(x_n)=\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n-x_m\|,$$

$$A_\lambda(x_n)=\limsup_{m\to\infty}\limsup_{n\to\infty}\max\{\|z\|:z\in M_\lambda(x_n,x_m)\},$$

$$M_\lambda(x,y)=\{z\in E:\max\{\|z-x\|,\|z-y\|\}\leq 1/2\|x-y\|(1+\lambda)\}.$$

Using again the sequence constructed in B_2 in the proof of Theorem 3.3 it is not difficult to check that this notion for a space $C(K)$ is also equivalent to weak orthogonality.

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