New presentations of surface braid groups

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Abstract

In this paper we give new presentations of the braid groups and the pure braid groups of a closed surface. We also give an algorithm to solve the word problem in these groups, using the given presentations.

1 Introduction

Let M be a closed surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of M. A *geometric braid* over M based at $\mathcal{P}$ is an n-tuple $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of paths, $\gamma_i : [0,1] \longrightarrow M$, such that

(1) $\gamma_i(0) = P_i$ for all $i = 1, \ldots, n$,

(2) $\gamma_i(1) \in \mathcal{P}$ for all $i = 1, \ldots, n$,

(3) $\{\gamma_1(t), \ldots, \gamma_n(t)\}$ are n distinct points of M for all $t \in [0,1]$.

For all $i = 1, \ldots, n$, we will call γ_i the *i-th string* of Γ.

Two geometric braids based at $\mathcal{P}$ are said to be *equivalent* if there exists a homotopy which deforms one of them into the other, provided that at any time we always have a geometric braid based at $\mathcal{P}$. We can naturally define the product of two braids as induced by the usual product of paths: for every $i = 1, \ldots, n$, we compose the string of the first braid which ends at P_i, with the i-th string of the second braid. This product is clearly well defined, and it endows the set of equivalence classes of braids with a group structure. This group is called the *braid group on n strings over M based at $\mathcal{P}$*, and is denoted by $B_n(M, \mathcal{P})$. This group does not depend, up to isomorphism, on the choice of $\mathcal{P}$, but only on the number of strings, so we may write $B_n(M)$ instead of $B_n(M, \mathcal{P})$.

A braid $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is said to be *pure* if $\gamma_i(1) = P_i$ for all $i = 1, \ldots, n$, that is, if all its strings are loops. The set of equivalence classes of pure braids forms a subgroup of

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$B_n(M, \mathcal{P})$ called *pure braid group on* n *strings over* M *based at* $\mathcal{P}$, and denoted $PB_n(M, \mathcal{P})$. Again, we may write $PB_n(M)$ since it does not depend on the choice of $\mathcal{P}$. Note that if $n = 1$, then $B_1(M) = PB_1(M) = \pi_1(M)$, the fundamental group of M.

There exists an interpretation of braid groups as fundamental groups of some spaces, called *configuration spaces*. Let F_nM denote the space of n-tuples of distinct points of M, that is, $F_nM = M^n \backslash \Delta$, where

$$\Delta = \{(x_1, \ldots, x_n) \in M^n \ / \ x_i = x_j \text{ for some } i \neq j\}.$$

It is clear that $PB_n(M) \simeq \pi_1(F_nM)$. Now consider the symmetric group on n elements, Σ_n. This group acts naturally on F_nM by permuting coordinates, so we can consider the *configuration space*:

$$\widehat{F}_nM = F_nM/\Sigma_n,$$

which can be seen as the space of embeddings of n points in M. We clearly have $B_n(M) \simeq \pi_1(\widehat{F}_nM)$.

This way to look at braids provides some useful exact sequences, derived from fibrations. The first one comes from the covering space map

$$F_nM \longrightarrow \widehat{F}_nM,$$

with fiber Σ_n. It induces the following exact sequence:

$$1 \longrightarrow PB_n(M) \xrightarrow{e} B_n(M) \xrightarrow{f} \Sigma_n \longrightarrow 1. \tag{1}$$

The homomorphism e is the natural inclusion, and f maps a given braid to the permutation that it induces on $\mathcal{P}$.

Now we consider the Fadell-Neuwirth fibration ([FN]): given $1 \leq m < n$, the map

$$\begin{array}{rccc} p: & F_nM & \longrightarrow & F_mM \\ & (x_1, \ldots, x_n) & \longmapsto & (x_{n-m+1}, \ldots, x_n) \end{array}$$

is a locally trivial fibration with fiber $F_{n-m}(M \backslash \{Q_1, \ldots, Q_m\})$, for any choice of the points $\{Q_1, \ldots, Q_m\}$. Set $\mathcal{P}' = \{P_2, \ldots, P_n\}$, take $m = n - 1$, and consider M different from the sphere and from the projective plane (so $\pi_2(M) = 1$). By the long exact sequence of homotopy groups of this fibration, we obtain

$$1 \longrightarrow \pi_1(M \backslash \mathcal{P}', P_1) \xrightarrow{u} PB_n(M, \mathcal{P}) \xrightarrow{v} PB_{n-1}(M, \mathcal{P}') \longrightarrow 1. \tag{2}$$

If $\gamma \in \pi_1(M \backslash \mathcal{P}', P_1)$, then $u(\gamma) = (\gamma, e_{P_2}, \ldots, e_{P_n})$, where e_{P_i} denotes the constant path on P_i, and, for $\Gamma = (\gamma_1, \ldots, \gamma_n) \in PB_n(M, \mathcal{P})$, one has $v(\Gamma) = (\gamma_2, \ldots, \gamma_n)$.

The goal of this paper is to determine new presentations of the braid groups of closed surfaces different from the sphere and from the projective plane. These presentations are much simpler than those which were known before ([S]). Moreover, the generators and the relations have an easy geometric interpretation. We also show that these presentations furnish an algorithm to solve the word problem for surface braid groups. Notice that similar presentations of the braid groups of the sphere and of the projective plane can be found in [FvB] and in [vB], respectively.

Our work is organized as follows. In Section 2 we state the results, introducing the generators and relations of our new presentations. Then we explain in Section 3 the method followed in the proofs, which we apply throughout Sections 4 and 5, for orientable and non-orientable surfaces, respectively. Finally, we describe in Section 6 an algorithm to solve the word problem in surface braid groups.

I would like to thank Luis Paris for giving me the idea of applying Lemma 3.1 to surface braid groups, and also for its valuable help in the writing of this paper.

2 Statements

The aim of this section is to state our presentations of surface braid groups, defining the generators and showing that the proposed relations are satisfied. We start with the case of an oriented surface different from the sphere.

Let M be a closed, orientable surface of genus $g \geq 1$. The first thing we want is to have a geometrical representation of a braid over M. We represent M as a polygon L of $4g$ sides, identified in the way of Figure 1 (See [M], page 34, ex. 8.9).

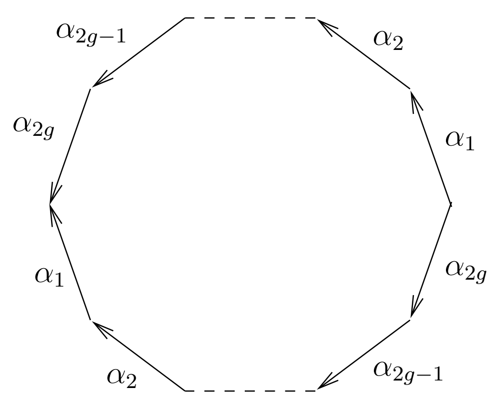

Figure 1: The polygon L representing M.

We could now take the cylinder $L \times I$ ($I = [0,1]$), and represent a braid Γ over M as it is usually done for the open disc, that is, in $L \times \{t\}$ we draw the n points $\gamma_1(t), \ldots, \gamma_n(t)$. But in this case a string could "go through a wall" of the cylinder and appear from the other side. Hence, if we look at the cylinder from the usual viewpoint, it would not be clear which are the "crossed walls" (see the left hand side of Figure 2).

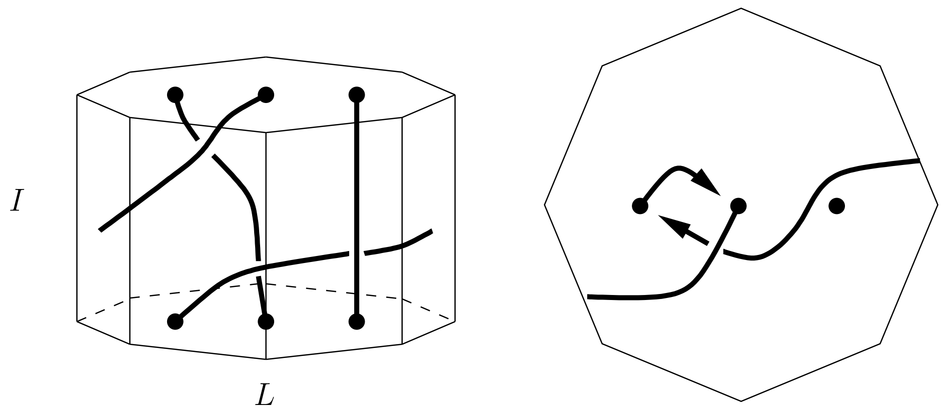

Figure 2: A braid over a surface of genus 2: two different viewpoints.

The solution we propose is to look at the cylinder from above, as in the right hand side of Figure 2. In this way, we get rid of the ambiguity, and moreover we see the strings again as paths in the surface. When two strings cross, we see passing above the one that reaches before the crossing point. Anyway, it is good to keep in mind the idea that we are looking to a cylinder, and to consider the paths as strings: in this manner, one can see more easily when two geometric braids are equivalent.

Now we can define the generators of $B_n(M)$. We choose the n base points along the horizontal diameter of L, as in Figure 3. Now given r, $1 \leq r \leq 2g$ we define the braid a_r as follows: its only nontrivial string is the first one, which goes through the r-th wall, in the way of Figure 3. That is, the first string will go upwards if r is odd, and downwards otherwise.

We also define, for all $i = 1, \ldots, n-1$, the braid σ_i as in Figure 3. Note that $\sigma_1, \ldots, \sigma_{n-1}$ are the classical generators of the braid group B_n of the disc.

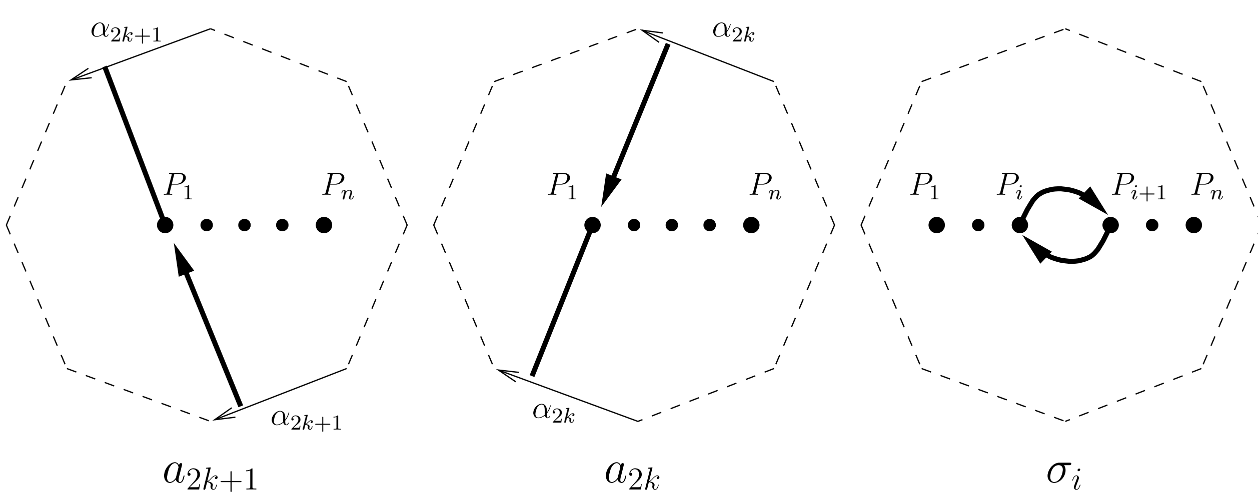

Figure 3: The generators of B_nM.

We will see later that the set $\{a_1, \ldots, a_{2g}, \sigma_1, \ldots, \sigma_{n-1}\}$ is a set of generators of $B_n(M)$. There are two relations between these generators that we can deduce as follows. Consider the interior of L. It is a subsurface D of M homeomorphic to a disc, so clearly every relation satisfied in the braid group $B_n = B_n(D)$ will be satisfied as well in $B_n(M)$ (the same homotopy can be used in both cases). In fact, since $g \geq 1$, it is known that B_n is a subgroup of $B_n(M)$ (see [PR]). Hence, from the classical presentation of B_n, we obtain two relations in $B_n(M)$:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geq 2),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \leq i \leq n-2).$$

Note also that if $i \in \{2, \ldots, n-1\}$ and $r \in \{1, \ldots, 2g\}$, then the non-trivial strings of σ_i and the one of a_r may be taken to be disjoint. This clearly implies that these two braids commute. Hence we have

$$a_r\sigma_i = \sigma_i a_r \qquad (1 \leq r \leq 2g;\ \ i \geq 2).$$

Now, in order to find more relations between the set of generators, we do the following construction. Denote by s_r the first string of a_r, for all $r = 1, \ldots, 2g$, and consider all the paths $s_1, \ldots, s_{2g}$. We can "cut" the polygon L along them, and "glue" the pieces along the paths $\alpha_1, \ldots, \alpha_{2g}$. We obtain another polygon of $4g$ sides which are labeled by $s_1, \ldots, s_{2g}$ (see in Figure 4 the case of a surface of genus 2; the general case is analogous). We will call this new polygon the P_1-*polygon* of M, since all of its vertices are identified to P_1, while L will be called the *initial polygon*. We obtain in this way a new representation of the surface M.

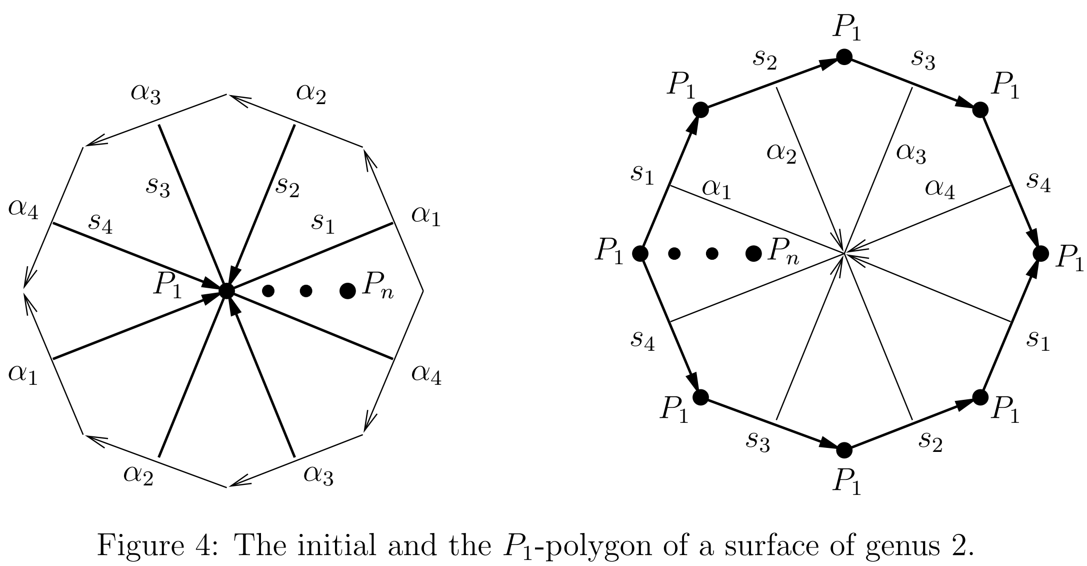

Figure 4: The initial and the P_1-polygon of a surface of genus 2.

We will use the P_1-polygon to show three more relations in $B_n(M)$. For instance, consider the braid $a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1}$. If we look at it in the P_1-polygon, it is clear that it is equivalent to the braid of Figure 5. But this one can be seen into the initial polygon as a braid that does not go through the walls, namely, an element of B_n, the braid group of the disc. Then we can easily show that it is equivalent to the braid $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1$. So we have:

$$a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1} = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1.$$

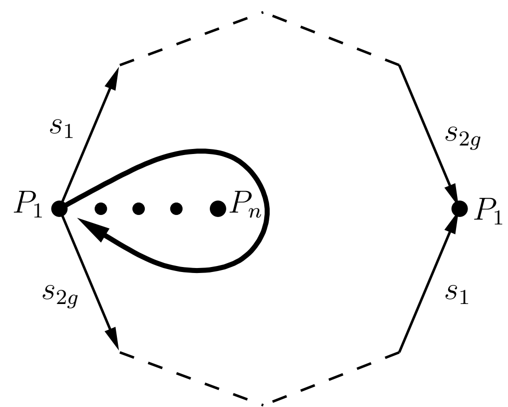

Figure 5: The braid $a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1}$.

Now we define, for each $r = 1, \ldots, 2g$, the braid

$$A_{2,r} = \sigma_1^{-1}\left(a_1 \cdots a_{r-1} a_{r+1}^{-1} \cdots a_{2g}^{-1}\right)\sigma_1^{-1}.$$

We will use the P_1-polygon to see how it looks like. In the left hand side of Figure 6, we can see a braid which is clearly equivalent to $A_{2,r}$ (if r is odd, the other case being analogous). If we "cut" and "glue" to see this braid in the P_1-polygon, we obtain the situation of the right hand side of Figure 6. That is, $A_{2,r}$ can be seen as a braid whose only nontrivial string is the second one, which goes upwards and crosses once the r-th wall s_r. Note that, unlike the case of a_r, $A_{2,r}$ always points upwards in the P_1-polygon, no matter the parity of r.

Therefore we have seen that the braid $A_{2,r}$ can be represented by a geometric braid, whose only non trivial string can be taken disjoint from all the paths s_t, $t \neq r$. This clearly implies that

$$a_t A_{2,r} = A_{2,r} a_t \qquad (1 \leq t, r \leq 2g;\ \ t \neq r).$$

Now we finish our set of relations by considering the commutator of the braids $(a_1 \cdots a_r)$ and $A_{2,r}$, for all $r = 1, \ldots, 2g$. In Figure 7 we can see a sketch of the homotopy which starts with this commutator and deforms it to a braid clearly equivalent to σ_1^2. Therefore, we obtain the relation:

$$(a_1 \cdots a_r)\, A_{2,r} = \sigma_1^2 A_{2,r}\, (a_1 \cdots a_r) \qquad (1 \leq r \leq 2g).$$

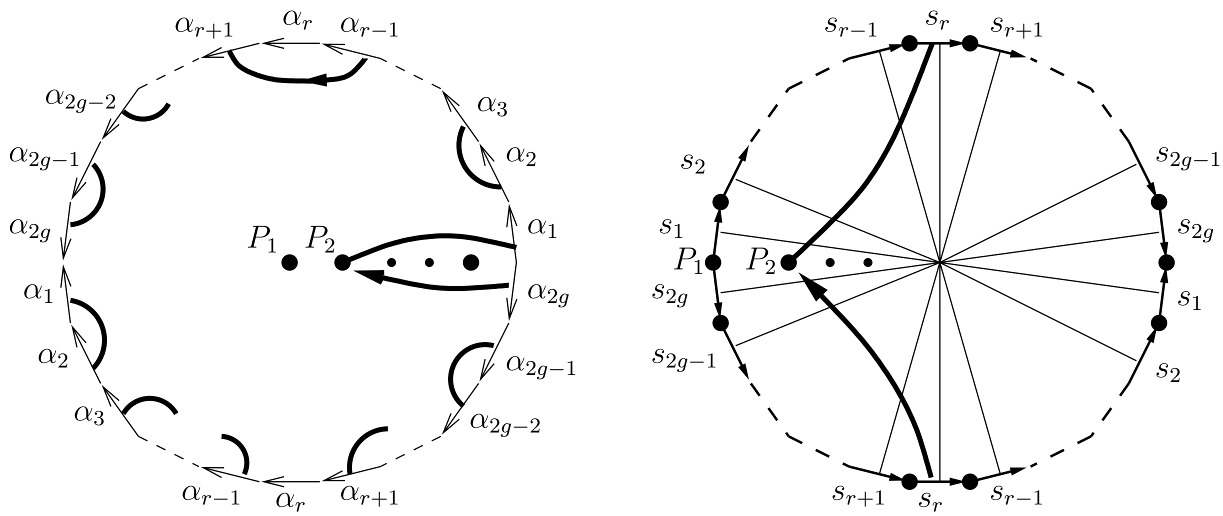

Figure 6: The braid $A_{2,r}$: In the P_1-polygon and in the initial one.

Now we claim that the six relations that we have considered form a complete set of defining relations of $B_n(M)$. In other words, we have the following result.

Theorem 2.1. *If M is a closed, orientable surface of genus $g \geq 1$, then $B_n(M)$ admits the following presentation:*

- *Generators:*

$$\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}.$$

- *Relations:*

(R1) $\sigma_i \sigma_j = \sigma_j \sigma_i \qquad (|i-j| \geq 2)$

(R2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad (1 \leq i \leq n-2)$

(R3) $a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$

(R4) $a_r A_{2,s} = A_{2,s} a_r \qquad (1 \leq r, s \leq 2g; \;\; r \neq s)$

(R5) $(a_1 \cdots a_r) \, A_{2,r} = \sigma_1^2 A_{2,r} \, (a_1 \cdots a_r) \qquad (1 \leq r \leq 2g)$

(R6) $a_r \sigma_i = \sigma_i a_r \qquad (1 \leq r \leq 2g; \;\; i \geq 2)$

where

$$A_{2,r} = \sigma_1^{-1} \left(a_1 \cdots a_{r-1} a_{r+1}^{-1} \cdots a_{2g}^{-1} \right) \sigma_1^{-1}.$$

Now we turn to the non-orientable case. Let M be a closed non-orientable surface of genus $g \geq 2$. To represent a braid in M we will also present the surface as a polygon, this time of $2g$ sides, as in Figure 8, and we make an additional cut: define the path e as in

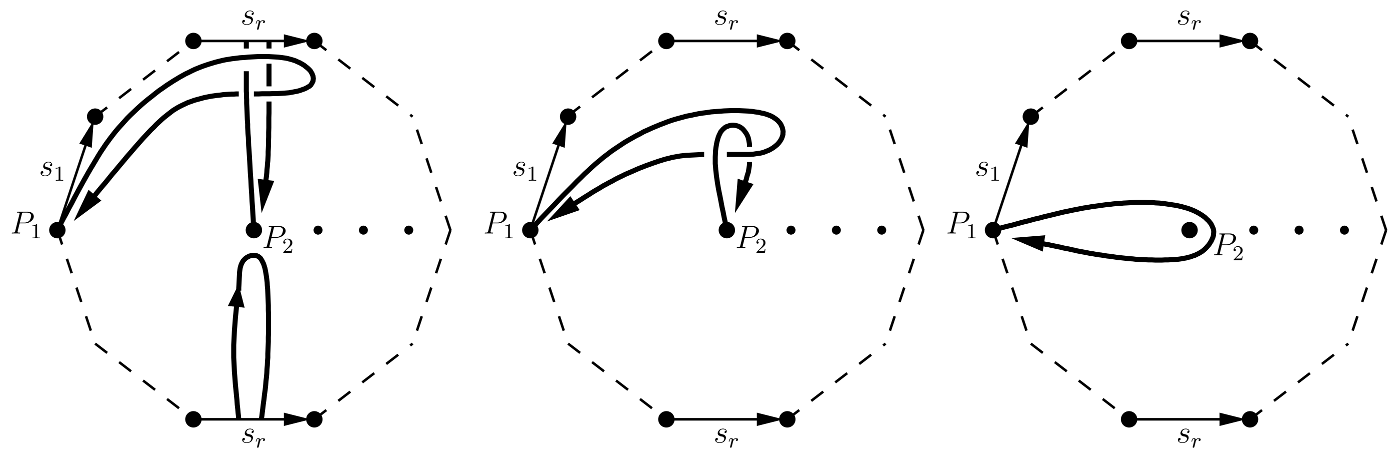

Figure 7: The braid $[a_1 \cdots a_r, A_{2,r}]$.

the left hand side of Figure 8, and cut the polygon along it. We get M represented as in the right hand side of the same figure, where we can also see how we choose the points $P_1, \dots, P_n$.

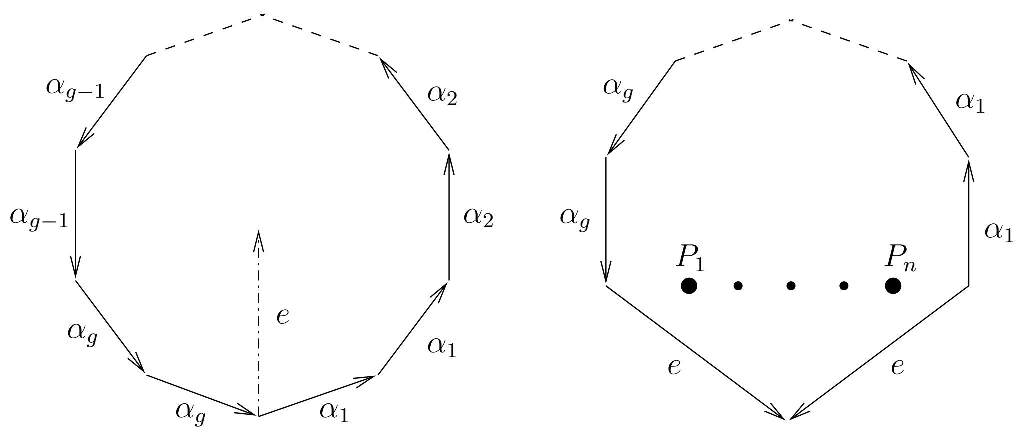

Figure 8: Representation of a non-orientable surface M.

We define now the generators of $B_n(M)$. They will be similar to those of the orientable surface braid groups. For all $i \in \{1, \dots n-1\}$, the braid σ_i will be the same as in the orientable case. For all $r \in \{1, \dots, g\}$, the braid a_r consists on the first string passing through the r-th wall, in the way of Figure 9, while the other strings are trivial paths.

There are six relations in the braid group of M that are analogous to those considered for an orientable surface. They can be shown to hold in the same way as in the orientable case; the only difference is the construction of the P_1-polygon. We denote by $s_1, \dots, s_g$ the first string of $a_1, \dots, a_g$, respectively, and in this case we define another path, e_1, which goes from P_1 to the final point of e (see Figure 9). Then we cut along the paths $s_1, \dots, s_g, e_1$ and glue along $\alpha_1, \dots, \alpha_g, e$. The result is the P_1-*polygon* of M whose sides, reading clockwise,

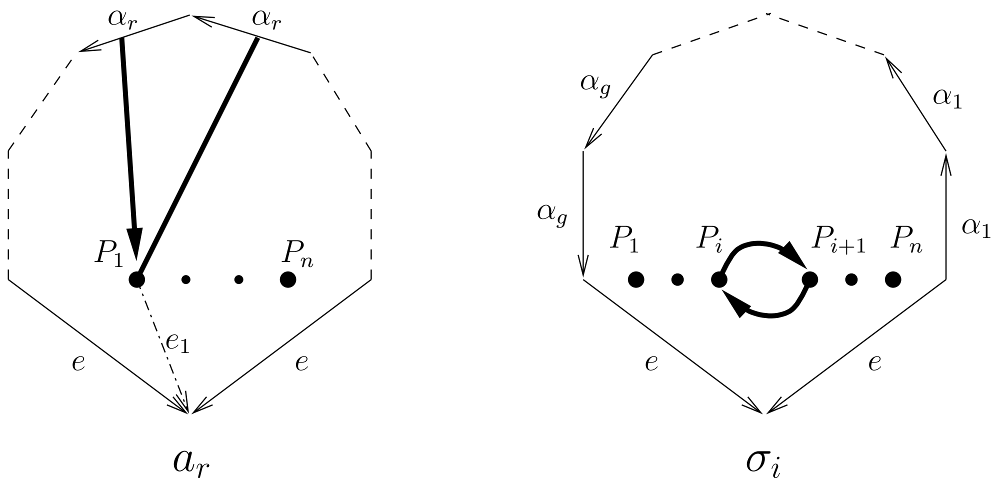

Figure 9: The generators of $B_n(M)$.

are labeled by $s_1, s_1, s_2, s_2, \ldots, s_g, s_g, e_1, e_1^{-1}$.

We claim that the six mentioned relations form a set of defining relations of $B_n(M)$. To be more precise, we claim the following.

Theorem 2.2. *If M is a closed, non-orientable surface of genus $g \geq 2$, then $B_n(M)$ admits the following presentation:*

- *Generators:*

$$\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_g.$$

- *Relations:*

(r1) $\sigma_i \sigma_j = \sigma_j \sigma_i \qquad (|i-j| \geq 2)$

(r2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad (1 \leq i \leq n-2)$

(r3) $a_1^2 \cdots a_g^2 = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$

(r4) $a_r A_{2,s} = A_{2,s} a_r \qquad (1 \leq r, s \leq g;\;\; r \neq s)$

(r5) $\left(a_1^2 \cdots a_{r-1}^2 a_r\right) A_{2,r} = \sigma_1^2 A_{2,r} \left(a_1^2 \cdots a_{r-1}^2 a_r\right) \qquad (1 \leq r \leq g)$

(r6) $a_r \sigma_j = \sigma_j a_r \qquad (1 \leq r \leq g;\;\; j \geq 2)$

where

$$A_{2,r} = \sigma_1^{-1} \left(a_1^2 \cdots a_{r-1}^2 a_r^{-1} a_{r-1}^{-2} \cdots a_1^{-2}\right) \sigma_1.$$

3 A method for finding presentations

Consider an exact sequence of groups

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 1,$$

where we suppose $A \subset B$, and i is the inclusion map. Suppose that A and C have presentations

$$A =< G_A;\ R_A >, \qquad C =< G_C;\ R_C > .$$

For each $y \in G_C$, we choose an element $\tilde{y} \in B$ such that $p(\tilde{y}) = y$, and for each relator $r = y_1 \dots y_m \in R_C$, we write $\tilde{r} = \tilde{y}_1 \dots \tilde{y}_m \in B$. Then it is clear that for every $r \in R_C$, there exists a word f_r over G_A such that $\tilde{r} = f_r$ in B.

On the other hand, for all $x \in G_A$ and $y \in G_C$, there exists a word $g_{x,y}$ over G_A such that $\tilde{y}\, x\, \tilde{y}^{-1} = g_{x,y}$ in B.

Lemma 3.1. *Under the above conditions, B admits the following presentation:*

- *Generators:* $\{G_A\} \cup \{\tilde{y};\ y \in G_C\}$
- *Relations:*
  - **Type 1:** $r_A = 1,$ *for all* $r_A \in R_A$.
  - **Type 2:** $\tilde{r} = f_r,$ *for all* $r \in R_C$.
  - **Type 3:** $\tilde{y}\, x\, \tilde{y}^{-1} = g_{x,y},$ *for all* $x \in G_A$, *and all* $y \in G_C$.

The proof of this lemma is left to the reader. The plan of the proofs of Theorems 2.1 and 2.2 is as follows:

Step 1. We will introduce an abstract group $\overline{PB_n(M)}$ given by its presentation, and define a homomorphism

$$\overline{PB_n(M)} \xrightarrow{\varphi} PB_n(M).$$

Step 2. We will prove by induction on n that φ is an isomorphism, applying Lemma 3.1 to the exact sequence (2):

$$1 \longrightarrow \pi_1(M \backslash \mathcal{P}', P_1) \xrightarrow{u} PB_n(M, \mathcal{P}) \xrightarrow{v} PB_{n-1}(M, \mathcal{P}') \longrightarrow 1.$$

Step 3. We denote by $\overline{B_n(M)}$ the abstract group given by the presentation of Theorem 2.1 if M is oriented, and by the presentation of Theorem 2.2 if M is non-oriented. It is shown in Section 2 that there is a well defined homomorphism

$$\overline{B_n(M)} \xrightarrow{\psi} B_n(M).$$

We will apply Lemma 3.1 to the exact sequence (1):

$$1 \longrightarrow PB_n(M) \xrightarrow{e} B_n(M) \xrightarrow{f} \Sigma_n \longrightarrow 1$$

to show that ψ is actually an isomorphism.

4 The braid groups of an orientable surface

In this section we prove Theorem 2.1 following the procedure given in Section 3. So, throughout the section, M is assumed to be an orientable surface of genus $g \geq 1$.

Step 1. Let $\overline{PB_n(M)}$ be the group given by the following presentation:

Presentation 1

- Generators: $\{a_{i,r};\ \ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{T_{j,k};\ \ 1 \leq j < k \leq n\}$.
- Relations:

(PR1) $a_{n,1}^{-1} a_{n,2}^{-1} \cdots a_{n,2g}^{-1} a_{n,1} a_{n,2} \cdots a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n}$.

(PR2) $a_{i,r} A_{j,s} = A_{j,s} a_{i,r}$ $\qquad (1 \leq i < j \leq n;\ 1 \leq r, s \leq 2g;\ r \neq s)$.

(PR3) $(a_{i,1} \cdots a_{i,r}) A_{j,r} \left(a_{i,r}^{-1} \cdots a_{i,1}^{-1}\right) A_{j,r}^{-1} = T_{i,j} T_{i,j-1}^{-1}$ $\qquad (1 \leq i < j \leq n;\ 1 \leq r \leq 2g)$.

(PR4) $T_{i,j} T_{k,l} = T_{k,l} T_{i,j}$ $\qquad (1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n)$.

(PR5) $T_{k,l} T_{i,j} T_{k,l}^{-1} = T_{i,k-1} T_{i,k}^{-1} T_{i,j} T_{i,l}^{-1} T_{i,k} T_{i,k-1}^{-1} T_{i,l}$ $\qquad (1 \leq i < k \leq j < l \leq n)$.

(PR6) $a_{i,r} T_{j,k} = T_{j,k} a_{i,r}$ $\qquad (1 \leq i < j < k \leq n$ or $1 \leq j < k < i \leq n)$, $(1 \leq r \leq 2g)$.

(PR7) $a_{i,r} \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}\right) = \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}\right) a_{i,r}$
$\qquad (1 \leq j < i \leq k \leq n)$.

(PR8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1} \cdots a_{i,1}^{-1} T_{i,j-1} T_{i,j}^{-1} a_{i,1} \cdots a_{i,2g}\right) a_{j,1} \cdots a_{j,2g} a_{j,1}^{-1} \cdots a_{j,2g}^{-1}$.

Where

$$A_{j,s} = a_{j,1} \cdots a_{j,s-1} a_{j,s+1}^{-1} \cdots a_{j,2g}^{-1}.$$

Later, we will make use of a different presentation of $\overline{PB_n(M)}$, based on the following lemma.

Lemma 4.1. *Let F be the free group freely generated by $\{x_1, \ldots, x_{2g}\}$. Set*

$$X_r = x_1 \cdots x_{r-1} x_{r+1}^{-1} \cdots x_{2g}^{-1}.$$

Then $\{X_1, \ldots X_{2g}\}$ is a free system of generators of F.

Proof: We only need to give the formulae of the change of generators, which are

$$x_k = \left(X_1X_2^{-1}\cdots X_{k-2}X_{k-1}^{-1}\right)\left(X_{k+1}X_{k+2}^{-1}\cdots X_{2g-1}^{-1}X_{2g}\right) \qquad \text{if } k \text{ is odd,}$$

$$x_k^{-1} = \left(X_1X_2^{-1}\cdots X_{k-2}^{-1}X_{k-1}\right)\left(X_{k+1}^{-1}X_{k+2}\cdots X_{2g-1}^{-1}X_{2g}\right) \qquad \text{if } k \text{ is even.}$$

□

As a direct consequence of this lemma, $\overline{PB_n(M)}$ admits the following presentation.

Presentation 2

- Generators: $\{A_{i,r};\ \ 1\leq i\leq n,\ 1\leq r\leq 2g\}\cup\{T_{j,k};\ \ 1\leq j<k\leq n\}$.
- Relations: The same of Presentation 1, where

$$a_{i,k} = \left(A_{i,1}A_{i,2}^{-1}\cdots A_{i,k-2}A_{i,k-1}^{-1}\right)\left(A_{i,k+1}A_{i,k+2}^{-1}\cdots A_{i,2g-1}^{-1}A_{i,2g}\right) \qquad \text{if } k \text{ is odd,}$$

$$a_{i,k}^{-1} = \left(A_{i,1}A_{i,2}^{-1}\cdots A_{i,k-2}^{-1}A_{i,k-1}\right)\left(A_{i,k+1}^{-1}A_{i,k+2}\cdots A_{i,2g-1}^{-1}A_{i,2g}\right) \qquad \text{if } k \text{ is even.}$$

According to Step 1, we must define a homomorphism

$$\overline{PB_n(M)} \xrightarrow{\varphi} PB_n(M).$$

By abuse of notation, we will still denote by $a_{i,r}$ and $T_{i,j}$ the braids that will be the images of $a_{i,r}$ and $T_{i,j}$, respectively, under the homomorphism φ. These braids are defined as follows.

- In $a_{i,r}$, the i-th string goes through the r-th wall, as in Figure 10. This string will go upwards if r is odd, and downwards otherwise. The other strings are trivial. Note that $a_{1,r}=a_r$ for all r.
- In $T_{i,j}$, the i-th string surrounds the points $P_{i+1},\ldots,P_j$, in the way of Figure 10, while the other strings are trivial paths. If $i=j$, we make $T_{i,j}$ to be the trivial braid.

We will denote by $s_{i,r}$ the i-th string of $a_{i,r}$, and by $t_{i,j}$ that of $T_{i,j}$. One can easily show that for any i, the set of paths $\{s_{i,1},\ldots,s_{i,2g}\}$ generates $\pi_1(M)$. Now, for any $i\in\{2,\ldots,n\}$ we can define the P_i-polygon as we defined the P_1-polygon in Section 2: we cut L along $s_{i,1},\ldots,s_{i,2g}$ and glue along $\alpha_1,\ldots,\alpha_{2g}$.

We define, for $2\leq j\leq n$ and $1\leq r\leq 2g$, the braid

$$A_{j,r} = a_{j,1}\cdots a_{j,r-1}a_{j,r+1}^{-1}\cdots a_{j,2g}^{-1}.$$

Like in the representation of $A_{2,r}$ in the P_1-polygon considered in Section 2, $A_{j,r}$ can be represented in the P_i-polygon (for $1\leq i<j$), as the braid of Figure 11, whose only

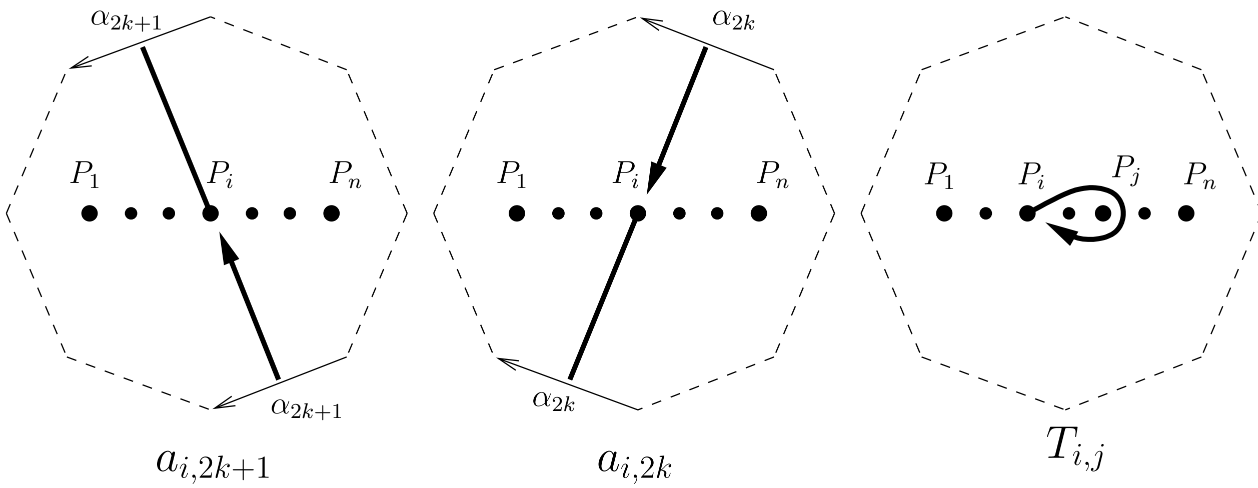

Figure 10: The generators of PB_nM.

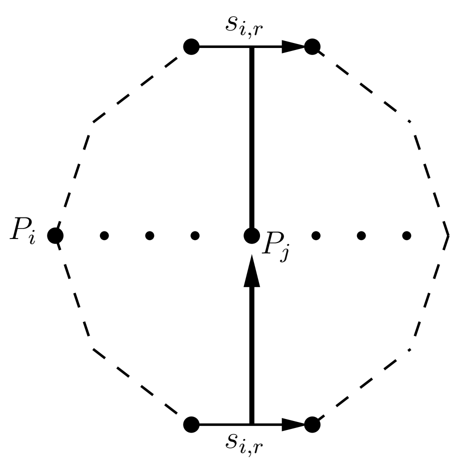

Figure 11: The braid $A_{j,r}$ in the P_i-polygon ($i < j$).

nontrivial string is the j-th one, which goes upwards and crosses once the r-th wall $s_{i,r}$. Note that this representation does not depend on i, but it is only valid when $i < j$.

Now we define φ in the obvious way. In order to show that it is a homomorphism, we must show that the relations of $\overline{PB_n(M)}$ still hold in $PB_n(M)$. Relations (PR4) and (PR5) can be easily checked, since they can be seen in the cylinder as if they were braids over a disc (the interior of L). Relation (PR6) is obvious, once we have drawn the corresponding braids. Relations (PR1), (PR2) and (PR3) are analogous to Relations (R3), (R4) and (R5) of Theorem 2.1, and can be verified in the same way. Relation (PR7) is easily checked in the P_j-polygon, and finally, to verify Relation (PR8) we need all the P_i-polygons for $i = 1, \dots, j$: If $i < j$, it is clear by looking at the P_i-polygon that

$$a_{i,2g}^{-1} \cdots a_{i,1}^{-1} T_{i,j-1} T_{i,j}^{-1} a_{i,1} \cdots a_{i,2g}$$

is equivalent to the braid on the left hand side of Figure 12, thus it is equivalent to that on the right hand side, represented in the P_j-polygon. Then Relation (PR8) is clear, drawing all the factors in the P_j-polygon.

Hence, we have shown that φ is a homomorphism, so this finishes the first step.

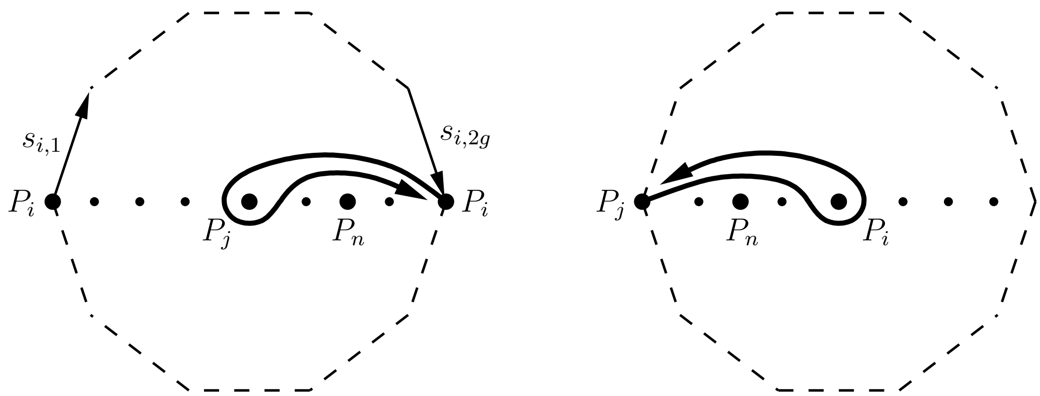

Figure 12: The braid $a_{i,2g}^{-1}\cdots a_{i,1}^{-1}T_{i,j-1}T_{i,j}^{-1}a_{i,1}\cdots a_{i,2g}$.

Step 2. We show by induction on n that φ is an isomorphism. The case $n=1$ is clear, since the presentation of $\overline{PB_1(M)}$ turns to be

$$\overline{PB_1(M)} = \left\langle \{a_{1,1},\ldots,a_{1,2g}\}\ ;\ a_{1,1}^{-1}a_{1,2}^{-1}\cdots a_{1,2g}^{-1}a_{1,1}a_{1,2}\cdots a_{1,2g}=1 \right\rangle,$$

and this is also a presentation of $\pi_1(M) = PB_1(M)$. Moreover, since $n=1$, one has $\varphi(a_{1,i}) = a_{1,i} = s_{1,i}$ for all $i=1,\ldots,2g$, so $\overline{PB_1(M)} \overset{\varphi}{\simeq} PB_1(M)$.

Now suppose $\overline{PB_{n-1}(M)} \overset{\varphi}{\simeq} PB_{n-1}(M)$, and recall the exact sequence (2):

$$1 \longrightarrow \pi_1(M\backslash\mathcal{P}',P_1) \xrightarrow{u} PB_n(M,\mathcal{P}) \xrightarrow{v} PB_{n-1}(M,\mathcal{P}') \longrightarrow 1.$$

In order to apply Lemma 3.1 we need to know presentations of the groups at both hand sides. For the group on the left hand side, we have the presentation

$$\pi_1(M\backslash\mathcal{P}',P_1) = \langle \{s_{1,1},\ldots,s_{1,2g},t_{1,2},\ldots,t_{1,n-1}\};\ \ \phi\ \ \rangle.$$

It will be good for our purposes to include $t_{1,n}$ among the generators, so we add a single relation which can be easily deduced from the pictures (using the P_1-polygon):

$$\pi_1(M\backslash\mathcal{P}',P_1) = \left\langle \{s_{1,1},\ldots,s_{1,2g},t_{1,2},\ldots,t_{1,n}\};\ \ t_{1,n} = s_{1,1}\cdots s_{1,2g}s_{1,1}^{-1}\cdots s_{1,2g}^{-1}\ \ \right\rangle.$$

We know as well, by the induction hypothesis, two presentations of $PB_{n-1}(M)$; we shall use Presentation 2 of $\overline{PB_{n-1}(M)}$. So we can apply Lemma 3.1 to the exact sequence (2).

Note that $v(a_{i,r}) = a_{i-1,r}$, for $i = 2, \ldots, n$, so $v(A_{i,r}) = A_{i-1,r}$, for $i = 2, \ldots, n$. Note also that $v(T_{i,j}) = T_{i-1,j-1}$, where $2 \leq i \leq j \leq n$. So we know pre-images by v of the generators of $PB_{n-1}(M, \mathcal{P}')$.

It is also clear that $u(s_{1,r}) = a_{1,r}$ and $u(t_{1,j}) = T_{1,j}$ for all possible r and j. Hence, we obtain immediately that a set of generators of $PB_n(M, \mathcal{P})$ is

$$\{a_{1,r};\ \ 1 \leq r \leq 2g\} \cup \{A_{i,r};\ \ 2 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{T_{j,k};\ \ 1 \leq j < k \leq n\}.$$

We can apply again Lemma 4.1 to have a new set of generators

$$\{a_{i,r};\ \ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{T_{j,k};\ \ 1 \leq j < k \leq n\}.$$

which is the image by φ of the generating set of $\overline{PB_n(M)}$. In particular, φ is surjective.

Now we prove that φ is an isomorphism by the following procedure.

First, we denote by G_A the set of generators of $\pi_1(M \backslash \mathcal{P}', P_1)$, and by G the set of generators of $\overline{PB_n(M)}$. We consider the unique relation in the presentation of $\pi_1(M \backslash \mathcal{P}', P_1)$, which we can consider via u as a relation in $PB_n(M)$. This will be the unique relation of Type 1 in the presentation of $PB_n(M)$. The procedure starts by showing that this relation holds when it is considered in $\overline{PB_n(M)}$, that is, we have a relation in $\overline{PB_n(M)}$ which maps by φ to the only relation in the presentation of $\pi_1(M \backslash \mathcal{P}', P_1)$.

Next, for each relator r of $PB_{n-1}(M)$, we consider the "canonical" pre-image by v of r, denoted by $\tilde{r}$, in the way of Lemma 3.1. Since $PB_n(M)$ and $\overline{PB_n(M)}$ have the "same" generators (via φ), we can also consider $\tilde{r}$ as a word over G. Now we find a word U over G such that the equality $\tilde{r} = U$ holds in $\overline{PB_n(M)}$, and such that $\varphi(U)$ is a word over G_A. This will give us the relations of Type 2 in the presentation of $PB_n(M)$.

Finally, for each $x \in G_A$ and each generator y of $PB_{n-1}(M)$, we find a word V over G such that the equality $\tilde{y}\, x\, \tilde{y}^{-1} = V$ holds in $\overline{PB_n(M)}$, where $\tilde{y}$ is the canonical pre-image by v of y, and such that $\varphi(V)$ is a word over G_A. This will give us the relations of Type 3 in the presentation of $PB_n(M)$.

In this way, we will have found all relations of Types 1, 2 and 3 of Lemma 3.1 and, therefore, a presentation of $PB_n(M)$, and, at the same time, we will have shown that φ is injective, and consequently, that φ is an isomorphism.

Let us start with the procedure. The unique relation in the presentation of $\pi_1(M \backslash \mathcal{P}', P_1)$ corresponds to Relation (PR8) of $\overline{PB_n(M)}$, for $j = 1$, so it holds in this group.

Relations of Type 2 are easy to find. First, (PR1) can be seen as follows:

$$a_{n,1}^{-1} a_{n,2}^{-1} \cdots a_{n,2g}^{-1} a_{n,1} a_{n,2} \cdots a_{n,2g} \left(\prod_{i=2}^{n-1} T_{i,n-1}^{-1} T_{i,n} \right)^{-1} = T_{1,n-1}^{-1} T_{1,n}.$$

Note that the left hand side maps by φ to $\tilde{r}$, where r is a relator of $PB_{n-1}(M)$ corresponding to (PR1), while the right hand side maps by φ to a word over G_A. Hence, U is equal to the right hand side of the equation, and this yields the first relation of Type 2. The remaining relations of Type 2 are also images by φ of relations in Presentation 1; namely (PR2), (PR3), (PR4) and (PR5) when $i \geq 2$, (PR6) when $i \geq 2$ and $j \geq 2$, and (PR7), (PR8) when $j \geq 2$. For all these relations, the word U is just the trivial word, except for (PR8), for which $U = a_{1,2g}^{-1} \cdots a_{1,1}^{-1} T_{1,j-1} T_{1,j}^{-1} a_{1,1} \cdots a_{1,2g}$.

Finally, we find the relations of Type 3. For $i = 1$, (PR2) becomes

$$A_{j,s} a_{1,r} A_{j,s}^{-1} = a_{1,r} \qquad (r \neq s),$$

so $V = a_{1,r}$. Next, using (PR2), Relation (PR3) turns to be equivalent to

$$A_{j,r} a_{1,r} A_{j,r}^{-1} = \left(a_{1,r-1}^{-1} \cdots a_{1,1}^{-1}\right) T_{1,j-1} T_{1,j}^{-1} \left(a_{1,1} \cdots a_{1,r}\right),$$

so V equals the right hand side of this equation. Relations of the form $T_{k,l} T_{1,j} T_{k,l}^{-1} = V$, where V is a word over G_A, follow from (PR4)-(PR5), while those of the form $T_{k,l} a_{1,r} T_{k,l}^{-1} = V$ follow from (PR6), when $i = 1$. Also, if $j > k$, we obtain from (PR6) the relations $A_{j,r} T_{1,k} A_{j,r}^{-1} = V$, where V is a word over G_A.

The only remaining relations are those of the form $A_{j,r} T_{1,k} A_{j,r}^{-1} = V$, when $1 < j \leq k$, which are deduced as follows: By (PR7), we know that $a_{j,s}$ commutes with the element

$$a_{1,2g}^{-1} \cdots a_{1,1}^{-1} T_{1,k} a_{1,2g} \cdots a_{1,1}$$

for $s = 1, \ldots, 2g$. This implies that $A_{j,r}$ commutes with the same element, so

$$\left(a_{1,2g}^{-1} \cdots a_{1,1}^{-1} T_{1,k} a_{1,2g} \cdots a_{1,1}\right) = A_{j,r} \left(a_{1,2g}^{-1} \cdots a_{1,1}^{-1} T_{1,k} a_{1,2g} \cdots a_{1,1}\right) A_{j,r}^{-1}$$

$$= \left(A_{j,r} a_{1,2g}^{-1} A_{j,r}^{-1}\right) \cdots \left(A_{j,r} a_{1,1}^{-1} A_{j,r}^{-1}\right) \left(A_{j,r} T_{1,k} A_{j,r}^{-1}\right) \left(A_{j,r} a_{1,2g} A_{j,r}^{-1}\right) \cdots \left(A_{j,r} a_{1,1} A_{j,r}^{-1}\right).$$

But using (PR2) and (PR3) we know how to write all the terms in the above product (except the middle one) as words over G_A, so we are done.

Hence, we have shown that $\overline{PB_n(M)} \overset{\varphi}{\simeq} PB_n(M)$ and therefore, we have proved:

Theorem 4.2. *If M is a closed, orientable surface of genus $g \geq 1$, then $PB_n(M)$ admits Presentation 1 (and also Presentation 2) as presentation.*

Step 3. Now we want to find a presentation of $B_n(M)$, for $g \geq 1$. We define then the group $\overline{B_n(M)}$, given by the presentation in Theorem 2.1.

This is the most reduced presentation we have found. But to show its validity we will modify it, obtaining a new one with more generators and relations, but equivalent to the first one.

First, we change our notation, and call $a_{1,r}$ the generators a_r, for $r = 1, \ldots, 2g$. Then we must simply add to the given presentation the generators

- $a_{i,r}$ $\quad i = 2, \ldots, n\,; \quad r = 1, \ldots, 2g,$
- $T_{j,k}$ $\quad 1 \le j < k \le n,$

and the relations

(R7) $a_{j+1,r} = \sigma_j a_{j,r} \sigma_j$ $\qquad (1 \le j \le n-1;\ 1 \le r \le 2g;\ r$ even).

(R8) $a_{j+1,r} = \sigma_j^{-1} a_{j,r} \sigma_j^{-1}$ $\qquad (1 \le j \le n-1;\ 1 \le r \le 2g;\ r$ odd).

(R9) $T_{j,k} = \sigma_j \sigma_{j+1} \cdots \sigma_{k-2} \sigma_{k-1}^2 \sigma_{k-2} \cdots \sigma_j$ $\qquad (1 \le j < k \le n).$

Clearly, both presentations define the same group, that is, $\overline{B_n(M)}$. Now we define $\psi : \overline{B_n(M)} \to B_n(M)$ in the natural way. It is an easy exercise to show, using the same methods as before, that Relations (R7), (R8) and (R9) map to relations in $B_n(M)$. Therefore, ψ is a well defined homomorphism.

Recall now the exact sequence (1):

$$1 \longrightarrow PB_n(M) \xrightarrow{e} B_n(M) \xrightarrow{f} \Sigma_n \longrightarrow 1.$$

We know by Theorem 4.2 a presentation of $PB_n(M)$ (say Presentation 1), and it is also known that a presentation of Σ_n is

- Generators: $\delta_1, \ldots, \delta_{n-1}$.
- Relations:
  - $\delta_i \delta_j = \delta_j \delta_i \qquad |i-j| \ge 2,$
  - $\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1} \qquad 1 \le i \le n-2,$
  - $\delta_i^2 = 1 \qquad 1 \le i \le n-1,$

where δ_i is the permutation $(i,\ i+1)$, for any i.

Now σ_i is clearly a pre-image by f of δ_i, so by Lemma 3.1 $\overline{B_n(M)}$ and $B_n(M)$ have the same generators, and ψ is surjective.

Similarly to what we did in Step 2, we show now that ψ is an isomorphism by the following procedure.

First, we denote by G_A the set of generators of $PB_n(M)$, and by G the set of generators of $\overline{B_n(M)}$. For each relation in the presentation of $PB_n(M)$, we consider it via e as a relation in $B_n(M)$, and we show that it also holds in $\overline{B_n(M)}$.

Next, for each relator r of Σ_n, we consider its canonical pre-image by f, denoted by $\tilde{r}$. Then we find a word U over G such that the equality $\tilde{r} = U$ holds in $\overline{B_n(M)}$, and such that $\psi(U)$ is a word over G_A.

Finally, for each $x \in G_A$ and each generator δ_i of Σ_n, we find a word V over G such that the equality $\sigma_i\, x\, \sigma_i^{-1} = V$ holds in $\overline{B_n(M)}$, and such that $\psi(V)$ is a word over G_A.

This gives us the relations of Types 1, 2, and 3 of Lemma 3.1 and, therefore, a presentation of $B_n(M)$, and, at the same time, this shows that ψ is injective, and, consequently, that ψ is an isomorphism.

Let us then verify in $\overline{B_n(M)}$ the relations of Type 1. In the case of (PR1), we start with (R3):

$$a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1. \tag{3}$$

Using (R7) and (R8), we see that the left hand side of Equation (3) becomes

$$\sigma_1 \cdots \sigma_{n-1} \left(a_{n,1} \cdots a_{n,2g}\right) \sigma_{n-1}^{-1} \cdots \sigma_1^{-2} \cdots \sigma_{n-1}^{-1} \left(a_{n,1}^{-1} \cdots a_{n,2g}^{-1}\right) \sigma_{n-1} \cdots \sigma_1.$$

On the other hand, from (R1), (R2) (braid relations) and (R9), we get

$$T_{i,n-1}^{-1} T_{i,n} \;=\; \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_i \;=\; \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{n-1}^{-1},$$

so

$$\prod_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n} = \sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1}.$$

Therefore, Equation (3) becomes

$$a_{n,1} \cdots a_{n,2g} \left(\prod_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n} \right)^{-1} a_{n,1}^{-1} \cdots a_{n,2g}^{-1} = 1,$$

which is clearly equivalent to (PR1).

We will use in what follows some relations of $\overline{B_n(M)}$ easily deduced from (R1)-(R9). From (R7) and (R8), we get

$$a_{i,r} = \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) a_{1,r} \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \quad \text{if } r \text{ is odd.} \tag{4}$$

$$a_{i,r} = \left(\sigma_{i-1} \cdots \sigma_1\right) a_{1,r} \left(\sigma_1 \cdots \sigma_{i-1}\right) \quad \text{if } r \text{ is even.} \tag{5}$$

$$A_{j,s} = \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1}\right) A_{2,s} \left(\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) = \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1}\right) A_{1,s} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right). \tag{6}$$

Also, from (R1) and (R2), we obtain

$$\sigma_j \left(\sigma_k \sigma_{k-1} \cdots \sigma_i\right) = \left(\sigma_k \sigma_{k-1} \cdots \sigma_i\right) \sigma_{j+1} \qquad (i \le j < k). \tag{7}$$

$$\sigma_j \left(\sigma_k^{-1} \sigma_{k-1}^{-1} \cdots \sigma_i^{-1}\right) = \left(\sigma_k^{-1} \sigma_{k-1}^{-1} \cdots \sigma_i^{-1}\right) \sigma_{j+1} \qquad (i \le j < k). \tag{8}$$

$$\sigma_i \cdots \sigma_{k-1}\sigma_k^2\sigma_{k-1}^{-1} \cdots \sigma_i^{-1} = \sigma_k^{-1} \cdots \sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1} \cdots \sigma_k. \tag{9}$$

Now using (6), (7), (8) and (R6), we see that if $1 \leq k \leq j-2$;

$$\begin{aligned}
\sigma_k A_{j,s} &= \sigma_k \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1}\right) A_{1,s} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right) \\
&= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1}\right) \sigma_{k+1} A_{1,s} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right) \\
&= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1}\right) A_{1,s}\sigma_{k+1} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right) = A_{j,s}\sigma_k.
\end{aligned} \tag{10}$$

In the same way, using (6), (R6) and (R4), we get

$$a_{1,r}A_{j,s} = a_{1,r}\left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1}\right) A_{2,s} \left(\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) = A_{j,s}a_{1,r},$$

if $r \neq s$ and $1 < j$.

Therefore, if $i < j$ and $r \neq s$, by (4) and (5) $a_{i,r}$ is a product of elements which commute with $A_{j,s}$, so we obtain

$$a_{i,r}A_{j,s} = A_{j,s}a_{i,r},$$

which shows that (PR2) holds in $\overline{B_n(M)}$.

Now we verify Relation (PR3). We will do the case when r is odd, the other case being analogous. It is clear which of the known relations of $\overline{B_n(M)}$ we are using at each of the following equalities:

$$\begin{aligned}
&(a_{i,1} \ldots a_{i,r}) A_{j,r} = \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) (a_{1,1} \ldots a_{1,r}) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) A_{j,r} \\
&= \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) (a_{1,1} \ldots a_{1,r}) A_{j,r} \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) (a_{1,1} \ldots a_{1,r}) \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1}\right) A_{2,r} \left(\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1}\right) (a_{1,1} \ldots a_{1,r}) A_{2,r} \left(\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1}\right) \sigma_1^2 A_{2,r} (a_{1,1} \ldots a_{1,r}) \left(\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_i \cdots \sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1} \cdots \sigma_1^{-1}\right) \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1} A_{2,r}\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) (a_{1,1} \ldots a_{1,r}) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_i \cdots \sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1} \cdots \sigma_1^{-1}\right) A_{j,r} (a_{1,1} \ldots a_{1,r}) \left(\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}\right) \\
&= \left(\sigma_i \cdots \sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1} \cdots \sigma_i^{-1}\right) A_{j,r} (a_{i,1} \ldots a_{i,r}) \\
&= T_{i,j}T_{i,j-1}^{-1} A_{j,r} (a_{i,1} \ldots a_{i,r}).
\end{aligned}$$

This shows the case of (PR3). Relations (PR4) and (PR5) are actually relations in the braid group of the disc, so they are a consequence of (R1) and (R2). (PR6) is obtained easily from (R9), (4), (5) and the braid relations (R1) and (R2). So we may turn to (PR7): It is clear that it suffices to show that in $\overline{B_n(M)}$, $A_{i,r}$ commutes with $\left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}\right)$ for $1 \leq j < i \leq k < n$. This is shown as follows (remember that we can already use (PB1)-(PB6)):

$$A_{i,r}\left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}\right)$$

$$= \left(a_{j,2g}^{-1} \cdots a_{j,r+1}^{-1}\right) A_{i,r} \left(a_{j,r}^{-1} \cdots a_{j,1}^{-1}\right) T_{j,k} a_{j,2g} \cdots a_{j,1}$$

$$= \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1}\right) T_{j,i} T_{j,i-1}^{-1} A_{i,r} T_{j,k} a_{j,2g} \cdots a_{j,1}$$

$$= \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1}\right) T_{j,i} T_{j,i-1}^{-1} A_{i,r} \left(\sigma_j \cdots \sigma_{k-1}^2 \cdots \sigma_j\right) a_{j,2g} \cdots a_{j,1}$$

$$= \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1}\right) T_{j,i} T_{j,i-1}^{-1} A_{i,r} \left(\sigma_j \cdots \sigma_{k-1}^2 \cdots \sigma_1\right) a_{1,2g} \cdots a_{1,1} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right)$$

(using (R3))

$$= \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1}\right) T_{j,i} T_{j,i-1}^{-1} A_{i,r} \left(\sigma_j \cdots \sigma_{k-1} \sigma_k^{-1} \cdots \sigma_{n-1}^{-2} \cdots \sigma_1^{-1}\right) a_{1,1} \cdots a_{1,2g} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right)$$

(by (R9) and (10))

$$= a_{j,2g}^{-1} \cdots a_{j,1}^{-1} \left(\sigma_j \cdots \sigma_{i-1}^2 A_{i,r} \sigma_{i-1} \cdots \sigma_{k-1} \sigma_k^{-1} \cdots \sigma_{n-1}^{-2} \cdots \sigma_1^{-1}\right) a_{1,1} \cdots a_{1,2g} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right)$$

$$= a_{j,2g}^{-1} \cdots a_{j,1}^{-1} \left(\sigma_j \cdots \sigma_{i-1} A_{i-1,r} \sigma_i \cdots \sigma_{k-1} \sigma_k^{-1} \cdots \sigma_{n-1}^{-2} \cdots \sigma_1^{-1}\right) a_{1,1} \cdots a_{1,2g} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right)$$

$$= a_{j,2g}^{-1} \cdots a_{j,1}^{-1} \left(\sigma_j \cdots \sigma_{k-1} \sigma_k^{-1} \cdots \sigma_{n-1}^{-2} \cdots \sigma_i^{-1} \sigma_{i-1} \sigma_{i-2}^{-1} \cdots \sigma_1^{-1}\right) A_{i,r} a_{1,1} \cdots a_{1,2g} \sigma_1^{-1} \cdots \sigma_{j-1}^{-1}$$

$$= a_{j,2g}^{-1} \cdots a_{j,1}^{-1} \left(\sigma_j \cdots \sigma_{k-1} \sigma_k^{-1} \cdots \sigma_{n-1}^{-2} \cdots \sigma_1^{-1}\right) a_{1,1} \cdots a_{1,2g} \left(\sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right) A_{i,r}$$

(by (R3) again)

$$= a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} \left(\sigma_{j-1} \cdots \sigma_1 a_{1,2g} \cdots a_{1,1} \sigma_1^{-1} \cdots \sigma_{j-1}^{-1}\right) A_{i,r}$$

$$= \left(a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}\right) A_{i,r}.$$

Finally, Relation (PR8) is verified using some intermediary results. The first is evident: by (R4) we see that in $\overline{B_n(M)}$, $A_{1,2g} A_{2,2g} = A_{2,2g} A_{1,2g}$, and moreover this braid commutes with σ_1, since

$$A_{1,2g}A_{2,2g}\sigma_1 = A_{1,2g}\sigma_1^{-1}A_{1,2g} = \sigma_1 A_{2,2g}A_{1,2g} = \sigma_1 A_{1,2g}A_{2,2g}.$$

Analogously, one shows that $(a_{1,2g}a_{2,2g})$ commutes with σ_1. The following result is a consequence of the previous ones and of (R5):

$$a_{1,2g}A_{2,2g}a_{1,2g}^{-1} = \left(a_{1,2g-1}^{-1}\cdots a_{1,1}^{-1}\right)\sigma_1^2 A_{2,2g}\left(a_{1,1}\cdots a_{1,2g-1}\right)$$

$$= A_{1,2g}^{-1}\sigma_1^2 A_{2,2g}A_{1,2g} = A_{1,2g}^{-1}A_{2,2g}A_{1,2g}\sigma_1^2 = A_{2,2g}\sigma_1^2,$$

so we obtain

$$a_{1,2g}^{-1}A_{2,2g} = A_{2,2g}a_{1,2g}^{-1}\sigma_1^{-2}. \tag{11}$$

Now we consider the factors in the right hand side of (PR8), and we see that

$$\left(a_{i,2g}^{-1}\cdots a_{i,1}^{-1}\right)T_{i,j-1}T_{i,j}^{-1}\left(a_{i,1}\cdots a_{i,2g}\right)$$

$$= \left(a_{i,2g}^{-1}\cdots a_{i,1}^{-1}\right)\sigma_i\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_i^{-1}\left(a_{i,1}\cdots a_{i,2g}\right)$$

$$= \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\left(a_{1,2g}^{-1}\cdots a_{1,1}^{-1}\right)\sigma_1\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\left(a_{1,1}\cdots a_{1,2g}\right)\sigma_1\cdots\sigma_{i-1}$$

(by (9))

$$= \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\left(a_{1,2g}^{-1}\cdots a_{1,1}^{-1}\right)\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-2}\sigma_2\cdots\sigma_{j-1}\left(a_{1,1}\cdots a_{1,2g}\right)\sigma_1\cdots\sigma_{i-1}$$

$$= \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\left(a_{1,2g}^{-1}\cdots a_{1,1}^{-1}\right)\sigma_1^{-2}\left(a_{1,1}\cdots a_{1,2g}\right)\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}$$

$$= \left(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\right)a_{1,2g}^{-1}A_{1,2g}^{-1}\sigma_1^{-2}A_{1,2g}a_{1,2g}\left(\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}\right)$$

(since $(A_{1,2g}A_{2,2g})$ commutes with σ_1)

$$= \left(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\right)a_{1,2g}^{-1}A_{2,2g}\sigma_1^{-2}A_{2,2g}^{-1}a_{1,2g}\left(\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}\right)$$

(by (11))

$$= \left(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\right)A_{2,2g}a_{1,2g}^{-1}\sigma_1^{-2}a_{1,2g}A_{2,2g}^{-1}\left(\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}\right)$$

(since, $(a_{1,2g}a_{2,2g})$ commutes with σ_1)

$$= \left(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\sigma_{j-1}^{-1}\cdots\sigma_2^{-1}\right)A_{2,2g}a_{2,2g}\sigma_1^{-2}a_{2,2g}^{-1}A_{2,2g}^{-1}\left(\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}\right)$$

$$= \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) A_{j,2g} a_{j,2g} \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1} \sigma_1^{-2} \sigma_2 \cdots \sigma_{j-1}\right) a_{j,2g}^{-1} A_{j,2g}^{-1} \left(\sigma_1 \cdots \sigma_{i-1}\right)$$

(by (10))

$$= A_{j,2g} a_{j,2g} \left(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\right) \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1} \sigma_1^{-2} \sigma_2 \cdots \sigma_{j-1}\right) \left(\sigma_1 \cdots \sigma_{i-1}\right) a_{j,2g}^{-1} A_{j,2g}^{-1}$$

(by (9))

$$= a_{j,1} \cdots a_{j,2g} \left(\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-2} \sigma_{i+1} \cdots \sigma_{j-1}\right) a_{j,2g}^{-1} \cdots a_{j,1}^{-1}.$$

And this clearly yields (PR8):

$$\left(\prod_{i=1}^{j-1} a_{i,2g}^{-1} \cdots a_{i,1}^{-1} T_{i,j-1} T_{i,j}^{-1} a_{i,1} \cdots a_{i,2g}\right) a_{j,1} \cdots a_{j,2g} a_{j,1}^{-1} \cdots a_{j,2g}^{-1}$$

$$= \left(\prod_{i=1}^{j-1} a_{j,1} \cdots a_{j,2g} \left(\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-2} \sigma_{i+1} \cdots \sigma_{j-1}\right) a_{j,2g}^{-1} \cdots a_{j,1}^{-1}\right) a_{j,1} \cdots a_{j,2g} a_{j,1}^{-1} \cdots a_{j,2g}^{-1}$$

$$= a_{j,1} \cdots a_{j,2g} \left(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \cdots \sigma_{j-1}^{-1}\right) a_{j,1}^{-1} \cdots a_{j,2g}^{-1}$$

$$= \left(\sigma_j \cdots \sigma_{n-1}\right) a_{n,1} \cdots a_{n,2g} \left(\sigma_{n-1}^{-1} \cdots \sigma_1^{-2} \cdots \sigma_{n-1}^{-1}\right) a_{n,1}^{-1} \cdots a_{n,2g}^{-1} \left(\sigma_{n-1} \cdots \sigma_j\right)$$

(by (R9) and (PR1))

$$= \left(\sigma_j \cdots \sigma_{n-1}\right) \left(\sigma_{n-1} \cdots \sigma_j\right) = T_{j,n}.$$

We have thus finished with relations of Type 1.

Consider now those of Type 2. For each relator in the presentation of Σ_n, we must find the word U mentioned above.

The first relator is $\delta_i \delta_j \delta_i^{-1} \delta_j^{-1}$, when $|i-j| \geq 2$ which, by (R1), yields in $\overline{B_n(M)}$ the relation

$$\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1 \qquad (|i-j| \geq 2).$$

Clearly, U is the trivial word.

The second relator, $\delta_i \delta_{i+1} \delta_i \delta_{i+1}^{-1} \delta_i^{-1} \delta_{i+1}^{-1}$ gives, by (R2),

$$\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1 \qquad (i = 1, \cdots, n-2),$$

so in this case U is also the trivial word.

Finally, by the third relator δ_i^2, we obtain, using (R9),

$$\sigma_i^2 = T_{i,i+1} \qquad (i = 1, \cdots, n-1),$$

hence $U = T_{i,i+1}$.

So we have obtained the relations in $\overline{B_n(M)}$ mapped by ψ to the relations of Type 2.

We finish the proof of Theorem 2.1 obtaining the relations of Type 3. They are very easy to deduce, using (10), (R1), (R2), (R7), (R8) and (R9). They are the following:

$$\begin{array}{ll}
\sigma_i a_{j,r} \sigma_i^{-1} = a_{j,r} & (j \neq i, i+1), \\
\sigma_i a_{i,r} \sigma_i^{-1} = a_{i+1,r} T_{i,i+1}^{-1} & \text{if } r \text{ is even}, \\
\sigma_i a_{i,r} \sigma_i^{-1} = T_{i,i+1} a_{i+1,r} & \text{if } r \text{ is odd}, \\
\sigma_i a_{i+1,r} \sigma_i^{-1} = T_{i,i+1} a_{i,r} & \text{if } r \text{ is even}, \\
\sigma_i a_{i+1,r} \sigma_i^{-1} = a_{i,r} T_{i,i+1}^{-1} & \text{if } r \text{ is odd}, \\
\sigma_i T_{j,k} \sigma_i^{-1} = T_{j,k} & (i \neq j-1, j, k), \\
\sigma_i T_{i+1,k} \sigma_i^{-1} = T_{i,k} T_{i,i+1}^{-1}, & \\
\sigma_i T_{i,k} \sigma_i^{-1} = T_{i,i+1} T_{i+1,k}, & \\
\sigma_i T_{j,i} \sigma_i^{-1} = T_{j,i-1} T_{j,i}^{-1} T_{j,i+1}. &
\end{array}$$

5 The braid groups of a non-orientable surface

This section is devoted to prove Theorem 2.2, using the same method as before. Thus, let M be a closed, non-orientable surface of genus $g \geq 2$.

Step 1. Denote by $\overline{PB_n(M)}$ the group defined by the following presentation.

Presentation 3

- Generators: $\{a_{i,r};\ \ 1 \leq i \leq n,\ 1 \leq r \leq g\} \cup \{T_{j,k};\ \ 1 \leq j < k \leq n\}$.
- Relations:

(Pr1) $a_{n,1}^2 \cdots a_{n,g}^2 = \prod_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n}$.

(Pr2) $a_{i,r} A_{j,s} = A_{j,s} a_{i,r}$ $\quad (1 \leq i < j \leq n;\ 1 \leq r, s \leq g;\ r \neq s)$.

(Pr3) $\left(a_{i,1}^2 \cdots a_{i,r-1}^2 a_{i,r}\right) A_{j,r} \left(a_{i,r}^{-1} a_{i,r-1}^{-2} \cdots a_{i,1}^{-2}\right) A_{j,r}^{-1} = T_{i,j} T_{i,j-1}^{-1}$ $\quad (1 \leq i < j \leq n;\ 1 \leq r \leq g)$.

(Pr4) $T_{i,j} T_{k,l} = T_{k,l} T_{i,j}$ $\quad (1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n)$.

(Pr5) $T_{k,l} T_{i,j} T_{k,l}^{-1} = T_{i,k-1} T_{i,k}^{-1} T_{i,j} T_{i,l}^{-1} T_{i,k} T_{i,k-1}^{-1} T_{i,l}$ $\quad (1 \leq i < k \leq j < l \leq n)$.

(Pr6) $a_{i,r}T_{j,k} = T_{j,k}a_{i,r}$ $\quad (1 \le i < j < k \le n \text{ or } 1 \le j < k < i \le n), (1 \le r \le g).$

(Pr7) $a_{i,r}\left(a_{j,g}^{-2}\cdots a_{j,1}^{-2}T_{j,k}\right) = \left(a_{j,g}^{-2}\cdots a_{j,1}^{-2}T_{j,k}\right)a_{i,r}$ $\quad (1 \le j < i \le k \le n).$

(Pr8) $T_{j,n} = a_{j,1}^2\cdots a_{j,g}^2\left(\prod_{i=1}^{j-1} T_{j-i,j}^{-1}T_{j-i,j-1}\right).$

Where

$$A_{j,r} = a_{j,1}^2\cdots a_{j,r-1}^2 a_{j,r}^{-1} a_{j,r-1}^{-2}\cdots a_{j,1}^{-2}.$$

We shall need, as in the orientable case, another presentation of $\overline{PB_n(M)}$, which is the following one.

Presentation 4

- Generators: $\{A_{i,r};\ 1 \le i \le n,\ 1 \le r \le g\} \cup \{T_{j,k};\ 1 \le j < k \le n\}$.
- Relations: the same as in Presentation 3, where

$$a_{i,r} = A_{i,1}^2\cdots A_{i,r-1}^2 A_{i,r}^{-1} A_{i,r-1}^{-2} A_{i,1}^{-2}.$$

It is clear that Presentation 3 and Presentation 4 are equivalent, in the same way as they were Presentation 1 and Presentation 2. We must now define the homomorphism

$$\overline{PB_n(M)} \xrightarrow{\varphi} PB_n(M),$$

by giving the image of the generators. They will be similar to those of the orientable surface. For all i and j such that $1 \le i \le j \le n$, the braid $T_{i,j}$ will be the same as in Section 4. For all i, r, such that $1 \le i \le n$ and $1 \le r \le g$, the braid $a_{i,r}$ will represent the i-th string passing through the r-th wall, in the way of Figure 13. We define as well the path e_i $(i = 1, \ldots, n)$, which goes from P_i to the final point of e, as in Figure 13.

Given $i \in \{1, \ldots, n\}$, denote by $s_{i,r}$ the i-th string of $a_{i,r}$. We can proceed as we did for the P_1-polygon in Section 2 to get the P_i-polygon: Cut along the paths e_i and $s_{i,1}, \ldots, s_{i,g}$, and glue along e and $\alpha_1, \ldots, \alpha_g$. The resulting P_i-polygon is labeled by the paths

$$s_{i,1}, s_{i,1}, s_{i,2}, s_{i,2}, \ldots, s_{i,g}, s_{i,g}, e_i, e_i^{-1},$$

reading clockwise. Now we can repeat the process of Section 2 to see that for $1 \le i < j$, the braid

$$A_{j,r} = a_{j,1}^2\cdots a_{j,r-1}^2 a_{j,r}^{-1} a_{j,r-1}^{-2}\cdots a_{j,1}^{-2}$$

can be represented in the P_i-polygon in the way of Figure 14.

The remainder of Step 1, that is to show that φ is a well defined homomorphism, is analogous to the orientable case. That is, Relations (Pr4), (Pr5) and (Pr6) are obvious; Relations (Pr1), (Pr2) and (Pr3) are analogous to Relations (r3), (r4) and (r5) of Theorem 2.2; and we can easily check Relations (Pr7) and (Pr8) in the P_j-polygon.

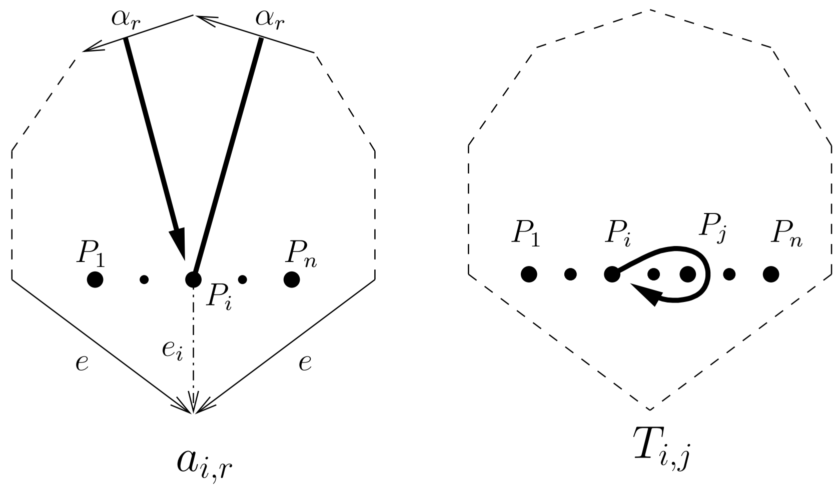

Figure 13: The generators of $PB_n(M)$.

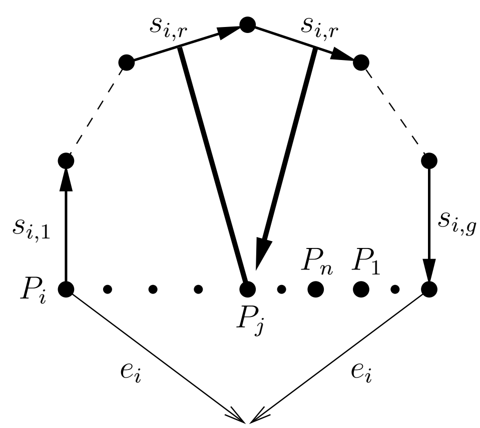

Figure 14: The braid $A_{j,r}$ in the P_i-polygon ($i < j$).

Step 2. This step parallels, up to evident substitutions, the corresponding one in Section 4, showing the following theorem:

Theorem 5.1. *If M is a closed, non-orientable surface of genus $g \geq 2$, then $PB_n(M)$ admits Presentation* 3 *(and also Presentation* 4*) as presentation.*

Step 3. Denote by $\overline{B_n(M)}$ the group defined by the presentation of Theorem 2.2. Call $a_{1,r}$ the elements a_r for $r = 1, \dots, g$, and then add the generators

- $a_{i,r}$ $\quad i = 2, \dots, n\,; \quad r = 1, \dots, g,$
- $T_{j,k}$ $\quad 1 \leq j < k \leq n,$

and the relations

(r7) $a_{j+1,r} = \sigma_j^{-1} a_{j,r} \sigma_j$ $\qquad (1 \leq j \leq n-1;\ 1 \leq r \leq g).$

(r8) $T_{j,k} = \sigma_j \sigma_{j+1} \cdots \sigma_{k-2} \sigma_{k-1}^2 \sigma_{k-2} \cdots \sigma_j$ $\qquad (1 \leq j < k \leq n).$

This provides an equivalent presentation of $\overline{B_n(M)}$, and the naturally defined function

$$\psi : \overline{B_n(M)} \longrightarrow B_n(M),$$

which is easily proved to be a well defined homomorphism.

Now it remains to apply Lemma 3.1 to the exact sequence (1), and then to find relations in $\overline{B_n(M)}$ mapping by ψ to those of Types 1, 2 and 3, as we did in Section 4. For those of Type 1 corresponding to (Pr1)-(Pr6), we can use almost the same calculations that in the previous section.

The relation mapping to (Pr7) is obtained as follows:

$$a_{1,r} \left(a_{j,g}^{-2} \cdots a_{j,1}^{-2} T_{j,k} \right)$$

$$= a_{i,r} \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} a_{1,g}^{-2} \cdots a_{1,1}^{-2} \sigma_1 \cdots \sigma_{k-1} \right) \sigma_{k-1} \cdots \sigma_j$$

(by (r3))

$$= a_{i,r} \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \right) \left(\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \right) \left(\sigma_{n-1}^{-1} \cdots \sigma_k^{-1} \right) \left(\sigma_{k-1} \cdots \sigma_j \right)$$

$$= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \right) \left(\sigma_1^{-1} \cdots \sigma_{i-2}^{-1} \right) a_{i,r} \sigma_{i-1}^{-1} \left(\sigma_i^{-1} \cdots \sigma_{n-1}^{-1} \right) \left(\sigma_{n-1}^{-1} \cdots \sigma_k^{-1} \right) \left(\sigma_{k-1} \cdots \sigma_j \right)$$

$$= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \right) \left(\sigma_1^{-1} \cdots \sigma_{i-2}^{-1} \right) \sigma_{i-1}^{-1} a_{i-1,r} \left(\sigma_i^{-1} \cdots \sigma_{n-1}^{-1} \right) \left(\sigma_{n-1}^{-1} \cdots \sigma_k^{-1} \right) \left(\sigma_{k-1} \cdots \sigma_j \right)$$

$$= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \right) \left(\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \right) \left(\sigma_{n-1}^{-1} \cdots \sigma_k^{-1} \right) \left(\sigma_{k-1} \cdots \sigma_i \right) a_{i-1,r} \sigma_{i-1} \sigma_{i-2} \cdots \sigma_j$$

$$= \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \right) \left(\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \right) \left(\sigma_{n-1}^{-1} \cdots \sigma_k^{-1} \right) \left(\sigma_{k-1} \cdots \sigma_j \right) a_{i,r}$$

$$= \left(a_{j,g}^{-2} \cdots a_{j,1}^{-2} T_{j,k} \right) a_{i,r},$$

and the relation mapping to (Pr8), comes from the following calculation:

$$a_{j,1}^2 \cdots a_{j,g}^2 \left(\prod_{i=1}^{j-1} T_{j-i,j}^{-1} T_{j-i,j-1} \right)$$

(by (9))

$$= a_{j,1}^2 \cdots a_{j,g}^2 \left(\sigma_{j-1}^{-1} \cdots \sigma_1^{-2} \cdots \sigma_{j-1}^{-1} \right)$$

$$= \sigma_{j-1}^{-1} \cdots \sigma_1^{-1} a_{1,1}^2 \cdots a_{1,g}^2 \sigma_1^{-1} \cdots \sigma_{j-1}^{-1}$$

(by (r3))

$$= \sigma_j \cdots \sigma_{n-1}^2 \cdots \sigma_j = T_{j,n}.$$

Finally, the relations mapping by ψ to those of Type 2, are identical to those for the orientable surfaces, and relations of Type 3 are equally easy to deduce. Therefore, we have finished the proof of Theorem 2.2.

6 The word problem

In this section we explain an algorithm to solve the word problem in the braid group of a surface, using our new presentations. We shall only explain the orientable case, remarking that the same method can be used in the non-orientable one.

Let ω be a word over the generators of $B_n(M)$, that is, over $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_{2g}$ and their inverses. The algorithm we propose shall give as output a word

$$\omega' = \omega_1 \cdots \omega_n s$$

equivalent to w, where ω_i will be a word over $\{a_{i,1}, \dots, a_{i,2g}, T_{i,i+1}, \dots, T_{i,n-1}\}$, and s will be a word over $\{\sigma_1, \dots, \sigma_{n-1}\}$ representing the permutation which ω induces on the strings. Moreover, we will show that this expression is unique, thus $\omega = 1$ if and only if ω' is the trivial word. This algorithm is analogous to the classical braid combing in the braid group of the disc.

First we need some previous results. Consider the homomorphism f in the exact sequence (1); it sends ω to its corresponding permutation. Now for any element of Σ_n, we can take a normal form as a word over $\{\delta_1, \dots, \delta_{n-1}\}$. For instance, we can use the normal forms in [H], where any element of Σ_n is written as a product

$$t_{1,k_1} t_{2,k_2} \cdots t_{n-1,k_{n-1}},$$

where $t_{m,0} = 1$ and $t_{m,k} = \delta_m \delta_{m-1} \cdots \delta_{m-k+1}$. If we replace in this normal form δ_i by σ_i for $i = 1, \dots, n-1$, we obtain a map $g : \Sigma_n \to W$, where W is the set of words over $\{\sigma_1, \dots, \sigma_{n-1}\}$ and their inverses.

Consider then the composition $\varepsilon = g \circ f$:

$$\varepsilon : \ B_n(M) \xrightarrow{f} \Sigma_n \xrightarrow{g} W.$$

This map sends any braid to a braid word inducing the same permutation on the strings. Moreover, the image of ε is finite, since so is Σ_n.

Now in order to apply the algorithm, we need to make a "dictionary", in the following way: for all braid words p in the image of ε, consider all braids of the form

$$p\ a_r^{\pm 1}\ p^{-1}, \qquad\qquad p\ \sigma_i^{\pm 1}\ \varepsilon(p\sigma_i)^{-1}.$$

Clearly, there is only a finite number of them, and they are all pure braids. It is not difficult to write these braids as words over $\{a_{i,r}, T_{j,k}\}$ using the relations of the given presentation of $B_n(M)$. These are the first words in our dictionary.

Now for $j = 1, \ldots, n$, we define the following sets:

$$W_j = \{a_{i,r}^{\pm 1};\ \ i = 1, \ldots, j,\ r = 1, \ldots, 2g\} \cup \{T_{i,k}^{\pm 1};\ \ i = 1, \ldots, j,\ k = i+1, \ldots, n-1\},$$

$$V_j = \{A_{j,r}^{\pm 1};\ \ r = 1, \ldots, 2g\} \cup \{T_{j,k}^{\pm 1};\ \ k = j+1, \ldots, n-1\}.$$

For each $x \in W_i$ and each $y \in V_j$, $i < j$, we want to add to our dictionary an expression of the form

$$y\ x\ y^{-1} = Z,$$

where Z is a word over W_i. If y is a positive letter, this expression is just a relation of Type 3. It may happen that in Z there is a letter of the form $T_{l,n}^{\pm 1}$ $(l \leq i)$, but we can replace it by a word over W_i using (PR8). If y is a negative letter, we can deduce the above expression in the same way that we did for relations of Type 3. So in any case, we can add all of them to our dictionary.

We still need one more result: Denote by $S_{n,r}$ the n-th string of $A_{n,r}$. Since $s_{n,1}, \ldots, s_{n,2g}$ generates $\pi_1(M, P_n)$, Lemma 4.1 clearly implies that $\{S_{n,1}, \ldots, S_{n,2g}\}$ is another set of generators. Moreover, applying the formulae of Lemma 4.1, one has

$$s_{n,1}^{-1}s_{n,2}^{-1}\cdots s_{n,2g}^{-1}s_{n,1}s_{n,2}\cdots s_{n,2g} = \left(S_{n,2g}^{-1}S_{n,2g-1}S_{n,2g-2}^{-1}\cdots S_{n,1}\right)\left(S_{n,2g}S_{n,2g-1}^{-1}S_{n,2g-2}\cdots S_{n,1}^{-1}\right).$$

Hence we obtain:

$$\pi_1(M, P_n) = \left\langle \{S_{n,1}, \ldots, S_{n,2g}\};\ \left(S_{n,2g}^{-1}S_{n,2g-1}\cdots S_{n,2}^{-1}S_{n,1}\right)\left(S_{n,2g}S_{n,2g-1}^{-1}\cdots S_{n,2}S_{n,1}^{-1}\right) = 1 \right\rangle.$$

We are now ready to start with the algorithm. Thus, let ω be a word over the generators of $B_n(M)$. Define the word $s = \varepsilon(\omega)$. Since the normal forms in Σ_n are unique, so is s. We obtain a word $\overline{\omega} = \omega s^{-1} \in PB_n(M)$ such that $\omega = \overline{\omega} s$.

Next we want to write $\overline{\omega}$ as a word over $\{a_{i,r}, T_{j,k}\}$ (where i, r, j and k take all possible values). Suppose that $\overline{\omega}$ has length m, that is, $\overline{\omega} = x_1 \cdots x_m$ where each x_i is a generator of $B_n(M)$ or its inverse. For all $i = 1, \ldots, m$ define $\overline{\omega}_i = x_1 \cdots x_i$. Since $\overline{\omega} \in PB_n(M)$, then $\varepsilon(\overline{\omega}_m) = \varepsilon(\overline{\omega}) = 1$, so one has:

$$\overline{\omega} = x_1 \cdots x_m = \left(1\ x_1\ \varepsilon(\overline{\omega}_1)^{-1}\right)\left(\varepsilon(\overline{\omega}_1)\ x_2\ \varepsilon(\overline{\omega}_2)^{-1}\right)\cdots\left(\varepsilon(\overline{\omega}_{m-1})\ x_m\ \varepsilon(\overline{\omega}_m)^{-1}\right).$$

But all factors on the right hand side of the equation are included in our dictionary, so we can use it to write all of them, and thus $\overline{\omega}$, as a word over the generators of $PB_n(M)$.

The next step is to replace in $\overline{\omega}$ all the letters of the form $a_{n,r}^{\pm 1}$ using the formula in Presentation 2,

$$a_{n,r}^{(-1)^{r+1}} = \left(A_{n,1}A_{n,2}^{-1}A_{n,3}\cdots A_{n,r-1}^{\pm 1}\right)\left(A_{n,r+1}^{\mp 1}\cdots A_{n,2g-1}^{-1}A_{n,2g}\right),$$

and all the letters of the form $T_{j,n}^{\pm 1}$, using (PR8). In this way we obtain $\overline{\omega}$ written as a word over $W_{n-1} \cup V_n$. We use again the dictionary to "move" to the right hand side of $\overline{\omega}$ all the letters in V_n. We will obtain $\overline{\omega} = X\, Y$, where X is a word over W_{n-1} and Y is a word over V_n.

Consider now the following exact sequence, coming also from the Fadell-Neuwirth fibration (see [B]).

$$1 \longrightarrow PB_{n-1}(M\backslash\{P_n\}) \xrightarrow{u} PB_n(M) \xrightarrow{v} \pi_1(M, P_n) \longrightarrow 1,$$

where for all $\Gamma = (\gamma_1, \ldots, \gamma_n) \in PB_n(M)$, $v(\Gamma) = \gamma_n$. Note that $v(\overline{\omega}) = Y \in \pi_1(M)$. Now in $\pi_1(M)$ we could apply Dehn's algorithm (see [LS]) to obtain a normal form of Y. At each step of Dehn's algorithm, a sub-word of Y would be replaced by a shorter one, using the relation

$$\left(S_{n,2g}^{-1}S_{n,2g-1}S_{n,2g-2}^{-1}\cdots S_{n,1}\right)\left(S_{n,2g}S_{n,2g-1}^{-1}S_{n,2g-2}\cdots S_{n,1}^{-1}\right) = 1.$$

Instead of this, we will do a similar process in $PB_n(M)$: each time that Dehn's algorithm replaces a sub-word of Y in $\pi_1(M)$, we replace the corresponding sub-word in $\overline{\omega} = XY \in PB_n(M)$ using

$$\left(A_{n,2g}^{-1}A_{n,2g-1}A_{n,2g-2}^{-1}\cdots A_{n,1}\right)\left(A_{n,2g}A_{n,2g-1}^{-1}A_{n,2g-2}\cdots A_{n,1}^{-1}\right) = \prod_{i=1}^{n-1} T_{i,n-1}^{-1}T_{i,n},$$

which is a relation equivalent to (PR1); then we remove the $T_{i,n}^{\pm 1}$ using (PR8) and we move again the letters in V_n to the right hand side of our word.

At the end of this process, we will obtain $\overline{\omega} = X_{n-1}\, \omega_n$, where ω_n is the normal form of $v(\overline{\omega})$ in $\pi_1(M)$, so it is unique, and X_{n-1} is a word over W_{n-1}.

The algorithm will end in $n-1$ steps: At each step, we have a word X_m over W_m, we replace the letters of the form $a_{m,r}^{\pm 1}$ by words over V_m, and then we move all the letters of V_m to the right hand side, using the dictionary. Then we remove all the sub-words of the form xx^{-1} or $x^{-1}x$, and we obtain $X_m = X_{m-1}\omega_m$, where X_{m-1} is a word over W_{m-1} and ω_m is a reduced word over V_m. If we prove that the word ω_m is unique, we will have the unique factorization $\omega = \omega_1 \cdots \omega_n s$ as the output of our algorithm.

Define $M_{n-m} = M\backslash\{P_{m+1}, \ldots, P_n\}$ for any $m = 1, \ldots, n-1$. In [B] we can find the following exact sequence, analogous to the previous one.

$$1 \longrightarrow PB_{m-1}(M_{n-m+1}) \xrightarrow{f} PB_m(M_{n-m}) \xrightarrow{g} \pi_1(M_{n-m}) \longrightarrow 1.$$

We only need to notice that $X_m \in PB_m(M_{n-m})$, and $g(X_m) = \omega_m$. Now since $\pi_1(M_{n-m})$ is a free group with free system of generators $\{a_{m,r};\ 1 \le r \le 2g\} \cup \{T_{m,j};\ m+1 \le j \le n-1\}$, and since ω_m is a reduced word, then it is unique, as we wanted to show.

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