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On stationary points of nonexpansive set-valued mappings

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Abstract
In this paper we deal with stationary points (also known as endpoints) of nonexpansive set-valued mappings and show that the existence of such points under certain conditions follows as a consequence of the existence of approximate stationary sequences. In particular we provide abstract extensions of well-known fixed point theorems.

MSC: 54C60; 47H09

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1 Introduction and preliminaries

Let X be a Banach space and C be a nonempty subset of X. A set-valued mapping $T : C \to 2^X \setminus \{\emptyset\}$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \quad (x, y \in C),$$

where $H(\cdot, \cdot)$ stands for the Hausdorff metric defined as

$$H(A, B) := \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}$$

and $d(a, B) := \inf_{x\in B} d(a, x)$. An element $x \in C$ is said to be a fixed point of T if $x \in Tx$ and a stationary point if $Tx = \{x\}$.

The first relevant work for existence of fixed points for nonexpansive set-valued mappings was provided by Markin [1] in 1968. Then a large and deep theory was developed by several authors (see, for instance, [2–6] or [7], Chapter 15, and references therein). This theory, however, is very far from as much advanced a theory as the corresponding one for single-valued nonexpansive mappings. The problem of the existence of stationary points has remained almost unexplored for nonexpansive mappings, it being the case that most results about them require contractive like conditions on the mapping as is the case in [8–11]. There has recently been some activity in this direction though. Several authors have begun the study of generalized set-valued nonexpansive mappings through an approach given by the properties of approximate sequences of fixed points where stationary points have appeared in a natural way. See, for instance, [12, 13] and the notion of a strong approximate fixed point sequence (which we call an approximate stationary point sequence here) in [14]. In the present work we show that some of the very well-known properties implying the existence of fixed points for nonexpansive single-valued mappings also imply the existence of stationary points in the set-valued case provided approximate stationary point sequences exist.

We will consider X as a Banach space and C a bounded, nonempty, closed, and convex subset of X. A closed ball of center x and radius r will be denoted $B_r[x]$. We denote by $F(X)$ the family of all nonempty closed subsets of X and by $K(X)$ the family of all nonempty compact subsets of X. For $x \in X$, the distance from x to C is given by

$$d(x, C) := \inf_{a \in C} d(x, a),$$

while the Chebyshev radius of C with respect to x is given by

$$r_x(C) := \sup_{z \in C} \|x - z\|.$$

The Chebyshev radius of C with respect to a set A will be given by

$$r_A(F) = \inf_{a \in A} r_a(C)$$

and denoted simply as $r(C)$ if $A = C$. The Chebyshev center of C, which may be empty, is then defined as

$$Z(C) := \left\{ x \in C : r_x(C) = \inf_{z \in C} r_z(C) \right\}.$$

Consider $T : C \to 2^X \setminus \{\emptyset\}$ to be a nonexpansive set-valued mapping. We say that a sequence (x_n) in C is an approximate stationary point sequence (a.s.p. sequence) of T if

$$\lim_{n \to \infty} r_{x_n}(Tx_n) = 0,$$

while (x_n) is an approximate fixed point sequence (a.f.p. sequence) if

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

Approximate fixed point sequences play a major role in metric fixed point theory for both single- and set-valued nonexpansive mappings. It is a very well-known fact, after Nadler's principle for set-valued contractions, that such sequences always exist provided $T : C \to F(C)$ is nonexpansive (see, for instance, [7], Chapter 15, [15], Theorem 8.23, or [14]). On the contrary, we see that approximate stationary point sequences do not need exist even if the mapping has a fixed point, its values are compact and X is a Hilbert space.

Example 1 Let B be the closed unit ball of ℓ_2. Define the nonexpansive set-valued mapping $T : B \to K(B)$ by

$$T(x_1, x_2, \dots) = \left\{ (1, 0, 0, \dots), (0, x_1, x_2, \dots) \right\}.$$

Trivially, the origin is a fixed point for T. Now, if (x_n) is a.s.p. sequence then it must be the case that $\operatorname{diam}(T(x_n))$ tends to 0 as n goes to ∞; however, it is immediate that $\operatorname{diam}(T(x)) \geq 1$ for any x in B.

Remark After the referee comments, the authors have learnt about references [16, 17] which deal with close problems to ours. That is, in these works the authors study questions related to stationary points of nonexpansive multivalued mappings provided the existence of approximate stationary point sequences is guaranteed. Although there is some overlapping among [16, 17] and the present work, their goals and ours are different. In [16] the author mainly focuses on conditions involving uniform convexity and wonders about the structure of the set of stationary points while [17] focuses on taking up some of the questions raised in [16]. In the present work, however, we deal with existence of stationary points on more general conditions than those studied in [16, 17] which, in particular, lead to more general versions of some results provided by these references (compare Theorems 3.1 and 3.4 in [16] to, respectively, Theorems 6 and 7 below). Notice also that Theorem 1 below gives an answer to Question 5.3, which remains out of scope of [17], raised in [16].

2 Main results

For our first result we will deal with two different notions of T-invariant sets.

Definition 1 Let $T : C \to 2^C \setminus \{\emptyset\}$, then:

- A subset D of C is said to be T-invariant if $Tx \subseteq D$ for all $x \in D$.
- A subset D of C is said to be weakly T-invariant if $Tx \cap D \neq \emptyset$ for all $x \in D$.

Remark 1 Notice that the two notions coincide for single-valued mappings.

A nonempty, closed, and convex subset D of C will be said to be a minimal T-invariant set (minimal weakly T-invariant set) if it does not contain any proper closed and convex subset which is T-invariant (weakly T-invariant).

Our first goal is to study the existence of stationary points for nonexpansive set-valued mappings under the conditions of a normal structure (see [7], Chapter 6).

Definition 2 A Banach space X is said to have a normal structure (resp., weak normal structure) if for every nonempty, bounded, closed (resp., weakly compact), and convex subset C of X with $\operatorname{diam}(C) > 0$ there exists $x \in C$ such that

$$r_x(C) < \operatorname{diam}(C).$$

Also, a nonempty convex subset C of a Banach space X is said to have a normal structure if the same happens for each nonempty, convex, and bounded subset D of C with $\operatorname{diam}(D) > 0$.

We will use the next two propositions.

Proposition 1 ([18], p. 152) *For every weakly compact convex subset C of a Banach space X, $Z(C)$ is a nonempty, closed, and convex subset of C.*

Proposition 2 ([18], p. 153) *Let X be a Banach space and C be a weakly compact convex subset of X with* $\operatorname{diam}(C) > 0$ *and the normal structure. Then*

$$\operatorname{diam}\bigl(Z(C)\bigr) < \operatorname{diam}(C).$$

The next result can be seen as an abstract extension of Kirk's fixed point theorem.

Theorem 1 *Let X be a Banach space and C a nonempty, weakly compact, and convex subset of X with normal structure. Then a nonexpansive mapping* $T\colon C \to K(C)$ *has a stationary point if and only if there is a nonempty, closed, and convex subset F of C which is minimal weakly T-invariant and minimal T-invariant.*

Proof It is obvious that if T has a stationary point x then $F = \{x\}$ fulfills all the requirements. Conversely, let F be the minimal set given by the statement. If $\operatorname{diam}(F) = 0$ then its element is a stationary point. Therefore we can assume that $\operatorname{diam}(F) > 0$. We will show that the set $Z(F)$ contradicts the minimality of F.

By Proposition 1, $Z(F)$ is a nonempty, weakly compact, and convex set. Let $x \in Z(F)$ and $y \in F$ be arbitrary. Fix $z \in Tx$ and let $w \in Ty$ be such that

$$\|z - w\| \leq H(Tx, Ty) \leq \|x - y\| \leq r_x(F) = r(F).$$

Since F is T-invariant we have $w \in F$ and so it is also in $B_{r(F)}[z] \cap F \cap Ty$, which is a nonempty set. Therefore, since $y \in F$ is arbitrary, $B_{r(F)}[z] \cap F$ is weakly T-invariant. On the other hand, F is minimal weakly T-invariant and $B_{r(F)}[z] \cap F$ is nonempty, weakly compact, and weakly T-invariant, so $B_{r(F)}[z] \cap F = F$. Therefore $F \subseteq B_{r(F)}[z]$ and so for each $x \in F$, $\|x - z\| \leq r(F)$. This implies that $r_z(F) \leq r(F)$ and hence $z \in Z(F)$. Since $z \in Tx$ was arbitrary, $Tx \subset Z(F)$, that is, $Z(F)$ is T-invariant. By Proposition 2 we have $\operatorname{diam}(Z(F)) < \operatorname{diam}(F)$, which leads to a contradiction with the minimality of F. □

Our next result, inspired by [19], Lemma 1, is a technical one which explores the properties of minimal sets of stationary point free mappings.

Theorem 2 *Let C be a nonempty, weakly compact, and convex subset of Banach space X and* $T: C \to 2^C \setminus \{\emptyset\}$ *be a stationary point free nonexpansive set-valued mapping. There exist* $\alpha > 0$ *and a minimal nonempty, weakly compact, and convex T-invariant subset E of C such that for every* $z \in E$ *and any a.s.p. sequence* $(x_n)_{n=1}^{\infty}$ *in E we have*

$$\limsup_{n\to\infty} \|x_n - z\| \geq \alpha.$$

Proof Suppose that Σ is the set of all nonempty, weakly compact, and convex T-invariant subsets D of C. The family $\Sigma \neq \emptyset$, because $C \in \Sigma$ and it can be partially ordered by set inclusion. An easy application of Zorn's lemma shows that the family Σ possesses a minimal element E. The diameter of E must be positive since otherwise the T invariancy of E would imply that T has a stationary point which is a contradiction.

We show by contradiction that there exists $\alpha > 0$ such that

$$\limsup_{n\to\infty} \|x_n - z\| \geq \alpha,$$

for each a.s.p. sequence (x_n) in E and each $z \in E$. Choose $\gamma > 0$ such that $2\gamma < \operatorname{diam}(E)$. There exists an a.s.p. sequence (x_n) in E such that

$$\limsup_{n\to\infty} \|x_n - z_0\| < \gamma,$$

for some point $z_0 \in E$. Put

$$D = \Bigl\{ z \in E : \limsup_{n\to\infty} \|x_n - z\| \le \gamma \Bigr\}.$$

D is nonempty, closed, and convex (therefore weakly compact) subset of E. We show D is T-invariant. Let $x \in D$ and $z \in Tx$ be arbitrary. For each $y \in Tx_n$, we have

$$\begin{aligned} \|x_n - z\| &\le \|x_n - y\| + \|y - z\| \\ &\le r_{x_n}(Tx_n) + \|y - z\|. \end{aligned}$$

Since $y \in Tx_n$ is arbitrary, we get

$$\begin{aligned} \|x_n - z\| &\le r_{x_n}(Tx_n) + d(z, Tx_n) \\ &\le r_{x_n}(Tx_n) + H(Tx, Tx_n) \\ &\le r_{x_n}(Tx_n) + \|x_n - x\|. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \|x_n - z\| \le 2\gamma.$$

Hence $z \in D$. This implies that $Tx \subseteq D$ and since $x \in D$ was arbitrary, D is T-invariant.

Hence $D \subsetneqq E$ is nonempty, weakly compact, and convex and T-invariant, *i.e.*, $D \in \Sigma$ and $\operatorname{diam}(D) < \operatorname{diam}(E)$, which is a contradiction because E is a minimal member of Σ. □

We have the following immediate corollaries.

Corollary 1 *If the Banach space X is reflexive, then the same holds true for any nonempty, closed, convex, and bounded subset C of X.*

Corollary 2 *Let C be a nonempty, weakly compact, and convex subset of Banach space X and $T : C \to 2^C \setminus \{\emptyset\}$ be a stationary point free nonexpansive set-valued mapping. There exist $\alpha > 0$ and a nonempty, convex, weakly compact, and T-invariant subset D of C such that for every compact subset A of D and any a.s.p. sequence $(x_n)_{n=1}^{\infty}$ in D we have*

$$\limsup_{n\to\infty} r_{x_n}(A) \ge \alpha.$$

Proof By Theorem 2 there exist $\alpha > 0$ and a nonempty, weakly compact, and convex T-invariant subset D of C such that for each a.s.p. sequence $(x_n) \subseteq D$ and for each $z \in D$

$$\limsup_{n\to\infty} \|x_n - z\| \ge \alpha.$$

We claim that for each (x_n) in D and each compact subset A of D we have

$$\limsup_{n\to\infty} r_{x_n}(A) \geq \alpha.$$

Suppose for contradiction that there exist a compact subset A of D and an a.s.p. sequence $(x_n) \subseteq D$ such that

$$\limsup_{n\to\infty} r_{x_n}(A) < \alpha.$$

Since A is compact, so for each $n \in \mathbb{N}$ there exists $y_n \in A$ such that $r_{x_n}(A) = \|x_n - y_n\|$. Again by compactness of A we can find a subsequence (y_{n_k}) of (y_n) convergent to some $z \in A$. Hence

$$\limsup_{k\to\infty} \|x_{n_k} - z\| = \limsup_{n\to\infty} r_{x_n}(A) < \alpha,$$

which is a contradiction, since any subsequence of an a.s.p. sequence is an a.s.p. sequence. □

We introduce next some elements that will be needed. Given a nonempty subset C of a Banach space X and a bounded sequence (x_n) in X, $r_a(C, \{x_n\})$ stands for the asymptotic radius of (x_n) with respect to C, $Z_a(C, \{x_n\})$ for the asymptotic center of (x_n) with respect to C and $\epsilon_0(X)$ is the characteristic of the convexity of the Banach space X, and they are defined as follows:

$$r_a\bigl(C, \{x_n\}\bigr) = \inf_{x\in C} \limsup_{n\to\infty} \|x_n - x\|,$$

$$Z_a\bigl(C, \{x_n\}\bigr) = \Bigl\{z \in C : \limsup_{n\to\infty} \|x_n - z\| = r_a\bigl(C, \{x_n\}\bigr)\Bigr\},$$

and

$$\epsilon_0(X) = \sup\bigl\{\epsilon > 0 : \delta_X(\epsilon) = 0\bigr\},$$

where $\delta_X : [0,2] \to [0,1]$ is the modulus of convexity of the Banach space X and it is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}.$$

The next technical well-known results will be needed.

Theorem 3 ([18], p. 131) *Let (x_n) be a bounded sequence in a Banach space X and C a nonempty, weakly compact, and convex subset of X. Then $Z_a(C, \{x_n\})$ is nonempty and convex.*

Theorem 4 ([18], p. 131) *Let C be a nonempty, closed, and convex subset of the Banach space X and (x_n) a bounded sequence in X. Then*

$$\operatorname{diam}\bigl(Z_a\bigl(C, \{x_n\}\bigr)\bigr) \leq \epsilon_0(X) r_a\bigl(C, \{x_n\}\bigr).$$

Theorem 5 ([7], p. 61) *If* $\epsilon_0(X) \leq 1$, *then X has normal structure*.

Definition 3 Given $T\colon C \to 2^C \setminus \{\emptyset\}$, with C a nonempty, closed, and convex subset of Banach space X, we will say that T has the approximate stationary point sequence property in C if T has an approximate stationary point sequence in any T-invariant, nonempty, closed, and convex subset of C.

Remember that, as Example 1 exhibits, the existence of such sequences is not guaranteed in general.

Theorem 6 *Let X be a Banach space with characteristic of convexity* $\epsilon_0(X) \leq 1$. *Let C be a nonempty, weakly compact, and convex subset of X and* $T\colon C \to 2^C \setminus \{\emptyset\}$ *a nonexpansive mapping. Then T has a stationary point if and only if T has the approximate stationary point sequence property.*

Proof We only need to prove that T has a stationary point provided it has the approximate stationary point sequence property. Suppose for contradiction that T is stationary point free. By Theorem 2, C contains a nonempty, weakly compact, and convex minimal T-invariant subset E. From the hypothesis there is (x_n) an a.s.p. sequence in E. From Theorem 5, X has normal structure and so we can fix $x_0 \in E$ such that $r_{x_0}(E) < \operatorname{diam}(E)$. By Theorem 3, $Z_a(E, \{x_n\})$ is nonempty. Then

$$r_a\bigl(E, \{x_n\}\bigr) \leq \limsup_{n\to\infty} \|x_n - x_0\| < \operatorname{diam}(E).$$

By Theorem 4 and the fact that $\epsilon_0(X) \leq 1$,

$$\operatorname{diam}\bigl(Z_a\bigl(E, \{x_n\}\bigr)\bigr) \leq r_a\bigl(E, \{x_n\}\bigr),$$

and, since $r_a(E, \{x_n\}) < \operatorname{diam}(E)$, we obtain

$$\operatorname{diam}\bigl(Z_a\bigl(E, \{x_n\}\bigr)\bigr) < \operatorname{diam}(E).$$

Now, the fact that $Z_a(E, \{x_n\})$ is T-invariant follows in a similar way as was shown for set D in the proof of Theorem 2. Therefore, since it is weakly compact and convex too, we meet a contradiction with the minimality of E. □

A Banach space X is said to satisfy the Opial property if for every weakly null sequence $\{x_n\}$ and every nonzero vector x in X, we have

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|.$$

It is well known that if X is a Banach space which satisfies the Opial property, then X has the weak normal structure (see *e.g.* [20]).

Theorem 7 *Let X be a Banach space which satisfies the Opial property. Let C be a nonempty, weakly compact, and convex subset of X and* $T\colon C \to 2^C \setminus \{\emptyset\}$ *a nonexpansive mapping. Then T has a stationary point if and only if T has the approximate stationary point sequence property.*

Proof We only need to prove that T has a stationary point provided it has the approximate stationary point sequence property. Suppose for contradiction that T is stationary point free. Let E be the minimal subset of C as obtained in the proof of Theorem 2 which is nonempty, weakly compact, and convex, T-invariant. From the hypothesis there is (x_n) an a.s.p. sequence in E, we can assume this sequence is weakly convergent to x. Since

$$Z_a(\{x_n\}, E) = \Big\{ z \in E : \limsup_{n\to\infty} \|x_n - z\| \le r_a(E, \{x_n\}) \Big\},$$

is a nonempty, weakly compact, convex, and T-invariant subset of E, then it must be the case that

$$E = Z_a(\{x_n\}, E).$$

Therefore, for each $z \in E$, we have

$$\limsup_{n\to\infty} \|x_n - z\| = r_a(E, \{x_n\}).$$

Since X has the Opial property, and $x_n \rightharpoonup x$, for each $z \in E$ where $z \neq x$, we have

$$r_a(E, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z\| = r_a(E, \{x_n\}),$$

which is a contradiction. □

In 1974 Lim [6] gave the following extension of the Markin fixed point theorem for uniformly convex spaces.

Theorem 8 *Let X be a uniformly convex Banach space, let C be a closed, bounded, and convex subset of X and $T : C \to K(C)$ be a nonexpansive mapping. Then T has a fixed point.*

In 1990 Kirk and Massa [4] gave the following partial generalization of Lim's theorem.

Theorem 9 *Let C be a closed, bounded, and convex subset of a Banach space X and $T : C \to KC(C)$ a nonexpansive mapping, where $KC(C)$ stands for the collection of nonempty, compact, and convex subsets of C. If the asymptotic center in C of each bounded sequence of X is nonempty and compact, then T has a fixed point.*

Motivated by these two theorems we have the following for stationary points.

Theorem 10 *Suppose that X is a Banach space such that for each closed convex bounded subset C of X the asymptotic center in C of each bounded sequence is nonempty and compact. Let $T: C \to 2^C \setminus \{\emptyset\}$ be a nonexpansive mapping with C weakly compact. Then T has a stationary point if and only if T has the approximate stationary point sequence property.*

Proof We only need to prove that T has a stationary point provided it has the approximate stationary point sequence property. Suppose for contradiction that T is stationary point free. Let (y_n) be an arbitrary sequence in C. The sequence (y_n) is bounded and, by the assumptions, $Z_a(\{y_n\}, C)$ is nonempty and compact. As in previous proofs, we know that the asymptotic center $Z_a(\{y_n\}, C)$ is a nonempty, weakly compact, and convex T-invariant subset of X. Again, by the assumptions,

$$\inf_{x \in Z_a(\{y_n\}, C)} r_x(Tx) = 0.$$

This, accompanied with the compactness of $Z_a(\{y_n\}, C)$, implies that there exists a convergent a.s.p. sequence (x_n) in $Z_a(\{y_n\}, C)$, say $x_n \to x$.

We show next that x is a stationary point of T. Let $z \in Tx$ and fix $n \in \mathbb{N}$. For each $y \in Tx_n$, we have

$$\begin{aligned} \|x - z\| &\leq \|x_n - x\| + \|x_n - y\| + \|z - y\| \\ &\leq \|x_n - x\| + r_{x_n}(Tx_n) + \|y - z\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|x - z\| &\leq \|x_n - x\| + r_{x_n}(Tx_n) + d(z, Tx_n) \\ &\leq \|x_n - x\| + r_{x_n}(Tx_n) + H(Tx, Tx_n) \\ &\leq r_{x_n}(Tx_n) + 2\|x_n - x\|. \end{aligned}$$

Since $z \in Tx$ is arbitrary, we obtain

$$r_x(Tx) \leq r_{x_n}(Tx_n) + 2\|x_n - x\|.$$

As $n \to \infty$, we obtain $r_x(Tx) = 0$ and therefore, $Tx = \{x\}$, which finally proves the statement by contradiction. □

In particular, the following corollary holds.

Corollary 3 *Let X be a uniformly convex Banach space. Then the same conclusions as in Theorem* 10 *hold.*

Recall that a bounded sequence (x_n) in a Banach space X is said to be regular with respect to a bounded subset C of X if the asymptotic radii (with respect to C) of all subsequences of (x_n) are the same, that is,

$$r_a\bigl(C, \{x_n\}\bigr) = r_a\bigl(C, \{x_{n_i}\}\bigr) \quad \text{for each subsequence } (x_{n_i}) \text{ of } (x_n).$$

A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset C of X and for every bounded sequence (x_n) in C which is regular with respect to C

$$r_C\bigl(Z_a\bigl(C, \{x_n\}\bigr)\bigr) \leq \lambda r_a\bigl(C, \{x_n\}\bigr)$$

where the Chebyshev radius of a bounded subset D of X relative to C is defined by

$$r_C(D) = \inf\bigl\{r_x(D) : x \in C\bigr\}.$$

Indeed, any Banach space with the (DL)-condition has the weak normal structure (see [21], Theorem 3.3).

Before giving the next result, we need the following proposition.

Proposition 3 ([6, 22]) *Let C be a nonempty bounded subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then $\{x_n\}$ has a subsequence that is regular with respect to C.*

Theorem 11 *Let X be a Banach space with the* (DL)*-condition. Let $T: C \to 2^C \setminus \{\emptyset\}$ be a nonexpansive mapping with C nonempty, weakly compact, and convex. Then T has a stationary point if and only if T has the approximate stationary point sequence property.*

Proof We only need to prove that T has a stationary point provided it has the approximate stationary point sequence property. Suppose for contradiction that T is stationary point free. From Theorem 2 there exist a minimal nonempty, weakly compact, and convex T-invariant subset E of C and $\alpha > 0$ such that for each a.s.p. sequence (z_n) in E

$$\limsup_{n\to\infty} \|z_n - z\| \geq \alpha, \quad z \in E.$$

Consider an a.s.p. sequence (x_n) in E, which exists by hypothesis. By Proposition 3 this sequence has a regular subsequence with respect to E, say again (x_n). Since X satisfies the property (DL) we have

$$r_E\big(Z_a\big(E,\{x_n\}\big)\big) \leq \lambda r_a\big(E,\{x_n\}\big) \tag{1}$$

for some $\lambda \in [0,1)$. Since

$$\limsup_{n\to\infty} \|x_n - z\| \geq \alpha, \quad z \in E,$$

$r_a(E,\{x_n\}) > 0$. Therefore,

$$\lambda r_a\big(E,\{x_n\}\big) < r_a\big(E,\{x_n\}\big).$$

Since $Z_a(E,\{x_n\})$ is nonempty, weakly compact, and convex, and a T-invariant set contained in E, by minimality, we have $Z_a(E,\{x_n\}) = E$. Therefore, in particular, from (1) we have

$$r_E(E) \leq \lambda r_a\big(E,\{x_n\}\big) < r_a\big(E,\{x_n\}\big),$$

which is a contradiction since $\{x_n\} \subseteq E$. □

Next we list some sufficient conditions that lead to the (DL) property.

Corollary 4 *Let X be a Banach space such that*

$$\rho'_X(0) < \frac{1}{2}.$$

Then the same conclusions as in Theorem 11 *hold.*

Proof See [23], Corollary 1. □

Corollary 5 *Let X be a Banach space such that it satisfies one of the following two equivalent conditions*:

1. $r_X(1) > 0$,
2. $\Delta_0(X) < 1$.

Then the same conclusions as in Theorem 11 *hold*.

Proof See [23], Corollary 2. □

Corollary 6 *Let X be a Banach space such that*

$$j(X) < 1 + \frac{1}{\mu(X)'}.$$

Then the same conclusions as in Theorem 11 *hold*.

Proof See [24], Corollary 3.2. □

Corollary 7 *Let X be a Banach space such that*

$$\rho_X'(0) < \frac{1}{\mu(X)}.$$

Then the same conclusions as in Theorem 11 *hold*.

Proof See [25], Theorem 4. □

Corollary 8 *Let X be a Banach space such that*

$$\delta_X\left(\frac{1}{R(X)} + \sqrt{1 - \frac{1}{R(X)} + \frac{1}{R(X)^2}}\right) > \frac{1}{2}\left(1 - \frac{1}{R(X)}\right).$$

Then the same conclusions as in Theorem 11 *hold*.

Proof See [26], Theorem 3.19. □

We close this work with a last case where we follow the approach provided in [19]. Let $(X, \|\cdot\|)$ be a Banach space. Assume that there exists a family $\{R_k : X \to [0,\infty)\}_{k\in\mathbb{N}}$ of seminorms on X such that for all $x \in X$ we have:

1. $R_1(x) = \|x\|$ and for all $k \geq 2$, $R_k(x) \leq \|x\|$.
2. $\lim_{k\to\infty} R_k(x) = 0$.
3. If $x_n \rightharpoonup 0$ (weakly convergent to zero), then for all $k \geq 1$

$$\limsup_{n\to\infty} R_k(x_n) = \limsup_{n\to\infty} \|x_n\|.$$

4. If $x_n \rightharpoonup 0$, then for all $k \geq 1$

$$\limsup_{n\to\infty} R_k(x_n + x) = \limsup_{n\to\infty} R_k(x_n) + R_k(x).$$

Let $(\gamma_k) \subseteq (0,1)$ be an arbitrary nondecreasing sequence such that $\lim_{k\to\infty} \gamma_k = 1$. An equivalent norm on X may be defined as

$$|||x||| := \sup_k \gamma_k R_k(x),$$

for each $x \in X$.

We restate Lemma 2 of [19] next.

Lemma 1 *Let X be a Banach space endowed with a family of seminorms $\{R_k(\cdot)\}$ satisfying properties* 1-4 *stated above, and $|||\cdot|||$ be the equivalent norm given above. If (x_n) and (y_n) are two bounded sequences in X with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then*

$$\limsup_m \limsup_{n\to\infty} |||x_n - y_m||| \geq \limsup_{n\to\infty} |||x_n - x||| + \limsup_m |||y_m - y|||.$$

Theorem 12 *Let X be a Banach space endowed with a family of seminorms $\{R_k(\cdot)\}$ satisfying Properties* 1-4 *stated above, and $|||\cdot|||$ be the equivalent norm given above. Let $T: C \to 2^C \setminus \{\emptyset\}$ be a nonexpansive mapping with respect to $|||\cdot|||$ and C a nonempty, weakly compact, and convex. Then T has a stationary point if and only if T has the approximate stationary point sequence property.*

Proof We only need to prove that T has a stationary point provided it has the approximate stationary point sequence property. Suppose for contradiction that T is stationary point free. By Theorem 2 there exist a minimal, nonempty, T-invariant, weakly compact, and convex subset E.

By hypothesis there exists a weakly convergent a.s.p. sequence (x_n) in E, say weakly convergent to $x \in E$. Put $\limsup_{n\to\infty} |||x_n - x||| = \gamma$. Theorem 2 implies that $\gamma > 0$. Set

$$D = \left\{ z \in E : \limsup_{n\to\infty} |||x_n - x||| \leq \gamma \right\}.$$

It is clear that D is a nonempty, weakly compact, and convex subset of D. Moreover, following the same reasoning as in Theorem 2, it can be shown that D is T-invariant subset of E. The minimality of E now implies that $E = D$.

Now, for each $m \in \mathbb{N}$ we have $\limsup_{n\to\infty} |||x_n - x_m||| \leq \gamma$. Hence by Lemma 1,

$$\gamma \geq \limsup_m \limsup_{n\to\infty} |||x_n - x_m||| \geq \limsup_{n\to\infty} |||x_n - x||| + \limsup_m |||x_m - x||| \geq 2\gamma,$$

which is a contradiction. □

Remark 2 Notice that, since approximate fixed point sequence always exists in the single-valued case for the theorems we have revisited, our results can be regarded as abstract extensions of corresponding results for single-valued nonexpansive mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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