arXiv:1403.7126v1 [math.NT] 27 Mar 2014

ON THE DISTRIBUTION (MOD 1) OF THE NORMALIZED ZEROS OF THE RIEMANN ZETA-FUNCTION

J. ARIAS DE REYNA

Abstract. We consider the problem whether the ordinates of the non-trivial zeros of $\zeta(s)$ are uniformly distributed modulo the Gram points, or equivalently, if the normalized zeros (x_n) are uniformly distributed modulo 1. Odlyzko conjectured this to be true. This is far from being proved, even assuming the Riemann hypothesis (RH, for short).

Applying the Piatetski-Shapiro 11/12 Theorem we are able to show that, for $0 < \kappa < 6/5$, the mean value $\frac{1}{N}\sum_{n\le N}\exp(2\pi i\kappa x_n)$ tends to zero. The case $\kappa = 1$ is especially interesting. In this case the Prime Number Theorem is sufficient to prove that the mean value is 0, but the rate of convergence is slower than for other values of κ. Also the case $\kappa = 1$ seems to contradict the behavior of the first two million zeros of $\zeta(s)$.

We make an effort not to use the RH. So our Theorems are absolute. We also put forward the interesting question: will the uniform distribution of the normalized zeros be compatible with the GUE hypothesis?

Let $\rho = \frac{1}{2} + i\alpha$ run through the complex zeros of zeta. We do not assume the RH so that α may be complex. For $0 < \kappa < \frac{6}{5}$ we prove that

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\operatorname{Re}\alpha\le T} e^{2i\kappa\vartheta(\alpha)} = 0$$

where $\vartheta(t)$ is the phase of $\zeta(\frac{1}{2}+it) = e^{-i\vartheta(t)}Z(t)$.

1. Introduction.

This paper deals with the distribution of the ordinates of the non-trivial zeros of $\zeta(s)$. This distribution has received much attention. On the assumption of the RH Rademacher [18] (also [17, p. 434–459]) showed that the sequence (γ_n) is uniformly distributed modulo 1. Here $\rho_n = \beta_n + i\gamma_n$ are the zeros of $\zeta(s)$ in the upper half plane, counted with multiplicity, ordered by increasing γ_n, and ties being broken by ordering β_n from smallest to largest.

Given a sequence of non-negative real numbers (γ_n) and another strictly increasing sequence of non-negative real numbers (g_n) with $\lim_n g_n = \infty$, it is said that the (γ_n) are *uniformly distributed modulo* (g_n) if the numbers y_n defined by

$$y_n = \frac{\gamma_n - g_m}{g_{m+1} - g_m}, \quad \text{where} \quad \gamma_n \in [g_m, g_{m+1})$$

are uniformly distributed in $[0,1]$, (compare [6, p. 4]).

Our problem is whether the ordinates (γ_n) of the zeros of $\zeta(s)$ are uniformly distributed modulo the Gram points (g_n). In a non published report, Odlyzko [14, p. 60] conjectured

Date: March 28, 2014.

Author supported by MINECO grant MTM2012-30748.

that the ordinates of the zeros γ_n are not related to the Gram points for γ_n large. Hence he conjectured that the normalized zeros will be uniformly distributed modulo 1. This is far from being proved, even assuming the RH.

Since the Gram points are defined by the equation $\pi n = \vartheta(g_n)$, our problem is equivalent to the question as to whether the *normalized zeros* $x_n := \frac{1}{\pi}\vartheta(\gamma_n) \approx \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}$ are uniformly distributed mod 1. By the asymptotic properties of $\vartheta(t)$ this is equivalent to the uniform distribution mod 1 of the numbers $\frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}$ (see Theorem 1.2 in [6, p.3]). The question is especially interesting because the normalized zeros have average spacing 1.

Hardy and Littlewood [7, p. 162–177] proved that for any $a > 0$, uniformly for $x \in J$, where J is any compact interval of positive numbers, we have

$$\sum_{0<\gamma<T} e^{a\rho\log(-i\rho)} x^\rho \rho^{-\frac{a}{2}} = \mathcal{O}(T^{\frac{1}{2}+\frac{a}{2}})$$

from which, assuming the RH, they derive that for any $a, \theta > 0$, we have

$$\sum_{0<\gamma<T} e^{ai\gamma\log(\gamma\theta)} = \mathcal{O}(T^{\frac{1}{2}+\frac{a}{2}}).$$

Recall that $g_n \sim 2\pi n/\log n$. Fujii [3] proved that for any $a > 0$ and $b > 0$, the sequence (γ_n) is uniformly distributed modulo $((\log n)^a \cdot bn/\log n)$.

In [4] Fujii proves, under the RH, that for any positive κ and a we have

$$\sum_{0<\gamma\le T} e^{i\kappa\gamma\log\frac{\kappa\gamma}{2\pi e a}} = -e^{\frac{\pi i}{4}}\frac{\sqrt{a}}{\kappa}\sum_{n\le(\frac{\kappa T}{2\pi a})^\kappa}\frac{\Lambda(n)}{n^{\frac{1}{2}-\frac{1}{2\kappa}}}e^{-2\pi i a n^{\frac{1}{\kappa}}} + \mathcal{O}(T^{\frac{\kappa}{2}}\log T\log\log T) + \mathcal{O}(T^{\frac{1}{2}-\frac{\kappa}{2}}\log T).$$

Our main result is an absolute version of this equation. Since we do not assume the RH, we define the numbers α in such a way that the zeros of zeta are given by $\rho = \frac{1}{2} + i\alpha$. Here the numbers α may be (non-real) complex numbers, if the Riemann hypothesis fails. Then we show without any assumption the following.

Theorem 3.1. *For $\kappa > 0$ we have*

$$(1)\quad \sum_{0<\operatorname{Re}\alpha\le T} e^{2i\kappa\vartheta(\alpha)} = -\frac{e^{\frac{\pi i}{4}(1-\kappa)}}{\sqrt{\kappa}}\sum_{n\le(T/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\frac{1}{2}-\frac{1}{2\kappa}}}e^{-2\pi i\kappa n^{1/\kappa}} + \mathcal{O}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \mathcal{O}_\kappa(T^{\frac{\kappa}{2}}\log^2 T).$$

Applying, for $\kappa > 1$, the Piatetski-Shapiro 11/12 Theorem, we get

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\operatorname{Re}\alpha\le T} e^{2i\kappa\vartheta(\alpha)} = 0, \qquad 0 < \kappa < \frac{6}{5}.$$

The case $\kappa = 1$ is especially interesting, since it seems to contradict the behavior of the first two million zeros of $\zeta(s)$.

Our results are also very similar to some of Schoißengeier [20], who extended the cited analysis of Hardy and Littlewood. In the case $\kappa = 1$ our formula gives the following

Corollary. *The Von Mangoldt function can be approximated in the following way*

$$\psi(T/2\pi) = \sum_{n\le T/2\pi} \Lambda(n) = -\sum_{0<\operatorname{Re}\alpha\le T} e^{2i\vartheta(\alpha)} + \mathcal{O}(T^{\frac{1}{2}}\log^2 T).$$

This is equivalent to one of the results of Schoißengeier, but obtained here by a simpler analysis.

1.1. **Computational data.** Our interest in the distribution of the γ_n with respect to the Gram points originates from the observation that the angle of the curve $\operatorname{Re}\zeta(s) = 0$ at a zero $\frac{1}{2}+i\gamma$ (on the critical line) with the positive real axis is equal to $\vartheta(\gamma) \bmod \pi$. Odlyzko's list of first $2,001,052$ zeros of zeta has been used to generate the following pictures of the distribution of $\frac{1}{\pi}\vartheta(\gamma) \bmod 1$.

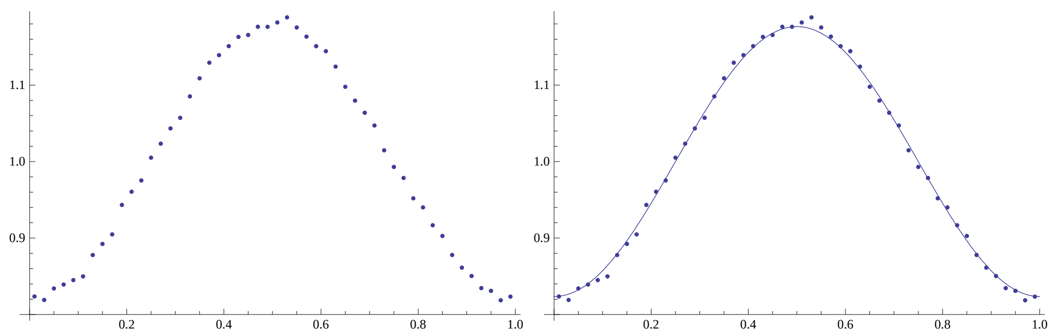

FIGURE 1. Distribution mod 1 of the normalized first 2 001 052 zeros.

The curve we have drawn, approximately fitting the data, is the density function $\rho(x) = 1 - \frac{3}{17}\cos(2\pi x)$. If the RH is true so that $\alpha_n = \gamma_n$ for all n and this initial distribution is maintained then we have

$$\lim_T \frac{1}{N(T)} \sum_{0<\operatorname{Re}\alpha\le T} e^{2i\vartheta(\alpha)} = \lim_N \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i x_n} \approx \int_0^1 e^{2\pi i x}\Bigl(1 - \frac{3}{17}\cos 2\pi x\Bigr)\,dx = -\frac{3}{34}.$$

Hence, our Corollary 3.2 shows that the trend in the behavior of γ_n seen in the above figures will not be maintained for larger values of T.

Odlyzko observed that the distribution of the normalized zeros is nearer to a uniform distribution for higher zeros. But this seems to be at odds with Titchmarsh [21, Theorem 10.6] who shows that the mean values of $Z(g_{2n})$ and $Z(g_{2n+1})$ are equal to 2 and -2, respectively.

Hence we ask the question: *It is true that the normalized zeros are uniformly distributed modulo* 1*?*

We also comment here about another conjecture regarding the distribution of the zeros of zeta. Assuming the RH, Montgomery [13] proved his result about the correlation of pairs of zeros and stated his *pair correlation conjecture.*

The differences $x_{n+1}-x_n$ of the normalized zeros satisfy

$$x_{n+1}-x_n=\frac{1}{\pi}\int_{\gamma_n}^{\gamma_{n+1}}\vartheta'(t)\,dt.$$

Since $\vartheta'(t)=\frac{1}{2}\log\frac{t}{2\pi}+\boldsymbol{\mathcal{O}}(t^{-2})$ we have

$$x_{n+1}-x_n\approx\frac{\gamma_{n+1}-\gamma_n}{\pi}\frac{1}{2}\log\frac{\gamma_n}{2\pi}=\delta_n.$$

So, Montgomery's conjecture is also a conjecture about our normalized zeros.

A natural question here is: *Is the pair correlation conjecture compatible with a uniform distribution (mod 1) of the normalized zeros?*

2. Notations and tools.

When possible we follow the standard notations. As in Titchmarsh [21, section 9.4] the zeros $\beta+i\gamma$ of $\zeta(s)$ with $\gamma>0$ are arranged in a sequence $\rho_n=\beta_n+i\gamma_n$ so that $\gamma_{n+1}\geq\gamma_n$. We will not assume the Riemann hypothesis (RH for short), and following Riemann [19] define α_n by $\rho_n=\frac{1}{2}+i\alpha_n$. The numbers α_n are the zeros of $\Xi(t)$ with positive real part. The RH is equivalent to the equality $\alpha_n=\gamma_n$ for all natural numbers n. We denote by $N(T)$, where $T>0$, the number of zeros of $\zeta(s)$ in the rectangle $0<\sigma<1$, $0\leq t\leq T$.

The functional equation of $\zeta(s)$ can be written as

$$\zeta(s)=\chi(s)\zeta(1-s),\quad \chi(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}=2^{s-1}\pi^s\sec\tfrac{1}{2}\pi s/\Gamma(s). \tag{2}$$

For t real we have $|\chi(\frac{1}{2}+it)|=1$, and there exist two real and real analytic functions $\vartheta(t)$ and $Z(t)$ (see Titchmarsh [21, section 4.17]) such that

$$\zeta(\tfrac{1}{2}+it)=e^{-i\vartheta(t)}Z(t),\quad \chi(\tfrac{1}{2}+it)=e^{-2i\vartheta(t)}. \tag{3}$$

The function $\chi(s)$ has poles for $s=2n+1$ with $n=0,1,\dots$ and zeros for $s=-2n$. Let Ω be the complex plane $\mathbf{C}$ with two cuts along the half-lines $(-\infty,0]$ and $[1,\infty)$. The function $\chi(s)$ is analytic on the simply connected Ω and does not vanish there. So we may define $\log\chi(s)$ in Ω in such a way that

$$-\log\chi(\tfrac{1}{2}+it)=2i\vartheta(t),\qquad \vartheta(0)=0. \tag{4}$$

The function $\vartheta(t)$ is extended in this way to all $-i(\Omega-\frac{1}{2})$ as an analytic function. Also we fix the meaning of

$$\chi(s)^{-\kappa}:=e^{-\kappa\log\chi(s)}=e^{2i\kappa\vartheta(\tau)}\quad\text{where}\quad s=\tfrac{1}{2}+i\tau\in\Omega. \tag{5}$$

Definition 2.1. For any non-trivial zero $\rho=\beta+i\gamma=\frac{1}{2}+i\alpha$ we define the normalized zero as

$$x=\frac{1}{\pi}\vartheta(\alpha). \tag{6}$$

Also let x_n be the normalized zero corresponding to $\rho_n=\beta_n+i\gamma_n$.

The function $\vartheta(t)$ is strictly increasing for $t > 6.28984\ldots$ For integral $k \geq -1$ the Gram point $t = g_k (> 7)$ is defined as the unique solution of $\vartheta(t) = k\pi$ (see [2, p. 126]).

In the interval $[0, T]$ there are approximately as many Gram points as zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$. Gram's "law" (to which there are many exceptions) states that in each Gram interval (g_n, g_{n+1}) there is a zero of $\zeta(s)$. If Gram's law were generally true, then the RH would be true, the zeros would be simple and γ_n would be an element of (g_{n-2}, g_{n-1}). Of course Gram's law is not true, but it is still a good heuristic to locate the zeros of $\zeta(s)$ for relatively small t which can be reached by our computers.

Also, it is well known that in each interval $(T, T+1)$, with $T \geq 2$ we can select a number T' such that if γ is the ordinate of any zero of $\zeta(s)$ then $|T' - \gamma| \gg \frac{1}{\log T}$.

From the book by Huxley we quote two lemmas which will be essential in our proof [9, p. 88].

Lemma 2.2 (First Derivative Test)**.** *Let $f(x)$ be real and differentiable on the open interval (α, β) with $f'(x)$ monotone and $f'(x) \geq \mu > 0$ on (α, β). Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\Big| \int_\alpha^\beta g(x) \exp(2\pi i f(x))\, dx \Big| \leq \frac{V}{\pi \mu}.$$

Lemma 2.3 (Second Derivative Test)**.** *Let $f(x)$ be real and twice differentiable on the open interval (α, β) with $f''(x) \geq \lambda > 0$ on (α, β). Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\Big| \int_\alpha^\beta g(x) \exp(2\pi i f(x))\, dx \Big| \leq \frac{4V}{\sqrt{\pi \lambda}}.$$

The next two lemmas can be inferred from Levinson [12] and Gonek [5].

Lemma 2.4. *Let $\kappa_0 > 0$ and $K \subset \mathbf{R}$ a compact set be given. Then there exist constants $c > 0$, $C > 0$ such that for any $r > 1$, $a \in K$ and $\kappa \geq \kappa_0$, we have*

$$\int_{r(1-c)}^{r(1+c)} x^a \exp\Big\{2\pi i \Big(\kappa x \log \frac{x}{er}\Big)\Big\}\, dx = \kappa^{-\frac{1}{2}} r^{a+\frac{1}{2}} e^{\frac{\pi i}{4}} e^{-2\pi i \kappa r} + R,$$

with $|R| \leq C r^a$.

Lemma 2.5. *Let $\kappa_0 > 0$ and $K \subset \mathbf{R}$ a compact set be given. Then there exist constants $c > 0$, $C > 0$ such that for any $r > 1$, $\kappa \geq \kappa_0$, $a \in K$ and $\frac{r}{2} \leq A < B \leq 2r$ we have*

$$\int_A^B x^a \exp\Big\{2\pi i \Big(\kappa x \log \frac{x}{er}\Big)\Big\}\, dx = I_0 + R_1 + R_2 + R_3,$$

where

$$|R_1| \leq C r^a, \quad |R_2| \leq C \frac{r^{a+1}}{|A - r| + r^{1/2}}, \quad |R_3| \leq C \frac{r^{a+1}}{|B - r| + r^{1/2}},$$

and where $I_0 = \kappa^{-\frac{1}{2}} r^{a+\frac{1}{2}} e^{\frac{\pi i}{4}} e^{-2\pi i \kappa r}$ for $A \leq r \leq B$ and 0 in all other cases.

Now we state the best zero-free region known. A proof can be found in the book of Ivić [11, Thm. 6.1].

Theorem 2.6. *There is an absolute constant $C > 0$ such that $\zeta(s) \neq 0$ for*

$$\sigma \geq 1 - C(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}} \qquad (t \geq t_0). \tag{7}$$

Lemma 2.7. *Let $\rho = \beta + i\gamma$ with $\beta \in (0,1)$ and $\gamma > 0$ and define α by $\rho = \frac{1}{2} + i\alpha$. Then for any $\kappa > 0$ we have*

$$e^{2i\kappa\vartheta(\alpha)} = \Bigl(\frac{\gamma}{2\pi}\Bigr)^{\kappa(\beta-\frac{1}{2})} \exp\Bigl\{i\Bigl(\kappa\gamma\log\frac{\gamma}{2\pi} - \kappa\gamma - \frac{\kappa\pi}{4}\Bigr)\Bigr\}(1 + \mathcal{O}_\kappa(\gamma^{-1})). \tag{8}$$

Proof. This follows easily from Titchmarsh [21, eq. (4.12.3)]. □

We will use the following Theorem of Piatetski-Shapiro [15].

Theorem 2.8. *For $\varepsilon > 0$, $\frac{2}{3} < \gamma < 1$ and all k with $1 \leq k \leq x^{1-\gamma}\log^2 x$, we have*

$$\sum_{p\leq x} e^{2\pi i k p^\gamma} \ll x^{\frac{11}{12}+\varepsilon}. \tag{9}$$

The exponent 11/12 in this Theorem has been improved, but with a smaller range of γ. For our needs the range is important. Therefore, we will use this theorem as stated.

3. Main theorem

Theorem 3.1. *For $\kappa > 0$*

(10)

$$\sum_{0<\operatorname{Re}\alpha<T} e^{2i\kappa\vartheta(\alpha)} = -\frac{e^{\frac{\pi i}{4}(1-\kappa)}}{\sqrt{\kappa}} \sum_{n<(T/2\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac{1}{2}-\frac{1}{2\kappa}}} e^{-2\pi i\kappa n^{1/\kappa}} + \mathcal{O}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \mathcal{O}_\kappa(T^{\frac{\kappa}{2}}\log^2 T).$$

Proof. There exists T' such that $T < T' < T+1$ with $|T' - \gamma| \gg 1/\log T$ for any ordinate γ of a zero of $\zeta(s)$ and such that

$$\sum_{0<\operatorname{Re}\alpha\leq T} e^{2i\kappa\vartheta(\alpha)} = \sum_{0<\operatorname{Re}\alpha\leq T'} e^{2i\kappa\vartheta(\alpha)} + \mathcal{O}_\kappa(T^{\frac{\kappa}{2}}).$$

In fact, here the difference between the two sums is composed of at most of $C\log T$ terms. Lemma 2.7 yields

$$\Bigl|\sum_{T<\operatorname{Re}\alpha\leq T'} e^{2i\kappa\vartheta(\alpha)}\Bigr| \leq \sum_{T<\gamma\leq T'} \Bigl(\frac{\gamma}{2\pi}\Bigr)^{\kappa(\beta-\frac{1}{2})}.$$

Applying Theorem 2.6 we have

$$\Bigl|\sum_{T<\operatorname{Re}\alpha\leq T'} e^{2i\kappa\vartheta(\alpha)}\Bigr| \leq C\log T\Bigl(\frac{T'}{2\pi}\Bigr)^{\kappa(\frac{1}{2}-c(\log T')^{-\frac{2}{3}}(\log\log T')^{-\frac{1}{3}})}$$

$$\ll_\kappa C(\log T)T^{\frac{\kappa}{2}}e^{-c\kappa(\log T')^{\frac{1}{3}}(\log\log T')^{-\frac{1}{3}}} = \mathcal{O}_\kappa(T^{\frac{\kappa}{2}}).$$

Also

$$\Big|\sum_{(T/2\pi)^\kappa<n\le(T'/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\frac12-\frac1{2\kappa}}}e^{-2\pi i\kappa n^{\frac1\kappa}}\Big|\ll_\kappa(\log T)T^{\frac12-\frac\kappa2}(T'^\kappa-T^\kappa)$$

$$\ll_\kappa(\log T)T^{\frac12+\frac\kappa2}\Big|1-\Big(\frac{T+1}{T}\Big)^\kappa\Big|\ll_\kappa(\log T)T^{\frac\kappa2-\frac12}.$$

Therefore, replacing T by T' if needed, we may assume for the rest of the proof that T satisfies $|T-\gamma|\gg 1/\log T$ for any ordinate γ of a zero of $\zeta(s)$.

Since (cf. Davenport [1, p. 80])

$$\frac{\zeta'(s)}{\zeta(s)}=B-\frac{1}{s-1}+\sum_{n=1}^\infty\Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)+\sum_\rho\Big(\frac{1}{s-\rho}+\frac1\rho\Big) \tag{11}$$

we have by Cauchy's Theorem,

$$U(T):=\sum_{0<\operatorname{Re}\alpha<T}e^{2i\kappa\vartheta(\alpha)}=\frac{1}{2\pi i}\int_{C_T}\frac{\zeta'(s)}{\zeta(s)}\chi(s)^{-\kappa}\,ds. \tag{12}$$

Here the path of integration C_T is the boundary of the rectangle $(\sigma_0,\sigma_1)\times(2\pi,T)$ with $1<\sigma_1<3/2$ and T with $|T-\gamma_n|\gg 1/\log T$ for all n. We will take $\sigma_1=1+\frac{1}{\log T}$, but we maintain the simpler notation σ_1. The restriction $\sigma_1<3/2$ allows one to obtain explicit bounds (independent of σ_1) on all pertinent inequalities.

The value of σ_0 depends on κ. We will take $\sigma_0=\frac12-\frac2\kappa$ for $0<\kappa<\frac43$ and $\sigma_0=-1$ when $\kappa\ge\frac43$. In this way $\sigma_0\le-1$ in all cases.

Then we have

$$U(T)=\frac{1}{2\pi i}\int_{\sigma_0+2\pi i}^{\sigma_1+2\pi i}\frac{1}{\chi(s)^\kappa}\frac{\zeta'(s)}{\zeta(s)}\,ds+\frac{1}{2\pi i}\int_{\sigma_1+2\pi i}^{\sigma_1+Ti}\cdots-\frac{1}{2\pi i}\int_{\sigma_0+Ti}^{\sigma_1+Ti}\cdots-\frac{1}{2\pi i}\int_{\sigma_0+2\pi i}^{\sigma_0+Ti}\cdots$$
$$:=U_1(T)+U_2(T)-U_3(T)-U_4(T).$$

Lemmas 4.1, 4.2 and 4.3 yield

$$U(T)=\mathcal{O}_\kappa(1)+U_2(T)+\mathcal{O}_\kappa(T^{\frac\kappa2}\log T)+\mathcal{O}_\kappa(1).$$

Now we apply Lemma 4.4 and we see that $U_2(T)$ is equal to the sum on the right in (10) plus the remainders $\mathcal{O}_\kappa(T^{\frac\kappa2}\log^2 T)$ and $\mathcal{O}_\kappa(T^{\frac{1-\kappa}{2}}\log T)$. Therefore, we have our result. □

Corollary 3.2. *For $0<\kappa<\frac65$ we have*

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\operatorname{Re}\alpha\le T}e^{2i\kappa\vartheta(\alpha)}=0. \tag{13}$$

For $\kappa=0$ the above limit is easily seen to be 1.

Proof. We have $N(T)=\operatorname{card}\{\alpha:0<\operatorname{Re}\alpha\le T\}=\mathcal{O}(T\log T)$. For $0<\kappa<1$, the trivial bound yields

$$\sum_{n\le(T/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\frac12-\frac1{2\kappa}}}e^{-2\pi i\kappa n^{1/\kappa}}=\mathcal{O}_\kappa(T^{\frac\kappa2+\frac12}\log T)$$

and the limit is easily shown to be 0.

In the case $\kappa = 1$ we apply Theorem 3.1, and observe that in this case

$$\Big|\frac{e^{\pi i/4}}{\sqrt{\kappa}}\sum_{n\le (T/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\frac12-\frac1{2\kappa}}}e^{-2\pi i\kappa n^{1/\kappa}}\Big| \le \sum_{n\le T/2\pi}\Lambda(n) = \mathcal{O}(T).$$

Therefore,

$$\frac{1}{N(T)}\sum_{0<\operatorname{Re}\alpha\le T} e^{2i\kappa\vartheta(\alpha)} = \mathcal{O}(1/\log T).$$

For $1<\kappa<\frac32$, the Piatetski-Shapiro Theorem 2.8 with $k=\kappa$, and $\gamma = 1/\kappa$ yields, for any $\varepsilon>0$,

$$\sum_{n\le x}\Lambda(n)e^{2\pi i\kappa n^\gamma} = \mathcal{O}(x^{\frac{11}{12}+\varepsilon}).$$

Partial summation then yields

$$\sum_{n\le (T/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\frac12-\frac1{2\kappa}}}e^{-2\pi i\kappa n^{1/\kappa}} = \mathcal{O}(T^{\frac{5\kappa}{12}+\frac12+\varepsilon\kappa}).$$

It follows that

$$\frac{1}{N(T)}\sum_{0<\operatorname{Re}\alpha\le T} e^{2i\kappa\vartheta(\alpha)} = \mathcal{O}(T^{\frac{5\kappa}{12}-\frac12+\varepsilon\kappa}/\log T) + \mathcal{O}(T^{\frac{\kappa}{2}-1}\log T).$$

So the limit is 0 for $1<\kappa<\frac65<\frac32$. □

4. Bounds.

4.1. Bound of the bottom integral.

Lemma 4.1. *Uniformly for all $\sigma_1\in(1,3/2)$*

$$U_1(T) = \frac{1}{2\pi i}\int_{\sigma_0+2\pi i}^{\sigma_1+2\pi i}\frac{1}{\chi(s)^\kappa}\frac{\zeta'(s)}{\zeta(s)}\,ds = \mathcal{O}_\kappa(1). \tag{14}$$

Proof. $U_1(T)$ is a well defined and continuous function of $\sigma_1\in[1,3/2]$ (it does not depend on T). □

4.2. Bound of the top integral.

Lemma 4.2. *Let T be such that $|T-\gamma_n|\gg 1/\log T$, and let $\sigma_1 = 1+\frac{1}{\log T}$. Then*

$$U_3(T) = \frac{1}{2\pi i}\int_{\sigma_0+iT}^{\sigma_1+iT}\frac{1}{\chi(s)^\kappa}\frac{\zeta'(s)}{\zeta(s)}\,ds = \mathcal{O}_\kappa(T^{\kappa/2}\log T). \tag{15}$$

Proof. We apply Titchmarsh [21, eq. (4.12.3)] so that

$$\chi(s) = \Big(\frac{2\pi}{t}\Big)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{1}{4}\pi)}\big\{1+\boldsymbol{O}(t^{-1})\big\}$$

on any strip $\alpha \le \sigma \le \beta$ and for $t \to +\infty$. Therefore, we will have

$$\chi(s)^{-\kappa} = \Big(\frac{t}{2\pi}\Big)^{\kappa(\sigma+it-\frac{1}{2})} e^{-i\kappa(t+\frac{1}{4}\pi)}\big\{1+\boldsymbol{O}_\kappa(t^{-1})\big\}. \tag{16}$$

It follows that for $s = \sigma + iT$ with $\sigma_0 < \sigma < \sigma_1$ we will have

$$|\chi(\sigma+iT)|^{-\kappa} \le CT^{\kappa(\sigma-\frac{1}{2})}.$$

We choose T satisfying $|T-\gamma| \gg 1/\log T$, so that by applying Theorem 9.6(A) of Titchmarsh we get on the segment $s = \sigma + iT$ with $-1 < \sigma < \sigma_1$ that

$$\frac{\zeta'(s)}{\zeta(s)} = \boldsymbol{O}(\log^2 T).$$

By Ingham [10, Theorem 27 p. 73] this extends to the entire segment $\sigma_0 < \sigma < \sigma_1$. Then, with the constant C depending on κ.

$$|U_3(T)| \le C\int_{\sigma_0}^{\sigma_1} T^{\kappa(\sigma-1/2)}\log^2(T)\,d\sigma \le C\frac{T^{\kappa(\sigma_1-1/2)}}{\kappa\log T}\log^2 T.$$

Taking $\sigma_1 = 1 + \frac{1}{\log T}$ we get

$$|U_3(T)| = \boldsymbol{O}_\kappa(T^{\kappa/2}\log T).$$

□

4.3. **Bound of the left integral.**

Lemma 4.3. *For* $0 \le \kappa$ *we have*

$$U_4(T) = \frac{1}{2\pi i}\int_{\sigma_0+2\pi i}^{\sigma_0+iT} \frac{1}{\chi(s)^\kappa}\frac{\zeta'(s)}{\zeta(s)}\,ds = \boldsymbol{O}_\kappa(1). \tag{17}$$

Proof. We integrate along the line $s = \sigma_0 + it$ with $2\pi < t < T$. So we may apply Ingham [10, Theorem 27 p. 73] so that

$$\frac{\zeta'(s)}{\zeta(s)} = \boldsymbol{O}(\log t).$$

Also, applying (16) we get

$$U_4(T) \ll_\kappa \int_{2\pi}^{T} t^{\kappa(\sigma_0-\frac{1}{2})}\log t\,dt. \tag{18}$$

The choice of σ_0 ($\sigma_0 = \frac{1}{2} - \frac{2}{\kappa}$ for $0 < \kappa < \frac{4}{3}$, and $\sigma_0 = -1$ when $\kappa \ge \frac{4}{3}$) guarantees that $\kappa(\sigma_0 - \frac{1}{2}) \le -2$. Therefore, the integral is bounded. □

4.4. **Bound of the right integral.**

Lemma 4.4. *Taking* $\sigma_1 = 1 + \frac{1}{\log T}$ *we have*

$$(19)\quad U_2(T) = \frac{1}{2\pi i}\int_{\sigma_1+2\pi i}^{\sigma_1+iT} \frac{1}{\chi(s)^\kappa}\frac{\zeta'(s)}{\zeta(s)}\,ds$$
$$= -\frac{e^{\frac{\pi i}{4}(1-\kappa)}}{\sqrt{\kappa}} \sum_{n<(T/2\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12-\frac{1}{2\kappa}}} e^{-2\pi i\kappa n^{1/\kappa}} + \mathcal{O}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \mathcal{O}_\kappa(T^{\frac{\kappa}{2}}\log^2 T).$$

Proof. We have taken $\sigma_1 > 1$ in order to apply the expression as a Dirichlet series. So

$$U_2(T) = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\frac{1}{2\pi i}\int_{\sigma_1+2\pi i}^{\sigma_1+iT} \frac{1}{\chi(s)^\kappa} e^{-it\log n}\,ds$$

with $s = \sigma_1 + it$. Therefore, by (16)

$$U_2(T) = -\frac{1}{2\pi}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\int_{2\pi}^{T} e^{-it\log n}\Big(\frac{t}{2\pi}\Big)^{\kappa(\sigma_1+it-\frac12)} e^{-i\kappa(t+\frac14\pi)}V(t)\,dt$$

where $V(t) = 1 + \mathcal{O}_\kappa(t^{-1})$. Then

$$U_2(T) = -\frac{1}{2\pi}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\int_{2\pi}^{T} e^{-it\log n}\Big(\frac{t}{2\pi}\Big)^{\kappa(\sigma_1+it-\frac12)} e^{-i\kappa(t+\frac14\pi)}\,dt + R$$

where R is the error term. Then

$$|R| \ll_\kappa \sum_{n=2}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\int_{2\pi}^{T} t^{\kappa(\sigma_1-\frac12)-1}\,dt \ll_\kappa \sum_{n=2}^\infty \frac{\Lambda(n)}{n^{\sigma_1}} T^{\kappa(\sigma_1-\frac12)} = -T^{\kappa(\sigma_1-\frac12)}\frac{\zeta'(\sigma_1)}{\zeta(\sigma_1)}.$$

Therefore, taking $\sigma_1 = 1 + \frac{1}{\log T}$ we get $R = \mathcal{O}_\kappa(T^{\kappa/2}\log T)$.

It remains to compute

$$V_2(T) := \frac{1}{2\pi}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\int_{2\pi}^{T} e^{-it\log n}\Big(\frac{t}{2\pi}\Big)^{\kappa(\sigma_1+it-\frac12)} e^{-i\kappa(t+\frac14\pi)}\,dt =$$
$$= e^{-i\kappa\pi/4}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\sigma_1}}\int_{1}^{T/2\pi} x^{\kappa(\sigma_1-\frac12)} \exp\Big\{2\pi i\Big(\kappa x\log x - \kappa x - x\log n\Big)\Big\}\,dx.$$

These integrals are classical stationary phase integrals. We will apply the theorems in Huxley [8], or [9], and Lemma 2.5. The stationary phase occurs when $\kappa\log x - \log n = 0$. That is for $x = n^{1/\kappa}$. We subdivide each integral in three parts, by dividing the interval of integration $I = (1, T/2\pi)$ in three parts

$$I_0 := I \cap (\tfrac12 n^{1/\kappa}, 2n^{1/\kappa}), \qquad I_1 := I \smallsetminus (\tfrac12 n^{1/\kappa}, +\infty), \qquad I_2 = I \smallsetminus (-\infty, 2n^{1/\kappa}).$$

It is easy to see that $I_0 \cup I_1 \cup I_2 = I$ is a partition. Some of these three intervals may be empty. The partition depends on n, and is different for each integral, but we prefer to maintain a simple notation.

In this way $V_2(T) = V_{2,0}(T) + V_{2,1}(T) + V_{2,2}(T)$ is subdivided in three parts

$$V_{2,j}(T) := e^{-i\kappa\pi/4}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_1}}\int_{I_j} x^{\kappa(\sigma_1-\frac12)}\exp\Big\{2\pi i\Big(\kappa x\log x-\kappa x-x\log n\Big)\Big\}\,dx.$$

4.4.1. *Bound of $V_{2,1}(T)$.* In this case the interval of integration $I_1 := I\setminus(\frac12 n^{1/\kappa},+\infty)$ does not contain the stationary point $n^{1/\kappa}$, and for any point x in this interval we have $x<\frac12 n^{\frac1\kappa}$ so that

$$f'(x)=\kappa\log x-\log n<-\kappa\log 2<0.$$

We may apply Lemma 2.2. Since $I_1\subset(1,T/2\pi)$, the constant V in the lemma is less than

$$V\le 2\cdot\Big(\frac{T}{2\pi}\Big)^{\kappa(\sigma_1-\frac12)}.$$

So, we get

$$|V_{2,1}(T)|\le\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_1}}\frac{2\left(\frac{T}{2\pi}\right)^{\kappa(\sigma_1-\frac12)}}{\pi\kappa\log 2}\ll_\kappa\Big(\frac{T}{2\pi}\Big)^{\kappa(\sigma_1-\frac12)}\frac{|\zeta'(\sigma_1)|}{\zeta(\sigma_1)}.$$

We now choose $\sigma_1=1+\frac{1}{\log T}$ and get

$$|V_{2,1}(T)|\ll_\kappa T^{\frac{\kappa}{2}}\log T.$$

4.4.2. *Bound of $V_{2,2}(T)$.* In this case the interval of integration is $I_2 = I\setminus(-\infty,2n^{1/\kappa})$. Hence, the stationary point $n^{\frac1\kappa}\notin I_2$ and we may apply Lemma 2.2 again. For $x\in I_2$ we have $x\ge 2n^{\frac1\kappa}$ so that

$$f'(x)=\kappa\log x-\log n\ge\kappa\log 2>0.$$

If I_2 is non empty we have $2n^{\frac1\kappa}\le\frac{T}{2\pi}$. The V of Lemma 2.2 is the same as in the previous case so that

$$|V_{2,2}(T)|\le\sum_{n\le(T/4\pi)^\kappa}\frac{\Lambda(n)}{n^{\sigma_1}}\frac{2\left(\frac{T}{2\pi}\right)^{\kappa(\sigma_1-\frac12)}}{\pi\kappa\log 2}$$

and, as before, we get the same bound

$$|V_{2,2}(T)|\ll_\kappa T^{\frac{\kappa}{2}}\log T.$$

4.4.3. *Bound of $V_{2,0}(T)$.* We will apply Lemma 2.5. In our case $a=\kappa(\sigma_1-\frac12)$ and $f(x)=\kappa x\log x-\kappa x-x\log n=\kappa x\log\frac{x}{en^{1/\kappa}}$ so that $r=n^{1/\kappa}$. Since the interval of integration is given by $I_0=(1,T/2\pi)\cap(\frac12 n^{1/\kappa},2n^{1/\kappa})$, we will have $r\in I_0$, only in the case $1<n^{\frac1\kappa}<\frac{T}{2\pi}$. For $n>(T/\pi)^\kappa$ the interval $I_0=\emptyset$. Therefore,

$$V_{2,0}(T)=e^{-i\kappa\pi/4}\sum_{n\le(T/\pi)^\kappa}\frac{\Lambda(n)}{n^{\sigma_1}}\int_{I_0}g_n(x)\exp(2\pi i f_n(x))\,dx$$

so that Lemma 2.5 yields

$$V_{2,0}(T)=e^{-i\kappa\pi/4}\sum_{n<(T/2\pi)^\kappa}\frac{\Lambda(n)}{n^{\sigma_1}}\kappa^{-\frac12}n^{\sigma_1-\frac12+\frac{1}{2\kappa}}e^{\frac{\pi i}{4}}e^{-2\pi i\kappa n^{1/\kappa}}+R \tag{20}$$

where R is bounded by

$$R \ll \sum_{n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\sigma_1}} \Big(n^{\sigma_1-\frac12} + \frac{n^{\sigma_1-\frac12+\frac1\kappa}}{|A_n - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} + \frac{n^{\sigma_1-\frac12+\frac1\kappa}}{|B_n - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} \Big)$$

where $(A_n, B_n) = (1, T/2\pi) \cap (\frac12 n^{1/\kappa}, 2n^{1/\kappa})$.

By partial summation we get

$$\sum_{n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} = \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}).$$

It is easy to see that $A_n = 1$ for $n \le 2^\kappa$ and $A_n = \frac12 n^{\frac1\kappa}$ for $n \ge 2^\kappa$. Then

$$\sum_{n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} \frac{n^{\frac1\kappa}}{|A_n - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} \ll_\kappa \sum_{n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} = \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}).$$

We have $B_n = 2n^{\frac1\kappa}$ if $n \le (T/4\pi)^\kappa$ and $B_n = T/2\pi$ if $n \ge (T/4\pi)^\kappa$. Therefore,

$$\sum_{n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} \frac{n^{\frac1\kappa}}{|B_n - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} \ll_\kappa$$

$$\ll_\kappa \sum_{n\le(T/4\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} + \sum_{(T/4\pi)^\kappa<n<(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} \frac{n^{\frac1\kappa}}{|T/2\pi - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} \ll_\kappa$$

$$\ll_\kappa T^{\frac\kappa2} + T^{\frac{1-\kappa}{2}} \log T + T^{\frac\kappa2} \log^2 T.$$

(See Lemma 4.5 for the last step.)

Hence, (20) implies

$$V_{2,0}(T) = \frac{e^{\frac{\pi i}{4}(1-\kappa)}}{\sqrt{\kappa}} \sum_{n<(T/2\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12-\frac{1}{2\kappa}}} e^{-2\pi i\kappa n^{1/\kappa}} + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}\log^2 T).$$

4.4.4. *End of the proof of Lemma 4.4.* We saw that $U_2(T) = V_2(T) + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}\log T)$, so that

$$V_2(T) = V_{2,0}(T) + V_{2,1}(T) + V_{2,2}(T) = \frac{e^{\frac{\pi i}{4}(1-\kappa)}}{\sqrt{\kappa}} \sum_{n<(T/2\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12-\frac{1}{2\kappa}}} e^{-2\pi i\kappa n^{1/\kappa}} +$$

$$+ \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}\log T) + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}\log^2 T).$$

□

Lemma 4.5. *We have*

$$S_\kappa(T) := \sum_{(T/4\pi)^\kappa<n\le(T/\pi)^\kappa} \frac{\Lambda(n)}{n^{\frac12}} \frac{n^{\frac1\kappa}}{|T/2\pi - n^{\frac1\kappa}| + n^{\frac{1}{2\kappa}}} = \boldsymbol{\mathcal{O}}_\kappa(T^{\frac{1-\kappa}{2}}\log T) + \boldsymbol{\mathcal{O}}_\kappa(T^{\frac\kappa2}\log^2 T). \tag{21}$$

Proof. We have

$$S_\kappa(T) \le \frac{\log\left(\frac{T}{\pi}\right)^\kappa}{\left(\frac{T}{4\pi}\right)^{\frac{\kappa}{2}}} \sum_{(T/4\pi)^\kappa < n \le (T/\pi)^\kappa} \frac{n^{\frac{1}{\kappa}}}{|T/2\pi - n^{\frac{1}{\kappa}}| + n^{\frac{1}{2\kappa}}}$$
$$\le \frac{\log\left(\frac{T}{\pi}\right)^\kappa}{\left(\frac{T}{4\pi}\right)^{\frac{\kappa}{2}}} \left(\frac{T}{\pi}\right) \sum_{(T/4\pi)^\kappa < n \le (T/\pi)^\kappa} \frac{1}{|T/2\pi - n^{\frac{1}{\kappa}}| + n^{\frac{1}{2\kappa}}}$$
$$\ll_\kappa T^{1-\frac{\kappa}{2}} \log T \sum_{(T/4\pi)^\kappa < n \le (T/\pi)^\kappa} \frac{1}{|T/2\pi - n^{\frac{1}{\kappa}}| + \sqrt{T/4\pi}}.$$

Since the function $(|A - x^{\frac{1}{\kappa}}| + B)^{-1}$ is increasing and then decreasing, a geometrical argument yields that the sum is bounded by two times the maximum plus an integral. Thus

$$\sum_{(T/4\pi)^\kappa < n \le (T/\pi)^\kappa} \frac{1}{|T/2\pi - n^{\frac{1}{\kappa}}| + \sqrt{T/4\pi}} \le 2\left(\frac{4\pi}{T}\right)^{\frac{1}{2}} + \int_{(T/4\pi)^\kappa}^{(T/\pi)^\kappa} \frac{dx}{|T/2\pi - x^{\frac{1}{\kappa}}| + \sqrt{T/4\pi}}.$$

We change variables $x = y^\kappa$ and this yields

$$\sum_{(T/4\pi)^\kappa < n \le (T/\pi)^\kappa} \frac{1}{|T/2\pi - n^{\frac{1}{\kappa}}| + \sqrt{T/4\pi}} \le 2\left(\frac{4\pi}{T}\right)^{\frac{1}{2}} + \int_{T/4\pi}^{T/\pi} \frac{\kappa y^{\kappa-1}\, dy}{|T/2\pi - y| + \sqrt{T/4\pi}}$$
$$\le 2\left(\frac{4\pi}{T}\right)^{\frac{1}{2}} + \kappa (T/\pi)^\kappa (4\pi/T) \int_{T/4\pi}^{T/\pi} \frac{dy}{|T/2\pi - y| + \sqrt{T/4\pi}}.$$

It can easily be shown that the last integral is of order $\log T$. In fact it is less than $\log T$ for $T \ge 3$. It follows that

$$0 \le S_\kappa(T) \ll_\kappa T^{1-\frac{\kappa}{2}} \log T (T^{-\frac{1}{2}} + T^{\kappa-1} \log T)$$

completing the proof of Lemma 4.5. □

5. Acknowledgement

I wish to express my thanks to Jan van de Lune (Hallum, The Netherlands) for his encouragements and linguistic assistance.

References

[1] H. Davenport *Multiplicative number theory*, 3rd edition, revised and with a preface by Hugh L. Montgomery, Springer-Verlag, New York, 2000.
[2] H. M. Edwards, *Riemann's zeta function*, Dover Publications Inc., Mineola, NY, 2001.
[3] A. Fujii, *On the uniformity of the distribution of the zeros of the Riemann zeta function*, J. Reine Angew. Math., **302**, (1978), 167–205.
[4] A. Fujii, *Some observations concerning the distribution of the zeros of the zeta functions. III*, Proc. Japan Acad. Ser. A Math. Sci., **68**, (1992), 105–110.
[5] S. M. Gonek, *Mean values of the Riemann zeta function and its derivatives*, Invent. Math., **75**, (1984), 123–141.
[6] L. Kuipers, & H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New York, 1974.

[7] G. H. Hardy, & J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math., **41**, (1920), 119–196.
[8] M. N. Huxley, *On stationary phase integrals*, Glasgow Math. J., **36**, (1994), 355–362.
[9] M. N. Huxley, *Area, lattice points, and exponential sums*, Oxford University Press, New York, 1996.
[10] A. E. Ingham, *The distribution of prime numbers*, reprint of the 1932 original, with a foreword by R. C. Vaughan, Cambridge University Press, Cambridge, 1990.
[11] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons Inc., New York, 1985.
[12] N. Levinson, *More than one third of zeros of Riemann's zeta-function are on* $\sigma = 1/2$, Advances in Math., **13**, (1974), 383–436.
[13] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, R.I. , 1973, 181–193.
[14] A. M. Odlyzko, *The* 10^{20}*-th zero of the Riemann zeta function and 175 millions of its neighbors*, 1992, 1–163.
[15] I. I. Pyateckiĭ-Šapiro, *On the distribution of prime numbers in sequences of the form* $[f(n)]$, Mat. Sbornik N.S., **33(75)**, (1953), 559–566, (translation contained in [16, p.39–47]).
[16] I. Piatetski-Shapiro, *Selected works of Ilya Piatetski-Shapiro*, Edited and with commentaries by James Cogdell, Simon Gindikin, Peter Sarnak, Pierre Deligne, Stephen Gelbart, Roger Howe and Stephen Rallis. American Mathematical Society, Providence, RI, 2000.
[17] H. Rademacher, *Collected papers of Hans Rademacher. Vol. II*, Edited and with a preface by Emil Grosswald, MIT Press, Cambridge, Mass., 1974.
[18] H. Rademacher, *Fourier Analysis in Number Theory*, Symposium on Harmonic Analysis and Related Integral Transforms, Cornell University, Ithaca, N.Y. , 1956, 1–25, (contained in [17, p.434–459]).
[19] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Berlin Akad, (1859), 671–680.
[20] J. Schoißengeier, *The connection between the zeros of the* ζ*-function and sequences* $(g(p))$*, p prime,* mod 1, Monatsh. Math., **87**, (1979), 21–52.
[21] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed. with a preface by D. R. Heath-Brown, Oxford University Press, New York, 1986.

Facultad de Matemáticas, Univ. de Sevilla, Apdo. 1160, 41080-Sevilla, Spain

E-mail address: arias@us.es