

Quantitative Weighted Mixed Weak-Type Inequalities for Classical Operators

SHELDY OMBROSI, CARLOS PÉREZ & JORGELINA RECCHI

ABSTRACT. We improve on several mixed weak-type inequalities both for the Hardy-Littlewood maximal function and for Calderón-Zygmund operators. These types of inequalities were considered by Muckenhoupt and Wheeden and later on by Sawyer estimating the $L^{1,\infty}(uv)$ norm of $v^{-1}T(fv)$ for special cases. The emphasis is made in proving new and more precise quantitative estimates involving the A_p or A_∞ constants of the weights involved.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Let M denote the usual Hardy-Littlewood maximal function; then, according to a fundamental result of B. Muckenhoupt [Mu], M is a bounded operator on the Lebesgue space $L^p(d\mu)$, $1 < p < +\infty$, if and only if $d\mu = w(x)dx$ and the weight w satisfies the simple geometric condition

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q in \mathbb{R}^n . This is the celebrated Muckenhoupt A_p condition. A similar result holds in the case $p = 1$: namely, M is of weak type $(1, 1)$ with respect to μ ; that is, $M : L^1(\mu) \rightarrow L^{1,\infty}(\mu)$ if and only if $d\mu = w(x) dx$ and the weight w satisfies the A_1 condition,

$$[w]_{A_1} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) (\text{ess inf}_Q w)^{-1} < \infty,$$

where, again, the supremum is taken over all cubes Q in \mathbb{R}^n .

Since the A_p theorem of Muckenhoupt plays a central role in modern harmonic analysis, different proofs besides the original in [Mu] have been considered in the literature. In particular, E. Sawyer tried in [Sa] the following approach based on the factorization theorem for A_p weights of P. Jones (see [GCRdF]). Recall that a weight w satisfies the A_p condition if and only if there are two A_1 weights u and v such that

$$w = uv^{1-p}.$$

Then, if the operator $Sf = \frac{M(vf)}{v}$ is defined, the boundedness of M on $L^p(w)$ may be rewritten as

$$(1.1) \quad \int_{\mathbb{R}^n} |Sf|^p uv \, dx \leq c \int_{\mathbb{R}^n} |f|^p uv \, dx.$$

Observe now that since $v \in A_1$, $Mv \leq [v]_{A_1} v$, and hence S is bounded in $L^\infty(uv)$. Therefore, if we show that S is of weak type $(1, 1)$ with respect to the measure $uv \, dx$, we can apply the Marcinkiewicz interpolation theorem to derive (1.1). This is precisely the statement of the following theorem from [Sa].

Theorem 1.1. *If $u, v \in A_1(\mathbb{R})$, then*

$$\left\| \frac{M(g)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \|g\|_{L^1(u)},$$

where c depends only on the A_1 constant of u and the A_1 constant of v . This shows that the operator $Sf = v^{-1}M(vf)$ is of weak type $(1, 1)$ with respect to the measure $vu \, dx$.

In the same article, Sawyer conjectured that this theorem should also hold for the maximal function in \mathbb{R}^n and for the Hilbert transform H instead of M .

The article of Sawyer was also very much motivated by a previous work of B. Muckenhoupt and R. Wheeden [MW]. This time, the main result of this paper holds for both the one-dimensional Hardy-Littlewood maximal function and the Hilbert transform. To be more precise, the main result proved in [MW] is the following.

Theorem 1.2. *Let $w \in A_1(\mathbb{R})$; there exists then a constant c such that*

$$\|M(fw^{-1})w\|_{L^{1,\infty}(\mathbb{R})} \leq c \|f\|_{L^1(\mathbb{R})}$$

and

$$\|H(fw^{-1})w\|_{L^{1,\infty}(\mathbb{R})} \leq c \|f\|_{L^1(\mathbb{R})}.$$

In [C-UMP1], the authors extended both Theorems 1.1 and 1.2 to \mathbb{R}^n , including in particular the conjectures formulated by Sawyer mentioned above. The precise result is the following.

Theorem 1.3. *Suppose $u \in A_1$ and that either $v \in A_1$ or $v \in A_\infty(u)$; then, there exists a constant c such that*

$$(1.2) \quad \left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \|f\|_{L^1(uv)}$$

and

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \|f\|_{L^1(uv)},$$

where M is the Hardy-Littlewood maximal operator and T is a Calderón-Zygmund operator.

We note that this result holds for T^* , the maximal singular integral operator, instead of T . Given weights u and v , by $v \in A_\infty(u)$, we mean that v satisfies the A_∞ condition defined with respect to the measure $u \, dx$ (as opposed to Lebesgue measure). A more precise definition is given in Section 2 below.

We emphasize that this theorem contains both Theorems 1.1 and 1.2 as particular cases. Indeed, the case of the first theorem is clear. For the second, if $w \in A_1$, we let $u = w$ and $v = w^{-1}$. Then, $uv = 1 \in A_\infty$, and thus $v \in A_\infty(u)$ by Lemma 2.1 and Observation 2.2.

To prove Theorem 1.3, the authors show that it suffices to prove the result for the dyadic maximal function M_d by proving an extrapolation-type theorem, Theorem 1.5 below, that allows one to replace T or M by M_d . To be more precise, the combination of the following two theorems from [C-UMP1] proves Theorem 1.3.

Theorem 1.4. *Suppose that $u \in A_1$ and that either $v \in A_1$ or $v \in A_\infty(u)$; then, there exists a constant c such that*

$$(1.3) \quad \left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \|f\|_{L^1(uv)}.$$

Theorem 1.5. *Given a family \mathcal{F} of a pair of functions, suppose that for some $p \in (0, \infty)$ and for every $w \in A_\infty$,*

$$\|f\|_{L^p(w)} \leq C \|g\|_{L^p(w)},$$

for all $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and where C depends only on the A_∞ constant of w . Then, for all weights $u \in A_1$ and $v \in A_\infty$,

$$\|fv^{-1}\|_{L^{1,\infty}(uv)} \leq C \|gv^{-1}\|_{L^{1,\infty}(uv)} \quad (f, g) \in \mathcal{F}.$$

Here, \mathcal{F} denotes a family of ordered pairs of non-negative, measurable functions (f, g) .

Theorem 1.5 from [C-UMP1] is used to pass from M to M_d , since by standard methods, for every $p \in (0, \infty)$ and every $w \in A_\infty$,

$$\|M(fv)\|_{L^p(w)} \leq c \|M_d(fv)\|_{L^p(w)},$$

where the constant c involves the A_∞ constant of w . However, there are recent results showing that Theorem 1.5 can be avoided in the transition from M to M_d . Indeed, using for instance [HP, p. 792], we have that

$$Mf \leq c_n \sum_{\alpha \in \{0,1/3\}^n} M_d^\alpha f,$$

where M_d^α is an appropriate shifted dyadic maximal function with similar properties as M_d . Thus, the expression on the left in (1.2) is bounded by a dimensional-constant multiple of the corresponding expression for M_d^α . Since each of these M_d^α has similar properties as M_d , the corresponding proof of (1.3) is exactly the same.

In [C-UMP1], the authors conjectured that Theorem 1.4 still holds under milder hypotheses on the weight v . To be more precise, the authors state what is now known as “Sawyer’s conjecture,” although E. Sawyer never asserted it. The conjecture is the following.

Conjecture 1.6. *Suppose that $u \in A_1$ and $v \in A_\infty$. Then, there exists a constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \|f\|_{L^1(uv)}.$$

Note that if $v \in A_\infty(u)$ (we always assume $u \in A_1$), then $v \in A_\infty$ (see Lemmas 2.1 and 2.3). This conjecture has been open for several years and has been studied by different authors.

In this paper, we try to understand the difficulties of this conjecture, and propose alternative ways to prove it. We will also study how the constants of the weights u and v are reflected in these inequalities; that is, we look for quantitative versions of this type of inequality.

The first question we pose concerning Sawyer’s theorem is the following:

What is the sharp dependence on the constants of the weights u and v when both are in A_1 ?

Following the proof given in [C-UMP1], which is an adaptation of the original proof given by Sawyer in [Sa] for the real line, we show the dependence on the weight constants. More specifically, we prove the following result.

Theorem 1.7. *If $u \in A_1$ and $v \in A_1$, there exists a dimensional constant c such that*

$$(1.4) \quad \left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c [u]_{A_1}^2 [v]_{A_1}^4 \|f\|_{L^1(uv)}.$$

The proof may be found in Section 7.

We believe that the dependency on the constants in inequality (1.4) is not sharp, since the method does not seem to be adequate. Trying to understand this issue, we will focus on the special case $u = 1$ that is interesting in its own right. The finiteness of the estimate in this special case is assured by Theorem 1.3 by

assuming even a weaker condition on v than A_1 , namely, $v \in A_\infty(u) = A_\infty$. The method we use is different from the one considered in the proof of Theorem 1.7 allowing us to obtain more precise estimates. In particular, we will prove the linearity of the constant bound of the weight v if we assume the stronger condition $v \in A_1$ and the result is sharp. Our theorem is the following.

Theorem 1.8. *Let $v \in A_1$. There exists a dimensional constant c , independent from $[v]_{A_1}$, such that*

$$(1.5) \quad \left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(v)} \leq c[v]_{A_1} \|f\|_{L^1(\mathbb{R}^n)}.$$

Furthermore, the linear dependence on $[v]_{A_1}$ is sharp.

However, we want to understand the more general case.

Problem 1.9. *Find an increasing function $\varphi : [1, \infty] \rightarrow [1, \infty]$ for which the following inequality holds whenever $v \in A_\infty$:*

$$(1.6) \quad \left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(v)} \leq c\varphi([v]_{A_\infty}) \|f\|_{L^1(\mathbb{R}^n)},$$

where c is a constant that depends on the dimension.

This problem is a special case of Conjecture 1.6 with $u = 1$, and it will be studied in Section 4. The best constant in (1.6), $\varphi([v]_{A_\infty})$, is finite by Theorem 1.3. Our goal is to determine the best dependence on the constant of the weight v , or, in other words, to find the smallest function φ . Recall that $A_\infty = \bigcup_{p \geq 1} A_p$, and that, if $w \in A_\infty$ we use the weight constant

$$(1.7) \quad [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) \, dx,$$

called the Fujii-Wilson constant in some recent papers. We could use instead the constant defined by Hrušev in [Hr], which is more natural; however, it was shown in [HP] that it is much larger than the one given by the functional (1.7).

We note here that a condition on the weight v in (1.5) or (1.6) must be taken into account. Indeed, there are estimates like

$$(1.8) \quad \left\| \frac{M(f)}{Mw} \right\|_{L^{1,\infty}(Mw)} \leq c \|f\|_{L^1(\mathbb{R}^n)},$$

namely with $v = Mw$, that are false for a general function w or measure. This will be shown in Section 5 where, furthermore, an interesting relationship with the two weight problem for singular integrals is implicit in the argument. In general, weights of the form Mw are not A_∞ weights but small perturbations, namely, when $v = (Mw)^\delta$, $\delta \in (0, 1)$, makes the inequality to be true, since

in this case $v \in A_1$ and Theorem 1.5 applies. It is interesting that in special situations and for large perturbations of the weight, the result is still true. Indeed, if $v(x) = |x|^{-nr} \approx (M\delta)^r$ with $r > 1$, then there is a finite constant c such that

$$\left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(v)} \leq c \|f\|_{L^1(\mathbb{R}^n)},$$

with the result being false in the case $r = 1$. This was proved in dimension one by Andersen and Muckenhoupt in [AM], and by Martín-Reyes, Ortega Salvador, and Sarrión Gavián [MOS] in higher dimensions. We note that these weights $v(x) = |x|^{-nr}$ are not A_∞ weights.

In view of Theorem 1.8 and the case $v = 1$, we state the following conjecture for the general case.

Conjecture 1.10. *Let $u \in A_1$ and $v \in A_1$; then, there exists a dimensional constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c [u]_{A_1} [v]_{A_1} \|f\|_{L^1(uv)}.$$

To see that the dependency cannot be better than $[u]_{A_1} [v]_{A_1}$, we prove the following result that strengthens our conjecture.

Theorem 1.11. *Let $u \in A_1, v \in A_1$. If*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq c \varphi([u]_{A_1}, [v]_{A_1}) \|f\|_{L^1(uv)},$$

then there is a constant c independent of the weights such that

$$\varphi([u]_{A_1}, [v]_{A_1}) \geq c [u]_{A_1} [v]_{A_1}.$$

Another related problem, partly intermediate between the previous two problems, would be to determine how the dependence on the constant $[v]_{A_p}$ is if we assume that $v \in A_p$ for some $p \geq 1$. We should also take into account that Theorem 1.8 gives the sharp dependence on the real line when assuming the stronger assumption $v \in A_1$. Based on this, we state the following conjecture.

Conjecture 1.12. *Let $v \in A_p, p \geq 1$; then, there exists a dimensional constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c [v]_{A_p} \|f\|_{L^1(v)}.$$

We were not able to prove this conjecture, but we have obtained the following result using an adequate Calderón-Zygmund decomposition that involves the A_∞ constant of the weight.

Theorem 1.13. *Let $v \in A_p$, $p \geq 1$; then, there exists a dimensional constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c[v]_{A_\infty} \max\{p, \log(e + [v]_{A_p})\} \|f\|_{L^1(v)}.$$

Corollary 1.14. *Let $v \in A_p$, $p \geq 1$; then, there exists a dimensional constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq C_n[v]_{A_p} \max\{p, \log(e + [v]_{A_p})\} \|f\|_{L^1(v)}.$$

We also try to improve the dependency on the weight constant using some other refined constants that were introduced in [HP] and formalized in the work of Lerner and Moen [LM].

Theorem 1.15. *Let $v \in A_p$, $p \geq 1$; then, there exists a dimensional constant c such that*

$$\left\| \frac{M_d(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq cp[v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}} \log(e + [v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}}) \|f\|_{L^1(v)}.$$

We defer to Section 2 for the definition of $[v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}}$.

In this paper, we will also study similar problems for Calderón-Zygmund operators instead of the Hardy-Littlewood maximal function. In particular, we improve the following theorem from [HP].

Theorem 1.16. *Suppose that T is a Calderón-Zygmund operator; then, there is a dimensional constant c such that, for any $v \in A_1$,*

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c[v]_{A_1} \log(e + [v]_{A_\infty}) \|f\|_{L^1(v)}.$$

This theorem improved the following result previously obtained in [LOP2]:

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c[v]_{A_1} \log(e + [v]_{A_1}) \|f\|_{L^1(v)}.$$

In Section 5, we give a version of Corollary 1.14 for Calderón-Zygmund operators. We prove the following result.

Theorem 1.17. *Suppose T is a Calderón-Zygmund Operator; then, there is a dimensional constant c such that, for any $v \in A_p$,*

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c[v]_{A_p} \max\{p, \log(e + [v]_{A_p})\} \|f\|_{L^1(v)}.$$

2. PRELIMINARIES

As usual, a weight will be a non-negative locally integrable function. Given a weight w , $p \in (1, \infty)$, and a cube Q , we denote

$$A_p(w; Q) := \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} = \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p},$$

where $\sigma = w^{-1/(p-1)}$. When $p = 1$ we define the limiting quantity as

$$A_1(w; Q) := \left(\frac{1}{|Q|} \int_Q w \right) (\inf_Q w)^{-1} = \lim_{p \rightarrow 1} A_p(w, Q).$$

For $p = \infty$, we consider two constants. The first constant is defined as a limit of the $A_p(w; Q)$ constants

$$A_\infty^{\text{exp}}(w; Q) := \left(\frac{1}{|Q|} \int_Q w \right) \exp \left(\frac{1}{|Q|} \int_Q \log w^{-1} \right) = \lim_{p \rightarrow \infty} A_p(w, Q).$$

To define the second constant, we let

$$A_\infty^W(w; Q) := \frac{1}{w(Q)} \int_Q M(\chi_Q w),$$

and define

$$[w]_{A_p} = \sup_Q A_p(w; Q), \quad \|w\|_{A_\infty} = \sup_Q A_\infty^{\text{exp}}(w; Q), \quad [w]_{A_\infty} = \sup_Q A_\infty^W(w; Q).$$

We write $w \in A_p$ if $[w]_{A_p} < \infty$ and $w \in A_\infty$ if $\|w\|_{A_\infty} < \infty$ or $[w]_{A_\infty} < \infty$. The constant $\|w\|_{A_\infty}$ was defined by Hrušćev in [Hr]. The constant $[w]_{A_\infty}$ was defined by Fujii in [F] and rediscovered by M. Wilson in [W1, W3], who also showed that both constants define the class A_∞ . In [HP], the authors proved the estimate

$$[w]_{A_\infty} \leq c_n \|w\|_{A_\infty},$$

and provided examples showing that $\|w\|_{A_\infty}$ can be exponentially larger than $[w]_{A_\infty}$.

We now define the mixed-type constants. Given $1 \leq p < \infty$ and $\alpha, \beta \geq 0$, motivated by some results for the two weighted estimates for the maximal function in [HP], Lerner and Moen in [LM] defined the mixed constants

$$[w]_{(A_p)^\alpha (A_r)^\beta} = \sup_Q A_p(w; Q)^\alpha A_r(w; Q)^\beta, \quad 1 \leq r < \infty,$$

the exponential mixed constants

$$(2.1) \quad [w]_{(A_p)^\alpha (A_\infty^{\text{exp}})^\beta} = \sup_Q A_p(w; Q)^\alpha A_\infty^{\text{exp}}(w; Q)^\beta,$$

and the Fujii-Wilson mixed constants

$$[w]_{(A_p)^\alpha(A_\infty^W)^\beta} = \sup_Q A_p(w; Q)^\alpha A_\infty^W(w; Q)^\beta.$$

If $\alpha > 0$, the class of weights that satisfies $[w]_{(A_p)^\alpha(A_\infty^W)^\beta} < \infty$ is simply the class A_p , since

$$\max([w]_{A_p}^\alpha, [w]_{A_\infty}^\beta) \leq [w]_{(A_p)^\alpha(A_\infty^W)^\beta} \leq [w]_{A_p}^{\alpha+\beta}.$$

Analogously, a weight w satisfies $[w]_{(A_p)^\alpha(A_\infty^{\text{exp}})^\beta} < \infty$ if and only if w is in A_p such that the inequality holds for the exponential mixed constant. In [LM], the authors show that if $0 < \alpha \leq \beta \leq 1$ and $w \in A_p$, then

$$(2.2) \quad [w]_{(A_p)^\alpha(A_\infty^{\text{exp}})^{1-\alpha}} \leq [w]_{(A_p)^\beta(A_\infty^{\text{exp}})^{1-\beta}}.$$

We finish this section by defining the generalized A_∞ class of weights $A_\infty(\mu)$ where μ is a doubling measure. To do this, we recall some well-known definitions about generalized Hardy-Littlewood maximal operators. For a complete account, we refer the reader to [D, GCRdF].

Given a doubling measure μ , we define the maximal operator M_μ by

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y).$$

For $1 < p < \infty$, given a weight w , we say that $w \in A_p(\mu)$ if, for all cubes Q ,

$$\left(\frac{1}{\mu(Q)} \int_Q w(x) \, d\mu(x) \right) \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'} \, d\mu(x) \right)^{p-1} \leq C.$$

We say that $w \in A_1(\mu)$ if $M_\mu w(x) \leq Cw(x)$. We denote the union of all the $A_p(\mu)$ classes by $A_\infty(\mu)$, that is,

$$A_\infty(\mu) = \bigcup_{p \geq 1} A_p(\mu).$$

Since μ is doubling, then M_μ is bounded on $L^p(w \, d\mu)$, $1 < p < \infty$, if and only if $w \in A_p(\mu)$. As usual, when μ is the Lebesgue measure we omit the subscript μ , and write simply M or A_p . Also, if μ is absolutely continuous given by the weight u , we then simply write $A_p(u)$, $1 \leq p \leq \infty$.

The next two lemmas were proved in [C-UMP2].

Lemma 2.1. *If $u \in A_1$ and $v \in A_\infty(u)$, then $uv \in A_\infty$. In particular, if $v \in A_p(u)$, $1 \leq p < \infty$, then $uv \in A_p$.*

Observation 2.2. *If $u \in A_1$, then $v \in A_\infty(u)$ if and only if $uv \in A_\infty$.*

Lemma 2.3. *If $u \in A_1$ and $uv \in A_\infty$, then $v \in A_\infty$.*

3. THE A_1 CASE

Proof of Theorem 1.8. As usual, we denote M^c , the centered Hardy-Littlewood maximal operators, and its corresponding centered weighted M_v^c maximal function. Now, by standard arguments,

$$\frac{M(fv)}{v} \approx \frac{M^c(fv)}{v} \leq \frac{M^c v}{v} M_v^c(f) \leq \frac{Mv}{v} M_v^c(f) \leq [v]_{A_1} M_v^c(f),$$

and then

$$\left\| \frac{M(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq c_n [v]_{A_1} \|M_v^c(f)\|_{L^{1,\infty}(v)} \leq c_n [v]_{A_1} \|f\|_{L^1(v)},$$

by the Besicovitch covering lemma.

The proof will be completed by showing that the linear exponent is the best possible. To see this, it is sufficient to consider $f(x) = (1/\delta)\chi_{(0,1)}(x)$ and $v(x) = |x|^{\delta-1}$, where $0 < \delta < 1$. Then, standard computations show $[v]_{A_1} \sim \frac{1}{\delta}$. On the other hand, we can compute

$$M(fv) \geq \begin{cases} \frac{1}{\delta} \frac{1}{x^{1-\delta}} & \text{if } x \in (0, 1), \\ \frac{1}{\delta^2} \frac{1}{x} & \text{if } x \in (1, \infty), \\ \frac{1}{\delta^2} \frac{1}{1-x} & \text{if } x \in (-\infty, 0), \end{cases}$$

and therefore $(0, \delta^{-2/\delta}) \subset \{x \mid M(fv) > v\}$. Continuing, we have

$$v \{x \mid M(fv) > v\} \geq v(0, \delta^{-2/\delta}) = \int_0^{\delta^{-2/\delta}} x^{\delta-1} dx = \frac{1}{\delta^3} = [v]_{A_1} \frac{1}{\delta^2},$$

but $\int_{\mathbb{R}} f(x)v(x) dx = \int_0^1 (1/\delta)x^{\delta-1} dx = 1/\delta^2$. □

Proof of Theorem 1.11. We let $f(x) = (1/\delta)\chi_{(0,1)}(x)$, and define $u(x) = \alpha\chi_{(0,1)}(x) + \chi_{(0,1)^c}(x)$, where $0 < \alpha < 1$ and $v(x) = |x|^{\delta-1}$, where $0 < \delta < 1$. Then, standard computations show that

$$[u]_{A_1} \sim \frac{1}{\alpha} \quad \text{and} \quad [v]_{A_1} \sim \frac{1}{\delta}.$$

Also, we have

$$M(fv) \geq \begin{cases} \frac{1}{\delta} \frac{1}{x^{1-\delta}} & \text{if } x \in (0, 1), \\ \frac{1}{\delta^2} \frac{1}{x} & \text{if } x \in (1, \infty) \\ \frac{1}{\delta^2} \frac{1}{1-x} & \text{if } x \in (-\infty, 0). \end{cases}$$

Then, $(0, \delta^{-2/\delta}) \subset \{x \mid M(fv) > v\}$, and then

$$uv \{x \mid M(fv) > v\} \geq uv(1, \delta^{-2/\delta}) = \int_1^{\delta^{-2/\delta}} x^{\delta-1} dx = \frac{1}{\delta}(\delta^{-2} - 1) \approx \frac{1}{\delta^3}.$$

On the other hand,

$$\int_{\mathbb{R}} f(x)u(x)v(x) dx = \frac{\alpha}{\delta} \int_0^1 x^{\delta-1} dx = \frac{\alpha}{\delta^2};$$

this proves $\varphi([u]_{A_1}, [v]_{A_1}) \gtrsim [u]_{A_1} [v]_{A_1}$. □

Observation 3.1. When considering the case $\alpha = \delta$, we can notice that $\varphi([u]_{A_1}, [v]_{A_1})$ cannot be $\max([u]_{A_1}, [v]_{A_1})$.

4. THE A_p CASE

Proof of Theorem 1.13. Without loss of generality, we may assume that f is non-negative and bounded with compact support. Let $v \in A_p$; then, $v \in A_r$, $r > p$ with $[v]_{A_r} \leq [v]_{A_p}$. Fix $t > 0$, and let $r > p$ be a parameter that will be chosen in a moment. Since $v \in A_r$, in particular, $v dx$ is a doubling weight. Therefore, we can form the Calderón-Zygmund decomposition of f at height t with respect to the measure $v(x) dx$. This yields a collection of disjoint dyadic maximal cubes $\{Q_j\}$ such that, for all Q_j ,

$$t < \frac{1}{v(Q_j)} \int_{Q_j} f(x)v(x) dx \leq \frac{v(Q'_j)}{v(Q_j)v(Q'_j)} \int_{Q'_j} f(x)v(x) dx \leq 2^{nr} [v]_{A_r} t,$$

where Q'_j is the ancestor of Q_j , and where the last inequality is obtained by using standard properties of the A_p weights (see Proposition 9.1.5 in [G]) and by using the maximality property of the Q_j .

Further, if we let $\Omega := \bigcup_j Q_j$, then $f(x) \leq t$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. We decompose f as $g + b$, where

$$g(x) = \begin{cases} \frac{1}{v(Q_j)} \int_{Q_j} f(x)v(x) dx & \text{if } x \in Q_j, \\ f(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and let $b(x) = \sum_j b_j(x)$, with

$$b_j(x) = \left(f(x) - \frac{1}{v(Q_j)} \int_{Q_j} f(x)v(x) dx \right) \chi_{Q_j}(x).$$

If we use this definition, we have that $g(x) \leq 2^{nr} [v]_{A_r} t$ for almost every $x \in \mathbb{R}^n$, and

$$\int_{Q_j} b_j(x)v(x) dx = 0.$$

Following [C-UMP1], if Q is a dyadic cube, then, for all $x \in Q$,

$$\begin{aligned} \frac{1}{|Q|} \int_Q f(x)v(x) dx &= \frac{1}{|Q|} \int_Q g(x)v(x) dx + \frac{1}{|Q|} \int_Q b(x)v(x) dx \\ &\leq M_d(gv)(x) + \tilde{M}_d(bv)(x), \end{aligned}$$

where

$$\tilde{M}_d(h)(x) = \sup_{x \in Q} \left| \frac{1}{|Q|} \int_Q h(y) dy \right|.$$

Then, if the supremum is taken over all dyadic cubes containing x , we have

$$M_d(fv) \leq M_d(gv) + \tilde{M}_d(bv).$$

Now,

$$\begin{aligned} &v \left(\left\{ x \in \mathbb{R}^n : \frac{M_d(fv)(x)}{v(x)} > t \right\} \right) \\ &\leq v \left(\left\{ x \in \mathbb{R}^n : \frac{M_d(gv)(x)}{v(x)} > \frac{t}{2} \right\} \right) + v \left(\left\{ x \in \Omega : \frac{\tilde{M}_d(bv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \\ &\quad + v \left(\left\{ x \in \mathbb{R}^n \setminus \Omega : \frac{\tilde{M}_d(bv)(x)}{v(x)} > \frac{t}{2} \right\} \right) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

To estimate I_1 we will use the following improvement of Buckley’s theorem (see [Bu]), whose proof can be found in [HP].

Lemma 4.1. *Let $1 < p < \infty$ and $v \in A_p$; then,*

$$\|M\|_{L^p(v)} \leq c_n p' [v]_{A_p}^{1/p} [v^{1-p'}]_{A_\infty}^{1/p},$$

where c_n is a dimensional constant.

We then have, after applying the Chebyshev inequality,

$$\begin{aligned} I_1 &\leq \frac{2^{r'}}{t^{r'}} \int_{\mathbb{R}^n} M(gv)^{r'} v^{1-r'} dx \\ &\leq \frac{C_n^{r'}}{t^{r'}} r^{r'} [v]_{A_r}^{r'-1} [v]_{A_\infty} \int_{\mathbb{R}^n} g^{r'} v dx. \end{aligned}$$

Since $g(x) \leq 2^{nr} [v]_{A_r} t$ and $[v]_{A_r} \leq [v]_{A_p}$, we have

$$\begin{aligned} I_1 &\leq \frac{C_n^{r'}}{t} r^{r'} [v]_{A_\infty} [v]_{A_r}^{2r'-2} \int_{\mathbb{R}^n} g(x)v(x) dx \\ &\leq \frac{C_n^{r'}}{t} r^{r'} [v]_{A_\infty} [v]_{A_p}^{2r'-2} \int_{\mathbb{R}^n} g(x)v(x) dx. \end{aligned}$$

Finally, if we let

$$r = 1 + \max\{p, \log(e + [v]_{A_p})\},$$

then

$$r' = 1 + \frac{1}{\max\{p, \log(e + [v]_{A_p})\}},$$

and a computation shows that $r^{r'}$ behaves like $\max\{p, \log(e + [v]_{A_p})\}$ and that $[v]_{A_p}^{2r'-2}$ is bounded. Therefore,

$$\begin{aligned} I_1 &\leq \frac{C_n}{t} [v]_{A_\infty} \max\{p, \log(e + [v]_{A_p})\} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus \Omega} f(x)v(x) dx + \sum_j \left(\frac{1}{v(Q_j)} \int_{Q_j} f(x)v(x) dx \right) v(Q_j) \right) \\ &\leq \frac{C_n}{t} [v]_{A_\infty} \max\{p, \log(e + [v]_{A_p})\} \int_{\mathbb{R}^n} f(x)v(x) dx. \end{aligned}$$

The estimate for I_2 follows immediately from the properties of the cubes Q_j :

$$I_2 \leq v(\Omega) = \sum_j v(Q_j) \leq \sum_j \frac{1}{t} \int_{Q_j} f(x)v(x) dx \leq \frac{1}{t} \int_{\mathbb{R}^n} f(x)v(x) dx.$$

Finally, we will prove that $I_3 = 0$. To see this, fix $x \in \mathbb{R}^n \setminus \Omega$; since b has support in Ω , to compute $\tilde{M}_d(bv)$ we only need to consider cubes which intersect Ω . Fix such a cube Q , and for each j , either $Q_j \subset Q$ or $Q \cap Q_j = \emptyset$. Then, since

$$\int_{Q_j} b_j(x)v(x) dx = 0,$$

we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q b(x)v(x) \, dx &= \frac{1}{|Q|} \sum_j \int_{Q \cap Q_j} b_j(x)v(x) \, dx \\ &= \frac{1}{|Q|} \sum_{Q_j \subset Q} \int_{Q_j} b_j(x)v(x) \, dx = 0. \end{aligned} \quad \square$$

We will use the following lemma for mixed A_p - A_∞ constants as defined in (2.1).

Lemma 4.2. *Let $p > 1$ and let $v \in A_p$; then,*

$$[v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}} \leq [v]_{A_p} \leq [v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}}^p.$$

Proof. The second inequality follows from a simple consequence of a Jensen inequality:

$$e^{(1/|Q|) \int_Q \log w(x) \, dx} \leq \frac{1}{|Q|} \int_Q w(x) \, dx,$$

which implies $[(1/|Q|) \int_Q w(x) \, dx] e^{(1/|Q|) \int_Q \log w^{-1}(x) \, dx}]^{p-1} \geq 1$, and then

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q w(x) \, dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx\right)^{p-1} \\ &\leq \left(\frac{1}{|Q|} \int_Q w(x) \, dx\right)^p \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx\right)^{p-1} \\ &\quad \times \left(e^{(1/|Q|) \int_Q \log w^{-1}(x) \, dx}\right)^{p-1}, \end{aligned}$$

whence we obtain

$$[v]_{A_p} \leq [v]_{(A_p)^{1/p}(A_\infty^{\text{exp}})^{1/p'}}^p.$$

The first inequality also follows from Jensen’s inequality in the form

$$e^{(1/|Q|) \int_Q \log w(x)^{-1} \, dx} \leq \left(\frac{1}{|Q|} \int_Q w(x)^{-\alpha} \, dx\right)^{1/\alpha}, \quad \alpha > 0,$$

if we consider the case $\alpha = p' - 1$. □

We also need the following lemma that will play an important role in the proof of Theorem 1.15. It is an improvement of Buckley’s theorem (see [Bu]), and the proof can be found in [HP].

Lemma 4.3. *Let $1 < p < \infty$ and $v \in A_p$; then,*

$$\|M\|_{L^p(v)} \leq c_n p' [v^{1-p'}]_{(A_{p'})^{1/p'}(A_\infty^{\text{exp}})^{1/p}},$$

where c_n is a dimensional constant.

Proof of Theorem 1.15. The structure of the proof is the same as that of Theorem 1.13. The only difference is in the analysis of I_1 . Indeed, combining the Chebyshev inequality with Lemma 4.3, we arrive at

$$I_1 \leq \frac{2^{r'}}{t^{r'}} \int_{\mathbb{R}^n} M(gv)^{r'} v^{1-r'} dx \leq \frac{2^{r'}}{t^{r'}} r^{r'} [v]_{(A_r)^{1/r} (A_\infty^{\text{exp}})^{1/r'}} \int_{\mathbb{R}^n} g^{r'} v dx,$$

and since $g(x) \leq 2^{nr} [v]_{A_r} t$, we have

$$I_1 \leq \frac{2^{r'(1+n)}}{t} r^{r'} [v]_{A_r}^{r'-1} [v]_{(A_r)^{1/r} (A_\infty^{\text{exp}})^{1/r'}} \int_{\mathbb{R}^n} g(x) v(x) dx.$$

As $r > p$, $[v]_{A_r} \leq [v]_{A_p}$ and by (2.2), $[v]_{(A_r)^{1/r} (A_\infty^{\text{exp}})^{1/r'}} \leq [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}}$. Finally, if we let

$$r = 1 + \max\{p, \log(e + [v]_{A_p})\},$$

then

$$r' = 1 + \frac{1}{\max\{p, \log(e + [v]_{A_p})\}}.$$

It is easy to see that $r^{r'}$ behaves like

$$\max\{p, \log(e + [v]_{A_p})\},$$

that $[v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}}$ behaves like $[v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}}$, and that $[v]_{A_p}^{r'-1}$ is bounded by a universal constant. Moreover, since $2^{r'(1+n)} \leq 2^{2(1+n)}$, we have that

$$I_1 \leq \frac{C_n}{t} [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}} \max\{p, \log(e + [v]_{A_p})\} \times \left(\int_{\mathbb{R}^n \setminus \Omega} f(x) v(x) dx + \sum_j \left(\frac{1}{v(Q_j)} \int_{Q_j} f(x) v(x) dx \right) v(Q_j) \right).$$

Now, by Lemma 4.2, we have

$$\begin{aligned} \max\{p, \log(e + [v]_{A_p})\} &\leq \max\{p, p \log(e + [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}})\} \\ &= p \log(e + [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}}), \end{aligned}$$

and then,

$$I_1 \leq \frac{C_n}{t} p [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}} \log(e + [v]_{(A_p)^{1/p} (A_\infty^{\text{exp}})^{1/p'}}) \int_{\mathbb{R}^n} f(x) v(x) dx.$$

This concludes the proof of the theorem. □

5. COUNTER-EXAMPLES

In this section, we show that inequality (1.8) is false. To do this, we proceed by contradiction, assuming that this inequality holds. We begin with the following duality argument for any weight w :

$$\|Tf\|_{L^p(w)} = \sup_{h: \|h\|_{L^{p'}(w)}=1} \left| \int_{\mathbb{R}^n} Tfhw \, dx \right|.$$

Fixing one of these h , we have

$$\int_{\mathbb{R}^n} Tfhw \, dx = \int_{\mathbb{R}^n} fT^t(hw) \, dx = \int_{\mathbb{R}^n} f \frac{T^t(hw)}{Mw} Mw \, dx$$

and then

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Tfhw \, dx \right| &\leq \|f\|_{L^p(Mw)} \left\| \frac{T^t(hw)}{Mw} \right\|_{L^{p'}(Mw)} \\ &= \|f\|_{L^p(Mw)} \|T^t f\|_{L^{p'}(Mw)^{1-p'}}. \end{aligned}$$

We now use the following lemma, which is a particular version of the classical estimate of Coifman-Fefferman for any Calderón-Zygmund operator T : let $p \in (0, \infty)$, and let $w \in A_\infty$; then, there is a constant c depending upon p, T and the A_∞ constant of w , such that

$$(5.1) \quad \|Tf\|_{L^p(w)} \leq c_{T,p,[w]_{A_\infty}} \|Mf\|_{L^p(w)}.$$

Then, as a consequence we have the following special situation. Let w be *any weight*, and let $p \in (1, \infty)$. Then, there is a constant depending only on p and T such that

$$(5.2) \quad \|Tf\|_{L^p(Mw)^{1-p}} \leq c \|Mf\|_{L^p(Mw)^{1-p}}.$$

This follows from (5.1) and the fact that $(Mw)^{1-p} \in A_\infty$. Indeed, since

$$(Mw)^{1-p} = (Mw)^{\delta(1-2p)} \in A_{2p} \quad \text{and} \quad \delta = \frac{p-1}{2p-1} < \frac{1}{2},$$

we have $[(Mw)^{1-p}]_{A_\infty} \leq [(Mw)^\delta]_{A_1}^{2p-1} \leq c_n^p$.

(It should be mentioned that (5.2) was improved in [LOP3] and later in [LOP1]. In these papers, the relevance was the sharpness of the constant c in terms of p , which behaves linearly in p , but is not important in our context. See also [R] for a similar estimate within the fractional integrals context.)

Then, since T^t is also a Calderón-Zygmund operator, we apply (5.2):

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Tfhw \, dx \right| &\leq c_{p,T} \|f\|_{L^p(Mw)} \|M(hw)\|_{L^{p'}(Mw)^{1-p'}} \\ &= c_{p,T} \|f\|_{L^p(Mw)} \left\| \frac{M(hw)}{Mw} \right\|_{L^{p'}(Mw)}. \end{aligned}$$

We now apply (1.8), which is equivalent to

$$\left\| \frac{M(fw)}{Mw} \right\|_{L^{1,\infty}(Mw)} \leq c \|f\|_{L^1(w)},$$

and since the operator $f \rightarrow M(fw)/(Mw)$ is trivially bounded on L^∞ with constant 1, we apply the Marcinkiewicz’s interpolation theorem to deduce

$$\left| \int_{\mathbb{R}^n} Tfhw \, dx \right| \leq c_p \|f\|_{L^p(Mw)} \|h\|_{L^{p'}(w)} = c_p \|f\|_{L^p(Mw)}.$$

Then, for any Calderón-Zygmund operator T and arbitrary weight w , we have produced the estimate

$$\|Tf\|_{L^p(w)} \leq c_p \|f\|_{L^p(Mw)}.$$

However, this inequality is well known to be false for any $p \in (1, \infty)$, as was shown by M. Wilson in [W2] for the simplest case, namely, the Hilbert transform.

6. CALDERÓN-ZYGMUND INTEGRAL OPERATOR

In this section, we show the following inequality:

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(v)} \leq C_n [v]_{A_p} \max\{p, \log(e + [v]_{A_p})\} \int_{\mathbb{R}^n} |f(x)|v(x) \, dx.$$

For the proof of this inequality, the following two results are needed. The first result was proved in [Hy], and the second result can be found in [GCRdF, p. 413].

Theorem 6.1. *Let $1 < p < \infty$, w an A_p -weight and T a Calderón-Zygmund operator; then, $\|Tf\|_{L^p(w)} \leq c_n p p' [w]_{A_p}^{\max(1, 1/(p-1))}$.*

Lemma 6.2. *Let w be a weight. There is a dimensional constant c_d such that, for any cube Q and for any function f supported in a cube Q with $\int_Q f(x) \, dx = 0$, the following inequality holds:*

$$\int_{\mathbb{R}^n \setminus 2Q} |Tf(y)|w(y) \, dy \leq c_n \int_Q |f(y)|Mw(y) \, dy.$$

The structure of the proof of Theorem 1.17 is similar to that of Theorem 1.13.

Proof of Theorem 1.17. Without loss of generality, we assume f is bounded and has compact support. Since $v \in A_p$, then for all $r > p$, we have $v \in A_r$, with $[v]_{A_r} \leq [v]_{A_p}$.

Fix $t > 0$. For now, let $r > p$ be arbitrary, as we will assign a specific value to r . Since $v \in A_r$, in particular, $v \, dx$ is a doubling weight. Therefore, we can

form the Calderón-Zygmund decomposition of f at height t with respect to the measure $\nu \, dx$. This yields a collection of disjoint dyadic maximal cubes $\{Q_j\}$, such that for all Q_j ,

$$t < \frac{1}{\nu(Q_j)} \int_{Q_j} f(x)\nu(x) \, dx \leq \frac{\nu(Q'_j)}{\nu(Q_j)\nu(Q'_j)} \int_{Q'_j} f(x)\nu(x) \, dx \leq 2^{nr} [\nu]_{A_r} t,$$

where as before Q'_j is the ancestor of Q_j , and where the last inequality is obtained by using standard properties of the A_p weights (see Proposition 9.1.5 in [G]), and by the maximal property of the Q_j . Further, if we let $\Omega := \cup_j Q_j$, then $f(x) \leq t$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. We decompose f as $g + b$, where

$$g(x) = \begin{cases} \frac{1}{\nu(Q_j)} \int_{Q_j} f(x)\nu(x) \, dx & \text{if } x \in Q_j, \\ f(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and we let $b(x) = \sum_j b_j(x)$, with

$$b_j(x) = \left(f(x) - \frac{1}{\nu(Q_j)} \int_{Q_j} f(x)\nu(x) \, dx \right) \chi_{Q_j}(x).$$

If we use these definitions, we have that $g(x) \leq 2^{nr} [\nu]_{A_r} t$ for almost every $x \in \mathbb{R}^n$, and

$$\int_{Q_j} b_j(x)\nu(x) \, dx = 0.$$

Then, since T is a sublinear operator, we have that

$$\begin{aligned} & \nu \left(\left\{ x \in \mathbb{R}^n : \frac{|T(f\nu)(x)|}{\nu(x)} > t \right\} \right) \\ & \leq \nu \left(\left\{ x \in \mathbb{R}^n : \frac{|T(g\nu)(x)|}{\nu(x)} > \frac{t}{2} \right\} \right) \\ & \quad + \nu \left(\left\{ x \in \tilde{\Omega} : \frac{|T(b\nu)(x)|}{\nu(x)} > \frac{t}{2} \right\} \right) \\ & \quad + \nu \left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : \frac{|T(b\nu)(x)|}{\nu(x)} > \frac{t}{2} \right\} \right) \\ & = I_1 + I_2 + I_3. \end{aligned}$$

For I_1 we combine the Chebyshev inequality with Theorem 6.1, bearing in mind that as $\nu \in A_r$, we have $\nu^{1-r'} \in A_{r'}$ with

$$[\nu^{1-r'}]_{A_{r'}} = [\nu]_{A_r}^{r'-1}.$$

Now, since the exponent of the constant $[v]_{A_r}$ in Lemma 6.1 is different if $p > 2$ or $p \leq 2$, we have divided the proof into two cases.

Case $p > 2$. In this case, as $r > 2$, we have $r' < 2$ and $\max(1, 1/(r' - 1)) = 1/(r' - 1)$. Thus,

$$I_1 \leq \frac{2^{r'}}{t^{r'}} \int_{\mathbb{R}^n} |T(gv)(x)|^{r'} v(x)^{1-r'} dx \leq \frac{C_n^{r'}}{t^{r'}} r^{r'} [v]_{A_r}^{r'} \int_{\mathbb{R}^n} g(x)^{r'} v(x) dx.$$

Then, since $g(x) \leq 2^{nr} [v]_{A_r} t$ and $[v]_{A_r} \leq [v]_{A_p}$, we obtain that

$$I_1 \leq \frac{C_n^{r'}}{t} r^{r'} [v]_{A_r}^{2r'-1} \int_{\mathbb{R}^n} g(x) v(x) dx \leq \frac{2^{r'(1+n)}}{t} r^{r'} [v]_{A_p}^{2r'-1} \int_{\mathbb{R}^n} g(x) v(x) dx.$$

As $r > p > 2$, we choose

$$r = 1 + \max\{p, \log(e + [v]_{A_p})\};$$

then,

$$2 > r' = 1 + \frac{1}{\max\{p, \log(e + [v]_{A_p})\}}.$$

For this reason, $r^{r'}$ behaves like $\max\{p, \log(e + [v]_{A_p})\}$ and $[v]_{A_p}^{2r'-1}$ like $[v]_{A_p}$:

$$\begin{aligned} I_1 &\leq \frac{C_n}{t} [v]_{A_p} \max\{p, \log(e + [v]_{A_p})\} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus \Omega} f(x) v(x) dx + \sum_j \left(\frac{1}{v(Q_j)} \int_{Q_j} f(x) v(x) dx \right) v(Q_j) \right) \\ &\leq \frac{C_n}{t} [v]_{A_p} \max\{p, \log(e + [v]_{A_p})\} \int_{\mathbb{R}^n} f(x) v(x) dx. \end{aligned}$$

Case $p \leq 2$. We choose $r = 1 + 2 \log(e + [v]_{A_p}) > 2 \geq p$; thus,

$$r' = 1 + \frac{1}{2 \log(e + [v]_{A_p})} < 2,$$

and $\max(1, 1/(r' - 1)) = 1/(r' - 1)$. We can now proceed analogously to the previous case:

$$I_1 \leq \frac{C_d^{r'}}{t} r^{r'} [v]_{A_p}^{2r'-1} \int_{\mathbb{R}^n} g(x) v(x) dx,$$

and therefore,

$$I_1 \leq \frac{C_d}{t} \log(e + [v]_{A_p}) \int_{\mathbb{R}^n} f(x) v(x) dx.$$

The estimate for I_2 follows immediately from the properties of the cubes Q_j and from the inequality $v(2Q) \leq 2^{np}[v]_{A_p} v(Q)$, as follows:

$$\begin{aligned} I_2 &\leq v(\tilde{\Omega}) \leq \sum_j v(2Q_j) \leq 2^{np}[v]_{A_p} \sum_j \frac{1}{t} \int_{Q_j} f(x)v(x) \, dx \\ &\leq 2^{np}[v]_{A_p} \frac{1}{t} \int_{\mathbb{R}^n} f(x)v(x) \, dx. \end{aligned}$$

Finally, to be able to estimate I_3 , we use Lemma 6.2 with $w \equiv 1$:

$$\begin{aligned} I_3 &\leq \frac{2}{t} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |T(bv)(x)| \, dx \leq \frac{2}{t} \sum_j \int_{\mathbb{R}^n \setminus 2Q_j} |T(b_j v)(x)| \, dx \\ &\leq \frac{2}{t} \sum_j \int_{Q_j} |b_j(x)|v(x) \, dx. \end{aligned}$$

If we have used the definition of b_j , we have that

$$I_3 \leq \frac{cd}{t} \|f v\|_{L^1(\mathbb{R}^n)}. \quad \square$$

7. AN ADAPTATION OF SAWYER’S PROOF WITH CONTROL OF THE CONSTANT

In this appendix, we prove Theorem 1.7 using a method similar to the one considered in [C-UMP1] for the proof of Theorem 1.4.

The statement of the theorem assumes that the weights belong to the A_1 class of weights. These weights satisfy a reverse Hölder inequality: namely, if $w \in A_1$, then there are two constants $r, c > 1$ such that

$$\left(\frac{1}{|Q|} \int_Q w^r \right)^{1/r} \leq \frac{c}{|Q|} \int_Q w.$$

However, we note that in the classical proofs there is a bad dependence on the constant $c = c(r, [w]_{A_1})$, and we need a more precise estimate to get our results.

Lemma 7.1. *Let $w \in A_1$, and let $r_w = 1 + 1/(2^{n+1}[w]_{A_1})$. Then, for any cube Q ,*

$$\left(\frac{1}{|Q|} \int_Q w^{r_w} \right)^{1/r_w} \leq \frac{2}{|Q|} \int_Q w.$$

As a consequence, we have that for any cube Q and for any measurable set $E \subset Q$,

$$\frac{w(E)}{w(Q)} \leq 2 \left(\frac{|E|}{|I|} \right)^{\varepsilon_w},$$

where $\varepsilon_w = 1/(1 + 2^{n+1}[w]_{A_1})$.

The proof of this reverse Hölder inequality can be found in [LOP1], and the consequence is an application of Hölder’s inequality.

Proof of Theorem 1.7. Fix $t > 0$, and define $g = fv/t$. Then, it is sufficient to show that

$$(7.1) \quad uv(\{x \in \mathbb{R}^n \mid M_d(g)(x) > v(x)\}) \leq C \int_{\mathbb{R}^n} |g(x)|u(x) dx,$$

for any function g bounded with compact support.

Fix $a > 2^n$. For each $k \in \mathbb{Z}$, let $\{I_j^k\}$ be the collection of maximal, disjoint dyadic cubes whose union is the set

$$\Omega_k = \{x \in \mathbb{R}^n \mid M_d v(x) > a^k\} \cap \{x \in \mathbb{R}^n \mid M_d g(x) > a^k\}.$$

This decomposition exists since g is bounded and has compact support, so the second set is contained in the union of maximal dyadic cubes. Define

$$\Gamma = \{(k, j) : |I_j^k \cap \{x : v(x) \leq a^{k+1}\}| > 0\}.$$

As $v \in A_1$, we have $Mv(x) \leq [v]_{A_1} v(x)$ almost everywhere. Hence, for $(k, j) \in \Gamma$,

$$(7.2) \quad \begin{aligned} \frac{a^k}{[v]_{A_1}} &\leq \frac{1}{[v]_{A_1}} \operatorname{ess\,inf}_{x \in I_j^k} M_d v(x) \leq \operatorname{ess\,inf}_{x \in I_j^k} v(x) \\ &\leq \frac{1}{|I_j^k|} \int_{I_j^k} v(x) dx \leq [v]_{A_1} a^{k+1}. \end{aligned}$$

(Intuitively, if $(k, j) \in \Gamma$, then I_j^k behaves like a cube from the Calderón-Zygmund decomposition of v at height a^k .) Then, up to a set of measure zero, we have the following inclusions: for each k ,

$$\{x \in \mathbb{R}^n \mid a^k < v(x) \leq a^{k+1}\} \cap \{x \in \mathbb{R}^n \mid M_d g(x) > v(x)\} \subset \bigcup_{j:(k,j) \in \Gamma} I_j^k.$$

Combining this with (7.2), we get that

$$uv(\{x \in \mathbb{R}^n \mid M_d g(x) > v(x)\}) \leq a[v]_{A_1} \sum_{(k,j) \in \Gamma} |I_j^k|^{-1} v(I_j^k) u(I_j^k).$$

Fix $N < 0$, and define $\Gamma_N = \{(k, j) \in \Gamma \mid k \geq N\}$. We will show that

$$\sum_{(k,j) \in \Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C \int_{\mathbb{R}^n} |g(x)|u(x) dx,$$

where the constant C does not depend on N . Inequality (7.1) then follows if we take the limit as $N \rightarrow -\infty$. To prove this, we replace the set of cubes $\{I_j^k\}$ by a subset with better properties. First, since $v \in A_1$, we can apply Lemma 7.1; and there exists $\varepsilon = (1 + 2^{n+1}[v]_{A_1})^{-1} > 0$ such that, given any cube I and $E \subset I$,

$$(7.3) \quad \frac{v(E)}{v(I)} \leq 2 \left(\frac{|E|}{|I|} \right)^\varepsilon.$$

Fix δ such that $0 < \delta < \varepsilon$. Define $\Delta_N = \{I_j^k \mid (k, j) \in \Gamma_N\}$. The cubes in Δ_N are all dyadic, so they are either pairwise disjoint or one is contained in the other. For $k > t$, since $\Omega_k \subset \Omega_t$ and since the cubes I_j^k are maximal in Ω_k , if $I_s^t \cap I_j^k \neq \emptyset$, then $I_j^k \subset I_s^t$. In particular, each cube $I_j^k \in \Delta_N$ is contained in $\bigcup_j I_j^N \subset \{x \mid M_d g(x) > a^N\}$. As we noted above, the last set is bounded, so Δ_N contains a maximal disjoint subcollection of cubes.

We form a sequence of sets $\{G_n\}$ by induction. Let G_0 be the set of all pairs $(k, j) \in \Gamma_N$ such that I_j^k is maximal in Δ_N . For $n \geq 0$, given the set G_n , define the set G_{n+1} to be the set of pairs $(k, j) \in \Gamma_N$ such that there exists $(t, s) \in G_n$ with $I_j^k \not\subset I_s^t$ and

$$(7.4) \quad \frac{1}{|I_j^k|} \int_{I_j^k} u(x) \, dx > a^{(k-t)\delta} \frac{1}{|I_s^t|} \int_{I_s^t} u(x) \, dx,$$

$$(7.5) \quad \frac{1}{|I_i^\ell|} \int_{I_i^\ell} u(x) \, dx \leq a^{(\ell-t)\delta} \frac{1}{|I_s^t|} \int_{I_s^t} u(x) \, dx,$$

whenever $(\ell, i) \in \Gamma_N$ and $I_j^k \not\subset I_i^\ell \subset I_s^t$.

Let $P = \bigcup_{n \geq 0} G_n$. Given $(s, t) \in P$, we refer to the cube I_s^t as a *principal cube*. Since every cube in Δ_N is contained in a maximal cube, every cube in Δ_N is contained in one or more principal cubes.

To continue, we divide the proof into several steps of the same form. We only look at the behavior of the A_1 -constants, and give the main ideas of the steps.

Step 1. We claim that

$$(7.6) \quad \sum_{(k,j) \in \Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C_\varepsilon \sum_{(k,j) \in P} |I_j^k|^{-1} v(I_j^k) u(I_j^k).$$

To prove this, fix $(t, s) \in P$, and let $Q = Q(t, s)$ be the set of indices $(k, j) \in \Gamma_N$ such that $I_j^k \subset I_s^t$ and I_s^t is the smallest principal cube containing I_j^k . In particular, each I_j^k is not a principal cube unless it equals I_s^t .

Thus, by (7.5), and since $I_j^k \subset \{x \mid M_d v(x) > a^k\}$,

$$\sum_{(k,j) \in Q} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq |I_s^t|^{-1} u(I_s^t) \sum_{k \geq t} a^{(k-t)\delta} v(I_s^t \cap \{x \mid M_d v(x) > a^k\}).$$

By (7.3) and (7.2), and since $v \in A_1$,

$$v(I_s^t \cap \{x \mid M_d v(x) > a^k\}) \leq 2[v]_{A_1}^{2\varepsilon} a^\varepsilon a^{(t-k)\varepsilon} v(I_s^t).$$

Combining these inequalities, we see that

$$\sum_{(k,j) \in Q} |J_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C_\varepsilon |I_s^t|^{-1} u(I_s^t) v(I_s^t),$$

where $C_\varepsilon = (a^{2\varepsilon-\delta}/(a^{\varepsilon-\delta} - 1))2[v]_{A_1}^{2\varepsilon}$.

If we now sum over all $(s, t) \in P$, we get (7.6), since $\bigcup_{(t,s) \in P} Q(t, s) = \Gamma_N$.

Step 2. For each k , let $\{J_i^k\}$ be the collection of maximal disjoint cubes whose union is $\{x \mid M_d g(x) > a^k\}$. Then,

$$a^k < \frac{1}{|J_i^k|} \int_{J_i^k} g(x) dx.$$

For each j , we have $I_j^k \subset \{x \mid M_d g(x) > a^k\}$, so there exists a unique $i = i(j, k)$ such that $I_j^k \subset J_i^k$. Hereafter, the index i will always be this function of (k, j) . Hence, by (7.6) and by (7.2),

$$\sum_{(k,j) \in \Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C_\varepsilon a[v]_{A_1} \int_{\mathbb{R}^n} h(x) g(x) dx,$$

where $h(x) = \sum_{(k,j) \in P} |J_i^k|^{-1} \chi_{J_i^k}(x) u(I_j^k)$.

To complete the proof, we show that for each x , $h(x) \leq C u(x)$. Fix $x \in \mathbb{R}^n$; without loss of generality, we may assume that $u(x)$ is finite. For each k , there exists at most one cube J_b^k such that $x \in J_b^k$. If it exists, denote this cube by J^k .

Define $P_k = \{(k, j) \in P \mid I_j^k \subset J^k\}$, and $G = \{k \mid P_k \neq \emptyset\}$. We form a sequence $\{k_m\}$ by induction. If $k \in G$, then $k \geq N$, so let k_0 be the least integer in G . Given K_m , $m \geq 0$, choose $k_{m+1} > k_m$ in G such that

$$(7.7) \quad \frac{1}{|J^{k_{m+1}}|} \int_{J^{k_{m+1}}} u(y) dy > \frac{2}{|J^{k_m}|} \int_{J^{k_m}} u(y) dy,$$

$$(7.8) \quad \frac{1}{|J^\ell|} \int_{J^\ell} u(y) dy \leq \frac{2}{|J^{k_m}|} \int_{J^{k_m}} u(y) dy,$$

where $k_m \leq \ell < k_{m+1}$, $\ell \in G$.

Since $u(x)$ is finite, the sequence $\{K_m\}$ only contains a finite number of terms. Then, by (7.8), we have

$$h(x) \leq \sum_m \frac{2}{|J^{k_m}|} \int_{J^{k_m}} u(y) dy \sum_{\ell \in G, k_m \leq \ell < k_{m+1}} \sum_{(\ell, j) \in P_\ell} \frac{u(I_j^\ell)}{u(J^\ell)}.$$

We claim that

$$(7.9) \quad \sum_{\ell \in G, k_m \leq \ell < k_{m+1}} \sum_{(\ell, j) \in P_\ell} \frac{u(I_j^\ell)}{u(J^\ell)} \leq C_9.$$

Given this, we are done: since the sequence $\{k_m\}$ is finite, let m be the largest index. Then, by (7.7) and (7.9),

$$h(x) \leq 2C_9 \left(2 - \left(\frac{1}{2}\right)^m\right) [u]_{A_1} u(x).$$

Therefore, to complete the proof we must show (7.9). We do this in two steps.

Step 3. The authors of [C-U~~M~~P1] proved that if $(\ell, j) \in P_\ell$, $k_m \leq \ell < k_{m+1}$, then

$$(7.10) \quad \frac{1}{|I_j^\ell|} \int_{I_j^\ell} u(y) \, dy > \frac{a^{(\ell-k_m)\delta}}{2[u]_{A_1}} \frac{1}{|J^\ell|} \int_{J^\ell} u(y) \, dy.$$

Step 4. We now prove (7.9). By (7.10), and again since $u \in A_1$, if $y \in I_j^\ell$, then

$$\lambda = \frac{a^{(\ell-k_m)\delta}}{2[u]_{A_1}} \frac{u(J^\ell)}{|J^\ell|} \frac{1}{[u]_{A_1}} < u(y);$$

hence,

$$\bigcup_{j: (\ell, j) \in P_\ell} I_j^\ell \subset \{x \in J^\ell \mid u(x) > \lambda\}.$$

For ℓ fixed, the cubes I_j^ℓ are disjoint. Therefore, since we have $u \in A_1$, there exist $\nu = (1 + 2^{n+1}[u]_{A_1})^{-1}$ such that

$$\sum_{j: (\ell, j) \in P_\ell} u(I_j^\ell) \leq 2^{1+\nu} u(J^\ell) [u]_{A_1}^{2\nu} a^{(k_m-\ell)\delta\nu}.$$

Therefore, we have that

$$\sum_{\ell \in G, k_m \leq \ell < k_{m+1}} \sum_{(\ell, j) \in P_\ell} \frac{u(I_j^\ell)}{u(J^\ell)} \leq C_9,$$

where $C_9 = 2^{1+\nu} [u]_{A_1}^{2\nu} a^{\delta\nu} / (a^{\delta\nu} - 1)$. Then, the constant C of Theorem 1.7 behaves like

$$\frac{a^{\varepsilon-\delta}}{a^{\varepsilon-\delta} - 1} 2^{3+\nu} a^{\varepsilon+2} [v]_{A_1}^{2\varepsilon+2} [u]_{A_1}^{2\nu+1} \frac{a^{\delta\nu}}{a^{\delta\nu} - 1} \left(2 - \left(\frac{1}{2}\right)^m\right) \approx [v]_{A_1}^4 [u]_{A_1}^2. \quad \square$$

Acknowledgements. The second author is supported by the Spanish government grant MTM-2014-53850-P, and the first and third authors are supported by Universidad Nacional del Sur and CONICET.

The authors would also like to thank the referee, whose suggestions and comments have been very helpful in improving the presentation of the paper.

REFERENCES

- [AM] K. F. ANDERSEN AND B. MUCKENHOUPT, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, *Studia Math.* **72** (1982), no. 1, 9–26. MR665888 (83k:42018).
- [Bu] S. M. BUCKLEY, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, *Trans. Amer. Math. Soc.* **340** (1993), no. 1, 253–272. <http://dx.doi.org/10.2307/2154555>. MR1124164 (94a:42011).
- [C-UMP1] D. V. CRUZ-URIBE, J. M. MARTELL, AND C. PÉREZ, *Weighted weak-type inequalities and a conjecture of Sawyer*, *Int. Math. Res. Not.* **30** (2005), 1849–1871. <http://dx.doi.org/10.1155/IMRN.2005.1849>. MR2172941 (2006i:42014).
- [C-UMP2] ———, *Weights, Extrapolation and the Theory of Rubio de Francia*, *Operator Theory: Advances and Applications*, vol. 215, Birkhäuser/Springer Basel AG, Basel, 2011. <http://dx.doi.org/10.1007/978-3-0348-0072-3>. MR2797562 (2012f:42001).
- [D] J. DUOANDIKOETXEA, *Fourier Analysis*, *Graduate Studies in Mathematics*, vol. 29, American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. MR1800316 (2001k:42001).
- [F] N. FUJII, *Weighted bounded mean oscillation and singular integrals*, *Math. Japon.* **22** (1977/78), no. 5, 529–534. MR0481968 (58 #2058).
- [GCRdF] J. GARCÍA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted Norm Inequalities and Related Topics*, *North-Holland Mathematics Studies*, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. *Notas de Matemática [Mathematical Notes]*, 104. MR807149 (87d:42023).
- [G] L. GRAFAKOS, *Modern Fourier Analysis*, 2nd ed., *Graduate Texts in Mathematics*, vol. 250, Springer, New York, 2009. <http://dx.doi.org/10.1007/978-0-387-09434-2>. MR2463316 (2011d:42001).
- [Hr] S. V. HRUŠČEV, *A description of weights satisfying the A_∞ condition of Muckenhoupt*, *Proc. Amer. Math. Soc.* **90** (1984), no. 2, 253–257. <http://dx.doi.org/10.2307/2045350>. MR727244 (85k:42049).
- [Hy] T. P. HYTÖNEN, *The sharp weighted bound for general Calderón-Zygmund operators*, *Ann. of Math. (2)* **175** (2012), no. 3, 1473–1506. <http://dx.doi.org/10.4007/annals.2012.175.3.9>. MR2912709.
- [HP] T. P. HYTÖNEN AND C. PÉREZ, *Sharp weighted bounds involving A_∞* , *Anal. PDE* **6** (2013), no. 4, 777–818. <http://dx.doi.org/10.2140/apde.2013.6.777>. MR3092729.
- [LM] A. K. LERNER AND K. MOEN, *Mixed A_p - A_∞ estimates with one supremum*, *Studia Math.* **219** (2013), no. 3, 247–267. <http://dx.doi.org/10.4064/sm219-3-5>. MR3145553.
- [LOP1] A. K. LERNER, S. OMBROSI, AND C. PÉREZ, *Sharp A_1 bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden*, *Int. Math. Res. Not. IMRN* **6** (2008), Art. ID rnm161, 11. <http://dx.doi.org/10.1093/imrn/rnm161>. MR2427454 (2009e:42030).
- [LOP2] ———, *Weak type estimates for singular integrals related to a dual problem of Muckenhoupt-Wheeden*, *J. Fourier Anal. Appl.* **15** (2009), no. 3, 394–403. <http://dx.doi.org/10.1007/s00041-008-9032-2>. MR2511869 (2010h:42021).
- [LOP3] ———, *A_1 bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden*, *Math. Res. Lett.* **16** (2009), no. 1, 149–156. <http://dx.doi.org/10.4310/MRL.2009.v16.n1.a14>. MR2480568 (2010a:42052).

- [Mu] B. MUCKENHOUP, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226. <http://dx.doi.org/10.1090/S0002-9947-1972-0293384-6>. MR0293384 (45 #2461).
- [MOS] F. J. MARTÍN-REYES, P. ORTEGA SALVADOR, AND M. D. SARRIÓN GAVILÁN, *Boundness of operators of Hardy type in $\Lambda^{p,q}$ spaces and weighted mixed inequalities for singular integral operators*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 1, 157–170. <http://dx.doi.org/10.1017/S0308210500023556>. MR1433089 (99d:42025).
- [MW] B. MUCKENHOUP AND R. L. WHEEDEN, *Some weighted weak-type inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Indiana Univ. Math. J. **26** (1977), no. 5, 801–816. <http://dx.doi.org/10.1512/iumj.1977.26.26065>. MR0447956 (56 #6266).
- [R] J. RECCHI, *Mixed A_1 - A_∞ bounds for fractional integrals*, J. Math. Anal. Appl. **403** (2013), no. 1, 283–296. <http://dx.doi.org/10.1016/j.jmaa.2013.02.018>. MR3035092.
- [Sa] E. SAWYER, *A weighted weak type inequality for the maximal function*, Proc. Amer. Math. Soc. **93** (1985), no. 4, 610–614. <http://dx.doi.org/10.2307/2045530>. MR776188 (86e:42033).
- [W1] J. M. WILSON, *Weighted inequalities for the dyadic square function without dyadic A_∞* , Duke Math. J. **55** (1987), no. 1, 19–50. <http://dx.doi.org/10.1215/S0012-7094-87-05502-5>. MR883661 (88d:42034).
- [W2] ———, *Weighted norm inequalities for the continuous square function*, Trans. Amer. Math. Soc. **314** (1989), no. 2, 661–692. <http://dx.doi.org/10.2307/2001403>. MR972707 (91e:42025).
- [W3] ———, *Weighted Littlewood-Paley Theory and Exponential-square Integrability*, Lecture Notes in Mathematics, vol. 1924, Springer, Berlin, 2008. MR2359017 (2008m:42034).

SHELDY OMBROSI & JORGELINA RECCHI:

Departamento de Matemáticas
 Universidad Nacional del Sur
 8000 Bahía Blanca, Argentina
 E-MAIL: sombrosi@uns.edu.ar; drecchi@uns.edu.ar

CARLOS PÉREZ:
 Department of Mathematics
 University of the Basque Country, Ikerbasque and BCAM
 Bilbao, Spain

KEY WORDS AND PHRASES: Maximal operators, Calderón-Zygmund operators, weighted estimates.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 42B20, 42B25, 46E30.

Received: August 26, 2014; revised: June 7, 2016.