

A Computational Approach to the \mathcal{D} -module of Meromorphic Functions

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Abstract

Let D be a divisor in \mathbf{C}^n . We present methods to compare the \mathcal{D} -module of the meromorphic functions $\mathcal{O}[*D]$ to some natural approximations. We show how the analytic case can be treated with computations in the Weyl algebra.

1 Introduction

Let us denote by $\mathcal{O} = \mathcal{O}_{\mathbf{C}^n}$ the sheaf of holomorphic functions on $X = \mathbf{C}^n$. Consider a point $p \in \mathbf{C}^n$. $Der(\mathcal{O}_p)$ is the \mathcal{O}_p -module of \mathbf{C} -derivations of \mathcal{O}_p . The elements in $Der(\mathcal{O}_p)$ are called *vector fields*.

Let $D \subset X$ be a divisor (i.e. a hypersurface) and $p \in D$. A vector field $\delta \in Der(\mathcal{O}_p)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_p$, where f is a local (reduced) equation of the germ $(D, p) \subset (\mathbf{C}^n, p)$. The \mathcal{O}_p -module of logarithmic vector fields (or logarithmic derivations) is denoted by $Der(\log D)_p$. This yields an \mathcal{O} -module sheaf denoted by $Der(\log D)$ (see [15]).

Let us denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf (of rings) of linear differential operators with holomorphic coefficients on $X = \mathbf{C}^n$. A local section P of \mathcal{D} (i.e. a linear differential operator) is a finite sum $P = \sum_{\alpha} a_{\alpha} \partial^{\alpha}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, a_{α} is a local section of \mathcal{O} and $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i = \frac{\partial}{\partial x_i}$ in some local chart.

For any divisor $D \subset \mathbf{C}^n$ we denote by $\mathcal{O}[*D]$ the sheaf of meromorphic functions with poles along D . It follows from the results of Bernstein-Björk ([1], [2]) on the existence of the b -function for each local equation f of D , that $\mathcal{O}[*D]$ is a left coherent \mathcal{D} -module. Kashiwara proved that the dimension of its characteristic variety is n and then that $\mathcal{O}[*D]$ is holonomic, [12].

We will consider some \mathcal{D} -modules associated to any divisor D :

- The (left) ideal $I^{\log D} \subset \mathcal{D}$ generated by the logarithmic vector fields $Der(\log D)$.
- The (left) ideal $\tilde{I}^{\log D} \subset \mathcal{D}$ generated by the set $\{\delta + a \mid \delta \in I^{\log D} \text{ and } \delta(f) = af\}$. More generally, the ideals $\tilde{I}^{(k)\log D}$ generated by the set

$$\{\delta + ka \mid \delta \in I^{\log D} \text{ and } \delta(f) = af\}$$

- The modules $M^{\log D} = \mathcal{D}/I^{\log D}$, $\widetilde{M}^{\log D} = \mathcal{D}/\widetilde{I}^{\log D}$ and more generally $\widetilde{M}^{(k)\log D} = \mathcal{D}/\widetilde{I}^{(k)\log D}$.

The inclusion $\widetilde{I}^{(k)\log D} \subset \text{Ann}_{\mathcal{D}}(1/f^k)$ yields to a natural morphism $\phi_D^k : \widetilde{M}^{(k)\log D} \rightarrow \mathcal{O}[\star D]$ defined by $\phi_D^k(\overline{P}) = P(1/f^k)$ where \overline{P} denotes the class of the operator $P \in \mathcal{D}$ modulo $\widetilde{I}^{(k)\log D}$. The image of ϕ_D^k is $\mathcal{D}\frac{1}{f^k}$, i.e. the \mathcal{D} -submodule of $\mathcal{O}[\star D]$ generated by $1/f^k$.

Considering the general ideals $\widetilde{I}^{(k)\log D}$ is a suggestion of Prof. Tajima. The point is the well known chain of inclusions

$$\mathcal{D} \cdot f^{-1} \subset \mathcal{D} \cdot f^{-2} \subset \dots \subset \mathcal{D} \cdot f^k = \mathcal{D} \cdot f^{k-1} = \dots = \mathcal{O}[\star D],$$

where k is least integer root of the b -function.

We are interested in the germ $(D, p) \subset (\mathbf{C}^n, p)$ for a fixed point $p \in D$. So we will work in the ring of germs of linear differential operators \mathcal{D}_p . We suppose that $p = 0 \in \mathbf{C}^n$ and from now on, we will denote $\mathcal{D} = \mathcal{D}_0$. In this context we will use the Weyl algebra $A_n(\mathbf{C})$ as a subring of \mathcal{D} .

Under a computational point of view the divisor D will be defined by a polynomial $f \in \mathbf{C}[x_1, \dots, x_n]$. The b -function of f is computable by [14] and we give a direct method to present $\mathcal{O}[\star D]$. If the calculation is intractable –as sometimes happens in the examples– we also present an indirect method to deduce that $\mathcal{O}[\star D]$ and the modules $\widetilde{M}^{(k)\log D}$ do not coincide. The method is strongly based in the following result of [13]:

Theorem 1.1. *The restriction to D of the sheaf $\text{Ext}_{\mathcal{D}}^i(\mathcal{O}[\star D], \mathcal{O})$ is zero for $i \geq 0$.*

More precisely, we prove that, under certain algorithmic conditions, some cohomology groups are not zero. The interest of this second method has been tested in [18], [9] and [10]. We point out how the algorithms presented in [17] –that only calculate cohomology groups in the algebraic case– could also be useful in some *analytic* situations.

It is important to underline that our methods manage the *analytic* case. As the inclusion $A_n(\mathbf{C}) \subset \mathcal{D}$ is flat, the computation of syzygies and free resolutions in the Weyl algebra yields to the analogous computations in \mathcal{D} .

2 Comparison algorithms

We propose in this section two methods to compare the logarithmic modules presented above. It is important to remark that the computation of the *analytic* $\text{Der}(\log D)$ can be made for a divisor D if its local equation is a polynomial $f \in \mathbf{C}[x_1, \dots, x_n]$. Simply compute, using Gröbner basis, a system of generators of (f_1, \dots, f_n, f) where $f_i = \frac{\partial f}{\partial x_i}$ because the inclusion of the Weyl algebra in \mathcal{D} is flat.

2.1 Direct comparison

The first method is complete but needs the calculation of the b -function.

Experimental evidences show that if the divisor is not locally *Euler homogeneous* (i.e. there is no $\delta \in \text{Der}(\log D)$ such that $\delta(f) = f$) the b -function is hard to compute. More precisely, the problem seems to be the calculation of $\text{Ann}_{\mathcal{D}[s]}(f^s)$ and the use of certain elimination orders during the calculation of Gröbner basis (here $\mathcal{D}[s]$ stands for the polynomial ring, with the indeterminate s commuting with \mathcal{D}). Nevertheless, the method is applicable to all the Euler homogeneous divisors we have considered (and to some not Euler homogeneous).

Algorithm 2.1. INPUT: A local equation $f = 0$ of a divisor D ;

1. Compute the b -function of f . Let $-\alpha_0$ be the least integer root.
2. Compute the ideal $\text{Ann}_{\mathcal{D}}(1/f^{\alpha_0})$.
3. Compute a set of generators $\{\mathbf{s}_1, \dots, \mathbf{s}_r\}$ of $\text{Syz}(f_1, \dots, f_n, f)$. The ideal $\tilde{I}^{(\alpha_0) \log D}$ is generated by the elements

$$\mathbf{s}_j \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \\ -\alpha_0 \end{pmatrix} \in \mathcal{D}.$$

4. Compare $\text{Ann}_{\mathcal{D}}(1/f^{\alpha_0})$ and $\tilde{I}^{(\alpha_0) \log D}$.

OUTPUT: $\mathcal{O}[*D] \simeq \widetilde{M}^{(\alpha_0) \log D} \Leftrightarrow \text{Ann}_{\mathcal{D}}(1/f^{\alpha_0}) = \tilde{I}^{(\alpha_0) \log D}$.

The correctness of the algorithm is obvious as

$$\mathcal{O}[*D] \simeq \mathcal{D} \frac{1}{f^{\alpha_0}} \simeq \mathcal{D} / \text{Ann}_{\mathcal{D}}(1/f^{\alpha_0}).$$

2.2 Indirect deduction: a sufficient condition

This second method is an alternative way when you can not obtain the b -function. In the worst case, it only needs the computation of a free resolution of $\widetilde{M}^{(\alpha) \log D}$ for any integer $\alpha \geq 1$. More precisely, the algorithm looks for a technical condition in some step of the free resolution. In many examples, it is enough to compute only the first syzygies¹.

Definition 2.2. If

$$0 \rightarrow \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \dots \rightarrow \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} M \rightarrow 0$$

is a free resolution of a module M , we will say that the *Successive Matrices Condition (SMC) holds at level i* if the two successive morphisms φ_i, φ_{i+1} have matrices verifying:

¹Taking into account that computing a complete free resolution can be a problem of great complexity, this option is very interesting.

1. The matrix of φ_{i+1} has no part in \mathcal{O} in some column j , i.e. all the elements in the j -th column are in the (left) ideal generated by $\partial_1, \dots, \partial_n$.
2. The matrix of φ_i has no constants in the in the j -th row. That is, for each P in the j -th row $P(1)$ is a function h with $h(0) = 0$.

Algorithm 2.3. INPUT: A local equation $f = 0$ of a divisor D ;

1. Compute a set of generators $\{\mathbf{s}_1, \dots, \mathbf{s}_r\}$ of $Syz(f_1, \dots, f_n, f)$. The ideal $\widetilde{I}^{(\alpha) \log D}$ is generated by the elements

$$\mathbf{s}_j \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \\ -\alpha \end{pmatrix} \in \mathcal{D}.$$

2. Compute a free resolution of $M = \widetilde{M}^{(\alpha) \log D}$

$$0 \rightarrow \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \dots \rightarrow \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} M \rightarrow 0.$$

OUTPUT: IF SMC holds **THEN** $\mathcal{O}[*D] \neq \widetilde{M}^{(\alpha) \log D}$.

We need a lemma to justify the algorithm. It explains the role of the SMC. The idea is obtaining an element in $Ker\varphi_{i+1}$ that is not in $Im\varphi_i$.

Lemma 2.4. *Let D be a divisor and*

$$0 \rightarrow \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \dots \rightarrow \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} \widetilde{M}^{(\alpha) \log D} \rightarrow 0 \quad (*)$$

a free resolution of $\widetilde{M}^{(\alpha) \log D}$ that verifies SMC at level i . Then

$$Ext_{\mathcal{D}}^i(\widetilde{M}^{(\alpha) \log D}, \mathcal{O}) \neq 0.$$

Proof. To obtain the Ext groups, we have to apply the functor $Hom_{\mathcal{D}}(-, \mathcal{O})$ to $(*)$. Using that

$$Hom_{\mathcal{D}}(\mathcal{D}^r, \mathcal{O}) \simeq \mathcal{O}^r$$

we obtain the complex

$$0 \rightarrow \mathcal{O} \xrightarrow{\varphi_1^t} \mathcal{O}^{r_1} \xrightarrow{\varphi_2^t} \mathcal{O}^{r_2} \rightarrow \dots \xrightarrow{\varphi_s^t} \mathcal{O} \xrightarrow{\varphi_s^t} \mathcal{O}^{r_s} \rightarrow 0,$$

where φ_i^t denotes the morphism with matrix the transposed of φ_i . The derivatives now act naturally.

Then

$$Ext_{\mathcal{D}}^i(\widetilde{M}^{(\alpha) \log D}, \mathcal{O}) = Ker\varphi_{i+1}^t / Im\varphi_i^t.$$

If the matrix of φ_{i+1} has no part in \mathcal{O} in the j -th row, then $\mathbf{e} = (0, \dots, 1, \dots, 0)$ –where 1 is in the j -th position– is in $Ker\varphi_{i+1}^t$, as the derivatives applied to 1 are zero.

This element can not be in $Im\varphi_i^t$ if the matrix has no constants. Applying the operators of the matrix it is not possible to obtain elements of degree 0. \square

We have the key to state the main result of this section: the correctness of 2.3.

Proposition 2.5. *Let D be a divisor with a free resolution of $\widetilde{M}^{(\alpha)\log D}$ that verifies SCM at some level. Then $\mathcal{O}[*D] \neq \widetilde{M}^{(\alpha)\log D}$.*

Proof. Evident from 2.4 and 1.1. □

A special case of SCM appears when you have a resolution of type

$$0 \rightarrow \mathcal{D}^{r_n} \xrightarrow{\varphi_n} \dots \rightarrow \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} M \rightarrow 0$$

of length n . Then

$$\text{Ext}_{\mathcal{D}}^n(M, \mathcal{O}) = \frac{\mathcal{O}^{r_n}}{\text{Im}\varphi_n^t},$$

and SCM means, at level n , that you can find in the matrix of φ_n a row with no constants.

Remark 2.6. Of course, natural generalizations of the SCM condition has to do with finding explicit elements in some $\text{Ker}\varphi_{i+1}^t$ with special properties. It is not that easy in general! Nevertheless the results of [17] can be applied in this situation as follows:

$$\text{Ext}_{A_n(\mathbf{C})}^0(R[*D], R) \neq 0 \Rightarrow \text{Ext}_{\mathcal{D}}^0(\mathcal{O}[*D], \mathcal{O}) \neq 0,$$

where R is the ring of polynomials and $R[*D]$ is its localization with respect to the equation f of D .

3 Application to the Spencer case.

In this section we explain how to apply the sufficient condition to a special case in which a tailored free resolution is provided.

Definition 3.1. ([15]) The divisor D is said to be *free at the point $p \in D$* if the \mathcal{O}_p -module $\text{Der}(\log D)_p$ is free. The divisor D is called *free* if it is free at each point $p \in D$.

Smooth divisors and normal crossing divisors are free. By [15] any reduced germ of plane curve $D \subset \mathbf{C}^2$ is a free divisor.

By Saito's criterium [15], $D \equiv (f = 0) \subset \mathbf{C}^n$ is free at a point p if and only if there exist n vector fields $\delta_i = \sum_{j=1}^n a_{ij}\partial_j$, $i = 1, \dots, n$, such that $\det(a_{ij}) = uf$ where u is a unit in \mathcal{O}_p . Here ∂_j is the partial derivative $\frac{\partial}{\partial x_j}$ and a_{ij} is a holomorphic function in \mathcal{O}_p .

Definition 3.2. We say that a free divisor D is *of Spencer type* if the complex

$$\mathcal{D} \otimes_{\mathcal{O}} \wedge^{\bullet} \text{Der}(\log D) \rightarrow M^{\log D} \rightarrow 0$$

(introduced in [4]) is a (locally) free resolution of $M^{\log D}$ and if this last \mathcal{D} -module is holonomic.

There are analogous resolutions for the family of modules $\widetilde{M}^{(k)\log D}$.

For this family of divisors, the solution complex $Sol(M^{\log D})$ (that is, the complex $\mathbf{R}\mathcal{H}om_{\mathcal{D}}(M^{\log D}, \mathcal{O})$) is naturally quasi-isomorphic to $\Omega^\bullet(\log D)$ (as we pointed in [10] as a deduction of [4]). On the other hand, a duality theorem proved in [10] has important consequences comparing $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$, namely

Theorem 3.3. ([18, 9]) *In dimension 2, the morphism ϕ_D^1 is an isomorphism if and only if D is a quasi-homogeneous plane curve.*

Theorem 3.4. [10] *Suppose the divisor $D \subset \mathbf{C}^n$ is free and locally quasi-homogeneous. Then the morphism ϕ_D^1 is an isomorphism (so, $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are isomorphic as \mathcal{D} -modules).*

The methods presented in section 2 give us computational tools to check the comparison between $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$.

Remark 3.5. Once you have the duality of [10], you also have a strategy to study the *Logarithmic Comparison Theorem (LCT)*, that is, the complex $\Omega^\bullet(\star D)$ of meromorphic differential forms and the complex $\Omega^\bullet(\log D)$ are quasi isomorphic. You have to travel round the following chain of isomorphisms:

$$\begin{aligned} \Omega^\bullet(\star D) &\simeq DR(\mathcal{O}[\star D]) \simeq DR(\mathcal{D}/Ann_{\mathcal{D}}(1/f)) \simeq DR(\widetilde{M}^{\log D}) \simeq \\ &\simeq DR((M^{\log D})^*) \simeq Sol(M^{\log D}) \simeq \Omega^\bullet(\log D), \end{aligned}$$

where for each coherent \mathcal{D} -module M we denote by $DR(M)$ the de Rham complex of M (see [13]).

Remark 3.6. There are two interesting experimental suggestions:

- We don't know examples of free divisors with integer roots of their b -function less than -1.
- We only know free divisors of Spencer type.

Finally, we have the following result:

Proposition 3.7. *Let $D \equiv (f = 0)$ be a Spencer divisor. Let $\delta_1, \dots, \delta_n$ be a basis of $Der(\log D)$ with $\delta_i = \sum_{j=1}^n a_{ij} \partial_j$ for $1 \leq j \leq n$. If $\sum_{j=1}^n \partial_j(a_{ij}) = 0$, then $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are not isomorphic.*

Proof. The last matrix of the Spencer free resolution of $\widetilde{M}^{\log D}$ is of a very special type . Its elements (due to the duality formulas of [10]) are of the form

$$\delta_i + \sum_{j=1}^n \partial_j(a_{ij}).$$

So, applying lemma 2.4 $Ext_{\mathcal{D}}^n(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$, so $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are not isomorphic. Thus there is no possible quasi-isomorphism between $DR(\mathcal{D}/Ann_{\mathcal{D}}(1/f))$ and $DR(\widetilde{M}^{\log D})$. \square

4 Examples

In the following examples, the computation of syzygies among polynomials have been made with CoCoA (see [11]).

The computations of syzygies in the Weyl Algebra, global b -functions and ideals of type $Ann_{\mathcal{D}}(1/f^\alpha)$ have been made with `kan/sm1`, [16], that is, using the algorithms of [14].

4.1 Example 1: $D \equiv (x(x^2 - y^3)(x^2 - zy^3) = 0)$

We will treat here the divisor $D \subset \mathbf{C}^3$ whose local equation at $(0,0,0)$ is given by $f = 0$ with

$$f = x(x^2 - y^3)(x^2 - zy^3).$$

This divisor is (globally) free and $\delta_1, \delta_2, \delta_3$ form a (global) basis of $Der(\log D)$, where

$$\begin{aligned} \delta_1 &= \frac{3}{2}x\partial_x + y\partial_y \\ \delta_2 &= (y^3z - x^2)\partial_z \\ \delta_3 &= (-\frac{1}{2}xy^2)\partial_x - \frac{1}{3}x^2\partial_y + (y^2z^2 - y^2z)\partial_z, \end{aligned}$$

whose coefficients verify that

$$\begin{vmatrix} \frac{3}{2}x & y & 0 \\ 0 & 0 & y^3z - x^2 \\ -\frac{1}{2}xy^2 & -\frac{1}{3}x^2 & y^2z^2 - y^2z \end{vmatrix} = -\frac{1}{2}f.$$

To begin with, we have to follow two steps:

- Step 1: Verify that $M^{\log D}$ is holonomic². The interest of this question is evident: if $M^{\log D}$ is not holonomic, the computation of its dual could not be managed as we do.
- Step 2: Compute a free resolution of $M^{\log D}$ with Gröbner basis computation of syzygies. Check if D has a free resolution of Spencer type. If this happens then duality holds by [10].

²This computation could be made with [16]

The example verifies these properties:

1. The module $Syz(\delta_1, \delta_2, \delta_3)$ is generated by the syzygies obtained from the commutators $[\delta_i, \delta_j]$. We have $Syz(\delta_1, \delta_2, \delta_3) = \langle \mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23} \rangle$ where

$$\begin{aligned}\mathbf{s}_{12} &= (-\delta_2, \delta_1 - 3, 0) \\ \mathbf{s}_{13} &= (-\delta_3, 0, \delta_1 - 2) \\ \mathbf{s}_{23} &= (0, -\delta_3 - y^2z, \delta_2).\end{aligned}$$

2. On the other hand, the module $Syz(\mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23})$ is generated by the element \mathbf{r} :

$$\mathbf{r} = (-y^2z^2\partial_z + y^2z\partial_z + \frac{1}{2}xy^2\partial_x - y^2z + \frac{1}{3}x^2\partial_y, \quad y^3z\partial_z - x^2\partial_z, \quad -y\partial_y - \frac{3}{2}x\partial_x + 5).$$

This is the element required to have the Spencer type resolution so, as we have said, duality holds.

We calculate the b -function of f . Its least integer root is -1, so

$$\mathcal{O}[\star D] \simeq \mathcal{D} \cdot 1/f \simeq \text{Ann}_{\mathcal{D}}(1/f).$$

To finish, we check that $\widetilde{I}^{\log D} = \text{Ann}_{\mathcal{D}}(1/f)$.

4.2 Example 2: $D \equiv (x^4 + y^4 + z^4 + x^2y^2z^2 = 0) \subset \mathbf{C}^3$

This divisor is not free. A set of generators of $\widetilde{M}^{\log D}$ is

$$\begin{aligned}\delta_1 &= (1/8x^2y^3z^2 + y) + (-1/32x^3y^3z^2 - 1/16xy^4z - 1/4xy)\partial_x + \\ &\quad + (-1/32x^2y^4z^2 + 1/8x^2z^2 - 1/4y^2)\partial_y + (-1/4yz)\partial_z,\end{aligned}$$

$$\begin{aligned}\delta_2 &= (-1/16x^4y^4z + z) + (1/64x^5y^4z + 1/32x^3y^2z^3 - 1/4xz)\partial_x + \\ &\quad + (1/64x^4y^5z - 1/16x^4yz - 1/4yz)\partial_y + (1/8x^2y^2 - 1/4z^2)\partial_z,\end{aligned}$$

$$\begin{aligned}\delta_3 &= (-1/16x^4y^3z^3 - 1/8x^2y^5z) + \\ &\quad + (1/64x^5y^3z^3 + 1/32x^3y^5z + 1/32x^3yz^5 + 1/16xy^3z^3)\partial_x + \\ &\quad + (1/64x^4y^4z^3 + 1/32x^2y^6z - 1/16x^4z^3 - 1/8x^2y^2z - 1/4z^3)\partial_y + \\ &\quad + (1/8x^2yz^2 + 1/4y^3)\partial_z,\end{aligned}$$

$$\begin{aligned}\delta_4 &= (-1/16xy^4z^4 + x) + (1/64x^2y^4z^4 + 1/32y^2z^6 + 1/8y^2z^2 - 1/4x^2)\partial_x + \\ &\quad + (1/64xy^5z^4 - 1/16xy^4z^4 - 1/4xy)\partial_y + (-1/4xz)\partial_z,\end{aligned}$$

$$\delta_5 = (1/8x^2y^2z + 1/4z^3)\partial_x + (-1/8xy^2z^2 - 1/4x^3)\partial_z,$$

$$\begin{aligned}\delta_6 &= (1/8xy^5z^2 + 1/4x^3y^3) + \\ &\quad + (-1/32x^2y^5z^2 - 1/16x^4y^3 - 1/16y^3z^4 - 1/8x^2yz^2 - 1/4y^3)\partial_x + \\ &\quad + (-1/32xy^6z^2 - 1/16x^3y^4 + 1/8xy^2z^2 + 1/4x^3)\partial_y.\end{aligned}$$

The free resolution is huge, but anyway computable with `kan/sm1`. It is of type

$$0 \rightarrow \mathcal{D} \xrightarrow{\varphi_3} \mathcal{D}^r \xrightarrow{\varphi_2} \mathcal{D}^s \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} M \rightarrow 0$$

To use proposition 2.4 you can check that there the elements in the last matrix φ_3 has no constants. So

$$\text{Ext}_{\mathcal{D}}^3(\widetilde{M}^{\log D}, \mathcal{O}) = \mathcal{O}/\text{im}(\varphi_3)^t \neq 0.$$

You have $\mathcal{O}[*D] \neq \widetilde{M}^{\log D}$.

4.3 Example 3: $D \equiv (x^4 + y^5 + xy^4 + zx^6 = 0) \subset \mathbf{C}^3$

In this example we detect that we have, in fact, a (non-Euler homogeneous) product that is a free divisor³. The basis of $\text{Der}(\log D)$ is

$$\begin{aligned} \delta_1 = & (x^6z^2 + 5/4x^5yz^2 - 2x^4z - x^2 - 5/4xy)\partial_x + \\ & +(5/4x^5yz^2 + 3/2x^4y^2z^2 - 5/2x^3yz - 1/2x^2y^2z - 3/4xy - y^2)\partial_y + \\ & +(2x^3z^2 - 2xz)\partial_z, \end{aligned}$$

$$\begin{aligned} \delta_2 = & (-4/5x^4yz - 8/5x^3y^2z - 3/4x^2y^3z + 25/4x^4z + 125/16x^3yz - xy^2 - 1/4y^3 + \\ & +125/16xy)\partial_x + (-x^3y^2z - 39/20x^2y^3z - 9/10xy^4z + 125/16x^3yz + 75/8x^2y^2z - \\ & -3/4y^3 + 1/4x^2 - 5/16xy + 25/4y^2)\partial_y + (-8/5xyz - 6/5y^2z + 25/2xz)\partial_z, \end{aligned}$$

$$\begin{aligned} \delta_3 = & (-3/10x^3y^2z - 3/8x^2y^3z + 25/8x^4z + \\ & +125/32x^3yz + 8/25x^3 - 1/8y^3 + 125/32xy)\partial_x + (-3/8x^2y^3z - 9/20xy^4z + \\ & +125/32x^3yz + 75/16x^2y^2z + 2/5x^2y - 1/50xy^2 + 1/40y^3 + 1/8x^2 - 5/32xy + \\ & +25/8y^2)\partial_y + (-3/5y^2z + 25/4xz + 16/25)\partial_z. \end{aligned}$$

From the monomial $16/25\partial_z$ we deduce by the Flow Theorem that D is a product. It is of Spencer type and the third matrix can be used to show that $\text{Ext}_{\mathcal{D}}^3(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$ so $\mathcal{O}[*D] \neq \widetilde{M}^{\log D}$.

4.4 Example 4: $D \equiv ((x+y)(xz+y)(x^4+y^5+xy^4) = 0) \subset \mathbf{C}^3$

In this last example the divisor $D \subset \mathbf{C}^3$ has as a local equation at $(0,0,0)$ the form

$$f = (x+y)(xz+y)(x^4+y^5+xy^4) = 0.$$

³By the way, this situation is impossible in dimension 2.

The divisor is globally free with $\delta_1, \delta_2, \delta_3$ as a global basis of $Der(\log D)$.

$$\delta_1 = (-x^2 - 5/4xy)\partial_x + (-3/4xy - y^2)\partial_y + (-1/4xz^2 + 1/4xz)\partial_z$$

$$\delta_2 = (xz + y)\partial_z$$

$$\begin{aligned} \delta_3 = & (2x^2y^2 + 5/2xy^3 + 1/2y^4 + 5/2x^3 - 7x^2y - 35/4xy^2 - 11x^2 - 55/4xy)\partial_x + \\ & +(3/2xy^3 + 3/2y^4 - 1/2x^3 + 2x^2y - 21/4xy^2 - 7y^3 - 33/4xy - 11y^2)\partial_y + \\ & +(1/2xy^2z^2 + 1/2y^3z^2 - 1/2xy^2z - 1/2y^3z - 1/4xyz^2 + 7/4xyz + 3/2y^2z + \\ & + 3/4xz^2 + 1/2x^2 + 1/2xy + 5/2xz + 7/2yz - 1/4y)\partial_z. \end{aligned}$$

In this case, D is of Spencer type and the third matrix has no constants so, again we have that $Ext_{\mathcal{D}}^3(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$.

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