Compositional frequent hypercyclicity on weighted Dirichlet spaces

Luis Bernal-González   Antonio Bonilla*

Dedicated to Professor José Rodríguez on the occasion of his 60th birthday

Abstract

It is proved that, in most cases, a scalar multiple of a linear-fractional generated composition operator λC_φ acting on a weighted Dirichlet space S_ν of holomorphic functions in the open unit disk is frequently hypercyclic if and only if it is hypercyclic. In fact, this holds for all triples (ν, λ, φ) with the possible exception of those satisfying $\nu \in [1/4, 1/2)$, $|\lambda| = 1$, $\varphi =$ a parabolic automorphism.

1 Introduction and terminology

The general context containing this paper is the dynamics of operators, while our specific setting will be the composition operators acting on weighted Dirichlet spaces.

As usual, $\mathbb{N}$ denotes the set of positive integers, while T^n ($n \in \mathbb{N}$) stand for the successive iterates of an operator T. We recall that a (continuous and linear) operator T on a topological vector space X is said to be *hypercyclic* if there exists a vector $x \in X$, also called *hypercyclic*, whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X.

Thus, x is a hypercyclic vector if its orbit meets every non-empty open subset U of X. Recently, F. Bayart and S. Grivaux ([2], [4]) have introduced the following new, stronger, quantified notion in the theory of hypercyclic operators.

Definition 1.1. Let X be a topological vector space and $T : X \to X$ an operator. Then a vector $x \in X$ is called *frequently hypercyclic* for T if, for every non-empty open subset U of X, the set

$$\{n \in \mathbb{N} : T^n x \in U\}$$

has positive lower density. The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector.

We recall that the *lower density* of a subset A of $\mathbb{N}$ is defined as

$$\underline{\mathrm{dens}}\,(A) = \liminf_{N \to \infty} \frac{\text{cardinality}\,\{n \in A : n \leq N\}}{N}.$$

The following statement, that is due to Bayart and Grivaux [4], furnishes a sufficient condition for frequent hypercyclity. Recall that an F-space is a metrizable complete topological vector space.

Theorem 1.2 (Frequent Hypercyclicity Criterion). *Let X be a separable F-space and $\|\cdot\|$ a complete F-norm on X defining its topology. Assume that T is an operator on X satisfying the following property: There exists a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that*

(i) $\sum_{n=1}^{\infty} \|T^n x\|$ *converges for all $x \in X_0$,*

(ii) $\sum_{n=1}^{\infty} \|S^n x\|$ *converges for all $x \in X_0$,*

(iii) $TSx = x$ *for all $x \in X_0$.*

Then T is frequently hypercyclic.

An operator T on an *F*-space X is said to satisfy the *Frequent Hypercyclicity Criterion* (in short, FHCC) provided that it possesses the property assumed in the last theorem. We point out that a weaker sufficient condition for frequent hypercyclicity has been recently obtained by Grosse-Erdmann and the second author in [7]. Such a weaker condition will not be used in this paper.

We have that an operator T on an *F*-space X is hypercyclic if and only if it is topologically transitive, that is, if for any pair of non-empty open subsets U, V of X there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, see [16]. Moreover, T is said to be *topologically mixing* if for any pair of non-empty open subsets U, V of X there exists some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. Thus, every topologically mixing operator is hypercyclic, but the converse is not true, see [9]. On the other hand, T is called *chaotic* if it is hypercyclic and it has a dense set of periodic points, that is, vectors $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$, see [13]. Inspired by an approach due to Taniguchi (see the next paragraph), Grosse-Erdmann and the second author have shown [7, Remark 2.2(b)] that if T satisfies

the FHCC then it is topologically mixing and chaotic. Nevertheless, Bayart and Grivaux [5, Corollary 5.2] have constructed a frequently hypercyclic operator that is not chaotic, while Badea and Grivaux [1, Corollary 4.4] have proved the existence of a frequently hypercyclic, chaotic but not mixing operator.

Denote by $\mathbb{D}$ the open unit disk $\{z : |z| < 1\}$ of the complex plane $\mathbb{C}$, and by $H(\mathbb{D})$ the class of holomorphic functions on $\mathbb{D}$. If $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map of $\mathbb{D}$, the composition operator C_φ generated by φ is defined by $C_\varphi f = f \circ \varphi$ $(f \in H(\mathbb{D}))$. It is well known (see [22]) that C_φ is a well-defined operator on each Hardy space $H^p(\mathbb{D})$. Taniguchi [21, Proposition 1] has shown there that under conditions that are stronger than those in Theorem 1.2 an operator on a separable Banach space is chaotic. He applies his criterion to deduce that for any hyperbolic (for the notions of hyperbolic and parabolic see below) automorphism φ of $\mathbb{D}$ the composition operator C_φ is chaotic on the Hardy space $H^p(\mathbb{D})$ for $1 \le p < \infty$, while for any parabolic automorphism φ the operator C_φ is chaotic on the Hardy space $H^p(\mathbb{D})$ for $1 \le p < 2$ (see also [17]). It follows from Theorem 1.2 that these operators are also frequently hypercyclic.

We add that Taniguchi [21, Theorem 3] has shown that if φ is a hyperbolic or parabolic automorphism of $\mathbb{D}$ then C_φ is chaotic on $H^p(\mathbb{D})$ for any $p \in (0, +\infty)$. Moreover, based on a clever eigenvalue criterion (see Lemma 2.4 below), Bayart and Grivaux [4, Corollary 3.7] have shown that every composition operator generated by any hyperbolic or parabolic automorphism of $\mathbb{D}$ is frequently hypercyclic on the Hilbert space $H^2(\mathbb{D})$.

Weaker cyclicity properties of composition operators on weighted or non-weighted Hardy spaces had been intensively investigated by Bourdon and Shapiro [8], Zorboska [23] and Gallardo and Montes [12].

For each sequence of positive numbers $\beta = \{\beta_n\}_0^\infty$ with $\limsup_{n \to \infty} \beta_n^{-1/n} \le 1$, the weighted Hardy space $\mathcal{H}^2(\beta)$ is defined as the Hilbert space of functions $f(z) = \sum_{n=0}^\infty a_n z^n$ analytic on $\mathbb{D}$ for which the norm $\sum_{n=0}^\infty |a_n|^2 \beta_n^2$ is finite (see [10, p. 16] or [12, p. 1]). This norm is induced by the inner product

$$\langle \sum_{n=0}^\infty a_n z^n, \sum_{n=0}^\infty b_n z^n \rangle = \sum_{n=0}^\infty a_n \overline{b_n} \beta_n^2.$$

Notice that the set of monomials $\{z^n/\beta_n\}_0^\infty$ forms a complete orthonormal system. In particular, the polynomials are dense in $\mathcal{H}^2(\beta)$. The weighted Hardy spaces are natural spaces in the sense that the norm convergence in $\mathcal{H}^2(\beta)$ implies uniform convergence on compact subsets of $\mathbb{D}$.

In the case in which the weights $\beta_n = (n+1)^\nu$, $\nu \in \mathbb{R}$, we obtain the so-called *weighted Dirichlet spaces* or $\mathcal{S}_\nu$ spaces. That is,

$$\mathcal{S}_\nu = \{f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^\infty |a_n|^2 (n+1)^{2\nu} < +\infty\}.$$

For instance, if $\nu = 0, -1/2, 1/2$, then $\mathcal{S}_\nu$ is, respectively, the classical Hardy space $H^2(\mathbb{D})$, the Bergman space $A^2(\mathbb{D})$, and the Dirichlet space $\mathcal{D}$.

Let *LFT* denote the family of linear fractional transformations $\varphi(z) = \frac{az+b}{cz+d}$ on the extended complex plane $\mathbb{C}_\infty$. A member of *LFT* different from the identity

can have either one or two fixed points in $\mathbb{C}_\infty$. A map $\varphi \in LFT$ is said to be parabolic if it has a unique fixed point in $\mathbb{C}_\infty$ or, equivalently, if it is conjugate to a translation $z \mapsto z + a$. If φ is neither the identity nor parabolic, then it is called hyperbolic (elliptic, resp.) whenever it is conjugate to a positive dilation $z \mapsto \alpha z$, $\alpha > 0$ (to a rotation $z \mapsto e^{i\theta}z$, resp.). The remainder of maps in LFT are called loxodromic. By $LFT(\mathbb{D})$ we denote the subfamily $\{\varphi \in LFT : \varphi(\mathbb{D}) \subset \mathbb{D}\}$. In particular, the derivative $\varphi'(z)$ exists and is finite for every $\varphi \in LFT(\mathbb{D})$ and every z in the unit circle $\mathbb{T}$. Let $\varphi \in LFT(\mathbb{D})$. If φ is parabolic then its fixed point η is in $\mathbb{T}$, and it satisfies $\varphi'(\eta) = 1$. If φ is a hyperbolic automorphism then its two fixed points are in $\mathbb{T}$, and one of them, say η, is attractive. If φ is a hyperbolic non-automorphism then it has a fixed attractive point η in $\mathbb{T}$, the other fixed point lying in $\{|z| > 1\} \cup \{\infty\}$. In the last two cases, we have $0 < \varphi'(\eta) < 1$. By using the Cayley transform $z \mapsto i\frac{1+z}{1-z}$ from $\mathbb{D}$ onto the right half plane Π, one can easily visualize to which translation (dilation, resp.) a parabolic (hyperbolic, resp.) transformation φ is conjugate, assuming that 1 is a fixed point for φ. Finally, if φ is elliptic (in this case φ is always an automorphism of $\mathbb{D}$) or loxodromic, then one fixed point is in $\mathbb{D}$ and the other one lies on $\{|z| > 1\}$ (see [20]).

According to a result by P.R. Hurst [18], the composition operator $C_\varphi : S_\nu \to S_\nu$ is bounded for any $\nu \in \mathbb{R}$ and any $\varphi \in LFT(\mathbb{D})$. E. Gallardo and A. Montes [12] have furnished a complete characterization of the hypercyclicity of λC_φ on S_ν in terms of λ, ν, φ. This characterization can be summarized as follows.

Theorem 1.3. *Let* $\lambda \in \mathbb{C}$, $\nu \in \mathbb{R}$ *and* $\varphi \in LFT(\mathbb{D})$. *Let* $C_\varphi : S_\nu \to S_\nu$ *be the composition operator generated by* C_φ. *We have:*
(a) *If* φ *is a hyperbolic automorphism and* η *is its attractive fixed point, then* λC_φ *is hypercyclic if and only if* $\nu < \frac{1}{2}$ *and* $\varphi'(\eta)^{\frac{1-2\nu}{2}} < |\lambda| < \varphi'(\eta)^{\frac{2\nu-1}{2}}$.
(b) *If* φ *is a parabolic automorphism, then* λC_φ *is hypercyclic if and only if* $\nu < \frac{1}{2}$ *and* $|\lambda| = 1$.
(c) *If* φ *is a hyperbolic non-automorphism and* η *is its boundary fixed point, then* λC_φ *is hypercyclic if and only if* $\nu \leq \frac{1}{2}$ *and* $\varphi'(\eta)^{\frac{1-2\nu}{2}} < |\lambda|$.
(d) *If* φ *is either an elliptic automorphism, or a loxodromic map, or a parabolic non-automorphism, or the identity, then* λC_φ *is never hypercyclic.*

In view of Theorem 1.3, and encouraged by Corollary 3.7 in [4], it is natural to pose the following question: *For which triples* $(\nu, \lambda, \varphi) \in \mathbb{R} \times \mathbb{C} \times LFT(\mathbb{D})$ *is it true that* λC_φ *is frequent hypercyclicity on* S_ν?

In this paper, we provide an a partial answer to the last question, namely, for *all* triples except perhaps for triples satisfying $\nu \in [1/4, 1/2)$, $|\lambda| = 1$, $\varphi = $ a parabolic automorphism. As a byproduct, Taniguchi's chaoticity results are extended to the weighted Dirichlet spaces.

2   Frequently hypercyclic composition operators

Here the result announced at the end of the previous section will be formally stated.

We need the following five auxiliary results. The first two of them are density results and can be found respectively in [12, Lemma 2.13 and Lemma 4.7]. The third one asserts that the all the dynamical properties considered in Section 1 are invariant under conjugation. Its proof is elementary, so it will be omitted. The fourth lemma contains an eigenvalue criterion for frequent hypercyclicity, that was developed by Bayart and Grivaux (see [2], [3], [4] and [5]) and inspired by Flytzanis [11]. This lemma adopts an heuristic idea due to Godefroy and Shapiro [13] (see also [6]), namely, rich supplies of eigenvectors associated to eigenvalues λ with $|\lambda| < 1$ and to eigenvalues λ with $|\lambda| > 1$ imply hypercyclicity. The fifth lemma can be found in [12, Lemma 1.2] and furnishes a useful renorming of the weighted Dirichlet spaces.

Lemma 2.1. *Assume that $\nu \leq 1/2$, that m_1 and m_2 are any positive integers and that α_1, α_2 are complex numbers with $|\alpha_i| \geq 1$ for $i = 1, 2$. Then the set of all polynomials that vanish at least m_1 times at α_1 and at least m_2 times at α_2 is dense in the space $\mathcal{S}_{\succeq}$.*

For $t \geq 0$, let e_t be the function

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right).$$

It is shown in [12, Proposition 3.10] (see also [19]) that $e_t \in S_\nu$ if and only $\nu < 1/4$.

Lemma 2.2. *Suppose that $\nu < 1/4$. Then*

$$\overline{\text{span}}\left\{e_t : t \geq 0\right\} = S_\nu.$$

Lemma 2.3. *Let X be a separable F-space. Assume that T is a hypercyclic (mixing, chaotic, frequently hypercyclic) operator on X satisfying the FHCC, and that R is an invertible operator on X. Then the operator RTR^{-1} is also hypercyclic (mixing, chaotic, frequently hypercyclic, respectively).*

Recall that a measure σ defined on the Borel σ-algebra generated by a topological space is said to be continuous if $\sigma(\{a\}) = 0$ for each singleton $\{a\}$.

Lemma 2.4. *Let T be an operator on a separable complex Banach space X. Assume that T has perfectly spanning set of eigenvectors associated to unimodular eigenvalues, that is, there is a continuous probability measure σ on $\mathbb{T} = \{z : |z| = 1\}$ such that*

$$\overline{\text{span}}\left(\bigcup_{\alpha \in A} \text{Ker}(T - \alpha I)\right) = X$$

for every Borel set $A \subset \mathbb{T}$ with $\sigma(A) = 1$. Then T is hypercyclic. Moreover, we have:
(a) If X is a Hilbert space then T is even frequently hypercyclic.
(b) If σ can be chosen to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$, then T is even mixing.

By $dA(z)$ we denote the normalized Lebesgue measure $\frac{dxdy}{\pi}$ $(z = x + iy)$ on $\mathbb{D}$.

Lemma 2.5. *If $v \in (-\infty, 1)$ then the expression*

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-2v} \, dA(z) \quad (f \in \mathcal{S}_v)$$

defines an equivalent norm on $\mathcal{S}_v$.

We are now ready to establish and prove our theorem.

Theorem 2.6. *Let $v \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $\varphi \in LFT(\mathbb{D})$, with*

$$(v, \lambda, \varphi) \notin [1/4, 1/2) \times \mathbb{T} \times \{\text{parabolic automorphisms of } \mathbb{D}\}.$$

Let $C_\varphi : \mathcal{S}_v \to \mathcal{S}_v$ be the composition operator generated by φ. Then the following statements are equivalent:
(a) λC_φ is frequently hypercyclic.
(b) λC_φ is topologically mixing.
(c) λC_φ is chaotic.
(d) λC_φ is hypercyclic.

Proof. The implications (a) $\Longrightarrow$ (d), (b) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (d) are trivial. Now, let $\lambda C_\varphi : \mathcal{S}_v \to \mathcal{S}_v$ be hypercyclic. By Theorem 1.3, φ is either a hyperbolic map or a parabolic automorphism. At this point we distinguish three cases. In the first two cases we will prove that λC_φ satisfies the FHCC. Then it is frequently hypercyclic and, according to [7, Remark 2.2(b)], it is also topologically mixing and chaotic, so (b) are (c) are fulfilled. In the third case we will use the eigenvalue criterion to prove the frequent hypercyclicity as well as the mixing property, while the chaoticity will be demonstrated directly. We denote $T = \lambda C_\varphi$.

Case 1: φ is a hyperbolic automorphism. In this case, φ has its two fixed points η, η' on $\mathbb{T}$. Let η be the attractive one. Take any automorphism σ of $\mathbb{D}$ satisfying $\sigma(\eta) = 1$, $\sigma(\eta') = -1$. Then $\varphi_0 := \sigma \circ \varphi \circ \sigma^{-1}$ is a hyperbolic automorphism of $\mathbb{D}$ with fixed points at $1, -1$, such the point 1 is the attractive one. Moreover, $T = C_\sigma \circ \lambda C_{\varphi_0} \circ C_{\sigma^{-1}}$. An application of Lemma 2.3 yields that it is enough to prove that λC_{φ_0} satisfies the FHCC. Consequently, we can assume without loss of generality that $1, -1$ are the fixed points of φ, the point 1 being attractive.

According to Theorem 1.3, we have that $v < 1/2$ and $\varphi'(1)^{\frac{1-2v}{2}} < |\lambda| < \varphi'(1)^{\frac{2v-1}{2}}$. We follow the proof of Theorem 3.5 in [12]. The explicit expression of φ is

$$\varphi(z) = \frac{(1 + \mu)z + 1 - \mu}{(1 - \mu)z + 1 + \mu},$$

where $\mu \in (0, 1)$ and, in fact, $\varphi'(1) = \mu$. Therefore

$$\mu^{\frac{1-2v}{2}} < |\lambda| < \mu^{\frac{2v-1}{2}}. \tag{1}$$

Choose $m \in \mathbb{N}$ with $m > 2 - 2v$ and

$$m > -\log|\lambda| / \log \mu. \tag{2}$$

Let X be the set of all holomorphic functions on a neighborhood of the closed disk $\overline{\mathbb{D}}$ that vanish at least m times at 1. Fix $f \in X$. It is proved in [12] that

$$\|T^n f\| \le C(|\lambda|^{2n}\mu^{2nm} + |\lambda|^{2n}\mu^{n(1-2\nu)}) \quad (n \in \mathbb{N}),$$

where C is a constant independent of n. From (1) and (2), we obtain that $|\lambda|\mu^m$ and $|\lambda|^2\mu^{1-2\nu}$ are less that 1, so

$$\sum_{n=1}^{\infty} \|T^n f\| < +\infty \quad \text{for all } f \in X. \tag{3}$$

Now, take $S := T^{-1} = \lambda^{-1}C_\varphi^{-1} = \lambda^{-1}C_{\varphi^{-1}}$ and consider the set Y of all holomorphic functions on a neighborhood of $\overline{\mathbb{D}}$ that vanish at least m times at -1. Observe that -1 is the attractive fixed point of φ^{-1} with $(\varphi^{-1})'(-1) = 1/\varphi'(-1) = \mu$ and that $\mu^{\frac{1-2\nu}{2}} < |\lambda| < \mu^{\frac{2\nu-1}{2}}$. Therefore, a similar argument leads to

$$\sum_{n=1}^{\infty} \|S^n f\| < +\infty \quad \text{for all } f \in Y. \tag{4}$$

If we set $X_0 := X \cap Y$, then we have $X_0 \supset \{$polynomials vanishing at least m times at 1 and $-1\}$, so X_0 is dense in S_ν by Lemma 2.1. Clearly, (3) and (4) hold for all $f \in X_0$. In addition, TS is the identity and X_0 is S-invariant, because φ^{-1} is conformal and fixes the points $1, -1$. Consequently, T satisfies the FHCC.

Case 2: φ is a hyperbolic non-automorphism. This time φ has two fixed points, one on $\mathbb{T}$ and the other one outside $\overline{\mathbb{D}}$. Choose an automorphism σ of $\mathbb{D}$ sending those points, respectively, to 1 and to certain $\alpha \in (-\infty, -1)$ (see [20, p. 114 and Exercise 10 on p. 125]). By using Lemma 2.3 as in the first part of Case 1, one can suppose without loss of generality that the fixed points of φ are $1, \alpha$. We follow the proof of Theorem 2.11 in [12]. The explicit expression of φ is

$$\varphi(z) = \frac{(\mu\alpha - 1)z + \alpha(1 - \mu)}{(\mu - 1)z + \alpha - \mu},$$

where $\alpha \in (-\infty, -1)$, $\mu \in (0, 1)$ and, in fact, $\varphi'(1) = \mu$. By Theorem 1.3, we must have $\nu \le 1/2$ and

$$\mu^{\frac{1-2\nu}{2}} < |\lambda|. \tag{5}$$

Choose $m \in \mathbb{N}$ satisfying $m > (1 - 2\nu)/2$ and (2). Denote by X (Y, resp.) the set of all polynomials that vanish at least m times at 1 (at α, resp.). This time, the inverse map $S = \lambda^{-1}C_{\varphi^{-1}}$ is not bounded on S_ν but it is well defined on the polynomials. It is proved in [12] that

$$\|T^n f\| \le M|\lambda\mu^m|^n \quad \text{for all } f \in X$$

and

$$\|S^n f\| \le C|\lambda|^{-n}\mu^{\frac{n(1-2\nu)}{2}} \quad \text{for all } f \in Y,$$

where the constants M, C are independent of n. By (2) and (5), both numbers $|\lambda\mu^m|$, $|\lambda|^{-1}\mu^{\frac{1-2\nu}{2}}$ are less than 1. Therefore, we obtain

$$\sum_{n=1}^{\infty} \|T^n f\| < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|S^n f\| < +\infty \quad \text{for all } f \in X \cap Y. \tag{6}$$

Now we define $X_0 := \bigcup_{n=0}^{\infty} S^n(X \cap Y)$. Then S is well defined on X_0, the set X_0 is dense in S_ν (by Lemma 2.1, because $X_0 \supset X \cap Y$) and S-invariant, TS is the identity on X_0 and (6) is satisfied for all $f \in X_0$ (this only carries a translation of the indexes n in both series). Again, conditions (i), (ii) and (iii) of Theorem 1.2 are fulfilled. Thus, T satisfies the FHCC.

Case 3: φ is a parabolic automorphism. According to Theorem 1.3 and the hypothesis, we must have $\nu < 1/4$ and $|\lambda| = 1$. Since φ is conjugate to a translation, we may suppose, after applying a similarity if necessary, that φ is conjugate to a translation $z \mapsto z + ia$ ($a \in \mathbb{R} \setminus \{0\}$), which is a self-map of the right half-plane $\mathbb{C}_+$. Then an appropriate linear fractional transformation mapping $\mathbb{D}$ onto $\mathbb{C}_+$ shows that we can assume (with a further application of Lemma 2.3) that φ has the form

$$\varphi(z) = \frac{(2 - ai)z + ai}{-aiz + 2 + ai},$$

with $a \in \mathbb{R} \setminus \{0\}$. Note that e_t ($t \geq 0$) is an eigenfunction for T associated to the eigenvalue λe^{-iat}. As in [4, Proof of Example 3.6], take σ to be the normalized length measure on $\mathbb{T}$, and let A be a measurable set of $\mathbb{T}$ with $\sigma(A) = 1$. Then $\sigma(\lambda^{-1}A) = 1$. If m is the Lebesgue measure on $[0, +\infty)$, we have

$$m(\{t \geq 0 : \lambda e^{-iat} \notin A\}) \leq \frac{1}{|a|} m(\{t \in \mathbb{R} : e^{it} \notin \lambda^{-1}A\}) = 0.$$

Hence the set $B := \{t \geq 0 : \lambda e^{-iat} \in A\}$ is dense in $[0, +\infty)$. Observe that

$$\bigcup_{\alpha \in A} \mathrm{Ker}(T - \alpha I) \supset \{e_t : t \in B\}. \tag{7}$$

Let $f \in S_\nu$ with $\langle f, e_t \rangle = 0$ for all $t \in B$. Since e_t depends continuously on t (this will be detailed at the end of the proof) and B is dense, we get $\langle f, e_t \rangle = 0$ for all $t \geq 0$, so $\langle f, g \rangle = 0$ for all $g \in \mathrm{span}\,\{e_t : t \geq 0\}$. By Lemma 2.2, the last span is dense in S_ν, whence $f = 0$. Consequently, $\{e_t : t \in B\}$ is total in S_ν. It follows from (7) that $\mathrm{span}\,(\bigcup_{\alpha \in A} \mathrm{Ker}(T - \alpha I))$ is dense in S_ν. Then Lemma 2.4 applies yielding that T is frequently hypercyclic and mixing.

Now, we prove that T is chaotic. Since T is already hypercyclic, our task is to demonstrate that the set $\mathcal{P}$ of T-periodic functions in S_ν is dense. For this, observe that

$$\mathcal{P} \supset \bigcup_{q \in \mathbb{Q}} \mathrm{Ker}\,(T - e^{i2\pi q}I) \supset \{e_t : t \in C\}, \tag{8}$$

where $\mathbb{Q}$ is the set of rational numbers, $C := [0, +\infty) \cap (a^{-1}(\beta + \pi\mathbb{Q}))$, with $\lambda = e^{i\beta}$. Note that C is dense in $[0, +\infty)$. An argument as in the above paragraph shows that $\{e_t : t \in C\}$ spans a dense set in S_ν. Since $\mathcal{P}$ is a linear manifold, it follows from (8) that $\mathcal{P}$ is dense in S_ν, as required.

Finally, we demonstrate that the map $E : t \in [0, +\infty) \mapsto e_t \in \mathcal{S}_\nu$ is continuous. To this end, we fix $u \geq 0$. Let $t \in [u/2, u + 1]$. By using the equivalent norm furnished by Lemma 2.5, one obtains

$$\|E(t) - E(u)\|^2 = |e^{-t} - e^{-u}|^2 + \int_{\mathbb{D}} |te_t(z) - ue_u(z)|^2 |\sigma'(z)|^2 (1 - |z|^2)^{1-2\nu}\, dA(z),$$

where $\sigma(z) := i\frac{1+z}{1-z}$ is the Cayley transform from $\mathbb{D}$ onto the upper half plane Π. Therefore

$$\|E(t) - E(u)\|^2 = |e^{-t} - e^{-u}|^2 + J(t),$$

where

$$J(t) := \frac{1}{\pi} \int_{\Pi} |te^{itw} - ue^{iuw}|^2 (1 - |\sigma^{-1}(w)|^2)^{1-2\nu} \, dA(w) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \psi(x,y,t) \, dxdy$$

and $\psi(x,y,t) := |te^{it(x+iy)} - ue^{iu(x+iy)}|^2 [\frac{4y}{x^2+(y+1)^2}]^{1-2\nu}$. As $t \to u$, we have $|e^{-t} - e^{-u}|^2 \to 0$ and $\psi(x,y,t) \to 0$ for all $x,y > 0$. But

$$|\psi(x,y,t)| \leq [(1+u)e^{-uy/2} + ue^{-uy}]^2 \left[\frac{4y}{x^2+(y+1)^2} \right]^{1-2\nu}$$
$$(x,y > 0; \ t \in [u/2, u+1]),$$

and the last function does not depend on t, and it is integrable on $(0, +\infty)^2$ because $2(1 - 2\nu) > 1$. From the Lebesgue Dominated Convergence Theorem, it follows that $J(t) \to 0$ $(t \to u)$, hence $\|E(t) - E(u)\| \to 0$ $(t \to u)$, so yielding the continuity of E. ∎

Let us make some comments about the refractory case

$$(\nu, \lambda, \varphi) \in [1/4, 1/2) \times \mathbb{T} \times \{\text{parabolic automorphisms of } \mathbb{D}\}.$$

Denote $T = \lambda C_\varphi$. The approach of the proof of Theorem 3.3 in [12] only leads to estimates of the form

$$\|T^n f\| \leq \frac{C}{n^{1-2\nu}} \tag{9}$$

for $\nu < 1/2$, that is not sufficient to apply Theorem 1.2. Nevertheless, since the so-called Hypercyclicity Criterion is satisfied for the entire sequence (n) of positive integers (see [14]), the operator T is in fact *mixing* for $\nu < 1/2$. Moreover, for $\nu \geq 1/4$ and $t > 0$, the functions e_t (the "most natural" eigenfunctions of C_φ) do not belong to S_ν. These functions seem to be the best candidate to be periodic functions (with appropriate values of t) for the operator C_φ. This leads us to conjecture that T is not chaotic for $\nu \in [1/4, 1/2)$. If this is the case, T would not satisfy the FHCC either.

Remark 2.7. The referee has kindly provided a new way to derive the frequent hypercyclicity of $T = C_\varphi$ in the parabolic case, with $|\lambda| = 1$, $\nu < 1/4$. This way does not use the eigenvalue criterion. Namely, following the proof of Theorem 3.3 in [12] we get estimates like (9) for $f \in X_0 := \{\text{holomorphic functions in some neighborhood of } \overline{\mathbb{D}} \text{ that vanish at least twice at } 1\}$, where we are assuming without loss of generality that 1 is the fixed point of φ. Similar estimates hold for $S := C_{\varphi^{-1}}$. Then $TS = I$ and both series $\sum_{n=1}^\infty \|T^n f\|^2$, $\sum_{n=1}^\infty \|S^n f\|^2$ converge for f in the dense set X_0. Since S_ν is a Hilbert space (so a Banach space with cotype 2), the conditions of the "random" Frequent Hypercyclicity Criterion, see [15, Theorem 2.1 and the subsequent comments] are satisfied, which implies that C_φ is frequently hypercyclic.

To conclude the paper, we want to pose the problem arising from the comments following the proof of Theorem 2.6.

Question. Assume that φ is a parabolic automorphism of $\mathbb{D}$, $|\lambda| = 1$ and $\nu \in [1/4, 1/2)$. Is λC_φ chaotic? Is λC_φ frequently hypercyclic?

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L. Bernal-González
Departamento de Análisis Matemático
Facultad de Matemáticas, Apdo. 1160
Avda. Reina Mercedes, s/n
41080 Sevilla, Spain
E-mail: lbernal@us.es

Antonio Bonilla
Departamento de Análisis Matemático
Universidad de La Laguna
C/Astrofísico Fco. Sánchez, s/n
38271 La Laguna, Tenerife, Spain
E-mail: abonilla@ull.es