THE MODIFICATION OF CLASSICAL HAHN POLYNOMIALS OF A DISCRETE VARIABLE.[1]

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Abstract

We consider a modification of moment functionals for the Hahn classical polynomials of a discrete variable by adding two mass points at the ends of the interval, i.e., in $x=0$ and $x=N-1$. We obtain the resulting orthogonal polynomials and identify them as hypergeometric functions. The corresponding three term recurrence relation and tridiagonal matrices are also studied.

§1 Introduction.

The study of orthogonal polynomials with respect to a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one or two delta Dirac measures has been performed by several authors [5] [6], [15]. A special emphasis is given to the modifications of classical linear functionals (Hermite [15], Laguerre [11], Jacobi [12] and Bessel [10]). Very recently appear some works related to modifications of the discrete classical measures, more concretly the Charlier, Kravchuk and Meixner measures, via addition of one delta Dirac measures at $x=0$ [1], [2], [3], [4] and [9].

In this paper we study the polynomials orthogonal with respect to the modification of the weight function of the classical Hahn polynomials via the addition of two different masses at the ends of the interval. In fact we find one expression for the *perturbed or generalized* monic Hahn polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x)$ as well as their representation in terms of the ${}_5F_4$ hypergeometric series. We also analyze the relation between tridiagonal matrices of these perturbed Hahn $\hat{h}_n^{A,B,\alpha,\beta}(x)$ and classical $\hat{h}_n^{\alpha,\beta}(x,N)$ polynomials.

The structure of the paper is as follows. In Section 2, we provide the basic properties of the classical orthogonal Hahn polynomials. In Section 3 we deduce expressions of the monic generalized Hahn polynomials in terms of the classical ones $\hat{h}_n^{\alpha,\beta}(x,N)$ and the first backward difference derivatives of the polynomials $\hat{h}_n^{\alpha-1,\beta}(x)$ and $\hat{h}_n^{\alpha,\beta-1}(x)$. In Section 4 we find their representation as hypergeometric functions ${}_5F_4$ and in Section 5 we analyze two particular cases: $A\neq 0$ $B=0$ and $A=0$, $B\neq 0$. Finally, in Section 6, from the three term recurrence relation (TTRR) of the classical orthogonal polynomials we find the TTRR which satisfy the perturbed ones and analyze the relation between tridiagonal matrices associated with the perturbed monic orthogonal polynomial sequence (PMOPS) $\{\hat{h}_n^{A,B,\alpha,\beta}(x,N)\}_{n=0}^{N-1}$ as a rank-one perturbation of the tridiagonal matrices associated with the classical monic orthogonal polynomial sequence (CMOPS) $\{\hat{h}_n^{\alpha,\beta}(x,N)\}_{n=0}^{N-1}$. The two special cases are also analyzed.

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§2 Some Preliminar Results.

Here we enclose some formulas for the classical Hahn polynomials which are useful in order to obtain the generalized polynomials orthogonal with respect to the linear functional $\mathcal{U}$ defined as a modification of the first ones troughtout the addition of two mass points. All the formulas for the classical Hahn polynomials can be found in a lot of books (see for instance the excellent monograph *Orthogonal Polynomials in Discrete Variables* by A.F. Nikiforov, S. K. Suslov, V. B. Uvarov [16], Chapter 2.)

The classical orthogonal polynomials of a discrete variable in the uniform lattice $x(s) = s$, where s belongs to the set of non-negative integers, are the polynomial solution of a second order linear difference equation of hypergeometric type

$$\sigma(x) \triangle \nabla P_n(x) + \tau(x) \triangle P_n(x) + \lambda_n P_n(x) = 0 , \tag{1}$$

where

$$\nabla f(x) = f(x) - f(x-1), \qquad \triangle f(x) = f(x+1) - f(x).$$

Here $\sigma(x)$ and $\tau(x)$ are polynomials in x of degree at most 2 and 1, respectively, and λ_n is a constant.

These polynomials are orthogonal with respect to the linear functional $\mathcal{H}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{H}, PQ > \ = \sum_{x \in \mathbb{N}} \rho(x)P(x)Q(x), \qquad \mathbb{N} = \{0,1,2,...\} , \tag{2}$$

where $\rho(x)$ is some non-negative function (weight function) supported in a countable set of the real line and such that

$$\triangle[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

The orthogonality relation is

$$\sum_{x \in \mathbb{N}} P_n(x)P_m(x)\rho(x) = \delta_{nm} d_n^2, \tag{3}$$

where d_n^2 denotes the square of the norm of these classical polynomials.

The polynomial solutions of equation (1) are uniquely determined, up to a normalized factor (B_n), by the difference analog of the Rodrigues formula (see [16] page 24 Eq.(2.2.7)):

$$P_n(x) = \frac{B_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right] . \tag{4}$$

They satisfy a three term recurrence relation of the form

$$\begin{gathered} xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \ \ n \geq 0 \\ P_{-1}(x) = 0 \qquad \text{and} \qquad P_0(x) = 1 \end{gathered} \tag{5}$$

and the Christoffel-Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m(x)P_m(y)}{d_m^2} = \frac{1}{x-y} \frac{a_{n-1}}{a_n} \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{d_{n-1}^2} \qquad n = 1,2,3,... \ . \tag{6}$$

Here a_n is the leading coefficient of the polynomial, i.e., the coefficient of the n-th power of x in the expansion:

$$P_n(x) = a_n x^n + b_n x^{n-1} + \dots \tag{7}$$

We will consider the classical Hahn polynomials $h_n^{\alpha,\beta}(x,N)$ (see [16] section **2.4** page 30 and table **2.1** page 42) which are solutions of the difference equation (1) and they are orthogonal with respect to the weight function $\rho(x)$ supported on $[0,N)$, with

$$\sigma(x) = x(x+\alpha-N), \quad \tau(x) = (\beta+1)(N-1) - x(\alpha+\beta+2) \quad , \quad \lambda_n = n(\alpha+\beta+N+1),$$

and

$$B_n = \frac{(-1)^n}{n!}, \qquad \rho(x) = \frac{\Gamma(\alpha+N-x)\Gamma(\beta+1+x)}{\Gamma(N-x)\Gamma(1+x)}, \qquad \alpha > -1, \quad \beta > -1.$$

His norm d_n^2 and his leading coefficient a_n are equal to.

$$d_n^2 = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n+1)n!(N-n-1)!\Gamma(\alpha+\beta+n+1)}, \qquad a_n = \frac{\Gamma(\alpha+\beta+2n+1)}{n!\Gamma(\alpha+\beta+n+1)}.$$

The coefficients of the TTRR (5) are

$$\begin{aligned}
\alpha_n &= \frac{(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}, \\
\beta_n &= \frac{\alpha-\beta+2N-2}{4} - \frac{(\beta^2-\alpha^2)(\alpha+\beta+2N}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \\
\gamma_n &= \frac{(\alpha+n)(\beta+n)(\alpha+\beta+N+n)(N-n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}.
\end{aligned} \tag{8}$$

These classical polynomials can be represented in terms of the hypergeometric function ${}_3\mathrm{F}_2$ (see [16] page 49, section **2.7**)

$$h_n^{\alpha,\beta}(x,N) = \frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{n!(N-n-1)!\Gamma(\beta+1)} {}_3\mathrm{F}_2\left({}^{-x,\alpha+\beta+n+1,-n}_{1-N,\beta+1}; 1\right), \tag{9}$$

where the hypergeometric function is defined by

$${}_pF_q\left({}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1), \qquad k = 1,2,3,\dots \quad .$$

Notice that the Hahn polynomials constitute a finite set of polynomials defined for $n = 0,1,\dots,N-1$ (see [7] or [16] .

As a consequence of these representations we can deduce

$$h_n^{\alpha,\beta}(0,N) = \frac{(-1)^n}{n!} \frac{\Gamma(\beta+n+1)(N-1)!}{\Gamma(\beta+1)(N-n-1)!}. \tag{10}$$

They satisfy the following differentiation formula:

$$\triangle\, h_n^{\alpha,\beta}(x,N) = (\alpha+\beta+n+1) h_{n-1}^{\alpha+1,\beta+1}(x,N-1), \tag{11}$$

and the *symmetry property*:

$$h_n^{\beta,\alpha}(N-1-x,N) = (-1)^n h_n^{\alpha,\beta}(x,N). \tag{12}$$

Let us now to prove the following Lemma:

Lemma 1 *The Classical Hahn polynomials satisfy the relation*

$$\frac{(\alpha+\beta+2n)(N-n-1)x}{\alpha+\beta+n} \nabla h_n^{\alpha-1,\beta}(x,N) = n(N-n-1)h_n^{\alpha,\beta}(x,N)+(n+\beta)h_{n-1}^{\alpha,\beta}(x,N). \tag{13}$$

Proof: Using the hypergeometric representation (9) for the Hahn polynomials we observe that the right side of (13) is equal to:

$$\frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(n-1)!(N-n-2)!\Gamma(\beta+1)}\left[\ {}_3\mathrm{F}_2\left({-x,\alpha+\beta+n+1,-n \atop 1-N,\beta+1};1\right) - {}_3\mathrm{F}_2\left({-x,\alpha+\beta+n,1-n \atop 1-N,\beta+1};1\right)\right] =$$

$$\frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(n-1)!(N-n-2)!\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{(-n)_k(-x)_k(\alpha+\beta+n)_k}{(1-N)_k(\beta+1)_k k!}\left[\frac{\alpha+\beta+n+k}{\alpha+\beta+n}-\frac{n-k}{n}\right] =$$

$$\frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(n-1)!(N-n-2)!\Gamma(\beta+1)}\frac{\alpha+\beta+2n}{n(\alpha+\beta+n)}\sum_{k=1}^{\infty}\frac{(-n)_k(-x)_k(\alpha+\beta+n)_k}{(1-N)_k(\beta+1)_k(k-1)!}.$$

Using the identity $(a)_k = a(a+1)_{k-1}$ the last expression becomes

$$\frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(n-1)!(N-n-2)!\Gamma(\beta+1)}\frac{x(\alpha+\beta+2n)}{(1-N)(\beta+1)}\sum_{k=0}^{\infty}\frac{(1-n)_k(1-x)_k(\alpha+\beta+n+1)_k}{(2-N)_k(\beta+2)_k k!} =$$

$$(\alpha+\beta+2n)(N-n-1)\frac{(-1)^{n-1}(N-2)!\Gamma(\beta+n+1)}{(n-1)!(N-n-1)!\Gamma(\beta+2)}{}_3\mathrm{F}_2\left({1-x,\alpha+\beta+n+1,1-n \atop 2-N,\beta+2};1\right).$$

Taking into account the representation

$$h_{n-1}^{\alpha,\beta+1}(x-1,N-1) = \frac{(-1)^{n-1}(N-2)!\Gamma(\beta+n+1)}{(n-1)!(N-n-1)!\Gamma(\beta+2)}{}_3\mathrm{F}_2\left({1-x,\alpha+\beta+n+1,1-n \atop 2-N,\beta+2};1\right),$$

and the difference equation (11), as well as the identity $\triangle f(x-1) = \nabla f(x)$ the lemma follows:

$$n(N-n-1)h_n^{\alpha,\beta}(x,N) - (n+\beta)h_{n-1}^{\alpha,\beta}(x,N) =$$

$$= (\alpha+\beta+2n)(N-n-1)h_{n-1}^{\alpha,\beta+1}(x-1,N-1) =$$

$$= \frac{(\alpha+\beta+2n)(N-n-1)}{\alpha+\beta+n}\nabla h_n^{\alpha-1,\beta}(x,N). \qquad \blacksquare$$

Next, we will obtain an useful property of the kernels $Ker_{n-1}^{\alpha,\beta}(x,0)$ and $Ker_{n-1}^{\alpha,\beta}(x,N-1)$. First at all, we have the following

$$Ker_{n-1}^{\alpha,\beta}(x,0) = \sum_{k=0}^{n-1}\frac{h_n^{\alpha,\beta}(x,N)h_n^{\alpha,\beta}(0,N)}{d_n^2} = \sum_{k=0}^{n-1}\frac{\hat{h}_n^{\alpha,\beta}(x,N)\hat{h}_n^{\alpha,\beta}(0,N)}{\hat{d}_n^2}, \tag{14}$$

where by $\hat{h}_n^{\alpha,\beta}(x,N)$ and $\hat{d}_n^2$ we denote the corresponding monic polynomials and its squared norm. Using this fact, the Christoffel-Darboux formula, as well as relation (13) we obtain the following expression for the kernels of the Hahn polynomials:

$$
\begin{aligned}
&Ker_{n-1}^{\alpha,\beta}(x,0) \equiv \sum_{m=0}^{n-1} \frac{h_m^{\alpha,\beta}(x,N)h_m^{\alpha,\beta}(0,N)}{d_m^2} = \frac{a_{n-1}}{x a_n d_{n-1}^2}\times \\
&\times \frac{(-1)^{n-1}(N-1)!\Gamma(\beta+n)}{n!(N-n-1)!\Gamma(\beta+1)}\Big[n(N-n-1)h_n^{\alpha,\beta}(x,N)+(n+\beta)h_{n-1}^{\alpha,\beta}(x,N)\Big] = \\
&= \frac{a_{n-1}}{a_n d_{n-1}^2}\frac{(-1)^{n-1}(N-1)!\Gamma(\beta+n)(\alpha+\beta+2n)}{n!(N-n-2)!\Gamma(\beta+1)(\alpha+\beta+n)} \bigtriangledown h_n^{\alpha-1,\beta}(x,N).
\end{aligned} \tag{15}
$$

To obtain a representation for the kernel $Ker_{n-1}^{\alpha,\beta}(x,N-1)$ we can use the above expression for the kernel $Ker_{n-1}^{\alpha,\beta}(x,0)$ and (12):

$$
\begin{aligned}
&Ker_{n-1}^{\alpha,\beta}(x,N-1) = \sum_{m=0}^{n-1} \frac{h_m^{\alpha,\beta}(x,N)h_m^{\alpha,\beta}(N-1,N)}{d_m^2} = \\
&= \sum_{m=0}^{n-1} \frac{h_m^{\beta,\alpha}(N-1-x,N)h_m^{\beta,\alpha}(0,N)}{d_m^2} = Ker_{n-1}^{\beta,\alpha}(N-x-1,0) = \\
&= \frac{a_{n-1}}{a_n d_{n-1}^2}\frac{(-1)^{n-1}(N-1)!\Gamma(\alpha+n)(\alpha+\beta+2n)}{n!(N-n-2)!\Gamma(\alpha+1)(\alpha+\beta+n)} \bigtriangledown h_n^{\beta-1,\alpha}(N-1-x,N) = \\
&= \frac{a_{n-1}}{a_n d_{n-1}^2}\frac{(-1)^{n-1}(N-1)!\Gamma(\alpha+n)(\alpha+\beta+2n)}{n!(N-n-2)!\Gamma(\alpha+1)(\alpha+\beta+n)}(-1)^{n-1} \bigtriangleup h_n^{\alpha,\beta-1}(x,N).
\end{aligned} \tag{16}
$$

If we denote by $\kappa_n(\alpha,\beta)$ the following quantities:

$$
\kappa_n(\alpha,\beta) = \frac{(-1)^{n-1}(N-1)!\Gamma(\alpha+\beta+2n)}{n!\Gamma(\beta+1)\Gamma(\alpha+n)\Gamma(\alpha+\beta+N+n)}, \tag{17}
$$

then the formulas (15) and (16) can be rewriten in the form:

$$
\begin{aligned}
&\sum_{m=0}^{n-1} \frac{\hat{h}_m^{\alpha,\beta}(x,N)\hat{h}_m^{\alpha,\beta}(0,N)}{\hat{d}_m^2} = \kappa_n(\alpha,\beta) \bigtriangledown \hat{h}_n^{\alpha-1,\beta}(x,N), \\
&\sum_{m=0}^{n-1} \frac{\hat{h}_m^{\alpha,\beta}(x,N)\hat{h}_m^{\alpha,\beta}(N-1,N)}{\hat{d}_m^2} = \kappa_n(\beta,\alpha)(-1)^{n-1} \bigtriangleup \hat{h}_n^{\alpha,\beta-1}(x,N).
\end{aligned} \tag{18}
$$

§3 The definition and orthogonal relation.

Consider the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients supported on the interval $[0,N)$ defined as

$$
<\mathcal{U},PQ> \;=<\mathcal{H},PQ> + AP(0)Q(0) + BP(N-1)Q(N-1), \quad x\in \mathbb{N}, \quad A,B\geq 0 \quad , \tag{19}
$$

where $\mathcal{H}$ is a classical moment functional (2) associated with the classical Hahn polynomials:

$$< \mathcal{H}, PQ > \; = \sum_{x=0}^{N-1} P(x)Q(x) \frac{\Gamma(\alpha+N-x)\Gamma(\beta+1+x)}{\Gamma(N-x)\Gamma(1+x)}, \qquad \alpha > -1, \quad \beta > -1. \tag{20}$$

We will determine the monic polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ which are orthogonal with respect to the functional $\mathcal{U}$ and prove that they exist for all positive values of the masses A and B.

Let us write the Fourier expansion of such generalized polynomials in terms of the classical monic orthogonal Hahn polynomials $\hat{h}_k^{\alpha,\beta}(x,N)$.

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) + \sum_{k=0}^{n-1} a_{n,k} \hat{h}_k^{\alpha,\beta}(x,N). \tag{21}$$

In order to obtain the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ with respect to $\mathcal{U}$, i.e.,

$$< \mathcal{U}, \hat{h}_n^{A,B,\alpha,\beta}(x,N)\hat{h}_k^{\alpha,\beta}(x,N) >= 0 \qquad 0 \le k < n.$$

Now putting (21) in (19) we find:

$$\begin{aligned} 0 = \; & < \mathcal{H}, \hat{h}_n^{A,B,\alpha,\beta}(x,N)\hat{h}_k^{\alpha,\beta}(x,N) > \; + \\ & + \; A\hat{h}_n^{A,B,\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(0,N) + \; B\hat{h}_n^{A,B,\alpha,\beta}(N-1,N)\hat{h}_k^{\alpha,\beta}(N-1,N). \end{aligned} \tag{22}$$

If we use the decomposition (21) and taking into account the orthogonality of the classical orthogonal polynomials with respect to the linear functional $\mathcal{H}$, then the coefficients $a_{n,k}$ are given by:

$$a_{n,k} = -A \; \frac{\hat{h}_n^{A,B,\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(0,N)}{\hat{d}_k^2} - B \; \frac{\hat{h}_n^{A,B,\alpha,\beta}(N-1,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2}. \tag{23}$$

Finally the equation (21) provides us the expression

$$\begin{aligned} \hat{h}_n^{A,B,\alpha,\beta}(x,N) = \; & \hat{h}_n^{\alpha,\beta}(x,N) - A\hat{h}_n^{A,B,\alpha,\beta}(0,N) \sum_{k=0}^{n-1} \frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(x,N)}{\hat{d}_k^2} - \\ & - B\hat{h}_n^{A,B,\alpha,\beta}(N-1,N) \sum_{k=0}^{n-1} \frac{\hat{h}_k^{\alpha,\beta}(N-1,N)\hat{h}_k^{\alpha,\beta}(x,N)}{\hat{d}_k^2}. \end{aligned} \tag{24}$$

To obtain the unknown values of $\hat{h}_n^{A,B,\alpha,\beta}(0,N)$ and $\hat{h}_n^{A,B,\alpha,\beta}(N-1,N)$ it is enough to evaluate (24) in $x=0$ and $x=N-1$ and solve the resulting system of two equations. The solution of these equations yields to:

$$\hat{h}_n^{A,B,\alpha,\beta}(0,N) = \frac{\begin{vmatrix} \hat{h}_n^{\alpha,\beta}(0,N) & BKer_{n-1}^{\alpha,\beta}(0,N-1) \\ \hat{h}_n^{\alpha,\beta}(N-1,N) & 1 + BKer_{n-1}^{\alpha,\beta}(N-1,N-1) \end{vmatrix}}{\begin{vmatrix} 1 + AKer_{n-1}^{\alpha,\beta}(0,0) & BKer_{n-1}^{\alpha,\beta}(0,N-1) \\ AKer_{n-1}^{\alpha,\beta}(0,N-1) & 1 + BKer_{n-1}^{\alpha,\beta}(N-1,N-1) \end{vmatrix}} \tag{25}$$

and

$$\hat{h}_n^{A,B,\alpha,\beta}(N-1,N) = \frac{\begin{vmatrix} 1+AKer_{n-1}^{\alpha,\beta}(0,0) & \hat{h}_n^{\alpha,\beta}(0,N) \\ AKer_{n-1}^{\alpha,\beta}(0,N-1) & \hat{h}_n^{\alpha,\beta}(N-1,N) \end{vmatrix}}{\begin{vmatrix} 1+AKer_{n-1}^{\alpha,\beta}(0,0) & BKer_{n-1}^{\alpha,\beta}(0,N-1) \\ AKer_{n-1}^{\alpha,\beta}(0,N-1) & 1+BKer_{n-1}^{\alpha,\beta}(N-1,N-1) \end{vmatrix}}. \tag{26}$$

From (24) and the last two expressions we can conclude that $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ exists for any positive value of the masses A and B. To prove it notice that the denominator is always positive:

$$\begin{vmatrix} 1+A\displaystyle\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(0,N)}{\hat{d}_k^2} & B\displaystyle\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2} \\ A\displaystyle\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2} & 1+B\displaystyle\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(N-1,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2} \end{vmatrix} =$$

$$= 1+A\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(0,N)}{\hat{d}_k^2} + B\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(N-1,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2} +$$

$$+AB\left[\sum_{k=0}^{n-1}\frac{(\hat{h}_k^{\alpha,\beta}(0,N))^2}{\hat{d}_k^2}\sum_{k=0}^{n-1}\frac{(\hat{h}_k^{\alpha,\beta}(N-1,N))^2}{\hat{d}_k^2} - \left(\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2}\right)^2\right].$$

Now if we take into account (12) and the Cauchy inequality $(\sum a_k b_k)^2 \le \sum a_k^2 \sum b_k^2$ the desired result follows.

In this way we have proved the following proposition:

Proposition 1 *The generalized Hahn polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ exist for all non-negative values of the masses and they admit a representation formula in terms of the classical ones as follows:*

$$\begin{aligned} \hat{h}_n^{A,B,\alpha,\beta}(x,N) = {} & \hat{h}_n^{\alpha,\beta}(x,N) - A\hat{h}_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta)\nabla \hat{h}_n^{\alpha-1,\beta}(x,N) - \\ & -B\hat{h}_n^{A,B,\alpha,\beta}(N-1,N)\kappa_n(\beta,\alpha)(-1)^{n-1}\Delta \hat{h}_n^{\alpha,\beta-1}(x,N). \end{aligned} \tag{27}$$

where $\kappa_n(\alpha,\beta)$, $\hat{h}_n^{A,B,\alpha,\beta}(0,N)$ and $\hat{h}_n^{A,B,\alpha,\beta}(N-1,N)$ are given in (17), (25) and (26), respectively, or

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) + \tau_{A,B}^{n,\alpha,\beta}\nabla \hat{h}_n^{\alpha-1,\beta}(x,N) - \tau_{B,A}^{n,\beta,\alpha}\Delta \hat{h}_n^{\alpha,\beta-1}(x,N), \tag{28}$$

where $\tau_{A,B}^{n,\alpha,\beta} = -A\hat{h}_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta)$ and $\tau_{B,A}^{n,\beta,\alpha} = -B\hat{h}_n^{B,A,\beta,\alpha}(0,N)\kappa_n(\beta,\alpha)$.

Proposition 2 *The orthogonal polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ satisfy the following symmetry property:*

$$h_n^{B,A,\beta,\alpha}(N-1-x,N) = (-1)^n h_n^{A,B,\alpha,\beta}(x,N).$$

Proof: Firstly, using the following two straightforward relations:

$$Ker_{n-1}^{\alpha,\beta}(0,0)=\sum_{k=0}^{n-1}\frac{(\hat{h}_k^{\alpha,\beta}(0,N))^2}{\hat{d}_k^2}=\sum_{k=0}^{n-1}\frac{(\hat{h}_k^{\beta,\alpha}(N-1,N))^2}{\hat{d}_k^2}=Ker_{n-1}^{\beta,\alpha}(N-1,N-1)$$

and

$$Ker_{n-1}^{\alpha,\beta}(0,N-1)=\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)\hat{h}_k^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2}=\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\alpha,\beta}(0,N)(-1)^k\hat{h}_k^{\beta,\alpha}(0,N)}{\hat{d}_k^2}=$$

$$=\sum_{k=0}^{n-1}\frac{\hat{h}_k^{\beta,\alpha}(N-1,N)\hat{h}_k^{\beta,\alpha}(0,N)}{\hat{d}_k^2}=Ker_{n-1}^{\beta,\alpha}(0,N-1),$$

and (25) and (26) we obtain

$$\hat{h}_n^{B,A,\beta,\alpha}(N-1,N)=(-1)^n\hat{h}_n^{A,B,\alpha,\beta}(0,N). \tag{29}$$

If we use the representation (28) where we interchange $\alpha\longleftrightarrow\beta$, $A\longleftrightarrow B$ and taking into account (29) and the symmetry property for the classical Hahn polynomials (12) we obtain

$$h_n^{B,A,\beta,\alpha}(N-1-x,N)=$$

$$=h_n^{\beta,\alpha}(N-1-x,N)-Bh_n^{B,A,\beta,\alpha}(0,N)\kappa_n(\beta,\alpha)\bigtriangledown h_k^{\beta-1,\alpha}(N-x-1,N)+$$

$$+Ah_n^{B,A,\beta,\alpha}(N-1,N)\kappa_n(\alpha,\beta)(-1)^n\bigtriangleup h_n^{\beta,\alpha-1}(N-x-1,N)=$$

$$=(-1)^n h_n^{\alpha,\beta}(x,N)-Ah_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta)(-1)^n\bigtriangledown h_n^{\alpha-1,\beta}(x,N)+$$

$$+Bh_n^{A,B,\alpha,\beta}(N-1,N)\kappa_n(\beta,\alpha)(-1)^n\bigtriangleup h_n^{\alpha,\beta-1}(x,N)=$$

$$=(-1)^n h_n^{A,B,\alpha,\beta}(x,N).$$

§4 The hypergeometric representation.

Now we can establish the following representation as hypergeometric function for the generalized Hahn polynomials :

Proposition 3 *The orthogonal polynomials $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ are, up to a constant factor, a generalized hypergeometric function ${}_5F_4$. More precisely*

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N)=\frac{(-1)^n(N-1)!\Gamma(\alpha+\beta+n+1)(\beta+1)_n}{(N-n-1)!\Gamma(\alpha+\beta+2n+1)}{}_5\mathrm{F}_4\left({-x,\alpha+\beta+n,-n,\gamma_0+1,\gamma_1+1 \atop 1-N,\beta+1,\gamma_0,\gamma_1};1\right),$$

Proof: The proof is very similar to the proof provided in [2]. To obtain the desired result we need to put the hypergeometric representation of these polynomials:

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N)=\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}\frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(N-n-1)!\Gamma(\beta+1)}{}_3\mathrm{F}_2\left({-x,\alpha+\beta+n+1,-n \atop 1-N,\beta+1};1\right),$$

in formula (28) and do some algebraic calculations. In fact

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) + \tau_{A,B}^{n,\alpha,\beta} \bigtriangledown \hat{h}_n^{\alpha-1,\beta}(x,N) - \tau_{B,A}^{n,\beta,\alpha} \bigtriangleup \hat{h}_n^{\alpha,\beta-1}(x,N)$$

where $\tau_{A,B}^{n,\alpha,\beta} = -A\hat{h}_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta)$. Then

$$h_n^{A,B,\alpha,\beta}(x,N) =$$

$$= \tfrac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)} \tfrac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(N-n-1)!\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{(-n)_k(-x)_k(\alpha+\beta+n+1)_k}{(1-N)_k(\beta+1)_k k!} + \tau_{A,B}^{n,\alpha,\beta}\times$$

$$\times \tfrac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+\beta+2n)} \tfrac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(N-n-1)!\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{(-n)_k(\alpha+\beta+n)_k}{(1-N)_k(\beta+1)_k k!} [(-x)_k - (1-x)_k] +$$

$$+\tau_{B,A}^{n,\beta,\alpha} \tfrac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+\beta+2n)} \tfrac{(-1)^n(N-1)!\Gamma(\beta+n)}{(N-n-1)!\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(-n)_k(\alpha+\beta+n)_k}{(1-N)_k(\beta)_k k!} [(-x-1)_k - (-x)_k] =$$

$$= \tfrac{(-1)^n(N-1)!\Gamma(\beta+n)\Gamma(\alpha+\beta+n)}{(N-n-1)!\Gamma(\beta+1)\Gamma(\alpha+\beta+2n+1)} \sum_{k=0}^{\infty} \frac{(-n)_k(\alpha+\beta+n)_k(-x-1)_k}{(1-N)_k(\beta+1)_k k!}\times$$

$$\times \left[\tfrac{(x+1-k)(\alpha+\beta+n+k)(\beta+n)}{x+1} + \tau_{A,B}^{n,\alpha,\beta}(\beta+n)(\alpha+\beta+2n)\tfrac{k(x+1-k)}{x(x+1)} - \tau_{B,A}^{n,\beta,\alpha}(\beta+k)(\alpha+\beta+2n)\tfrac{k}{x+1} \right].$$

Here we use the identities: $[(-x)_k - (1-x)_k] = \dfrac{k}{x}(-x)_k$ and

$$(a+1)_k = \frac{a+k}{a}(a)_k \qquad \text{or} \qquad (k+a) = a\frac{(a+1)_k}{(a)_k}. \tag{30}$$

As the expression inside the square brackets is a polynomial of second order, i.e., ak^2+bk+c, then it can be factorized in the form

$$a(k+\gamma_0)(k+\gamma_1), \qquad \text{with} \qquad \gamma_0\gamma_1 = \frac{(\beta+n)(\alpha+\beta+n)}{a}.$$

Using this and (30) we can rewrite the expression inside the square brackets in a form:

$$(\alpha+\beta+n)(\beta+n)\frac{(\gamma_0+1)_k(\gamma_1+1)_k}{(\gamma_0)_k(\gamma_1)_k}$$

and then

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)} \frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{(N-n-1)!\Gamma(\beta+1)}\times$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k(-x)_k(\alpha+\beta+n)_k(\gamma_0+1)_k(\gamma_1+1)_k}{(1-N)_k(\beta+1)_k(\gamma_0)_k(\gamma_1)_k k!} =$$

$$= \frac{(-1)^n(N-1)!\Gamma(\alpha+\beta+n+1)(\beta+1)_n}{(N-n-1)!\Gamma(\alpha+\beta+2n+1)} {}_5\mathrm{F}_4\left({-x,\alpha+\beta+n,-n,\gamma_0+1,\gamma_1+1 \atop 1-N,\beta+1,\gamma_0,\gamma_1} ;1\right). \qquad \blacksquare$$

Here the coefficients γ_0 and γ_1 are, in general, complex numbers. In the case when they are nonpositive integers we need to take the analytic continuation of the hypergeometric series.

It is straightforward to show that for $A = B = 0$ the hypergeometric functions of the proposition 3 yield to classical polynomials (9).

§5 Some special cases.

We will start from the representation formula (28)

$$\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) + \tau_{A,B}^{n,\alpha,\beta} \bigtriangledown \hat{h}_n^{\alpha-1,\beta}(x,N) - \tau_{B,A}^{n,\beta,\alpha} \bigtriangleup \hat{h}_n^{\alpha,\beta-1}(x,N),$$

where the coefficients $\tau_{A,B}^{n,\alpha,\beta}$ and $\tau_{B,A}^{n,\beta,\alpha}$ are equal to

$$\tau_{A,B}^{n,\alpha,\beta} = -A\hat{h}_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta) \qquad \tau_{B,A}^{n,\beta,\alpha} = -B\hat{h}_n^{B,A,\beta,\alpha}(0,N)\kappa_n(\beta,\alpha),$$

respectively. We will consider the following two special cases:

1. The case when we add only one mass at the point $x=0$, i.e.,$A>0$ and $B=0$.
2. The case when we add only one mass at the end point $x=N-1$, i.e., $A=0$ and $B>0$.

1. The case $B=0$. For the first case we deduce that

$$\tau_{A,0}^{n,\alpha,\beta} = \frac{A}{(1+AKer_{n-1}^{\alpha,\beta}(0,0))}\frac{\Gamma(\alpha+\beta+n+1)\Gamma(\beta+n+1)(\alpha+\beta+2n)^{-1}}{n!(N-n-1)!\Gamma(\alpha+\beta+n+N)\Gamma(\alpha+n)}\left[\frac{(N-1)!}{\Gamma(\beta+1)}\right]^2,$$

$$\tau_{0,A}^{n,\beta,\alpha} = 0.$$

Then the following representation formulas hold:

$$\hat{h}_n^{A,0,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) + \tau_{A,0}^{n,\alpha,\beta} \bigtriangledown \hat{h}_n^{\alpha-1,\beta}(x,N).$$

and

$$\hat{h}_n^{A,0,\alpha,\beta}(x,N) = \frac{(-1)^n(N-1)!(\beta+1)_n\Gamma(\alpha+\beta+n+1)}{(N-n-1)!\Gamma(\alpha+\beta+2n+1)}{}_4\mathrm{F}_3\left({-x,\alpha+\beta+n,-n,\gamma_0+1 \atop 1-N,\beta+1,\gamma_0};1\right),$$

where $\gamma_0 = -\dfrac{x(\alpha+\beta+n)}{x+\tau_{A,0}^{n,\alpha,\beta}(\alpha+\beta+2n)}$.

2. The case $A=0$. For this case we have that

$$\tau_{0,B}^{n,\alpha,\beta} = 0,$$

$$\tau_{B,0}^{n,\beta,\alpha} = \frac{B}{(1+BKer_{n-1}^{\beta,\alpha}(0,0))}\frac{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)(\alpha+\beta+2n)^{-1}}{n!(N-n-1)!\Gamma(\alpha+\beta+n+N)\Gamma(\alpha+\beta)}\left[\frac{(N-1)!}{\Gamma(\alpha+1)}\right]^2,$$

and then, we obtain the representation formulas:

$$\hat{h}_n^{0,B,\alpha,\beta}(x,N) = \hat{h}_n^{\alpha,\beta}(x,N) - \tau_{B,0}^{n,\beta,\alpha} \bigtriangledown \hat{h}_n^{\alpha,\beta-1}(x,N).$$

and

$$\hat{h}_n^{0,B,\alpha,\beta}(x,N) = \frac{(-1)^n(N-1)!\Gamma(\alpha+\beta+n+1)(\beta+1)_n}{(N-n-1)!\Gamma(\alpha+\beta+2n+1)}{}_5\mathrm{F}_4\left({-x,\alpha+\beta+n,-n,\gamma_0+1,\gamma_1+1 \atop 1-N,\beta+1,\gamma_0,\gamma_1};1\right),$$

where $-\gamma_0$ and $-\gamma_1$ are the solutions of the equation:

$$\left[\frac{(x+1-k)(\alpha+\beta+n+k)(\beta+n)}{x+1} - \tau_{B,0}^{n,\beta,\alpha}(\beta+k)(\alpha+\beta+2n)\frac{k}{x+1}\right] = 0.$$

§6 The Three Term Recurrence Relations and Relation between tridiagonal matrices T_{n+1} and $T_{n+1}^{A,B}$.

The generalized polynomials satisfy a three term recurrence relation (TTRR) of the form

$$\begin{gathered} x\hat{h}_n^{A,B,\alpha,\beta}(x,N) = \hat{h}_{n+1}^{A,\alpha,\beta}(x,N) + \beta_n^{A,B}\hat{h}_n^{A,B,\alpha,\beta}(x,N) + \gamma_n^{A,B}\hat{h}_{n-1}^{A,\alpha,\beta}(x,N) \quad , n \geq 0 \\ \hat{h}_{-1}^{A,B,\alpha,\beta}(x,N) = 0 \qquad \text{and} \qquad \hat{h}_0^{A,B,\alpha,\beta}(x,N) = 1. \end{gathered} \tag{31}$$

This is a simple consequence of their orthogonality with respect to a positive linear functional (see [7] or [16]). To obtain the explicit formula for the recurrence coefficients we can compare the coefficients of x^n in the two sides of (31). Let $b_n^{A,B}$ be the coefficient of x^{n-1} in the expansion $\hat{h}_n^{A,B,\alpha,\beta}(x,N) = x^n + b_n^{A,B}\,x^{n-1} + ...$, then:

$$\beta_n^A = b_n^{A,B} - b_{n+1}^{A,B}.$$

To calculate $\gamma_n^{A,B}$ is sufficient to evaluate (31) in $x=0$ and remark that $\hat{h}_{n-1}^{A,B,\alpha,\beta}(0,N) \neq 0$.

In order to obtain a general expression for the coefficient $\beta_n^{A,B}$ we can use the representation formula (28) for the generalized polynomials. Doing some algebraic calculations we find that $b_n^{A,B} = b_n + n\tau_{A,B}^{n,\alpha,\beta} - n\tau_{B,A}^{n,\beta,\alpha}$, where b_n denotes the coefficient of the $n-1$ power in the classical monic orthogonal Hahn polynomials $\hat{h}_n^{\alpha,\beta}(x,N) = x^n + b_n\,x^{n-1} + ...$

Using these formulas and (8) we obtain for generalized Hahn polynomials the following TTRR coefficients:

$$\begin{aligned} \beta_n^{A,B} = {} & \frac{\alpha-\beta+2N-2}{4} - \frac{(\beta^2-\alpha^2)(\alpha+\beta+2N}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)} + \\ & + n(\tau_{A,B}^{n,\alpha,\beta} - \tau_{B,A}^{n,\beta,\alpha}) - (n+1)(\tau_{A,B}^{n+1,\alpha,\beta} - \tau_{B,A}^{n+1,\beta,\alpha}), \\ \gamma_n^{A,B} = {} & -\frac{\hat{h}_{n+1}^{A,B,\alpha,\beta}(0,N)}{\hat{h}_{n-1}^{A,B,\alpha,\beta}(0,N)} - \beta_n^{A,B}\frac{\hat{h}_n^{A,B,\alpha,\beta}(0,N)}{\hat{h}_{n-1}^{A,B,\alpha,\beta}(0,N)}. \end{aligned}$$

Now we will find the relation between the tridiagonal matrices T_{n+1} corresponding to the classical polynomials $\hat{h}_n^{\alpha,\beta}(x,N)$ and the tridiagonal matrices $T_{n+1}^{A,B}$ corresponding to the perturbed ones $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$.

If we rewrite the TTRR (31) in the matrix form we obtain

$$x\ \mathbf{h_n^{A,B}} = T_{n+1}^{A,B}\ \mathbf{h_n^{A,B}} + \hat{h}_{n+1}^{A,B,\alpha,\beta}(x,N)\mathbf{e}_{\mathbf{n+1}}^{\mathbf{(n+1)}}\ , \tag{32}$$

where the corresponding tridiagonal matrix and the *perturbed polynomial vector* are denoted by $T_{n+1}^{A,B}$ and $\mathbf{h_n^{A,B}}$, respectively $(n \geq 0)$:

$$T_{n+1}^{A,B} = \begin{pmatrix} \beta_0^A & 1 & 0 & 0 & \dots & 0 & 0 \\ \gamma_1^A & \beta_1^A & 1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_2^A & \beta_2^A & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \gamma_n^A & \beta_n^A \end{pmatrix} \quad \text{and} \quad \mathbf{h_n^{A,B}} = \begin{pmatrix} \hat{h}_0^{A,B,\alpha,\beta}(x,N) \\ \hat{h}_1^{A,B,\alpha,\beta}(x,N) \\ \hat{h}_2^{A,B,\alpha,\beta}(x,N) \\ \vdots \\ \hat{h}_n^{A,B,\alpha,\beta}(x,N) \end{pmatrix},$$

and for a given positive integer $n \geq 1$ by $\mathbf{e}\ _{\mathbf{j}}^{\mathbf{(n+1)}}$ $(0 \leq j \leq n+1)$, we denote :

$$\mathbf{e}\ _{\mathbf{j}}^{\mathbf{(n+1)}} := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^T \quad \in \mathbb{R}^{n+1}.$$

A similar notation will be used for the tridiagonal matrix T_{n+1}, with the β_i's and γ_i's replaced by the corresponding coefficients of the three term recurrence relation for $\hat{h}_n^{\alpha,\beta}(x,N)$ and for the *polynomial vector* $\mathbf{h_n}$:

$$T_{n+1} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \gamma_n & \beta_n \end{pmatrix} \quad \text{and} \quad \mathbf{h_n} = \begin{pmatrix} \hat{h}_0^{\alpha,\beta}(x,N) \\ \hat{h}_1^{\alpha,\beta}(x,N) \\ \hat{h}_2^{\alpha,\beta}(x,N) \\ \vdots \\ \hat{h}_n^{\alpha,\beta}(x,N) \end{pmatrix}.$$

From the relation (24) and using (25) and (26), we deduce

$$\hat{h}_{n+1}^{A,B,\alpha,\beta}(x,N) = \hat{h}_{n+1}^{\alpha,\beta}(x,N) + \sum_{j=0}^{n} a_{n+1,j} \hat{h}_j^{\alpha,\beta}(x,N)\,, \tag{33}$$

or in the matrix form :

$$\mathbf{h_n^{A,B}} = R_{n+1} \ \ \mathbf{h_n}\,, \tag{34}$$

where R_{n+1} denotes the lower triangular matrix with 1 entries in the main diagonal :

$$R_{n+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{1,0} & 1 & 0 & 0 & \dots & 0 & 0 \\ a_{2,0} & a_{2,1} & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n-1} & 1 \end{pmatrix} \,,$$

and $a_{m,j}^{A,B}$ for all j in $0 \leq j \leq m-1$ are equal to:

$$a_{m,j}^{A,B} = -A \frac{\hat{h}_m^{A,B,\alpha,\beta}(0,N)\hat{h}_j^{\alpha,\beta}(0,N)}{\hat{d}_k^2} - B \frac{\hat{h}_m^{A,B,\alpha,\beta}(N-1,N)\hat{h}_j^{\alpha,\beta}(N-1,N)}{\hat{d}_k^2}.$$

Now putting (34) in (32) and using (33) we find

$$xR_{n+1} \ \ \mathbf{h_n} = T_{n+1}^{A,B} R_{n+1} \ \ \mathbf{h_n} + \left(\hat{h}_{n+1}^{\alpha,\beta}(x,N) + \sum_{j=0}^{n} a_{n+1,j} \hat{h}_j^{\alpha,\beta}(x,N) \right) \mathbf{e}\ _{\mathbf{n+1}}^{\mathbf{(n+1)}}\,,$$

from where, using the TTRR in the matrix form for the classical Hahn polynomials $x \ \ \mathbf{h_n} = T_{n+1} \ \ \mathbf{h_n} + \hat{h}_{n+1}^{\alpha,\beta}(x,N)\mathbf{e}\ _{\mathbf{n+1}}^{\mathbf{(n+1)}}$, we find

$$T_{n+1} \ \ \mathbf{h_n} = R_{n+1}^{-1} T_{n+1}^{A,B} R_{n+1} \ \ \mathbf{h_n} + \sum_{j=0}^{n} a_{n+1,j} P_j(x) \mathbf{e}\ _{\mathbf{n+1}}^{\mathbf{(n+1)}}\ .$$

Finally from this equation and using the fact that $R_{n+1}\Lambda_{n+1} = \Lambda_{n+1}$ we obtain the following relation between the tridiagonal matrices T_{n+1} and $T_{n+1}^{A,B}$

$$T_{n+1}^{A,B} = R_{n+1} \left(T_{n+1} + \Lambda_{n+1} \right) R_{n+1}^{-1}\,, \tag{35}$$

where Λ_{n+1} is the rank-one matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{n+1,0} & a_{n+1,1} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n-1} & a_{n+1,n} \end{pmatrix}.$$

Then we can conclude that $T_{n+1}^{A,B}$ is a rank-one perturbation of T_{n+1}.

Finally, we obtain explicitly the corresponding elements of the matrices R_{n+1} and Λ_{n+1} for the perturbed Hahn orthogonal polynomials in the special cases when $B=0$ or $A=0$:

$$\begin{aligned} a_{m,k}^{A,0} = \; & A\frac{(-1)^{m+k+1}}{(1+Ker_{m-1}^{\alpha,\beta}(0,0))}\left[\frac{(N-1)!}{\Gamma(\beta+1)}\right]^2 \times \\ & \times\frac{\Gamma(m+\beta+1)\Gamma(\alpha+\beta+2k+2)\Gamma(\alpha+\beta+m+1)}{k!\Gamma(\alpha+k+1)\Gamma(\alpha+\beta+2m+2)\Gamma(\alpha+\beta+N+k+1)(N-m-1)!}. \end{aligned} \tag{36}$$

For the second case using the symmetric propreties for the $\hat{h}_n^{A,B,\alpha,\beta}(x,N)$ and $\hat{h}_n^{\alpha,\beta}(x,N)$ polynomials we find

$$\begin{aligned} a_{m,k}^{0,B} = \; & B\frac{-1}{(1+Ker_{m-1}^{\beta,\alpha}(0,0))}\left[\frac{(N-1)!}{\Gamma(\alpha+1)}\right]^2 \times \\ & \times\frac{\Gamma(m+\alpha+1)\Gamma(\alpha+\beta+2k+2)\Gamma(\alpha+\beta+m+1)}{k!\Gamma(\beta+k+1)\Gamma(\alpha+\beta+2m+2)\Gamma(\alpha+\beta+N+k+1)(N-m-1)!}. \end{aligned} \tag{37}$$

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