ON THE PROPERTIES FOR MODIFICATIONS OF CLASSICAL ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLES.[1]

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Abstract

We consider a modification of moment functionals for some classical polynomials of a discrete variable by adding a mass point at $x = 0$. We obtain the resulting orthogonal polynomials, identify them as hypergeometric functions and derive the second order difference equation which these polynomials satisfy. The corresponding tridiagonal matrices and associated polynomials were also studied.

§1 Introduction.

The study of orthogonal polynomials with respect to a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one or two delta Dirac measures has been performed by several authors. In particular, Chihara [5] has considered some properties of such polynomials in terms of the location of the mass point with respect to the support of a positive measure. More recently Marcellán and Maroni [10] analyzed a more general situation for regular (quasi-definite) linear functionals, i.e., such that the principal submatrices of the corresponding infinite Hankel matrices associated with the moment sequences are nonsingular.

A special emphasis is given to the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel). Koornwinder [9] considered a system of polynomials orthogonal with respect to the classical weigtht function for Jacobi polynomials with two extra point masses added at $x = -1$ and $x = 1$. For generalized Laguerre polynomials $\{L_n^{\alpha,A}(x)\}_{n=0}^{\infty}$ that are orthogonal on $[0,\infty)$ with respect to the linear functional $\mathcal{C}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{C}, P > \; = \int_0^{\infty} P(x)\, x^{\alpha} e^{-x} dx + AP(0) \quad , \alpha > -1,\, A \geq 0\,,$$

Koekoek and Koekoek [8] found a differential equation of the form

$$A\sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny = 0,$$

where the coefficients $a_i(x)$, $i \in \{1,2,3,...\}$, are independent of n and a_0 depends on n but is independent of x. In the above paper, representation formulas for the new orthogonal

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polynomial sequences, as well as the second order differential equation that such polynomials satisfy were deduced.

In the open problem section of the Proceedings of the Third International Symposium on Orthogonal Polynomials and their Applications held in Erice (Italy), R. Askey raised the following question [1]:

Consider the Meixner Polynomials $M_n^{\gamma,\mu}(x)$, add or substract a mass point at $x = 0$ and find the resulting polynomials. Identify them as hypergeometric functions and show that these polynomials satisfy a difference equation in x.

In [3] Bavinck and van Haeringen gave the solution to the problem of finding the second order difference equation for generalized Meixner polynomials, as well as the infinite order difference equation which these polynomials satisfy. For generalized Charlier polynomials Bavinck and Koekoek [4] found the corresponding infinite order difference equation.

In the paper [2] we obtained the representation for such generalized Meixner polynomials as an hypergeometric function ${}_3F_2$, as well as the corresponding second order difference equation. Now we generalize this result for the Kravchuk and Charlier polynomials and continue the algebraic approach presented by Godoy, Marcellán, Salto, Zarzo (see [7]) in the framework of a more general theory based in the addition of a delta Dirac measure to a discrete semiclassical linear functional. We analyze the relation between tridiagonal matrices of the *perturbed or generalized* $P_n^A(x)$ and classical $P_n(x)$ polynomials, as well as the associated polynomials corresponding to the sequence $\{P_n^A(x)\}_{n=0}^{\infty}$.

The structure of the paper is as follows. In Section 2, we provide the basic properties of the classical orthogonal polynomials of discrete variable which will be needed, as well as the main data for the Meixner, Kravchuk and Charlier polynomials. In Section 3 we deduce expressions of the generalized Meixner, Kravchuk and Charlier polynomials and its first difference derivatives, as well as their representation as hypergeometric functions in the direction raised by Askey. In Section 4, we find the second order difference equation which these generalized polynomials satisfy. In Section 5, from the three term recurrence relation (TTRR) of the classical orthogonal polynomials we find the TTRR which satisfy the perturbed ones. In Section 6, from the relation of the perturbed polynomials $P_n^A(x)$ as a linear combination of the classical ones, we find the tridiagonal matrices associated with the perturbed monic orthogonal polinomial sequence (PMOPS) $\{P_n^A(x)\}_{n=0}^{\infty}$ as a rank-one perturbation of the tridiagonal matrices associated with the classical monic orthogonal polynomial sequence (CMOPS) $\{P_n(x)\}_{n=0}^{\infty}$. Finally, in Section 7 we find the associated polynomials $P_n^{(1),A}(x)$ corresponding to $\{P_n^A(x)\}_{n=0}^{\infty}$ in terms of the associated polynomials $P_n^{(1)}(x)$ corresponding to $\{P_n(x)\}_{n=0}^{\infty}$ and the classical ones.

§2 Some Preliminar Results.

Firstly, we enclose some formulas for the classical Meixner, Kravchuk and Charlier polynomials which are useful in order to obtain the generalized polynomials orthogonal with respect to the linear functional $\mathcal{U}$ defined as a modification of the first ones troughtout the addition of a mass point. All the formulas and other properties for the classical Meixner, Kravchuk and Charlier polynomials can be found in a lot of books (see for instance the excellent monograph *Orthogonal Polynomials in Discrete Variables* by A.F. Nikiforov, S. K. Suslov, V. B. Uvarov [11], Chapter 2.)

We will use monic polynomials, i.e., polynomials with leading coefficient equal to 1. The classical orthogonal polynomials of a discrete variable in the uniform lattice $x(s) = s$ are the

polynomial solution of a second order linear difference equation of hypergeometric type

$$\sigma(x) \triangle \nabla P_n(x) + \tau(x) \triangle P_n(x) + \lambda_n P_n(x) = 0\,, \tag{1}$$

where

$$\nabla f(x) = f(x) - f(x-1), \qquad \triangle f(x) = f(x+1) - f(x).$$

Here $\sigma(x)$ and $\tau(x)$ are polynomials in x of degree at most 1 and 2, respectively, and λ_n is a constant.

These polynomials are orthogonal with respect to the linear functional $\mathcal{L}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{L}, P > \; = \sum_{x \in I\!N} \rho(x) P(x), \qquad I\!N = \{0, 1, 2, ...\}\,, \tag{2}$$

where $\rho(x)$ is some non-negative function (weight function) supported in a countable set of the real line and such that

$$\triangle[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

The orthogonality relation is

$$\sum_{x \in I\!N} P_n(x) P_m(x) \rho(x) = \delta_{nm} d_n^2, \tag{3}$$

where d_n^2 denotes the square of the norm of these classical polynomials.

The polynomial solutions of equation (1) are uniquely determined, up to a normalized factor (R_n), by the difference analog of the Rodrigues formula (see [11] page 24 Eq.(2.2.7)):

$$P_n(x) = \frac{R_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right]. \tag{4}$$

They satisfy a three term recurrence relation of the form

$$\begin{gathered} x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \;\; n \geq 0 \\ P_{-1}(x) = 0 \qquad \text{and} \qquad P_0(x) = 1. \end{gathered} \tag{5}$$

and the Christoffel-Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m(x) P_m(y)}{d_m^2} = \frac{1}{x-y} \frac{P_n(x) P_{n-1}(y) - P_n(y) P_{n-1}(x)}{d_{n-1}^2} \qquad n = 1, 2, 3, ... \quad . \tag{6}$$

We will consider the following three classical monic orthogonal polynomials (CMOP) which are solutions of the difference equation (1).

I. The Meixner polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0, \infty)$, with

$$\sigma(x) = x, \quad \tau(x) = \gamma\mu - x(1-\mu) \quad , 0 < \mu < 1, \quad \gamma > 0, \quad \lambda_n = n(1-\mu),$$

and

$$R_n = \frac{1}{(\mu-1)^n}, \qquad \rho(x) = \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(\gamma)\Gamma(1+x)}, \qquad d_n^2 = \frac{n!(\gamma)_n \mu^n}{(1-\mu)^{\gamma+2n}}.$$

II. The Kravchuk polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0, N]$, with $n \leq N$

$$\sigma(x) = x, \quad \tau(x) = \frac{Np - x}{1-p} \quad , 0 < p < 1, \quad \lambda_n = \frac{n}{1-p},$$

and

$$R_n = (p-1)^n, \qquad \rho(x) = \frac{p^x N!(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(1+x)}, \qquad d_n^2 = \frac{n!N!p^n(1-p)^n}{(N-n)!}.$$

III. The Charlier polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,\infty)$, with

$$\sigma(x) = x, \quad ,\tau(x) = \mu - x \quad ,\mu > 0, \quad \lambda_n = n,$$

and

$$R_n = (-1)^n, \qquad \rho(x) = \frac{\mu^x e^{-\mu}}{\Gamma(1+x)}, \qquad d_n^2 = n!\mu^n.$$

They satisfy the so called structure relations

$$\frac{x}{n} \bigtriangledown M_n^{\gamma,\mu}(x) = \frac{\mu(1-\gamma-n)}{\mu-1} M_{n-1}^{\gamma,\mu}(x) + M_n^{\gamma,\mu}(x), \tag{7}$$

$$\frac{x}{n} \bigtriangledown K_n^p(x) = p(N-n+1)K_{n-1}^p(x) + K_n^p(x), \tag{8}$$

$$\frac{x}{n} \bigtriangledown C_n^\mu(x) = \mu C_{n-1}^\mu(x) + C_n^\mu(x). \tag{9}$$

These classical polynomials can be represented as hypergeometric functions (see [11] page 49, section **2.7**)

$$M_n^{\gamma,\mu}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n} \quad {}_2F_1\left(\begin{matrix}-n,-x\\ \gamma\end{matrix}\,\middle|\, 1-\frac{1}{\mu}\right), \tag{10}$$

$$K_n^p(x) = \frac{(-p)^n N!}{(N-n)!} \quad {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix}\,\middle|\, \frac{1}{p}\right), \tag{11}$$

$$C_n^\mu(x) = (-\mu)^n \quad {}_2F_0\left(\begin{matrix}-n,-x\\ -\end{matrix}\,\middle|\, -\frac{1}{\mu}\right), \tag{12}$$

where the hypergeometric function is defined by

$${}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\,|x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!},$$

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1), \qquad k = 1,2,3,\ldots$$

As a consequence of these representations we can deduce

$$M_n^{\gamma,\mu}(0) = \frac{\mu^n}{(\mu-1)^n}\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}, \qquad K_n^p(0) = \frac{(-p)^n N!}{(N-n)!}, \qquad C_n^\mu(0) = (-\mu)^n. \tag{13}$$

We have proved (see [2] formula (26)) the following property for the kernels of the Meixner polynomials.

$$Ker_{n-1}^M(x,0) \equiv \sum_{m=0}^{n-1} \frac{M_m^{\gamma,\mu}(x)M_m^{\gamma,\mu}(0)}{d_m^2} = \frac{(-1)^{n-1}(1-\mu)^{n+\gamma-1}}{n!} \bigtriangledown M_n^{\gamma,\mu}(x). \tag{14}$$

It is straightforward to show that the kernels of $K_n^p(x)$ and $C_n^\mu(x)$ verify the following relations:

$$Ker_{n-1}^K(x,0) \equiv \sum_{m=0}^{n-1} \frac{K_m^p(x)K_m^p(0)}{d_m^2} = \frac{(p-1)^{1-n}}{n!} \bigtriangledown K_n^p(x), \tag{15}$$

$$Ker_{n-1}^C(x,0) \equiv \sum_{m=0}^{n-1} \frac{C_m^\mu(x)C_m^\mu(0)}{d_m^2} = \frac{(-1)^{n-1}}{n!} \bigtriangledown C_n^\mu(x). \tag{16}$$

§3 The definition, orthogonal relation and representation as hypergeometric series.

Consider the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{U}, P > \; = < \mathcal{L}, P > + AP(0), \qquad x \in \mathbb{N}, \qquad A \geq 0 \quad , \tag{17}$$

where $\mathcal{L}$ is a classical moment functional (2) associated to some classical polynomials of a discrete variable.

We will determine the monic polynomials $P_n^A(x)$ which are orthogonal with respect to the functional $\mathcal{U}$ and prove that they exist for all positive A (see (22) from below).

To obtain this, we can write the Fourier expansion of such generalized polynomials

$$P_n^A(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k} P_k(x), \tag{18}$$

where P_n denotes the classical monic orthogonal polynomial (CMOP) of degree n.

In order to find the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $P_n^A(x)$ with respect to $\mathcal{U}$, i.e.,

$$< \mathcal{U}, P_n^A(x) P_k(x) >= 0 \qquad \forall k < n.$$

Now putting (18) in (17) we find:

$$< \mathcal{U}, P_n^A(x) P_k(x) > \; = \; < \mathcal{L}, P_n^A(x) P_k(x) > \quad + \quad A P_n^A(0) P_k(0). \tag{19}$$

If we use the decomposition (18) and taking into account the orthogonality of the classical orthogonal polynomials with respect to the linear functional $\mathcal{L}$, then the coefficients $a_{n,k}$ are given by:

$$a_{n,k} = -A \; \frac{P_n^A(0) P_k(0)}{d_k^2}. \tag{20}$$

Finally the equation (18) provides us the expression

$$P_n^A(x) = P_n(x) - A P_n^A(0) \sum_{k=0}^{n-1} \frac{P_k(0) P_k(x)}{d_k^2}. \tag{21}$$

From (21) we can conclude that the representation of $P_n^A(x)$ exists for any positive value of the mass A. To obtain this it is enough to evaluate (21) in $x = 0$,

$$\left(1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} \right) P_n^A(0) = P_n(0) \neq 0, \tag{22}$$

and use the fact that

$$1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} \quad > \quad 0 \qquad n = 1, 2, 3, ...$$

From (22) we can deduce the values of $P_n^A(0)$ as follows

$$P_n^A(0) = \frac{P_n(0)}{1 + A\sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2}}. \tag{23}$$

Now in order to obtain an explicit expression for these polynomials we need some properties of the kernels of the CMOP. In [2] we solved this problem for the classical Meixner polynomials. In this work we will prove a similar result for Charlier and Kravchuk polynomials.

Doing some algebraic calculations in (21) and taking into account formulas (14)-(16) we obtain the following three expressions for the generalized polynomials :

For Meixner polynomials:

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) - AM_n^{\gamma,\mu,A}(0)\frac{(-1)^{n-1}(1-\mu)^{n+\gamma-1}}{n!} \bigtriangledown M_n^{\gamma,\mu}(x). \tag{24}$$

For Kravchuk polynomials:

$$K_n^{p,A}(x) = K_n^{p}(x) - AK_n^{p,A}(0)\frac{(p-1)^{n-1}}{n!} \bigtriangledown K_n^{p}(x). \tag{25}$$

For Charlier polynomials:

$$C_n^{\mu,A}(x) = C_n^{\mu}(x) - AC_n^{\mu,A}(0)\frac{(-1)^{n-1}}{n!} \bigtriangledown C_n^{\mu}(x). \tag{26}$$

In the above formula the values of the polynomials in $x = 0$ could be deduced from (23). Then, we obtain the following analytic expression for the Perturbed Monic Orthogonal Polynomials (PMOP) $P_n^A(x)$ for $x \neq 0$ (when $x = 0$ we can use (23))

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) + B_n \bigtriangledown M_n^{\gamma,\mu}(x) = (I + B_n\bigtriangledown)M_n^{\gamma,\mu}(x), \tag{27}$$

$$K_n^{p,A}(x) = K_n^{p}(x) + A_n \bigtriangledown K_n^{p}(x) = (I + A_n\bigtriangledown)K_n^{p}(x), \tag{28}$$

$$C_n^{\mu,A}(x) = C_n^{\mu}(x) + D_n \bigtriangledown C_n^{\mu}(x) = (I + D_n\bigtriangledown)C_n^{\mu}(x), \tag{29}$$

where A_n, B_n and D_n are constants given by:

$$B_n = A\frac{\mu^n(1-\mu)^{\gamma-1}(\gamma)_n}{n!(1 + AKer_{n-1}^M(0,0))},$$

$$A_n = A\frac{N!}{n!(N-n)!}\frac{p^n(1-p)^{1-n}}{(1 + AKer_{n-1}^K(0,0))},$$

$$D_n = A\frac{\mu^n}{n!(1 + AKer_{n-1}^C(0,0))}.$$

Remark: Using the Rodrigues formula (4), some extension of it follows in a straightforward way.

$$M_n^{\gamma,\mu,A}(x) = (I + B_n\bigtriangledown)\left[\frac{\mu^n}{(1-\mu)^n}\frac{\Gamma(x+1)}{\mu^x\Gamma(\gamma+x)} \bigtriangledown^{(n)} \frac{\mu^x\Gamma(x+\gamma+n)}{\Gamma(x+1)}\right],$$

$$K_n^{p,A}(x) = (I + A_n\bigtriangledown)\left[\frac{(-1)^n\Gamma(x+1)\Gamma(N-x+1)}{p^x\,(1-p)^{-x}} \bigtriangledown^{(n)} \frac{p^x\,(1-p)^{-x}}{\Gamma(x+1)\Gamma(N-x-n+1)}\right],$$

$$C_n^{\mu,A}(x) = (I + D_n\nabla)\left[\frac{(-1)^n\Gamma(x+1)}{\mu^x}\,\nabla^{(n)}\,\frac{\mu^{x+n}}{\Gamma(x+1)}\right].$$

Now we can establish the following representation as hypergeometric functions for the generalized polynomials:

Proposition 1 *The orthogonal polynomials $M_n^{\gamma,\mu,A}(x)$, $K_n^{p,A}(x)$ and $C_n^{\mu,A}(x)$ are, up to a constant factor, generalized hypergeometric functions. More precisely*

$$M_n^{\gamma,\mu,A}(x) = (\gamma)_n\frac{\mu^n}{(\mu-1)^n}\quad {}_3F_2\left(\begin{matrix}-n, -x, 1+xB_n^{-1}\\ \gamma, \quad xB_n^{-1}\end{matrix}\,\middle|\, 1-\frac{1}{\mu}\right), \tag{30}$$

$$K_n^{p,A}(x) = n!(-p)^n\frac{N!}{n!(N-n)!}\quad {}_3F_2\left(\begin{matrix}-n, -x, 1+xA_n^{-1}\\ -N, \quad xA_n^{-1}\end{matrix}\,\middle|\, \frac{1}{p}\right), \tag{31}$$

$$C_n^{\mu,A}(x) = (-\mu)^n\quad {}_3F_1\left(\begin{matrix}-n, -x, 1+xD_n^{-1}\\ xD_n^{-1}\end{matrix}\,\middle|\, -\frac{1}{\mu}\right). \tag{32}$$

Sketch of the Proof: The proof of this Proposition is similar to the proof for the Meixner case (see [2]). To obtain the desired result we need to put the hypergeometric representation of these polynomials in formulas (27), (28) and (29) and do some algebraic calculations. Here the coefficients xA_n^{-1},xB_n^{-1} and xC_n^{-1} are real numbers. In the case when they are nonpositive integers we need to take the analytic continuation of the hypergeometric series (30), (31) and (32). ■

It is straightforward to show that for $A = 0$ the hypergeometric functions (30), (31) and (32) yield to classical polynomials (10), (11) and (12).

§4 A second order difference equation.

In [2] we proved that the Meixner polynomials satisfy a second order difference equation. To prove this result we only used that in the difference equation of hypergeometric type for Meixner polynomials the function $\sigma(x)$ is equal to x. Taking into account that, for Charlier and Kravchuk polynomials, $\sigma(x) = x$, then the following Theorem holds:

Theorem 1 *The polynomials $M_n^{\gamma,\mu,A}(x)$, $K_n^{p,A}(x)$ and $C_n^{\mu,A}(x)$ satisfy a second order linear difference equation*

$$\begin{aligned}&[x + C(C\lambda_n + \lambda_n - \tau(x))](x-1)\triangle\nabla P_n^A(x) + (x-1)\tau(x)\triangle P_n^A(x) +\\ &+C[(\tau(x) - C\lambda_n)(\lambda_n - 1 - \tau(x-1)) + \lambda_n(\lambda_n + C) + (x + C\lambda_n)\triangle\tau(x)]\triangle P_n^A(x)\\ &+(x-1)\lambda_n P_n^A(x) + C\lambda_n[\lambda_n - 1 - \tau(x-1) + C(\triangle\tau(x) + \lambda_n)]P_n^A(x) = 0\quad ,\end{aligned} \tag{33}$$

where

$$x = 0, 1, 2, ..., \qquad \nabla f(x) = f(x) - f(x-1), \qquad \triangle f(x) = f(x+1) - f(x),$$

and by P_n^A we denote the generalized Meixner, Kravchuk or Charlier polynomials and C is the constant B_n, A_n or D_n, respectively, which is a function of n (see Section 3).

The proof of this Theorem for the Meixner case was given in [2]. Here we provide the proof for both three cases.

Proof: We will start from the representations (27), (28) and (29) for the generalized polynomials $P_n^A(x) = P_n(x) + C \bigtriangledown P_n(x)$. Multiplying this expression by x, using the second order difference equation that these classical polynomials satisfy

$$x \bigtriangleup \bigtriangledown P_n(x) + \tau(x) \bigtriangleup P_n(x) + \lambda_n P_n(x) = 0\,, \tag{34}$$

and using the identity $\bigtriangledown P_n(x) = \bigtriangleup P_n(x) - \bigtriangleup \bigtriangledown P_n(x)$ we obtain

$$xP_n^A(x) = (x + C\lambda_n)P_n(x) + C(x + \tau(x)) \bigtriangleup P_n(x). \tag{35}$$

Now if we apply the operator $\bigtriangleup$ to (35), from (34) the equation

$$x \bigtriangleup P_n^A(x) = [x - C\tau(x)] \bigtriangleup P_n(x) - C\lambda_n P_n(x) \tag{36}$$

follows. In the same way if we apply in (36) the operator $\bigtriangledown$ and using (34) we find

$$\begin{aligned} &x(x-1) \bigtriangleup \bigtriangledown P_n^A(x) = \\ &= -[(x-1)\tau(x) + C\tau(x)(\lambda_n - \tau(x-1) - 1) + Cx(\lambda_n + \bigtriangleup\tau(x))] \bigtriangleup P_n(x) - \\ &-[x - 1 + C(\lambda_n - \tau(x-1) - 1)]\lambda_n P_n(x). \end{aligned} \tag{37}$$

Now from (35),(36) and (37) the following determinant vanishes

$$\begin{vmatrix} xP_n^A(x) & a(x) & b(x) \\ x \bigtriangleup P_n^A(x) & c(x) & d(x) \\ x(x-1) \bigtriangleup \bigtriangledown P_n^A(x) & e(x) & f(x) \end{vmatrix} = 0\,, \tag{38}$$

where

$$\begin{aligned} &a(x) = (x + C\lambda_n), \qquad b(x) = C(x + \tau(x)), \qquad c(x) = -C\lambda_n, \\ &d(x) = x - C\tau(x), \qquad e(x) = -[x - 1 + C(\lambda_n - 1 - \tau(x-1))]\lambda_n, \\ &f(x) = -[(x-1)\tau(x) + C[\tau(x)(\lambda_n - 1 - \tau(x-1)) + x(\lambda_n + \bigtriangleup\tau(x))]. \end{aligned}$$

Expanding the determinant in (38) by the first column and dividing by x^2, the Theorem follows.■

The difference equation of the previous theorem (33) takes the form:

Meixner case $M_n^{\gamma,\mu,A}(x)$

$$\begin{aligned} &\{x + B_n[(1-\mu)(x + n + nB_n) - \gamma\mu]\}(x-1) \bigtriangleup \bigtriangledown M_n^{\gamma,\mu,A}(x) + \\ &(x-1)[\gamma\mu - x(1-\mu)] \bigtriangleup M_n^{\gamma,\mu,A}(x) + \\ &+B_n\{(1-\mu)[\gamma\mu(n + nB_n + 2x - 1) + (1-\mu)(x + n^2 - (x + nB_n)(x + n)) + \\ &+2nB_n] - \gamma\mu(1 + \gamma\mu)\} \bigtriangleup M_n^{\gamma,\mu,A}(x) + (x-1)n(1-\mu)M_n^{\gamma,\mu,A}(x) + \\ &nB_n(1-\mu)[(1-\mu)(x + n + nB_n - B_n - 1) - 1 - \gamma\mu]M_n^{\gamma,\mu,A}(x) = 0 \quad . \end{aligned} \tag{39}$$

Kravchuk case $K_n^{p,A}(x)$

$$\{x + A_n[\frac{x+n+nA_n}{1-p} - \frac{Np}{1-p}]\}(x-1) \triangle \nabla K_n^{p,A}(x) +$$

$$(x-1)[\frac{Np-x}{1-p}] \triangle K_n^{p,A}(x) +$$

$$+A_n\{\frac{1}{1-p}[\frac{Np}{1-p}(n+nA_n+2x-1) + \frac{x+n^2-(x+nA_n)(x+n)}{1-p} + \tag{40}$$

$$+2nA_n] - \frac{Np}{1-p}(1+\frac{Np}{1-p})\} \triangle K_n^{p,A}(x) + \frac{n}{1-p}(x-1)K_n^{p,A}(x) +$$

$$\frac{nA_n}{1-p}[\frac{x+n+nA_n-A_n-1-Np}{1-p} - 1]K_n^{p,A}(x) = 0 \quad .$$

Charlier case $C_n^{\mu,A}(x)$

$$[x + D_n(x+n+nD_n-\mu)](x-1) \triangle \nabla C_n^{\mu,A}(x) +$$

$$(x-1)(\mu-x) \triangle C_n^{\mu,A}(x) +$$

$$+D_n[n^2+x-2\mu+2nD_n-(n-x-\mu)(x-\mu+nD_n)] \triangle C_n^{\mu,A}(x) + \tag{41}$$

$$+(x-1)nC_n^{\mu,A}(x) + nD_n(x+n-\mu-2+nD_n-D_n)C_n^{\mu,A}(x) = 0 \quad .$$

§5 Three Term Recurrence Relations.

The generalized polynomials satisfy a three term recurrence relation (TTRR) of the form

$$xP_n^A(x) = P_{n+1}^A(x) + \beta_n^A P_n^A(x) + \gamma_n^A P_{n-1}^A(x) \quad , n \geq 0$$
$$P_{-1}^A(x) = 0 \qquad \text{and} \qquad P_0^A(x) = 1. \tag{42}$$

This is a simple consequence of their orthogonality with respect to a positive functional (see [6] or [11]). To obtain the explicit formula for the recurrence coefficients we can compare the coefficients of x^n in the two sides of (42). Let b_n^A be the coefficient of x^{n-1} in the expansion $P_n^A(x) = x^n + b_n^A\, x^{n-1} + ...$, then : $\beta_n^A = b_n^A - b_{n+1}^A$. To calculate γ_n^A is sufficient to evaluate (42) in $x = 0$ and remark that $P_n^A(0) \neq 0$.

In order to obtain a general expression for the coefficient β_n^A we can use the formulas (27), (28) and (29) for the generalized polynomials $P_n^A(x) = P_n(x) + C \nabla P_n(x)$, where $C = B_n$, A_n or D_n respectively. Doing some algebraic calculations we find that $b_n^A = b_n + nC$, where b_n denotes the coefficient of the $n-1$ power in the classical polynomials $P_n(x) = x^n + b_n x^{n-1} + \ldots$.

Using these formulas and the main data [11] for classical polynomials we obtain for generalized Meixner, Kravchuk and Charlier polynomials the following TTRR coefficients :

I Meixner polynomials:

$$b_n = \frac{n\mu}{\mu-1}(\gamma + \frac{n-1}{2}\frac{\mu+1}{\mu}) \,, \qquad \beta_n = \frac{n+\mu(n+\gamma)}{1-\gamma} \,,$$

$$b_n^A = \frac{n\mu}{\mu-1}(\gamma + \frac{n-1}{2}\frac{\mu+1}{\mu}) + A\frac{\mu^n(1-\mu)^{\gamma-1}(\gamma)_n}{(n-1)!(1+AKer_{n-1}^M(0,0))} \,,$$

and then

$$\beta_n^A = \frac{n+\mu(n+\gamma)}{1-\gamma} - A\frac{\mu^n(1-\mu)^{\gamma-1}(\gamma)_n}{n!}\left[\frac{\mu(\gamma+n)}{(1+AKer_n^M(0,0)} - \frac{n}{1+AKer_{n-1}^M(0,0)}\right],$$

$$\gamma_n^A = -\frac{M_{n+1}^{\gamma,\mu,A}(0)}{M_{n-1}^{\gamma,\mu,A}(0)} - \beta_n^A\frac{M_n^{\gamma,\mu,A}(0)}{M_{n-1}^{\gamma,\mu,A}(0)}.$$

II Kravchuk polynomials

$$b_n = -n[Np + (n-1)(\frac{1}{2}-p)], \qquad \beta_n = n + p(N-2n),$$

$$b_n^A = -n[Np + (n-1)(\frac{1}{2}-p)] + A\frac{N!}{(n-1)!(N-n)!}\frac{p^n(1-p)^{1-n}}{(1+AKer_{n-1}^K(0,0))},$$

and then

$$\beta_n^A = n + p(N-2n) + A\frac{N!p^n(1-p)^{-n}}{n!(N-n)!}\left[\frac{n(1-p)}{1+AKer_{n-1}^K(0,0)} - \frac{p(N-n)}{1+AKer_n^K(0,0)}\right],$$

$$\gamma_n^A = -\frac{K_{n+1}^{p,A}(0)}{K_{n-1}^{p,A}(0)} - \beta_n^A\frac{K_n^{p,A}(0)}{K_{n-1}^{p,A}(0)}.$$

III Charlier polynomials

$$b_n = -n(\mu + \frac{n-1}{2}), \qquad \beta_n = n + \mu,$$

$$b_n^A = -n(\mu + \frac{n-1}{2}) + A\frac{\mu^n}{(n-1)!(1+AKer_{n-1}^C(0,0))},$$

and then

$$\beta_n^A = n + \mu + A\frac{\mu^n}{n!}\left[\frac{n}{1+AKer_{n-1}^C(0,0)} - \frac{\mu}{1+AKer_n^K(0,0)}\right],$$

$$\gamma_n^A = -\frac{C_{n+1}^{\mu,A}(0)}{C_{n-1}^{\mu,A}(0)} - \beta_n^A\frac{C_n^{\mu,A}(0)}{C_{n-1}^{\mu,A}(0)}.$$

§6 Relation between tridiagonal matrices J_{n+1} and J_{n+1}^A.

In this section we are interested to find the relation between the tridiagonal matrices corresponding to the CMOP $P_n(x)$ and generalized polynomials $P_n^A(x)$, which we denote by J_{n+1} and J_{n+1}^A, respectively. It is well-known the crucial role that the tridiagonal matrices play in the numerical study of zeros of orthogonal polynomials defined on the real line, because they are its eigenvalues.

If we rewrite the TTRR (42) for PMOP in the matrix form we obtain

$$x\mathbf{P_n^A} = J_{n+1}^A\mathbf{P_n^A} + P_{n+1}^A(x)\mathbf{e}_{\mathbf{n+1}}^{\mathbf{(n+1)}}, \tag{43}$$

where the corresponding tridiagonal matrix and the *perturbed polynomial vector* are denoted by J_{n+1}^A and $\mathbf{P_n^A}$, respectively $(n \geq 0)$:

$$J_{n+1}^A = \begin{pmatrix} \beta_0^A & 1 & 0 & 0 & \dots & 0 & 0 \\ \gamma_1^A & \beta_1^A & 1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_2^A & \beta_2^A & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \gamma_n^A & \beta_n^A \end{pmatrix} \quad \text{and} \quad \mathbf{P_n^A} = \begin{pmatrix} P_0^A(x) \\ P_1^A(x) \\ P_2^A(x) \\ \vdots \\ P_n^A(x) \end{pmatrix},$$

and for a given integer $n \geq 1$ by $\mathbf{e}_{\mathbf{j}}^{(\mathbf{n+1})}$ $(0 \leq j \leq n+1)$, we denote :

$$\mathbf{e}_{\mathbf{j}}^{(\mathbf{n+1})} := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^T \quad \in \mathbb{R}^{n+1}.$$

A similar notation will be used for the tridiagonal matrix J_{n+1}, with the β_i's and γ_i's replaced by the corresponding coefficients of the three term recurrence relation for $P_n(x)$ and for the *polynomial vector* $\mathbf{P_n}$:

$$J_{n+1} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \gamma_n & \beta_n \end{pmatrix} \quad \text{and} \quad \mathbf{P_n} = \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_n(x) \end{pmatrix}$$

From the relation (21) and using (23), we deduce

$$P_{n+1}^A(x) = P_{n+1}(x) + \sum_{j=0}^{n} a_{n+1,j} P_j(x)\,, \tag{44}$$

or in the matrix form :

$$\mathbf{P_n^A} = R_{n+1}\mathbf{P_n}\,, \tag{45}$$

where R_{n+1} denotes the lower triangular matrix with 1 entries in the main diagonal :

$$R_{n+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{1,0} & 1 & 0 & 0 & \dots & 0 & 0 \\ a_{2,0} & a_{2,1} & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n-1} & 1 \end{pmatrix} \quad ,$$

and $a_{m,j}$ are:

$$a_{m,j} = -A \frac{P_m(0)}{1 + A \sum_{k=0}^{m-1} \frac{(P_k(0))^2}{d_k^2}} \frac{P_j(0)}{d_j^2}, \qquad 0 \leq j \leq m-1.$$

Now putting (45) in (43) and using (44) we find

$$xR_{n+1}\mathbf{P_n} = J_{n+1}^A R_{n+1}\mathbf{P_n} + (P_{n+1}(x) + \sum_{j=0}^{n} a_{n+1,j} P_j(x))\mathbf{e}_{\mathbf{n+1}}^{(\mathbf{n+1})}$$

from where, using the TTRR in the matrix form for the classical polynomials $P_n(x)$, we find

$$J_{n+1}\mathbf{P_n} = R_{n+1}^{-1} J_{n+1}^A R_{n+1}\mathbf{P_n} + \sum_{j=0}^{n} a_{n+1,j} P_j(x)\mathbf{e}_{\mathbf{n+1}}^{(\mathbf{n+1})} \quad .$$

Finally from this equation we obtain the following relation between tridiagonal matrices J_{n+1} and J_{n+1}^A

$$J_{n+1}^A = R_{n+1}\left(J_{n+1} + \Lambda_{n+1}\right) R_{n+1}^{-1}\,, \tag{46}$$

where Λ_{n+1} is a rank-one matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{n+1,0} & a_{n+1,1} & a_{n+1,2} & a_{n+1,3} & \dots & a_{n+1,n-1} & a_{n+1,n} \end{pmatrix}.$$

We conclude that J^A_{n+1} is a rank-one perturbation of J_{n+1}. Finally, we obtain the corresponding elements of the matrices R_{n+1} and Λ_{n+1} in the three studied cases, i.e., the Meixner, Kravchuk and Charlier perturbed orthogonal polynomials.

Meixner case :

$$a^M_{m,k} = A\frac{(-1)^{m+k+1}\mu^m(\gamma)_m}{k!(1-\mu)^{m-k-\gamma}(1+Ker^M_{m-1}(0,0))}. \tag{47}$$

Kravchuk case :

$$a^K_{m,k} = A\frac{(-1)^{m+k+1}p^m N!}{k!(1-p)^k(N-m)!(1+Ker^K_{m-1}(0,0))}. \tag{48}$$

Charlier case :

$$a^C_{m,k} = A\frac{(-1)^{m+k+1}\mu^m}{k!(1+Ker^C_{m-1}(0,0))}. \tag{49}$$

§7 Associated Polynomials.

In this section we study the monic associated polynomials corresponding to PMOPS $\{P^A_n(x)\}_{n=0}^{\infty}$. They are defined as follows [6]

$$P^{(1),A}_n(x) = \frac{1}{c^A_0}\mathcal{U}\left\{\frac{P^A_{n+1}(x)-P^A_{n+1}(y)}{x-y}\right\}, \tag{50}$$

where it is understood $\mathcal{U}$ operates on y and c^A_0 is the first moment of the functional, i.e., $c^A_0 = \mathcal{U}(1)$. Taking into account that $\mathcal{U}\{P^A_n(x)\} = \mathcal{L}\{P^A_n(x)\} + AP^A_n(0)$ and using the relation (44) for the generalized polynomials, as well as the linearity of the classical functional $\mathcal{L}$, we obtain the following relation:

$$P^{(1),A}_n(x) = \frac{c_0}{c^A_0}\left[P^{(1)}_n(x) - AP^A_{n+1}(0)\sum_{k=0}^{n}\frac{P_k(0)P^{(1)}_{k-1}(x)}{d^2_k}\right] + \frac{A}{c^A_0}\frac{P^A_{n+1}(x)-P^A_{n+1}(0)}{x}. \tag{51}$$

Lemma 1 *For all integer* $n \geq 1$

$$(x-y)\sum_{k=0}^{n}\frac{P_k(x)P^{(1)}_{k-1}(y)}{d^2_k} = \frac{P_{n+1}(x)P^{(1)}_{n-1}(y) - P_n(x)P^{(1)}_n(y)}{d^2_n} + \frac{1}{\gamma_1}.$$

Proof: We will use the recurrence relation (5) for the classical polynomials $P_n(x)$

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),\ n \geq 1$$

$$P_{-1}(x) = 0 \qquad \text{and} \qquad P_0(x) = 1$$

and the corresponding for the associated (see [6] page 85, Eq.(4.3))

$$yP_{n-1}^{(1)}(y) = P_n^{(1)}(y) + \beta_n P_{n-1}^{(1)}(y) + \gamma_n P_{n-2}^{(1)}(y),\ n \geq 1$$

$$P_{-1}^{(1)}(x) = 0 \qquad \text{and} \qquad P_0^{(1)}(x) = 1.$$

If we multiply the first equation by $P_{n-1}^{(1)}(y)$ and the second one by $P_n(x)$, subtract both and divide the resulting expression by d_n^2, then

$$(x-y)\frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = \frac{P_{k+1}(x)P_{k-1}^{(1)}(y) - P_k(x)P_k^{(1)}(y)}{d_k^2} - \frac{P_k(x)P_{k-2}^{(1)}(y) - P_{k-1}(x)P_{k-1}^{(1)}(y)}{d_{k-1}^2}.$$

Summing from $k=2$ to n we find:

$$(x-y)\sum_{k=2}^{n}\frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = \frac{P_{n+1}(x)P_{n-1}^{(1)}(y) - P_n(x)P_n^{(1)}(y)}{d_n^2} - \frac{P_2(x) - P_1(x)P_1^{(1)}(y)}{d_1^2}.$$

But

$$(x-y)\sum_{k=0}^{n}\frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = (x-y)\frac{P_1(x)P_0^{(1)}(y)}{d_1^2} + (x-y)\sum_{k=2}^{n}\frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2}.$$

Using the previous expression for the last sum, the identities $xP_1(x) = P_2(x) + \beta_1 P_1(x) + \gamma_1$ and $P_1^{(1)}(y) = y - \beta_1$ and doing some straightforward computations, the lemma follows.■

Lemma 2 *If for the generalized polynomials the following relation holds*

$$P_n(x)P_{n+1}(0) - P_{n+1}(x)P_n(0) = xD \bigtriangledown P_n(x),\ n \geq 1$$

where D is some constant independent of x (but in general it could be a function of n), then

$$P_{n-1}^{(1)}(x)P_{n+1}(0) - P_n^{(1)}(x)P_n(0) = xD \bigtriangledown P_{n-1}^{(1)}(x).$$

Proof: From the hypothesis of the lemma, we have for $\forall n \geq 1$

$$P_n(x)P_{n+1}(0) - P_{n+1}(x)P_n(0) = xD(P_n(x) - P_n(x-1)),$$

$$P_n(y)P_{n+1}(0) - P_{n+1}(y)P_n(0) = yD(P_n(y) - P_n(y-1)).$$

Substracting them, dividing by $x-y$, applying the functional $\mathcal{L}$ and taking into account the identities ($n \geq 1$)

$$\mathcal{L}\left\{\frac{xP_n(x) - yP_n(y)}{x-y}\right\} = xP_{n-1}^{(1)}(x) \quad \text{and} \quad \mathcal{L}\left\{\frac{xP_n(x-1) - yP_n(y-1)}{x-y}\right\} = xP_{n-1}^{(1)}(x-1)$$

the lemma follows. ■

Now we can consider the former expression for the monic associated polynomials corresponding to PMOPS $\{P_n^A(x)\}_{n=0}^{\infty}$. From (51) and using the previous lemmas we find

$$P_n^{(1),A}(x) = \frac{c_0}{c_0^A}\left[P_n^{(1)}(x) - AP_{n+1}^A(0)\left(\frac{P_{n+1}(0)P_{n-1}^{(1)}(x) - P_n(0)P_n^{(1)}(x)}{d_n^2} + \frac{1}{\gamma_1}\right)\right] + \\ + \frac{A}{c_0^A}\frac{P_{n+1}^A(x) - P_{n+1}^A(0)}{x}, \tag{52}$$

or

$$P_n^{(1),A}(x) = \frac{c_0}{c_0^A}\left[P_n^{(1)}(x) - AP_{n+1}^A(0)xCd_n^2 \bigtriangledown P_{n-1}^{(1)}(x)\right] + \frac{A}{c_0^A}\frac{P_{n+1}^A(x) - P_{n+1}^A(0)}{x}, \qquad (53)$$

where $C = B_n$, A_n and D_n (27)-(29) for the Meixner, Kravchuk and Charlier orthogonal polynomials respectively. These two equations are valid when $x \neq 0$ and $n \geq 1$. In the case $x = 0$ we must make use of the definitions :

$$P_n^{(1),A}(0) = \frac{1}{c_0^A}\mathcal{U}\left\{\frac{P_n^A(y) - P_{n+1}^A(0)}{y}\right\},$$

or, alternatively, the formula (52) and the limit when $x \to 0$.

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