ON THE GENERALIZED ASKEY-WILSON POLYNOMIALS

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Abstract. In this paper a generalization of Askey-Wilson polynomials is introduced. These polynomials are obtained from the Askey-Wilson polynomials via the addition of two mass points to the weight function of them at the points ± 1. Several properties of such new family are considered, in particular the three-term recurrence relation and the representation as basic hypergeometric series.

Dedicated to Paco Marcellán on the occasion of his 60th birthday

1. Introduction

The Krall-type polynomials are orthogonal with respect to a linear functional $\widetilde{\mathfrak{u}}$ obtained from a quasi-definite functional $\mathfrak{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures. These polynomials appear as eigenfunctions of a fourth order linear differential operator with polynomial coefficients that do not depend on the degree of the polynomials. They were firstly considered by Krall in [23] (for a more recent reviews see [4] and [22, chapter XV]). In fact, H. L. Krall discovered that there are only three extra families of orthogonal polynomials apart from the classical polynomials of Hermite, Laguerre and Jacobi that satisfy such a fourth order differential equation which are orthogonal with respect to measures that are not absolutely continuous with respect to the Lebesgue measure. This result motivated the study of the polynomials orthogonal with respect to the more general weight functions [18, 20] that could contain more instances of orthogonal polynomials being eigenfunctions of higher-order differential equations [22, chapters XVI, XVII].

In the last years the study of such polynomials have been considered by many authors (see e.g. [2, 5, 15, 16, 24, 26] and the references therein) with a special emphasis on the case when the starting functional $\mathfrak{u}$ is a classical continuous, discrete or q-linear functional with the linear type lattices (for more details see [3, 5] and references therein). In fact, for the q-case some examples related with the q-Laguerre and the little q-Jacobi polynomials were constructed by Haine and Grünbaum in [15] using the Darboux transformation. Later on, in [26], Vinet and Zhedanov presented a more complete study for the little q-Jacobi polynomials. In these both cases, the

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q-Krall polynomials satisfy a higher order q-difference equations with polynomial coefficients independent of n. For the discrete case the problem was solved very recently by A. Durán using a new method (see [12, 13], for details).

For the general q-quadratic lattice only few results were known. An important contribution to this case was done in [17] where the authors considered a generalized Askey-Wilson polynomials by adding mass points. They showed that the resulting orthogonal polynomials satisfy a higher order q-difference equation with polynomial coefficients independent of n, only if the masses are added at very specific points out of the interval of orthogonality $[-1, 1]$. Another contribution to this problem was done in [6], where a general theory of the the Krall-type polynomials on non-uniform lattices was developed. In fact, in [6] the authors studied the polynomials $\widetilde{P}_n(s)_q$ which are orthogonal with respect to the linear functionals $\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{k=1}^{N} A_k \delta_{x_k}$ defined on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ and considered, as a representative example, the Krall-type Racah polynomials (see also [7]). In fact, in [6, §5], we posed the problem of obtaining a generalization of the Askey-Wilson polynomials by adding two mass points at the end of the interval of orthogonality, motivated by the results in [17].

Thus our main aim here is to study the orthogonal polynomials obtained via the addition of two mass points at the end of the interval of orthogonality of the Askey-Wilson polynomials. The structure of the paper is as follows. In Section 2, some preliminary results on the Askey-Wilson polynomials are presented as well as the most general expression for the kernels on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. Our main results are in section 3, where we introduce a detailed study of the generalized Askey-Wilson polynomials obtained from the classical Askey-Wilson polynomials by adding two mass points at ∓ 1.

2. Preliminary results

Here we include some results of the theory of orthogonal polynomials on the non-uniform lattice (for more details see e.g., [1, 25])

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3. \tag{1}$$

The polynomials on non-uniform lattices $P_n(s)_q := P_n(x(s))$ are the polynomial solutions of the second order linear difference equation (SODE) of hypergeometric type

$$\begin{gathered} A_s y(s+1) + B_s y(s) + C_s y(s-1) + \lambda_n y(s) = 0, \\ A_s = \frac{\sigma(s) + \tau(s)\Delta x(s-\frac{1}{2})}{\Delta x(s)\Delta x(s-\frac{1}{2})}, \;\; C_s = \frac{\sigma(s)}{\nabla x(s)\Delta x(s-\frac{1}{2})}, \;\; B_s = -A_s - C_s, \end{gathered} \tag{2}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of degree at most 2 and exactly 1, respectively, and λ_n is a constant. They are orthogonal with respect to the linear functional $\mathfrak{u} : \mathbb{P}_q \mapsto \mathbb{C}$, where $\mathbb{P}_q$ denotes the space of polynomials on the lattice (1),

$$\langle \mathfrak{u}, P_n P_m \rangle = \delta_{m\,n} d_n^2, \quad \langle \mathfrak{u}, P \rangle = \int_{x_0}^{x_1} P(x)_q \rho(x) dx \tag{3}$$

where ρ is the *weight* function and $d_n^2 := \langle \mathfrak{u}, P_n^2 \rangle$.

Since the polynomials $P_n(s)_q$ are orthogonal with respect to a linear functional, they satisfy a three-term recurrence relation (TTRR) [1, 11]

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad n = 0, 1, 2, ..., \tag{4}$$

with the initial conditions $P_0(s)_q = 1, P_{-1}(s)_q = 0$, and also the differentiation formulas [1, Eqs. (5.65) and (5.67)] (or [8, Eqs. (24) and (25)]

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \overline{\alpha}_n P_{n+1}(s)_q + \overline{\beta}_n(s) P_n(s)_q, \tag{5}$$

$$\Phi(s)\frac{\Delta P_n(s)_q}{\Delta x(s)} = \widehat{\alpha}_n P_{n+1}(s)_q + \widehat{\beta}_n(s) P_n(s)_q, \tag{6}$$

where $\Phi(s) = \sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$, and

$$\overline{\alpha}_n = \widehat{\alpha}_n = -\frac{\alpha_n \lambda_{2n}}{[2n]_q}, \quad \overline{\beta}_n(s) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'}, \quad \widehat{\beta}_n(s) = \overline{\beta}_n(s) - \lambda_n \Delta x(s - \tfrac{1}{2}).$$

Notice that from (6) and the TTRR (4) the following useful relation follows

$$P_{n-1}(s)_q = \Theta(s, n) P_n(s)_q + \Xi(s, n) P_n(s + 1)_q, \tag{7}$$

where

$$\Theta(s, n) = \frac{\alpha_n}{\widehat{\alpha}_n \gamma_n}\left[\frac{\Phi(s)}{\Delta x(s)} - \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n) + \widehat{\beta}_n(s)\right], \ \Xi(s, n) = -\frac{\alpha_n}{\widehat{\alpha}_n \gamma_n}\frac{\Phi(s)}{\Delta x(s)}.$$

Using the Christoffel-Darboux formula for the n-th reproducing kernels

$$\mathrm{K}_n(x(s_1), x(s_2)) := \sum_{k=0}^{n} \frac{P_k(s_1)_q P_k(s_2)_q}{d_k^2} = \frac{\alpha_n}{d_n^2}\frac{P_{n+1}(s_1)_q P_n(s_2)_q - P_{n+1}(s_2)_q P_n(s_1)_q}{x(s_1) - x(s_2)},$$

and the relations (5) and (6), respectively, to eliminate P_{n+1}, we obtain the following two expressions

$$\begin{aligned} \mathrm{K}_n(x(s), x(s_0)) = \frac{\alpha_n P_n(s_0)_q}{\overline{\alpha}_n d_n^2} & \left\{ \frac{\overline{\beta}_n(s_0) - \overline{\beta}_n(s)}{x(s) - x(s_0)} P_n(s)_q + \frac{\sigma(s)}{x(s) - x(s_0)}\frac{\nabla P_n(s)_q}{\nabla x(s)} \right\} \\ & - \frac{\alpha_n}{\overline{\alpha}_n d_n^2}\frac{\sigma(s_0)}{x(s) - x(s_0)}\frac{\nabla P_n(s_0)_q}{\nabla x(s_0)} P_n(s)_q, \end{aligned} \tag{8}$$

$$
\begin{aligned}
\mathrm{K}_n(x(s),x(s_0)) = \frac{\alpha_n P_n(s_0)_q}{\widehat{\alpha}_n d_n^2}&\left\{\frac{\widehat{\beta}_n(s_0)-\widehat{\beta}_n(s)}{x(s)-x(s_0)}P_n(s)_q+\frac{\Phi(s)}{x(s)-x(s_0)}\frac{\Delta P_n(s)_q}{\Delta x(s)}\right\}\\
&-\frac{\alpha_n}{\widehat{\alpha}_n d_n^2}\frac{\Phi(s_0)}{x(s)-x(s_0)}\frac{\Delta P_n(s_0)_q}{\Delta x(s_0)}P_n(s)_q.
\end{aligned}
\tag{9}
$$

Let us mention here that the above two formulas generalize to an arbitrary value s_0 the Eqs. (9) and (10) obtained in [6, page 184].

Next, we introduce the Askey-Wilson polynomials defined by the following basic series [9] (for the definition and properties of basic series see e.g. [14])

$$
\begin{aligned}
P_n(x(s))_q := P_n(x(s),a,b,c,d|q) = &\frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}\\
&\times {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, aq^s, aq^{-s}\\ ab, ac, ad\end{matrix}\,\middle|\, q\,,\,q\right).
\end{aligned}
\tag{10}
$$

Notice that the Askey-Wilson polynomials are defined on the lattice $x(s)=\frac{q^s+q^{-s}}{2}$, $q^s=e^{i\theta}$ [10], which is a particular case of (1) when $c_1=c_2=1/2$ and $c_3=0$. Their main characteristics (see Eqs. (3), (4), (5), (6)) are given in Table 1.

Using the identity [1, page 156] (see also [6, page 201])

$$
(aq^s;q)_k(aq^{-s};q)_k=(-1)^k a^k q^{k\binom{k-1}{2}}\prod_{i=0}^{k-1}\Big[2x(s)-(aq^i+a^{-1}q^{-i})\Big],
$$

we can rewrite (10) as

$$
\begin{aligned}
P_n(x(s))_q = &\frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{n-1};q)_k}{(ab,ac,ad,q;q)_k}q^k\\
&\times(-1)^k a^k q^{k\binom{k-1}{2}}\prod_{i=0}^{k-1}\Big[2x(s)-(aq^i+a^{-1}q^{-i})\Big].
\end{aligned}
$$

Notice that for $x(s_0)=-1$ and $x(s_1)=1$, we obtain, respectively,

$$
\begin{aligned}
P_n(-1)_q &= \frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}{}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, -a, -a\\ ab, ac, ad\end{matrix}\,\middle|\, q\,,\,q\right),\\
P_n(1)_q &= \frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}{}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, a, a\\ ab, ac, ad\end{matrix}\,\middle|\, q\,,\,q\right).
\end{aligned}
$$

Table 1. Main data of the monic Askey-Wilson polynomials [19]

$P_n(s)$	$P_n(x(s),a,b,c,d\vert q)\,,\quad x(s)=\frac{q^s+q^{-s}}{2},\ \ q^s=e^{i\theta}\ ,\quad \Delta x(s)=\frac{q-1}{2}[q^s-q^{-s-1}]$
$\rho(s)$	$\dfrac{(q,ab,ac,ad,bc,bd,cd;q)_\infty h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{2\pi\sqrt{1-x^2}(abcd;q)_\infty h(x,a)h(x,b)h(x,c)h(x,d)},\ h(x,\alpha)=\prod_{k=0}^{\infty}\left(1-2\alpha x q^k+\alpha^2q^{2k}\right)$ $x_0=-1,\ x_1=1,\ a,b,c,d\in\mathbb{R}$ or complex conjugate pairs if $a,b,c,d\in\mathbb{C}$ and $\max(\vert a\vert,\vert b\vert,\vert c\vert,\vert d\vert)<1$
$\sigma(s)$	$q^{-4s}(q^s-a)(q^s-b)(q^s-c)(q^s-d)$
$\Phi(s)$	$q^{4s}(q^{-s}-a)(q^{-s}-b)(q^{-s}-c)(q^{-s}-d)$
$\tau(s)$	$\frac{4}{q^{1/2}-q^{-1/2}}(ab+ac+ad+bc+bd+cd)x(s)-\frac{2}{(q^{1/2}-q^{-1/2})}(a+b+c+d)$
$\tau_n(s)$	$\frac{4q^n}{q^{1/2}-q^{-1/2}}(ab+ac+ad+bc+bd+cd)x(s+\frac{n}{2})-\frac{2q^{n/2}}{(q^{1/2}-q^{-1/2})}(a+b+c+d)$
λ_n	$4q^{-n+1}(1-q^n)(1-abcdq^{n-1})$
d_n^2	$\dfrac{2^{-2n}(q,ab,ac,ad,bc,bd,cd,abcdq^{2n};q)_\infty}{(abcdq^{n-1};q)_n(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n,abcd;q)_\infty}$
β_n	$\dfrac{1}{2}\Big[-\dfrac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}-\dfrac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}+a+a^{-1}\Big]$
γ_n	$\dfrac{1}{4}\dfrac{(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})(1-abcdq^{n-2})}{(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$
$\overline{\alpha}_n=\widehat{\alpha}_n$	$4q^{-n+1}(q^{1/2}-q^{-1/2})(1-abcdq^{2n-1})$
$\overline{\beta}_n(s)$	$-\frac{2q^{-\frac{3n}{2}+1}(q^{1/2}-q^{-1/2})(1-abcdq^{n-1})}{ab+ac+ad+bc+bd+cd}\Big[2q^n(ab+ac+ad+bc+bd+cd)x(s+\frac{n}{2})-q^{n/2}(a+b+c+d)\Big]$
$\widehat{\beta}_n(s)$	$\overline{\beta}_n(s)-4q^{-n+1}(1-q^n)(1-abcdq^{n-1})\Delta x(s-\frac{1}{2})$

In a similar fashion we get

$$\Delta P_n(-1)_q=P_n(x(s_0+1))-P_n(x(s_0))=a(1-q^{-1})(1-a)\frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}\times {}_4\varphi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},aq,aq\\ ab,ac,ad\end{matrix}\,\middle|\,q\,,\,q\right),$$

$$\Delta P_n(1)_q=P_n(x(s_1+1))-P_n(x(s_1))=-a(1-q^{-1})(1+a)\frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}\times {}_4\varphi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},-aq,-aq\\ ab,ac,ad\end{matrix}\,\middle|\,q\,,\,q\right).$$

By inserting the values of Askey-Wilson polynomials given in Table 1 into (7) we arrive at the following identity

$$P_{n-1}(x(s))_q=\Theta(s,n)P_n(x(s))_q+\Xi(s,n)P_n(x(s+1))_q, \tag{11}$$

where

$$\Xi(s,n) = -\frac{q^{n-1}(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2}{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})} \times \frac{\Phi(s)}{(1-abcdq^{n-2})(q^{1/2}-q^{-1/2})(1-q^n)\Delta x(s)},$$

$$\begin{aligned}\Theta(s,n) = {} & \frac{q^{n-1}(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2}{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})} \\ & \times \frac{1}{(1-abcdq^{n-2})(q^{1/2}-q^{-1/2})(1-q^n)}\Bigg\{\frac{\Phi(s)}{\Delta x(s)} + 2q^{-n+1}(q^{1/2}-q^{-1/2})(1-abcdq^{2n-1}) \\ & \times \Big[2x(s)-a-a^{-1}+(1-abcdq^{n-1})\frac{(1-abq^n)(1-acq^n)(1-adq^n)}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}+a(1-q^n) \\ & \times \frac{(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}\Big] - \frac{2q^{-\frac{3n}{2}+1}(q^{1/2}-q^{-1/2})(1-abcdq^{n-1})}{ab+ac+ad+bc+bd+cd} \\ & \times \Big\{2q^n(ab+ac+ad+bc+bd+cd)x(s+\tfrac{n}{2}) - q^{n/2}(a+b+c+d)\Big\} \\ & - 4q^{-n+1}(1-q^n)(1-abcdq^{n-1})\Delta x(s-\tfrac{1}{2})\Bigg\}.\end{aligned}$$

3. The generalized Askey-Wilson polynomials

In this section we consider the modification of the Askey-Wilson polynomials (10) by adding two mass points, i.e., the polynomials orthogonal with respect to the functional $\widetilde{\mathfrak{u}} = u + A\delta(x(s)-x(s_0)) + B\delta(x(s)-x(s_1))$, where $\mathfrak{u}$ is defined in (3), $x_0 := x(s_0) = -1$ and $x_1 := x(s_1) = 1$.

By using [6, §3] the representation of the modified Askey-Wilson polynomials can be constructed

(12)
$$\widetilde{P}_n^{A,B}(x(s))_q = P_n(x(s))_q - A\widetilde{P}_n^{A,B}(-1)_q \mathrm{K}_{n-1}(x(s),-1) - B\widetilde{P}_n^{A,B}(1)_q \mathrm{K}_{n-1}(x(s),1),$$

then the system of two equations in the two unknowns $\widetilde{P}_n^{A,B}(-1)_q$ and $\widetilde{P}_n^{A,B}(1)_q$ becomes

$$\begin{aligned}\widetilde{P}_n^{A,B}(-1)_q &= P_n(-1)_q - A\widetilde{P}_n^{A,B}(-1)_q\mathrm{K}_{n-1}(-1,-1)) - B\widetilde{P}_n^{A,B}(1)_q\mathrm{K}_{n-1}(-1,1), \\ \widetilde{P}_n^{A,B}(1)_q &= P_n(1)_q - A\widetilde{P}_n^{A,B}(-1)_q\mathrm{K}_{n-1}(1,-1) - B\widetilde{P}_n^{A,B}(1)_q\mathrm{K}_{n-1}(1,1),\end{aligned}$$

whose solution is

$$\begin{pmatrix}\widetilde{P}_n^{A,B}(-1)_q \\ \widetilde{P}_n^{A,B}(1)_q\end{pmatrix} = \begin{pmatrix}1+A\mathrm{K}_{n-1}(-1,-1) & B\mathrm{K}_{n-1}(-1,1) \\ A\mathrm{K}_{n-1}(1,-1) & 1+B\mathrm{K}_{n-1}(1,1)\end{pmatrix}^{-1}\begin{pmatrix}P_n(-1)_q \\ P_n(1)_q\end{pmatrix}.$$

Notice that $\forall A, B > 0$,

$$\kappa_{n-1}(-1,1) := \det \begin{vmatrix} 1 + A\mathrm{K}_{n-1}(-1,-1) & B\mathrm{K}_{n-1}(-1,1) \\ A\mathrm{K}_{n-1}(1,-1) & 1 + B\mathrm{K}_{n-1}(1,1) \end{vmatrix} > 0. \tag{13}$$

Thus, by [6, Proposition 1] the polynomials $\widetilde{P}_n^{A,B}(s)_q$ are well defined for all values $A, B > 0$. Furthermore,

$$\begin{aligned} \widetilde{P}_n^{A,B}(-1)_q &= \frac{(1 + B\mathrm{K}_{n-1}(1,1))P_n(-1)_q - B\mathrm{K}_{n-1}(-1,1)P_n(1)_q}{\kappa_{n-1}(-1,1)}, \\ \widetilde{P}_n^{A,B}(1)_q &= \frac{(1 + A\mathrm{K}_{n-1}(-1,-1))P_n(1)_q - A\mathrm{K}_{n-1}(1,-1)P_n(-1)_q}{\kappa_{n-1}(-1,1)}, \end{aligned} \tag{14}$$

where $\kappa_{n-1}(-1,1)$ is given in (13).

The modified Askey-Wilson polynomials satisfy the following orthogonality relation

$$\begin{aligned} \int_{-1}^{1} \widetilde{P}_n^{A,B}(x)_q \widetilde{P}_m^{A,B}(x)_q \rho(x)dx + A\widetilde{P}_n^{A,B}(-1)_q \widetilde{P}_m^{A,B}(-1)_q \\ + B\widetilde{P}_n^{A,B}(1)_q \widetilde{P}_m^{A,B}(1)_q = \delta_{n,m}\widetilde{d}_n^2, \end{aligned}$$

where ρ and d_n denote the weight function and the norm of the Askey-Wilson polynomials (see Table 1[1]) and

$$\widetilde{d}_n^2 = \langle \widetilde{\mathfrak{u}}, \widetilde{P}_n^2(x) \rangle = d_n^2 + A\widetilde{P}_n^{A,B}(-1)_q P_n(-1)_q + B\widetilde{P}_n^{A,B}(1)_q P_n(1)_q.$$

Representation formulas for the generalized Askey-Wilson polynomials. Consider the representation formula (12) where the n-th kernel can be computed by the formulas (8) and (9). In fact, by using the main datas of Askey-Wilson polynomials (see Table 1) in (9), we obtain

$$\mathrm{K}_{n-1}(x(s),-1) = \varkappa_{-1}(s,n)P_{n-1}(x(s))_q + \overline{\varkappa}_{-1}(s,n)\frac{\Delta P_{n-1}(x(s))_q}{\Delta x(s)}, \tag{15}$$

[1]We have chosen $\rho(s)$ in such a way that $\int_{x=-1}^{1} \rho(x)dx = 1$, i.e., to be a probability measure.

where

$$\varkappa_{-1}(s,n)=\frac{(abcdq^{n-2};q)_{n-1}(abcd,\ q^{n},abq^{n-1},acq^{n-1},adq^{n-1},bcq^{n-1},bdq^{n-1},cdq^{n-1};q)_\infty}{2^{-2n+2}q(q^{1/2}-q^{-1/2})(abcdq^{2n-3};q)_\infty(q,ab,ac,ad,bc,bd,cd;q)_\infty}$$
$$\times\Bigg\{\Big[\frac{q^{\frac{n+1}{2}}(q^{1/2}-q^{-1/2})(1-abcdq^{n-2})[\frac{q^{\frac{n-1}{2}}+q^{-\frac{n-1}{2}}}{2}+x(s+\frac{n-1}{2})]}{x(s)+1}$$
$$+\frac{(1-q^n)(1-abcdq^{n-1})\Delta x(s-\frac{1}{2})}{x(s)+1}\Big]P_{n-1}(-1)$$
$$+\frac{2(1+a)(1+b)(1+c)(1+d)}{(q+q^{-1}-2)[x(s)+1]}\Delta P_{n-1}(-1)\Bigg\},$$

$$\overline{\varkappa}_{-1}(s,n)=\frac{(abcdq^{n-2};q)_{n-1}(abcd,\ q^{n},abq^{n-1},acq^{n-1},adq^{n-1},bcq^{n-1},bdq^{n-1},cdq^{n-1};q)_\infty}{2^{-2n+4}q^{-n+2}(q^{1/2}-q^{-1/2})(abcdq^{2n-3};q)_\infty(q,ab,ac,ad,bc,bd,cd;q)_\infty}$$
$$\times\frac{\Phi(s)}{x(s)+1}P_{n-1}(-1),$$

$$\mathrm{K}_{n-1}(x(s),1)=\varkappa_1(s,n)P_{n-1}(x(s))_q+\overline{\varkappa}_1(s,n)\frac{\Delta P_{n-1}(x(s))_q}{\Delta x(s)},\tag{16}$$

where

$$\varkappa_{1}(s,n)=\frac{(abcdq^{n-2};q)_{n-1}(abcd,\ q^{n},abq^{n-1},acq^{n-1},adq^{n-1},bcq^{n-1},bdq^{n-1},cdq^{n-1};q)_\infty}{2^{-2n+2}q(q^{1/2}-q^{-1/2})(abcdq^{2n-3};q)_\infty(q,ab,ac,ad,bc,bd,cd;q)_\infty}$$
$$\times\Bigg\{\Big[-\frac{q^{\frac{n+1}{2}}(q^{1/2}-q^{-1/2})(1-abcdq^{n-2})[\frac{q^{\frac{n-1}{2}}+q^{-\frac{n-1}{2}}}{2}-x(s+\frac{n-1}{2})]}{x(s)-1}$$
$$+\frac{(1-q^n)(1-abcdq^{n-1})\Delta x(s-\frac{1}{2})}{x(s)-1}\Big]P_{n-1}(1)$$
$$-\frac{2(1-a)(1-b)(1-c)(1-d)}{(q+q^{-1}-2)[x(s)-1]}\Delta P_{n-1}(1)\Bigg\},$$

$$\overline{\varkappa}_{1}(s,n)=\frac{(abcdq^{n-2};q)_{n-1}(abcd,\ q^{n},abq^{n-1},acq^{n-1},adq^{n-1},bcq^{n-1},bdq^{n-1},cdq^{n-1};q)_\infty}{2^{-2n+4}q^{-n+2}(q^{1/2}-q^{-1/2})(abcdq^{2n-3};q)_\infty(q,ab,ac,ad,bc,bd,cd;q)_\infty}$$
$$\times\frac{\Phi(s)}{x(s)-1}P_{n-1}(1).$$

By substituting (15) and (16) into (12), one finds

$$\widetilde{P}_n^{A,B}(x(s))_q=P_n(x(s))_q+\overline{A}(s,n)P_{n-1}(x(s))_q+\overline{B}(s,n)\frac{\Delta P_{n-1}(x(s))_q}{\Delta x(s)},\tag{17}$$

$$\overline{A}(s,n) = -\,A\widetilde{P}_n^{A,B}(-1)_q \varkappa_{-1}(s,n) - B\widetilde{P}_n^{A,B}(1)_q \varkappa_1(s,n),$$

$$\overline{B}(s,n) = -\,A\widetilde{P}_n^{A,B}(-1)_q \overline{\varkappa}_{-1}(s,n) - B\widetilde{P}_n^{A,B}(1)_q \overline{\varkappa}_1(s,n),$$

where $\widetilde{P}_n^{A,B}(-1)_q$ and $\widetilde{P}_n^{A,B}(1)_q$ are given in (14). Notice that the involved functions $\overline{A}$ and $\overline{B}$ as well as $\Delta P_{n-1}(s)_q/\Delta x(s)$ in (17) are not, in general, polynomials in $x(s)$. Thus, it is not easy to see that $\widetilde{P}_n^{A,B}(s)_q$ in (17) is a polynomial of degree n in $x(s)$ which is even a simple consequence of (12). Notice that if we use (8) instead of (9) we can obtain a formula similar to (17) but in terms of the backward difference operator.

From the Christoffel Darboux formula and the TTRR for the Askey-Wilson polynomials another representation formula for the modified Askey-Wilson polynomials follows (see e.g. §3 in [6])

$$\phi(s)\widetilde{P}_n^{A,B}(x(s))_q = A(s;n)P_n(x(s))_q + B(s;n)P_{n-1}(x(s))_q, \tag{18}$$

with the coefficients

$$\phi(s) = [x(s)^2 - 1],$$

$$\begin{aligned} A(s,n) = \phi(s) - \frac{1}{d_{n-1}^2}\Big\{ & A\widetilde{P}_n^{A,B}(-1)_q P_{n-1}(-1)_q [x(s)-1] \\ & + B\widetilde{P}_n^{A,B}(1)_q P_{n-1}(1)_q [x(s)+1]\Big\}, \end{aligned} \tag{19}$$

$$B(s,n) = \frac{1}{d_{n-1}^2}\Big\{A\widetilde{P}_n^{A,B}(-1)_q P_n(-1)_q[x(s)-1] + B\widetilde{P}_n^{A,B}(1)_q P_n(1)_q[x(s)+1]\Big\},$$

where $\widetilde{P}_n^{A,B}(-1)_q$ and $\widetilde{P}_n^{A,B}(1)_q$ are defined in (14).

Furthermore, there is one more representation formula for the modified Askey-Wilson families which can be obtained by substituting the relation (11) in (18)

$$\phi(s)\widetilde{P}_n^{A,B}(x(s))_q = a(s;n)P_n(x(s))_q + b(s;n)P_n(x(s+1))_q, \tag{20}$$

where $a(s;n) = A(s;n) + B(s;n)\Theta(s;n)$, $b(s;n) = B(s;n)\Xi(s;n)$, and A, B and Θ, Ξ are given by (19) and (11), respectively.

If we change in (20) s by $s+1$ and s by $s-1$ and then use (2) to eliminate $P_n(x(s+2))_q$ and $P_n(x(s-2))_q$, respectively, we obtain

$$u(s)\widetilde{P}_n(x(s+1))_q = c(s,n)P_n(x(s))_q + d(s,n)P_n(x(s+1))_q, \tag{21}$$

and

$$v(s)\widetilde{P}_n(x(s-1))_q = e(s,n)P_n(x(s))_q + f(s,n)P_n(x(s+1))_q, \tag{22}$$

respectively, where $u(s) = A_{s+1}\phi(s+1)$, $c(s,n) = -C_{s+1}b(s+1,n)$, and $d(s,n) = A_{s+1}a(s+1,n) - b(s+1,n)(\lambda_n + B_{s+1})$, $v(s) = C_s\phi(s-1)$, $e(s,n) = C_s b(s-1,n) -$

$a(s-1,n)(\lambda_n+B_s)$, and $f(s,n)=-A_s a(s-1,n)$. Then (20), (21) and (22) lead to

$$\begin{vmatrix} \phi(s)\widetilde{P}_n(x(s))_q & a(s,n) & b(s,n) \\ u(s)\widetilde{P}_n(x(s+1))_q & c(s,n) & d(s,n) \\ v(s)\widetilde{P}_n(x(s-1))_q & e(s,n) & f(s,n) \end{vmatrix} = 0. \tag{23}$$

Expanding the determinant (23) by the first column, we get the following second order linear difference equation for $\widetilde{P}_n^{A,B}(x(s))_q$

$$\widetilde{\phi}(s,n)\widetilde{P}_n(x(s-1))_q+\widetilde{\varphi}(s,n)\widetilde{P}_n(x(s))_q+\widetilde{\xi}(s,n)\widetilde{P}_n(x(s+1))_q=0, \tag{24}$$

$$\begin{aligned} \widetilde{\phi}(s,n) &= v(s)\Big[a(s,n)d(s,n)-b(s,n)c(s,n)\Big], \\ \widetilde{\varphi}(s,n) &= \phi(s)\Big[c(s,n)f(s,n)-d(s,n)e(s,n)\Big], \\ \widetilde{\xi}(s,n) &= u(s)\Big[b(s,n)e(s,n)-a(s,n)f(s,n)\Big]. \end{aligned}$$

Thus, the generalized Askey-Wilson polynomials satisfy a second order linear difference equation (24) with polynomial coefficients which explicitly depend on n.

Moreover one can obtain the TTRR of the monic generalized Askey-Wilson polynomials with two mass points (for details see Eqs. (20), (21) in [6])

$$x(s)\widetilde{P}_n^{A,B}(x(s))_q=\widetilde{P}_{n+1}^{A,B}(x(s))_q+\widetilde{\beta}_n\widetilde{P}_n^{A,B}(x(s))_q+\widetilde{\gamma}_n\widetilde{P}_{n-1}^{A,B}(x(s))_q, \quad n\in\mathbb{N},$$

where

$$\begin{aligned} \widetilde{\beta}_n &= \beta_n - A\left(\frac{\widetilde{P}_n^{A,B}(-1)_q P_{n-1}(-1)_q}{d_{n-1}^2}-\frac{\widetilde{P}_{n+1}^{A,B}(-1)_q P_n(-1)_q}{d_n^2}\right) \\ &\quad - B\left(\frac{\widetilde{P}_n^{A,B}(1)_q P_{n-1}(1)_q}{d_{n-1}^2}-\frac{\widetilde{P}_{n+1}^{A,B}(1)_q P_n(1)_q}{d_n^2}\right), \\ \widetilde{\gamma}_n &= \gamma_n\frac{1+\Delta_n^{A,B}}{1+\Delta_{n-1}^{A,B}},\ \Delta_n^{A,B}=\frac{A\widetilde{P}_n^{A,B}(-1)_q P_n(-1)_q}{d_n^2}+\frac{B\widetilde{P}_n^{A,B}(1)_q P_n(1)_q}{d_n^2}. \end{aligned}$$

Representation of $\widetilde{P}_n^{A,B}(x(s))_q$ in terms of basic series. In this section, we obtain an explicit formula for $\widetilde{P}_n^{A,B}(x(s),a,b,c,d|q)$ in terms of basic hypergeometric series. In fact, substituting (10) into (18) we obtain

$$\phi(s)\widetilde{P}_n^{A,B}(x(s))_q=\frac{(ab,ac,ad;q)_{n-1}}{(2a)^{n-1}(abcdq^{n-2};q)_{n-1}}\sum_{k=0}^{\infty}\frac{(q^{-n},abcdq^{n-2},aq^s,aq^{-s};q)_k}{(ab,ac,ad,q;q)_k}q^k\Pi_1(q^k),$$

where $\phi(s)$, $A(s,n)$ and $B(s,n)$ are given in (19) and

$$\begin{aligned}\Pi_1(q^k) = A(s,n)&\frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-abcdq^{n+k-2})}{2a(1-abcdq^{2n-3})(1-abcdq^{2n-2})}\\ &+B(s,n)\frac{(1-q^{-n+k})}{(1-q^{-n})}\\ &=-\Big\{A(s,n)abcdq^{n-2}\vartheta_n^{a,b,c,d}+B(s,n)q^{-n}\Big\}\frac{(q^k-q^{\kappa(s)})}{1-q^{-n}},\end{aligned} \tag{25}$$

where

$$q^{\kappa(s)} = \frac{A(s,n)\vartheta_n^{a,b,c,d}+B(s,n)}{A(s,n)abcdq^{n-2}\vartheta_n^{a,b,c,d}+B(s,n)q^{-n}},$$
$$\vartheta_n^{a,b,c,d} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-q^{-n})}{2a(1-abcdq^{2n-3})(1-abcdq^{2n-4})}.$$

By taking into account the identity $(q^k-q^z)(q^{-z};q)_k=(1-q^z)(q^{1-z};q)_k$ we obtain

$$\phi(s)\widetilde{P}_n^{A,B}(x(s))_q = D_n^{a,b,c,d}(s)\,{}_5\varphi_4\left(\begin{matrix} q^{-n}, abcdq^{n-2}, aq^s, aq^{-s}, q^{1-\kappa(s)} \\ ab, ac, ad, q^{-\kappa(s)}\end{matrix}\,\middle|\, q,q\right), \tag{26}$$

where

$$D_n^{a,b,c,d}(s) = \frac{-(ab,ac,ad;q)_{n-1}}{(2a)^{n-1}(abcdq^{n-2};q)_{n-1}}\frac{1-q^{\kappa(s)}}{1-q^{-n}}\Big\{A(s,n)abcdq^{n-2}\vartheta_n^{a,b,c,d}+B(s,n)q^{-n}\Big\}.$$

Remark 1. *Notice that $\phi(s)\widetilde{P}_n^{A,B}(x(s))_q$, in the left hand side of (26), is a polynomial of degree $n+2$ in $x(s)$ which follows from (18) and (19). In order to see that formula (26) gives a polynomial of degree $n+2$ it is sufficient to notice that the function Π_1 defined in (25) is a polynomial in $x(s)$, which follows from that fact that $A(s,n)$ and $B(s,n)$ are polynomial of degree 2 and 1 in $x(s)$, respectively, (19).*

Remark 2. *We note that the properties of the modified Askey-Wilson polynomials with one mass point at $x=\pm 1$ can be obtained from the ones with two mass points by putting $A=0$ or $B=0$, respectively.*

Remark 3. *We remark that the relation between Askey-Wilson $P_n(x;a,b,c,d;q)$ polynomials defined in (10) and q-Racah polynomials $u_n^{\alpha,\beta}(\mu(t),\widetilde{a},\widetilde{b})$ defined on the lattice $\mu(t)=[t]_q[t+1]_q=c_1(q^t+q^{-t-1})+c_3$, $c_1=q^{1/2}(q^{1/2}-q^{-1/2})^{-2}$ and $c_3=-q^{-1/2}(1+q)(q^{1/2}-q^{-1/2})^{-2}$ follows*

$$\frac{2^n}{(q^{1/2}-q^{-1/2})^{2n}}P_n(2c_1q^{-1/2}x+c_3, q^{\widetilde{a}+\frac12}, q^{\beta-\widetilde{a}+\frac12}, q^{\alpha+\widetilde{b}+\frac12}, q^{-\widetilde{b}+\frac12};q) = u_n^{\alpha,\beta}(\mu(t),\widetilde{a},\widetilde{b})$$

by setting $e^{2i\theta} = q^{2s} = q^{2t+1}$, *where*

$$u_n^{\alpha,\beta}(\mu(t), \widetilde{a}, \widetilde{b}) = q^{-\frac{n}{2}(2\widetilde{a}+1)} \frac{(q^{\widetilde{a}-\widetilde{b}+1}, q^{\beta+1}, q^{\widetilde{a}+\widetilde{b}+\alpha+1}; q)_n}{(q^{1/2} - q^{-1/2})^{2n}(q^{\alpha+\beta+n+1}; q)_n}$$
$$\times {}_4\varphi_3 \left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{\widetilde{a}-t}, q^{t+\widetilde{a}+1} \\ q^{\widetilde{a}-\widetilde{b}+1}, q^{\beta+1}, q^{\widetilde{a}+\widetilde{b}+\alpha+1} \end{array} \middle| \, q \, , \, q \right).$$

Combining the above limit, with the ones considered in [6, §4.2] we can construct the analog of q-Askey Tableau for the Krall-type polynomials. For more details on how one should take the limits we refer to the paper [21].

Concluding remarks

In this paper we have constructed a generalized Askey-Wilson polynomials by adding two mass points at the end of the interval of orthogonality and obtained some of their properties, as the TTRR and the representation as basic hypergeometric series. In particular, we have showed that they satisfy a second order linear q-difference equation on the lattice $x(s) = (q^s + q^{-s})/2$ (see (24)). This equation has the form (2) but with coefficients that explicitely depend on n, the degree of the polynomials. In general, they will not satisfy a higher order difference equation with coefficients independent of n. An example of such polynomials satisfying a higher order difference equation with coefficients independent of n was constructed in [26].

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