SOME EXTENSION OF THE BESSEL-TYPE ORTHOGONAL POLYNOMIALS.

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Abstract

We consider the perturbation of the classical Bessel moment functional by the addition of the linear functional $M_0\delta(x) + M_1\delta'(x)$, where M_0 and $M_1 \in \mathbb{R}$. We give necessary and sufficient conditions in order for this functional to be a quasi-definite functional. In such a situation we analyze the corresponding sequence of monic orthogonal polynomials $B_n^{\alpha,M_0,M_1}(x)$. In particular, a hypergeometric representation $({}_4F_2)$ for them is obtained.

Furthermore, we deduce a relation between the corresponding Jacobi matrices, as well as the asymptotic behavior of the ratio $B_n^{\alpha,M_0,M_1}(x)/B_n^{\alpha}(x)$, outside of the closed contour Γ containing the origin and the difference between the new polynomials and the classical ones, inside Γ.

1   Introduction.

In this paper we will study a generalization of the Bessel polynomials [8], [6], which results when we perturb the classical Bessel moment functional by the addition of the linear functional $M_0\delta(x) + M_1\delta'(x)$, where M_0 and $M_1 \in \mathbb{R}$. If $M_1 = 0$, the corresponding sequence of orthogonal polynomials has been studied by E. Hendriksen [7] in the framework of rational approximation and by F. Marcellán and P. Maroni [10] in the theory of semi-classical orthogonal polynomials. There the corresponding second order differential equation is deduced.

When $M_0 = 0$ and $M_1 > 0$, a general theory is introduced in [5], but unfortunately only one example based in Hermite polynomials is considered.

Later on, in [1]-[2] the perturbation of the Laguerre linear functional by the addition of a linear functional $M_0\delta(x) + M_1\delta'(x)$ is analyzed. There, the authors study asymptotic properties of the related orthogonal polynomials, the distribution of their zeros, the hypergeometric representation, and the second order differential equation that such polynomials satisfy.

In [3] the bounded case is studied, i.e., the perturbation of the Jacobi moment functional in the form,

$$< \mathcal{U}, P > = \int_{-1}^{1} P(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx + A_0 P(1) + B_0 P(-1) + A_1 P'(1) + B_1 P'(-1), \qquad (1)$$

where $P \in \mathbb{P}$ (linear space of polynomials with real coefficients), $\alpha, \beta > -1$ and $A_0, B_0, A_1, B_1 \in \mathbb{R}$. We give necessary and sufficient conditions in order for $\mathcal{U}$ to be a quasi-definite moment functional. In such a situation we analyze the corresponding sequence of monic orthogonal polynomials $P_n^{\alpha,\beta,A_0,B_0,A_1,B_1}(x)$. In particular, a hypergeometric representation $({}_6F_5)$, the three-term recurrence relation, the second

order differential equation and a symmetry property are also obtained.

The interest of the Bessel-type orthogonal polynomials is twofold.

First, they are related with a quasi-definite moment functional. As a consequence, positivity does not play the main role as in the standard cases. Thus, the existence of a sequence of orthogonal polynomials is not guaranteed.

Second, it is well known that zeros of orthogonal polynomials are eigenvalues of a Jacobi symmetric matrix. Now, the real symmetry is lost and the properties of zeros change very much in the sense that a new distribution appears.

The structure of the paper is as follows. In Section 2 we list some of the main properties of the classical Bessel polynomials which will be used later on. In Section 3 we define the generalized polynomials and find some of their properties. In Section 4 we obtain the representation of the generalized Bessel polynomials in terms of the hypergeometric functions. In Section 5 we obtain an asymptotic formula for these polynomials and in Section 6 we establish their quasi-orthogonality. Finally, in Sections 7 and 8 we obtain the three-term recurrence relation that such polynomials satisfy as well as the corresponding Jacobi matrices.

2 Bessel polynomials.

In this section we have enclosed some formulas for the Bessel polynomials $\{B_n^\alpha(x)\}_0^\infty$ which will be useful to obtain the generalized polynomials orthogonal with respect to the quasi-definite moment functional (see Eq. (21) below). All the formulas and properties for the classical Bessel polynomials can be found in [6] and [12]. In this work we will use monic polynomials, i.e., polynomials with leading coefficients equal to 1.
The Bessel polynomials are the polynomial solution of the second order linear differential equation (SODE) of hypergeometric type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \tag{2}$$

where

$$\sigma(x) = x^2, \quad \tau(x) = [(\alpha + 2)x + 2], \quad \lambda_n = -n(n + \alpha + 1). \tag{3}$$

They are orthogonal with respect to the quasi-definite moment functional $\mathcal{L}_0$, associated with the weight function ρ_0^α, on the linear space $\mathbb{P}$ of polynomials with real coefficients defined by

$$< \mathcal{L}_0, P(x) > = \int_\Gamma P(z)\rho_0^\alpha(z)dz, \quad P \in \mathbb{P}. \tag{4}$$

Here the integration is around the unit circle Γ or around some closed contour in $\mathbb{C}$ containing the origin, and

$$\rho_0^\alpha(z) = 2^{\alpha+1} \sum_{m=0}^\infty \frac{(-2)^m}{\Gamma(m + \alpha + 1)z^m}, \quad \alpha \neq -1, -2, -3, \dots. \tag{5}$$

The orthogonality relation is

$$\frac{1}{2\pi i} \int_\Gamma B_n^\alpha(z)B_m^\alpha(z)\rho_0^\alpha(z)dz = d_n^2 \delta_{nm}, \tag{6}$$

where

$$d_n^2 = ||B_n^\alpha(x)||^2 = \frac{(-1)^{n+1}2^{2n+\alpha+1}\Gamma(n + \alpha + 1)n!}{(2n + \alpha + 1)\Gamma(2n + \alpha + 2)}. \tag{7}$$

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They satisfy the differential relation

$$\frac{d}{dx} B_n^\alpha(x) = n B_{n-1}^{\alpha+2}(x). \tag{8}$$

Now, from the *structure relation*

$$x^2 \frac{d}{dx} B_n^\alpha(x) = n \left[x - \frac{2}{2n+\alpha} \right] B_n^\alpha(x) + \frac{4n(n+\alpha)}{(2n+\alpha)^2(2n+\alpha-1)} B_{n-1}^\alpha(x), \tag{9}$$

and *the three-term recurrence relation*,

$$x B_n^\alpha(x) = B_{n+1}^\alpha(x) + \beta_n^\alpha B_n^\alpha(x) + \gamma_n^\alpha B_{n-1}^\alpha(x), \qquad n \geq 1, \tag{10}$$

with

$$\beta_n^\alpha = -\frac{2\alpha}{(2n+\alpha)(2n+\alpha+2)}, \quad \gamma_n^\alpha = -\frac{4n(n+\alpha)}{(2n+\alpha+1)(2n+\alpha)^2(2n+\alpha-1)}, \tag{11}$$

and

$$B_0^\alpha(x) = 1, \quad B_1^\alpha(x) = x + \frac{2}{\alpha+2},$$

we deduce

$$B_{n-1}^\alpha(x) = \frac{(2n+\alpha)^2(2n+\alpha-1)}{4n(n+\alpha)} x^2 (B_n^\alpha)'(x) + \frac{(2n+\alpha)^2(2n+\alpha-1)}{4(n+\alpha)} \left[\frac{2}{2n+\alpha} - x \right] B_n^\alpha(x), \tag{12}$$

which allows us to express the Bessel kernels in a polynomial form. If we take derivatives in the above formula we obtain,

$$(B_{n-1}^\alpha)'(x) = -\frac{(2n+\alpha)^2(2n+\alpha-1)}{4(n+\alpha)} \left[B_n^\alpha(x) + \left(x - \frac{2x}{n} - \frac{2}{(2n+\alpha)} \right) (B_n^\alpha)'(x) - \frac{x^2}{n} (B_n^\alpha)''(x) \right]. \tag{13}$$

The Christoffel-Darboux formula for Bessel polynomials $B_n^\alpha(x)$ is

$$K_{n-1}^\alpha(x,y) := \sum_{k=0}^{n-1} \frac{B_k^\alpha(x) B_k^\alpha(y)}{d_k^2} = \frac{1}{x-y} \frac{B_n^\alpha(x) B_{n-1}^\alpha(y) - B_{n-1}^\alpha(x) B_n^\alpha(y)}{d_{n-1}^2}, \quad n = 1, 2, 3, \dots . \tag{14}$$

Through the work we will denote by

$$K_n^{\alpha(p,q)}(x,y) = \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_n^\alpha(x,y) = \sum_{m=0}^n \frac{(B_m^\alpha)^{(p)}(x) (B_m^\alpha)^{(q)}(y)}{d_m^2}, \tag{15}$$

the kernels of the Bessel polynomials, as well as their derivatives with respect to x and y, respectively.

The explicit representation of these polynomials is

$$B_n^\alpha(x) = \frac{2^n}{(\alpha+n+1)_n} \sum_{k=0}^n \binom{n}{k} (n+\alpha+1)_k \left(\frac{x}{2}\right)^k, \quad B_n^\alpha(0) = \frac{2^n}{(\alpha+n+1)_n} > 0 \tag{16}$$

or, equivalently, in terms of hypergeometric series

$$B_n^\alpha(x) = \frac{2^n}{(\alpha+n+1)_n} \, {}_2F_0 \left(\begin{array}{c} -n, n+\alpha+1 \\ - \end{array} \Bigg| -\frac{x}{2} \right), \tag{17}$$

with

$$_pF_q \left(\begin{array}{c} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{array} \Bigg| x \right) = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \tag{18}$$

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Here $(\alpha)_k$ denotes the *shifted factorial* or Pochhammer's symbol:

$$(\alpha)_0 := 1, \quad (\alpha)_k := \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k = 1,2,3,\dots\ .$$

From the above properties, we deduce the following formulas which will be useful later on:

$$K_{n-1}^\alpha(x,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)B_n^\alpha(0)}{4n(n+\alpha)d_{n-1}^2}\left[nB_n^\alpha(x) - x(B_n^\alpha(x))'\right],$$

$$K_{n-1}^{\alpha(0,1)}(x,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)B_n^\alpha(0)}{8n(n+\alpha)d_{n-1}^2}\times$$

$$\times\left[n^2(n+\alpha+1)B_n^\alpha(x) - (2+n(n+\alpha+1)x)(B_n^\alpha(x))'\right], \tag{19}$$

$$K_{n-1}^{\alpha(1,1)}(x,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)B_n^\alpha(0)}{8n(n+\alpha)d_{n-1}^2}\times$$

$$\times\left[n(n-1)(n+\alpha+1)(B_n^\alpha(x))' - (2+n(n+\alpha+1)x)(B_n^\alpha(x))''\right]$$

and

$$K_{n-1}^\alpha(0,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)(B_n^\alpha(0))^2}{4(n+\alpha)d_{n-1}^2},$$

$$K_{n-1}^{\alpha(0,1)}(0,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)(n-1)B_n^\alpha(0)(B_n^\alpha)'(0)}{4n(n+\alpha)d_{n-1}^2}, \tag{20}$$

$$K_{n-1}^{\alpha(1,1)}(0,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)\left[n(n-2)(n+\alpha+1)+\alpha+2\right]B_n^\alpha(0)(B_n^\alpha)'(0)}{8n(n+\alpha)d_{n-1}^2},$$

where

$$B_n^\alpha(0) = \frac{2^n}{(n+\alpha+1)_n}, \quad (B_n^\alpha)'(0) = \frac{n\,2^{n-1}}{(n+\alpha+2)_{n-1}}.$$

3 The definition and orthogonal relation.

Let $\mathcal{L}_0$ be the quasi-definite moment functional (4) and $\{B_n^\alpha(x)\}_0^\infty$ the monic Bessel orthogonal polynomial sequence (MOPS) with respect to $\mathcal{L}_0$. Consider the perturbation $\mathcal{L}_1$ of $\mathcal{L}_0$ given by

$$< \mathcal{L}_1, P(x) > = \int_\Gamma P(z)\rho_0^\alpha(z)dz + M_0 P(0) - M_1 P'(0), \quad P \in \mathbb{P}. \tag{21}$$

Proposition 1 *The perturbed moment functional $\mathcal{L}_1$ is quasi-definite if and only if the determinant of the matrix $\mathbb{K}_n$, denoted by $\triangle_n^{\alpha,M_0,M_1}$, does not vanish for every $n \geq 0$, where*

$$\mathbb{K}_n = \begin{pmatrix} 1 + M_0 K_n^\alpha(0,0) - M_1 K_n^{\alpha(0,1)}(0,0) & -M_1 K_n^\alpha(0,0) \\ M_0 K_n^{\alpha(0,1)}(0,0) - M_1 K_n^{\alpha(1,1)}(0,0) & 1 - M_1 K_n^{\alpha(0,1)}(0,0) \end{pmatrix}, \quad n \geq 0.$$

Proof: Assume $\mathcal{L}_1$ is quasi-definite, and let $\{B_n^{\alpha,M_0,M_1}(x)\}_0^\infty$ be the MOPS with respect to $\mathcal{L}_1$. Then, we will prove that

$$< \mathcal{L}_1, (B_n^{\alpha,M_0,M_1}(x))^2 > = \frac{\triangle_n^{\alpha,M_0,M_1}}{\triangle_{n-1}^{\alpha,M_0,M_1}} < \mathcal{L}_0, (B_n^\alpha(x))^2 >, \quad n \geq 0, \quad (\triangle_{-1}^{\alpha,M_0,M_1} = 1), \tag{22}$$

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as well as

$$B_n^{\alpha,M_0,M_1}(x) = B_n^\alpha(x) - M_0 B_n^{\alpha,M_0,M_1}(0)K_{n-1}^\alpha(x,0) + M_1(B_n^{\alpha,M_0,M_1})'(0)K_{n-1}^\alpha(x,0)+$$

$$+M_1 B_n^{\alpha,M_0,M_1}(0)K_{n-1}^{\alpha(0,1)}(x,0), \tag{23}$$

where $B_n^{\alpha,M_0,M_1}(0)$ and $(B_n^{\alpha,M_0,M_1})'(0)$ are given by

$$\begin{pmatrix} B_n^{\alpha,M_0,M_1}(0) \\ (B_n^{\alpha,M_0,M_1})'(0) \end{pmatrix} = \mathbb{K}_{n-1}^{-1} \begin{pmatrix} B_n^\alpha(0) \\ (B_n^\alpha)'(0) \end{pmatrix}, \quad n \geq 1, \tag{24}$$

i.e.,

$$B_n^{\alpha,M_0,M_1}(0) = \frac{2B_n^\alpha(0)\left[2 + M_1(n+\alpha+1)K_{n-1}^\alpha(0,0)\right]}{4 + n(n+\alpha+1)\left[M_1 K_{n-1}^\alpha(0,0)\right]^2 + 4\left[M_0 + M_1(n-1)(n+\alpha+1)\right]K_{n-1}^\alpha(0,0)},$$

$$(B_n^{\alpha,M_0,M_1})'(0) = \frac{(n+\alpha+1)B_n^\alpha(0)\left[2n + (2M_0 - M_1\left[n^2+n\alpha+n-\alpha-1\right])K_{n-1}^\alpha(0,0)\right]}{4 + n(n+\alpha+1)\left[M_1 K_{n-1}^\alpha(0,0)\right]^2 + 4\left[M_0 + M_1(n-1)(n+\alpha+1)\right]K_{n-1}^\alpha(0,0)}.$$

In fact, we can write the Fourier expansion of the generalized Bessel polynomials in terms of the Bessel polynomials

$$B_n^{\alpha,M_0,M_1}(x) = B_n^\alpha(x) + \sum_{k=0}^{n-1} a_{n,k}B_k^\alpha(x), \quad n \geq 1. \tag{25}$$

Then we can find the coefficients $a_{n,k}$ using the orthogonality of the polynomials $B_n^{\alpha,M_0,M_1}(x)$ with respect to $\mathcal{L}_1$, i.e.,

$$< \mathcal{L}_0, B_n^{\alpha,M_0,M_1}(x)B_j^\alpha(x) > \ =< \mathcal{L}_0, B_n^\alpha(x)B_j^\alpha(x) > + \sum_{k=0}^{n-1} a_{n,k} < \mathcal{L}_0, B_j^\alpha(x)B_k^\alpha(x) >$$

$$= a_{n,j} < \mathcal{L}_0, (B_j^\alpha(x))^2 >, \quad 0 \leq j \leq n-1, \tag{26}$$

whence

$$a_{n,j} = \frac{< \mathcal{L}_0, B_n^{\alpha,M_0,M_1}(x)B_j^\alpha(x) >}{< \mathcal{L}_0, (B_j^\alpha(x))^2 >}, \quad 0 \leq j \leq n-1. \tag{27}$$

Since

$$< \mathcal{L}_1, B_n^{\alpha,M_0,M_1}(x)B_j^\alpha(x) >= 0, \qquad 0 \leq j < n,$$

$$< \mathcal{L}_0, B_n^{\alpha,M_0,M_1}(x)B_j^\alpha(x) > \ =< \mathcal{L}_1, B_n^{\alpha,M_0,M_1}(x)B_j^\alpha(x) > -M_0 B_n^{\alpha,M_0,M_1}(0)B_j^\alpha(0)-$$

$$-M_1\left((B_n^{\alpha,M_0,M_1})'(0)B_j^\alpha(0) + B_n^{\alpha,M_0,M_1}(0)(B_j^\alpha)'(0)\right) =$$

$$= -M_0 B_n^{\alpha,M_0,M_1}(0)B_j^\alpha(0) + M_1(B_n^{\alpha,M_0,M_1})'(0)B_j^\alpha(0)+ \tag{28}$$

$$+M_1 B_n^{\alpha,M_0,M_1}(0)(B_j^\alpha)'(0).$$

If we use the decomposition (25) and substitute the coefficients $a_{n,j}$ according to (27) we obtain an explicit expression for the generalized Bessel polynomial $B_n^{\alpha,M_0,M_1}(x)$

$$B_n^{\alpha,M_0,M_1}(x) = B_n^\alpha(x) + \sum_{k=0}^{n-1} \frac{B_k^\alpha(x)}{< \mathcal{L}_0, (B_k^\alpha(x))^2 >} \times$$

$$\times \left(-M_0 B_n^{\alpha,M_0,M_1}(0)B_k^\alpha(0) + M_1(B_n^{\alpha,M_0,M_1})'(0)B_k^\alpha(0) + M_1 B_n^{\alpha,M_0,M_1}(0)(B_k^\alpha)'(0)\right) = \tag{29}$$

$$= B_n^\alpha(x) - M_0 B_n^{\alpha,M_0,M_1}(0)K_{n-1}^\alpha(x,0) + M_1(B_n^{\alpha,M_0,M_1})'(0)K_{n-1}^\alpha(x,0)+$$

$$+M_1 B_n^{\alpha,M_0,M_1}(0)K_{n-1}^{\alpha(0,1)}(x,0).$$

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Thus (23) follows. In particular,

$$
\left\{
\begin{aligned}
B_n^{\alpha,M_0,M_1}(0) &= B_n^\alpha(0) - M_0 B_n^{\alpha,M_0,M_1}(0) K_{n-1}^\alpha(0,0) + M_1 (B_n^{\alpha,M_0,M_1})'(0) K_{n-1}^\alpha(0,0) + \\
&\quad + M_1 B_n^{\alpha,M_0,M_1}(0) K_{n-1}^{\alpha(0,1)}(0,0), \\
(B_n^{\alpha,M_0,M_1})'(0) &= B_n^\alpha(0) - M_0 B_n^{\alpha,M_0,M_1}(0) K_{n-1}^{\alpha(1,0)}(0,0) + M_1 (B_n^{\alpha,M_0,M_1})'(0) K_{n-1}^{\alpha(1,0)}(0,0) + \\
&\quad + M_1 B_n^{\alpha,M_0,M_1}(0) K_{n-1}^{\alpha(1,1)}(0,0),
\end{aligned}
\right.
\tag{30}
$$

or, equivalently,

$$
\mathbb{K}_{n-1} \cdot \begin{pmatrix} B_n^{\alpha,M_0,M_1}(0) \\ (B_n^{\alpha,M_0,M_1})'(0) \end{pmatrix} = \begin{pmatrix} B_n^\alpha(0) \\ (B_n^\alpha)'(0) \end{pmatrix}.
\tag{31}
$$

Now,

$$
\begin{aligned}
\mathbb{K}_n &= \mathbb{K}_{n-1} + \begin{pmatrix} M_0 \dfrac{(B_n^\alpha(0))^2}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} - M_1 \dfrac{(B_n^\alpha)'(0) B_n^\alpha(0)}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} & -M_1 \dfrac{(B_n^\alpha(0))^2}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} \\[3ex] M_0 \dfrac{(B_n^\alpha)'(0) B_n^\alpha(0)}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} - M_1 \dfrac{(B_n^\alpha)'(0)(B_n^\alpha)'(0)}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} & -M_1 \dfrac{(B_n^\alpha)'(0) B_n^\alpha(0)}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} \end{pmatrix} = \\[3ex]
&= \mathbb{K}_{n-1} + \frac{1}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >} \begin{pmatrix} B_n^\alpha(0) \\ (B_n^\alpha)'(0) \end{pmatrix} \cdot \begin{pmatrix} M_0 B_n^\alpha(0) - M_1 (B_n^\alpha)'(0), & -M_1 B_n^\alpha(0) \end{pmatrix}.
\end{aligned}
\tag{32}
$$

Substituting (31) in (32) and computing the determinant on both sides of the equality we get

$$
\frac{\triangle_n^{\alpha,M_0,M_1}}{\triangle_{n-1}^{\alpha,M_0,M_1}} = 1 + \frac{B_n^{\alpha,M_0,M_1}(0) \left[M_0 B_n^\alpha(0) - M_1 (B_n^\alpha)'(0) \right] - (B_n^{\alpha,M_0,M_1})'(0) M_1 B_n^\alpha(0)}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >}.
\tag{33}
$$

On the other hand,

$$
\begin{aligned}
< \mathcal{L}_1, (B_n^{\alpha,M_0,M_1}(x))^2 > &= \mathcal{L}_1 \left[B_n^{\alpha,M_0,M_1}(x) B_n^\alpha(x) \right] = \\
&= < \mathcal{L}_0, B_n^{\alpha,M_0,M_1}(x) B_n^\alpha(x) > + M_0 B_n^\alpha(0) B_n^{\alpha,M_0,M_1}(0) - \\
&\quad - M_1 B_n^\alpha(0) (B_n^{\alpha,M_0,M_1})'(0) - M_1 (B_n^\alpha)'(0) B_n^{\alpha,M_0,M_1}(0) = \\
&= < \mathcal{L}_0, (B_n^\alpha(x))^2 > + M_0 B_n^\alpha(0) B_n^{\alpha,M_0,M_1}(0) - \\
&\quad - M_1 (B_n^\alpha)'(0) B_n^{\alpha,M_0,M_1}(0) - M_1 B_n^\alpha(0) (B_n^{\alpha,M_0,M_1})'(0).
\end{aligned}
\tag{34}
$$

From (33)-(34), we deduce

$$
\triangle_n^{\alpha,M_0,M_1} < \mathcal{L}_0, (B_n^\alpha(x))^2 > = \triangle_{n-1}^{\alpha,M_0,M_1} < \mathcal{L}_1, (B_n^{\alpha,M_0,M_1}(x))^2 >, \quad n \geq 1.
\tag{35}
$$

When $n = 0$, since $< \mathcal{L}_1, 1 > = < \mathcal{L}_0, 1 > + M_0 \neq 0$, and

$$
\det \mathbb{K}_0 = \det \begin{pmatrix} 1 + \dfrac{M_0}{< \mathcal{L}_0, 1 >} & -\dfrac{M_1}{< \mathcal{L}_0, 1 >} \\[3ex] 0 & 1 \end{pmatrix} = 1 + \frac{M_0}{< \mathcal{L}_0, 1 >} = \frac{< \mathcal{L}_1, 1 >}{< \mathcal{L}_0, 1 >} \neq 0.
\tag{36}
$$

If we get $\det \mathbb{K}_{-1} = 1$, then (35) holds also for $n = 0$. Thus, from (35) it follows by induction that $\det \mathbb{K}_n \neq 0$, $n \geq 0$ and

$$
\triangle_n^{\alpha,M_0,M_1} = \triangle_{n-1}^{\alpha,M_0,M_1} \frac{< \mathcal{L}_1, (B_n^{\alpha,M_0,M_1}(x))^2 >}{< \mathcal{L}_0, (B_n^\alpha(x))^2 >}, \quad n \geq 0.
\tag{37}
$$

Conversely, assume that $\det \mathbb{K}_n \neq 0$, $n \geq 0$ and define $B_n^{\alpha, M_0, M_1}(x)$ by (23) and (24). Then $\left\{ B_n^{\alpha, M_0, M_1}(x) \right\}_0^\infty$ is a MOPS. Indeed,

$$
\begin{aligned}
< \mathcal{L}_1, B_n^{\alpha, M_0, M_1}(x) B_j^\alpha(x) > &=< \mathcal{L}_0, B_n^{\alpha, M_0, M_1}(x) B_j^\alpha(x) > +M_0 B_n^{\alpha, M_0, M_1}(0) B_j^\alpha(0) - \\
&\quad -M_1 \left((B_n^{\alpha, M_0, M_1})'(0) B_j^\alpha(0) + B_n^{\alpha, M_0, M_1}(0)(B_j^\alpha)'(0) \right) \\
&=< \mathcal{L}_0, B_n^\alpha(x) B_j^\alpha(x) > -M_0 B_n^{\alpha, M_0, M_1}(0) < \mathcal{L}_0, K_{n-1}^\alpha(x,0) B_j^\alpha(x) > \\
&\quad +M_1 (B_n^{\alpha, M_0, M_1})'(0) < \mathcal{L}_0, K_{n-1}^\alpha(x,0) B_j^\alpha(x) > \\
&\quad +M_1 B_n^{\alpha, M_0, M_1}(0) < \mathcal{L}_0, K_{n-1}^{\alpha(0,1)}(x,0) B_j^\alpha(x) > +M_0 B_n^{\alpha, M_0, M_1}(0) B_n^\alpha(0) \\
&\quad -M_1 \left((B_n^{\alpha, M_0, M_1})'(0) B_j^\alpha(0) + B_n^{\alpha, M_0, M_1}(0)(B_j^\alpha)'(0) \right).
\end{aligned}
\tag{38}
$$

Hence

$$
< \mathcal{L}_1, B_n^{\alpha, M_0, M_1}(x) B_j^\alpha(x) > = \begin{cases} 0 & \text{if} \quad 0 \leq j \leq n-1, \\[2mm] \begin{aligned} &< \mathcal{L}_0, (B_n^\alpha(x))^2 > + \\ &+M_0 B_n^{\alpha, M_0, M_1}(0) B_j^\alpha(0) - \\ &-M_1 (B_n^{\alpha, M_0, M_1})'(0) B_j^\alpha(0) - \\ &-M_1 B_n^{\alpha, M_0, M_1}(0)(B_j^\alpha)'(0) \end{aligned} & \text{if} \quad j = n, \end{cases}
\tag{39}
$$

$$
= \begin{cases} 0 & \text{if} \quad 0 \leq j \leq n-1, \\[2mm] \dfrac{\triangle_n^{\alpha, M_0, M_1}}{\triangle_{n-1}^{\alpha, M_0, M_1}} < \mathcal{L}_0, (B_n^\alpha(x))^2 > \neq 0 & \text{if} \quad j = n. \end{cases}
$$

By (33)

$$
< \mathcal{L}_1, (B_n^{\alpha, M_0, M_1}(x) B_j^{\alpha, M_0, M_1}(x) > = \begin{cases} 0 & \text{if} \quad 0 \leq j \leq n-1, \\[2mm] \neq 0 & \text{if} \quad j = n. \end{cases}
\tag{40}
$$

Thus $\{B_n^{\alpha, M_0, M_1}(x)\}_0^\infty$ is a MOPS with respect to $\mathcal{L}_1$ and so $\mathcal{L}_1$ is quasi-definite. $\blacksquare$

4 Representation as hypergeometric series.

The explicit expression of the Bessel kernels (19)-(20) allows us to write $B_n^{\alpha, M_0, M_1}(x)$ in terms of the classical Bessel polynomials and their first derivatives. In fact, substituting (19)-(20) into (23) we have

$$
B_n^{\alpha, M_0, M_1}(x) = [1 - n \zeta_n^{\alpha, M_0, M_1}] B_n^\alpha(x) + [\eta_n^{\alpha, M_0, M_1} + x \zeta_n^{\alpha, M_0, M_1}](B_n^\alpha(x))',
\tag{41}
$$

where

$$\zeta_n^{\alpha,M_0,M_1} = \frac{(2n+\alpha)^2(2n+\alpha-1)B_n^\alpha(0)}{4(n+\alpha)d_{n-1}^2} \times$$

$$\times \left[M_0 B_n^{\alpha,M_0,M_1}(0) - M_1 (B_n^{\alpha,M_0,M_1})'(0) - \frac{1}{2}n(n+\alpha+1)M_1 B_n^{\alpha,M_0,M_1}(0) \right], \qquad (42)$$

$$\eta_n^{\alpha,M_0,M_1} = -\frac{(2n+\alpha)^2(2n+\alpha-1)B_n^\alpha(0)}{4(n+\alpha)d_{n-1}^2} M_1 B_n^{\alpha,M_0,M_1}(0).$$

Taking into account (8) the equation (41) becomes

$$B_n^{\alpha,M_0,M_1}(x) = [1 - n\zeta_n^{\alpha,M_0,M_1}]B_n^\alpha(x) + [\eta_n^{\alpha,M_0,M_1} + x\zeta_n^{\alpha,M_0,M_1}]nB_{n-1}^{\alpha+2}(x). \qquad (43)$$

On the other hand (see [9, Table VI, page 302]),

$$B_n^\alpha(x) = B_n^{\alpha+2}(x) + \frac{4n\, B_{n-1}^{\alpha+2}(x)}{(2n+\alpha)(2n+\alpha+2)} + \frac{4(n-1)n\, B_{n-2}^{\alpha+2}(x)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}, \qquad (44)$$

and

$$xB_{n-1}^{\alpha+2}(x) = B_n^{\alpha+2}(x) - \frac{2(\alpha+2)\, B_{n-1}^{\alpha+2}(x)}{(2n+\alpha)(2n+\alpha+2)} - \frac{4(n-1)(n+\alpha+1)\, B_{n-2}^{\alpha+2}(x)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}. \qquad (45)$$

Thus, we deduce

$$B_n^{\alpha,M_0,M_1}(x) = B_n^{\alpha+2}(x) + a_n^{\alpha,M_0,M_1} B_{n-1}^{\alpha+2}(x) + b_n^{\alpha,M_0,M_1} B_{n-2}^{\alpha+2}(x), \qquad (46)$$

where

$$a_n^{\alpha,M_0,M_1} = \frac{4n}{(2n+\alpha)(2n+\alpha+2)} \left[1 + \frac{\eta_n^{\alpha,M_0,M_1}(2n+\alpha)(2n+\alpha+2)}{4} - \frac{(2n+\alpha+2)\zeta_n^{\alpha,M_0,M_1}}{2} \right],$$

$$b_n^{\alpha,M_0,M_1} = \frac{4(n-1)n}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)} \left[1 - (2n+\alpha+1)\zeta_n^{\alpha,M_0,M_1} \right]. \qquad (47)$$

Now, we will prove the following proposition.

Proposition 2 *The orthogonal polynomial $B_n^{\alpha,M_0,M_1}(x)$ is, up to a constant factor, a generalized hypergeometric series. More precisely,*

$$B_n^{\alpha,M_0,M_1}(x) = \frac{2^{n-2}\pi_2(0)}{(n+\alpha+1)_{n+2}n(n-1)} \, _4F_2 \left(\begin{matrix} -n,\ n+\alpha+1,\ \beta_0+1,\ \beta_1+1 \\ \beta_0,\ \beta_1 \end{matrix} \middle| -\frac{x}{2} \right), \qquad (48)$$

where $\pi_2(0)$ is given in (53) and the coefficients $-\beta_0$ and $-\beta_1$ are the solutions of the quadratic equation in k (see formula (50) below) and they are, in general, complex numbers. In the case when for some $i = 0, 1$, $-\beta_i$ is a negative integer number we need to take the analytic continuation of the hypergeometric series (48).

(48) can be considered as a generalization of the representation of the Bessel polynomials as hypergeometric series.

Proof: Substituting the hypergeometric representation of the Bessel polynomials (17) into (46) we

8

find

$$B_n^{\alpha,M_0,M_1}(x) \quad = \frac{2^{n-2}}{(n+\alpha+1)_{n+2}n(n-1)} \sum_{k=0}^{\infty} \Big[4n(n-1)(k+n+\alpha+1)(k+n+\alpha+2)-$$

$$-2a_n^{\alpha,M_0,M_1}(k+n+\alpha+1)(k-n)(n-1)(2n+\alpha+1)(2n+\alpha+2)+$$

$$+b_n^{\alpha,M_0,M_1}(k-n+1)(k-n)(2n+\alpha-1)(2n+\alpha)(2n+\alpha+1)(2n+\alpha+2)\Big]\times$$

$$\times \frac{(-n)_k(n+\alpha+1)_k}{k!}\left(-\frac{x}{2}\right)^k. \tag{49}$$

Taking into account that the expression inside the quadratic brackets is a polynomial in k of degree 2 and denoting it $\pi_2(k)$, we can write

$$B_n^{\alpha,M_0,M_1}(x) = \frac{2^{n-2}a_n}{(n+\alpha+1)_{n+2}n(n-1)} \sum_{k=0}^{\infty} \frac{(-n)_k(n+\alpha+1)_k(k+\beta_0)(k+\beta_1)}{k!}\left(-\frac{x}{2}\right)^k, \tag{50}$$

where a_n is the leading coefficient of $\pi_2(k)$:

$$a_n \quad = 4n(n-1) - 2a_n^{\alpha,M_0,M_1}(n-1)(2n+\alpha+1)(2n+\alpha+2)+$$

$$+b_n^{\alpha,M_0,M_1}(2n+\alpha-1)(2n+\alpha)(2n+\alpha+1)(2n+\alpha+2), \tag{51}$$

and $\beta_i = \beta_i(n,\alpha,\beta,A_1,A_2,B_1,B_2)$ $(i=0,1)$ are the solutions of the quadratic equation in k (see formula (49)). Since $(k+\beta_i) = \dfrac{\beta_i(\beta_i+1)_k}{(\beta_i)_k}$, $i=0,1$ (50) becomes

$$B_n^{\alpha,M_0,M_1}(x) = \frac{2^{n-2}\pi_2(0)}{(n+\alpha+1)_{n+2}n(n-1)} \sum_{k=0}^{\infty} \frac{(-n)_k(n+\alpha+1)_k(1+\beta_0)_k(1+\beta_1)_k}{k!\,(\beta_0)_k\,(\beta_1)_k}\left(-\frac{x}{2}\right)^k, \tag{52}$$

where

$$\pi_2(0) \quad = 4n(n-1)(n+\alpha+1)(n+\alpha+2)+$$

$$+2a_n^{\alpha,M_0,M_1}n(n-1)(n+\alpha+1)(2n+\alpha+1)(2n+\alpha+2)+$$

$$+b_n^{\alpha,M_0,M_1}n(n-1)(2n+\alpha-1)(2n+\alpha)(2n+\alpha+1)(2n+\alpha+2), \tag{53}$$

which, by using the definition (18), is nothing else but the hypergeometric representation (48). ∎

5   Some Asymptotic Formulas.

In this section we will study some asymptotic formulas for the generalized Bessel polynomials. More precisely, the ratio $B_n^{\alpha,M_0,M_1}(x)/B_n^{\alpha}(x)$, outside of the closed contour Γ containing the origin and the difference between the new polynomials and the classical ones, inside Γ. To obtain the first one we will rewrite (41) in the form

$$\frac{B_n^{\alpha,M_0,M_1}(x)}{B_n^{\alpha}(x)} = 1 - n\zeta_n^{\alpha,M_0,M_1} + [\eta_n^{\alpha,M_0,M_1} + x\zeta_n^{\alpha,M_0,M_1}]\frac{(B_n^{\alpha}(x))'}{B_n^{\alpha}(x)}. \tag{54}$$

In order to obtain them we will rewrite the kernels (20) as follows:

$$K_{n-1}^{\alpha}(0,0) \quad = \frac{(2n+\alpha)^2(2n+\alpha-1)(B_n^{\alpha}(0))^2}{4(n+\alpha)d_{n-1}^2};$$

$$K_{n-1}^{\alpha(0,1)}(0,0) \quad = \frac{1}{2}(n-1)(n+\alpha+1)K_{n-1}^{\alpha}(0,0);$$

$$K_{n-1}^{\alpha(1,1)}(0,0) \quad = \frac{1}{4}(n+\alpha+1)\left[n(n-2)(n+\alpha+1)+\alpha+2\right]K_{n-1}^{\alpha}(0,0);$$

(55)

and also we will use the expression $(B_n^{\alpha})'(0) = \frac{1}{2}n(n+\alpha+1)B_n^{\alpha}(0)$.

Now, using the asymptotic formula for the Gamma function (see [13, formula 8.16, page 88], [14])

$$\Gamma(ax+b) \sim \sqrt{2\pi}e^{-ax}(a\,x)^{a\,x+b-\frac{1}{2}}, \qquad x \gg 1, \qquad a,b,x \in \mathbb{R},$$

(56)

we find

$$K_{n-1}^{\alpha}(0,0) \sim \frac{1}{\pi 2^{2\alpha+1}n^{\alpha-1}}\left(\frac{e}{2\,n}\right)^{4n}, \qquad \frac{[B_n^{\alpha}(0)]^2}{d_{n-1}^2} \sim \frac{(-1)^n}{2^{3\alpha-1}n^{\alpha-2}}\left(\frac{e}{2\,n}\right)^{4n}.$$

Using the explicit values for $B_n^{\alpha,M_0,M_1}(0)$ and $(B_n^{\alpha,M_0,M_1})'(0)$ we find

$$\left[M_0 B_n^{\alpha,M_0,M_1}(0) - M_1(B_n^{\alpha,M_0,M_1})'(0) - \frac{1}{2}n(n+\alpha+1)M_1 B_n^{\alpha,M_0,M_1}(0)\right] =$$

$$= \frac{\left(4\,M_0 - M_1\,(1+\alpha+n)\,\left[(1+\alpha)\,K_{n-1}^{\alpha}(0,0)\,M_1 + 4\,n\right]\right)\,B_n^{\alpha}(0)}{4 + K_{n-1}^{\alpha}(0,0)^2\,{M_1}^2\,n\,(1+\alpha+n) + 4\,K_{n-1}^{\alpha}(0,0)\,(M_0 - M_1\,(n-1)\,(1+\alpha+n))}.$$

From the above expression and (42) we obtain

$$\zeta_n^{\alpha,M_0,M_1} \sim \widetilde{C}(n,\alpha,M_0,M_1)\frac{(-1)^n}{2^{3\alpha+3}n^{\alpha-1}}\left(\frac{e}{2\,n}\right)^{4n}\left[1+O(n^{-1})\right],$$

$$\eta_n^{\alpha,M_0,M_1} \sim M_1\frac{(-1)^n}{2^{3\alpha+3}n^{\alpha-1}}\left(\frac{e}{2\,n}\right)^{4n}\left[1+O(n^{-1})\right],$$

(57)

where

$$\widetilde{C}(n,\alpha,M_0,M_1) = \left[2\,n^3\left(4M_0 + M_1\left[2-2\,\alpha-9\,\alpha^2\right]\right) - 4\,M_1 n^4\,(1+5\,\alpha) - 8\,M_1\,n^5\right].$$

Finally, from the asymptotics of the Bessel polynomials [6, Eq. (5) page 124]

$$B_n^{\alpha}(z) = \left(\frac{2nz}{e}\right)^n 2^{\alpha+\frac{1}{2}}e^{1/z}\left[1 - \frac{1+6\alpha(\alpha+1+2z^{-1})+6z^{-2}}{24n}+O(n^{-2})\right], \qquad z \in \mathbb{C}\backslash\{0\}.$$

we find outside of the closed contour Γ containing the origin (see formula (4))

$$\frac{(B_n^{\alpha})'(z)}{B_n^{\alpha}(z)} = \frac{2}{z}\left[1 - \frac{2+(3+2\alpha)z}{2nz}\right]+O\left(n^{-2}\right).$$

(58)

Then, from (54) and using all the above asymptotic formulas we obtain the following estimates for the ratio

$$\frac{B_n^{\alpha,M_0,M_1}(z)}{B_n^{\alpha}(z)} = 1 + \frac{(-1)^n e^{4n}}{n^{4n+\alpha-5}2^{4n+3\alpha-4}}\left[\frac{M_1(2-\alpha z+9\alpha^2 z)-8M_0 z}{z}+10M_1\alpha n - 4M_1 n^2 + O(n^{-1})\right],$$

where $z \in \mathbb{C}\backslash\{\Omega\}$, $\Omega = \text{Int}\,\Gamma$.

The difference between the new polynomials and the classical ones, when z is inside of the closed contour Γ containing the origin, is

$$B_n^{\alpha,M_0,M_1}(z) - B_n^{\alpha}(z) = \widetilde{C}(n,\alpha,M_0,M_1)\frac{(-1)^{n+1}}{2^{3\alpha+3}n^{\alpha-1}}\left(\frac{2nz}{e}\right)^n z^{\alpha+2}e^{1/z}\left[1+O(n^{-1})\right].$$

(59)

6 Second order differential equation.

The generalized Bessel polynomials can be given in terms of the Bessel polynomials $B_n^\alpha(x)$ and their first derivatives (see formula (41)) by means of the representation formula

$$B_n^{\alpha,M_0,M_1}(x) = c(x;n)B_n^\alpha(x) + d(x;n)\frac{d}{dx}B_n^\alpha(x), \tag{60}$$

where $c(x;n), d(x;n)$ are polynomials of bounded degree in x with coefficients depending on n:

$$c(x;n) = [1 - n\zeta_n^{\alpha,M_0,M_1}], \quad d(x;n) = [\eta_n^{\alpha,M_0,M_1} + x\zeta_n^{\alpha,M_0,M_1}]. \tag{61}$$

Taking derivatives in the above expression and using (2) and (60) we find

$$\sigma(x)\frac{d}{dx}B_n^{\alpha,M_0,M_1}(x) = e(x;n)B_n^\alpha(x) + f(x;n)\frac{d}{dx}B_n^\alpha(x), \tag{62}$$

where

$$e(x;n) = \sigma(x)c'(x;n) - \lambda_n d(x;n), \quad f(x;n) = \sigma(x)[c(x;n) + d'(x;n)] - \tau(x)d(x;n). \tag{63}$$

Analogously, if we take derivatives in (62) and use (2) and (60), we obtain

$$\sigma^2(x)\frac{d^2}{dx^2}B_n^{\alpha,M_0,M_1}(x) = g(x;n)B_n^\alpha(x) + h(x;n)\frac{d}{dx}B_n^\alpha(x), \tag{64}$$

where

$$\begin{aligned}
g(x;n) &= \sigma(x)e'(x;n) - \sigma'(x)e(x;n) - \lambda_n f(x;n), \\
h(x;n) &= \sigma(x)[e(x;n) + f'(x;n)] - 2\sigma'(x)f(x;n) - \tau(x)f(x;n).
\end{aligned} \tag{65}$$

The above expressions (60), (62), and (64) yield

$$\begin{vmatrix}
B_n^{\alpha,M_0,M_1}(x) & c(x;n) & d(x;n) \\
\sigma(x)\dfrac{d}{dx}B_n^{\alpha,M_0,M_1}(x) & e(x;n) & f(x;n) \\
\sigma^2(x)\dfrac{d^2}{dx^2}B_n^{\alpha,M_0,M_1}(x) & g(x;n) & h(x;n)
\end{vmatrix} = 0. \tag{66}$$

Expanding the determinant in (66) by the first column, we find the polynomials $B_n^{\alpha,M_0,M_1}(x)$ satisfy a second order differential equation of the form

$$\tilde{\sigma}(x;n)\frac{d^2}{dx^2}B_n^{\alpha,M_0,M_1}(x) + \tilde{\tau}(x;n)\frac{d}{dx}B_n^{\alpha,M_0,M_1}(x) + \tilde{\lambda}(x;n)B_n^{\alpha,M_0,M_1}(x) = 0, \tag{67}$$

where

$$\begin{aligned}
\tilde{\sigma}(x;n) = \quad -x^2 \Big\{ &\left(\zeta_n^{\alpha,M_0,M_1} - 1\right)x^2 + \Big[\zeta_n^{\alpha,M_0,M_1}(2n+\alpha) + \eta_n^{\alpha,M_0,M_1}\left(2 + \alpha + (2n+\alpha)n\zeta_n^{\alpha,M_0,M_1}\right) \\
&-2\zeta_n^{\alpha,M_0,M_1}\left(n\zeta_n^{\alpha,M_0,M_1} - 1\right)\Big]x + \Big[n\eta_n^{\alpha,M_0,M_1}(n+\alpha+1) + 2\left(1 - n\zeta_n^{\alpha,M_0,M_1}\right)\Big] \Big\},
\end{aligned}$$

$$\tilde{\tau}(x;n) = -(\alpha+2)\left[\zeta_n^{\alpha,M_0,M_1} - 1 + (2n+\alpha)\zeta_n^{\alpha,M_0,M_1}\right]x^3$$

$$+\left\{2\left[1 - \zeta_n^{\alpha,M_0,M_1}\left(4+2\alpha+2n - 3n\zeta_n^{\alpha,M_0,M_1} - \alpha n\zeta_n^{\alpha,M_0,M_1}\right)\right]\right.$$

$$\left. -\eta_n^{\alpha,M_0,M_1}(\alpha+3)\left(2+\alpha+\alpha n\zeta_n^{\alpha,M_0,M_1} + 2n^2\zeta_n^{\alpha,M_0,M_1}\right)\right\}x^2$$

$$-\left[4\eta_n^{\alpha,M_0,M_1}\left(3+\alpha - 2n\zeta_n^{\alpha,M_0,M_1} + n^2\zeta_n^{\alpha,M_0,M_1}\right) + 4\zeta_n^{\alpha,M_0,M_1}\left(n\zeta_n^{\alpha,M_0,M_1} - 1\right)\right.$$

$$\left. +\eta_n^{\alpha,M_0,M_1\,2}n(n+\alpha+1)(\alpha+4)\right]x$$

$$-2\eta_n^{\alpha,M_0,M_1}\left[\eta_n^{\alpha,M_0,M_1}n(n+\alpha+1) - 2n\zeta_n^{\alpha,M_0,M_1} + 2\right]$$

$$\tilde{\lambda}(x;n) = n(n+\alpha+1)\left\{\eta_n^{\alpha,M_0,M_1\,2}(n-1)(n+\alpha+2) + 2\eta_n^{\alpha,M_0,M_1}\left(\zeta_n^{\alpha,M_0,M_1}+1\right)\right.$$

$$-2n\eta_n^{\alpha,M_0,M_1}\zeta_n^{\alpha,M_0,M_1} + \left[2\zeta_n^{\alpha,M_0,M_1}\left(\zeta_n^{\alpha,M_0,M_1} - n\zeta_n^{\alpha,M_0,M_1} + 1\right)\right.$$

$$\left. +\eta_n^{\alpha,M_0,M_1}\left(4+\alpha - \alpha\zeta_n^{\alpha,M_0,M_1} - 2n\zeta_n^{\alpha,M_0,M_1} + \alpha n\zeta_n^{\alpha,M_0,M_1} + 2n^2\zeta_n^{\alpha,M_0,M_1}\right)\right]x$$

$$\left. +\left[\zeta_n^{\alpha,M_0,M_1}(2n+\alpha+1) - 1\right]\right\}.$$

The explicit expression for the coefficients in (67) has been obtained by using the algorithm developed in [4]. A simple calculation shows that for M_0, M_1 equal to zero, we recover from the above expressions the SODE for the classical Bessel polynomials.

7   Three-term recurrence relation for $B_n^{\alpha,M_0,M_1}(x)$.

Theorem 1 *Assume that $\triangle_n^{\alpha,M_0,M_1} \neq 0$, $n \geq 0$, i.e., $\mathcal{L}_1$ is quasi-definite. Then the MOPS $\{B_n^\alpha(x)\}_0^\infty$ related to $\mathcal{L}_1$ satisfies a three-term recurrence relation (TTRR)*

$$B_{n+1}^{\alpha,M_0,M_1}(x) = \left(x - \beta_n^{\alpha,M_0,M_1}\right)B_n^{\alpha,M_0,M_1}(x) - \gamma_n^{\alpha,M_0,M_1}B_{n-1}^{\alpha,M_0,M_1}(x), \quad n \geq 1. \qquad (68)$$

Proof: Since $xB_n^{\alpha,M_0,M_1}(x)$ is a polynomial of degree $n+1$, we have

$$xB_n^{\alpha,M_0,M_1}(x) = B_{n+1}^{\alpha,M_0,M_1}(x) + \beta_n^{\alpha,M_0,M_1}B_n^{\alpha,M_0,M_1}(x) +$$

$$+ \gamma_n^{\alpha,M_0,M_1}B_{n-1}^{\alpha,M_0,M_1}(x) + \sum_{k=0}^{n-2}C_k^n x^k,$$

where C_k^n, $k = 1, 2, ..., n-2$, are real coefficients. Taking the indefinite inner product $(\cdot, \cdot)$ associated with the functional $\mathcal{L}_1$ $(\alpha \neq -1, -2, -3, ...)$

$$(p,q) = \int_\Gamma p(z)q(z)\rho_0^\alpha(z)\,dz + M_0 p(0)q(0) + M_1\left(p(z)q(z)\right)'\big|_{z=0}, \qquad (69)$$

multiplying by x^m both sides of the above expression, and using the orthogonality of the generalized polynomials $B_n^{\alpha,M_0,M_1}(x)$ we find

$$0 = (B_n^{\alpha,M_0,M_1}(x),\, x^{m+1}) = \sum_{k=0}^{n-2} C_k^n (1, x^{m+k}), \qquad m = 0, 1, ..., n-2.$$

Since the determinant of the the above linear system in C_k^n is different from zero (it is, basically the Gram determinant of order $n-1$ for the indefinite inner product (69), see [14, Section **2.2** pages 25-28]), then we deduce that $C_k^n = 0$ for all $k = 0, 1, ..., n-2$.
Thus,

$$xB_n^{\alpha,M_0,M_1}(x) = B_{n+1}^{\alpha,M_0,M_1} + \beta_n^{\alpha,M_0,M_1} B_n^{\alpha,M_0,M_1}(x) + \gamma_n^{\alpha,M_0,M_1} B_{n-1}^{\alpha,M_0,M_1}(x). \tag{70}$$

Let us to obtain the coefficients β_n^{α,M_0,M_1} and $\gamma_n^{\alpha,M_0,M_1}$ in (70). Let $\tilde{b}_n$ be the coefficient of x^{n-1} in the expansion $B_n^{\alpha,M_0,M_1} = x^n + \tilde{b}_n\, x^{n-1} + ...$. Then, comparing the coefficients of x^n in the two hand sides of (70) we find $\beta_n^{\alpha,M_0,M_1} = \tilde{b}_n - \tilde{b}_{n+1}$. To calculate $\gamma_n^{\alpha,M_0,M_1}$ it is sufficient to evaluate (70) in $x = 0$ and remark that $B_{n+1}^{\alpha,M_0,M_1}(0) \neq 0$ ($n >> 1$).

In order to obtain a general expression for the coefficient β_n^{α,M_0,M_1} we can use the representation formula (41) for the generalized polynomials, in terms of $B_n^{\alpha}(x)$ and $(B_n^{\alpha}(x))'$

$$B_n^{\alpha,M_0,M_1}(x) = [1 - n\zeta_n^{\alpha,M_0,M_1}]B_n^{\alpha}(x) + [\eta_n^{\alpha,M_0,M_1} + x\zeta_n^{\alpha,M_0,M_1}](B_n^{\alpha}(x))', \tag{71}$$

Notice that the constants ζ_n and η_n depend on n, α, M_0, M_1 (see formula (42)) . Doing some algebraic calculations we find that

$$\tilde{b}_n = (1 - \zeta_n)b_n + \eta_n,$$

where b_n denotes the coefficient of x^{n-1} for the classical monic Bessel polynomials $B_n^{\alpha}(x)$ (see formula (16)), i.e.,

$$b_n = \frac{2(n-1)}{2n + \alpha}.$$

Thus, we obtain the following TTRR coefficients for generalized polynomials

$$\beta_n^{\alpha,M_0,M_1} = (1 - \zeta_n)b_n - (1 - \zeta_{n+1})b_{n+1} + (\eta_n - \eta_{n+1}), \tag{72}$$

and

$$\gamma_n^{\alpha,M_0,M_1} = -\frac{B_{n+1}^{\alpha,M_0,M_1}(0)}{B_{n-1}^{\alpha,M_0,M_1}(0)} - \beta_n^{\alpha,M_0,M_1}\frac{B_n^{\alpha,M_0,M_1}(0)}{B_{n-1}^{\alpha,M_0,M_1}(0)}. \tag{73}$$

∎

8   Quasi-orthogonality and zeros of $B_n^{\alpha,M_0,M_1}(x)$.

In this section we will study the quasi-orthogonality of the generalized Bessel polynomials as well as some properties of their zeros.

8.1   The Quasi-orthogonality.

Because of $x^2\mathcal{L}_1 = x^2\mathcal{L}_0$, we get

Proposition 3

$$x^2 B_n^{\alpha,M_0,M_1}(x) = \sum_{j=n-2}^{n+2} c_{n,j} B_j^{\alpha}(x), \quad n \geq 2, \tag{74}$$

where

$$c_{n,j} = \frac{<\mathcal{L}_0, x^2 B_n^{\alpha,M_0,M_1}(x)B_j^{\alpha}(x)>}{<\mathcal{L}_0, (B_j^{\alpha}(x))^2>} = \frac{<\mathcal{L}_1, x^2 B_n^{\alpha,M_0,M_1}(x)B_j^{\alpha}(x)>}{<\mathcal{L}_0, (B_j^{\alpha}(x))^2>}, \quad n-2 \leq j \leq n+2, \tag{75}$$

$$c_{n,n-2} \neq 0.$$

Proof: From

$$x^2 B_n^{\alpha,M_0,M_1}(x) = \sum_{j=0}^{n+2} c_{n,j} B_j^\alpha(x), \tag{76}$$

we deduce

$$c_{n,k} < \mathcal{L}_0, (B_k^\alpha(x))^2 > = < \mathcal{L}_0, x^2 B_n^{\alpha,M_0,M_1}(x) B_k^\alpha(x) > = < x^2 \mathcal{L}_0, B_n^{\alpha,M_0,M_1}(x) B_k^\alpha(x) > =$$

$$= < x^2 \mathcal{L}_1, B_n^{\alpha,M_0,M_1}(x) B_k^\alpha(x) > = \tag{77}$$

$$= < \mathcal{L}_1, B_n^{\alpha,M_0,M_1}(x) x^2 B_k^\alpha(x) > = 0, \quad k < n-2. \qquad \blacksquare$$

8.2 Some properties of the zeros.

Let us denote $(\alpha_{k,n})_{k=1}^n$ the zeros of the polynomials $B_n^{\alpha,M_0,M_1}(x)$ for n large enough.

Lemma 1 (Kakeya [6, page 77]) *Let* $p_n(z) = \sum_{j=0}^n b_j z^j$ *be a polynomial of degree* n *with positive coefficients and set* $\min_j \dfrac{b_j}{b_{j+1}} = a_1$, $\max_j \dfrac{b_j}{b_{j+1}} = a_2$. *Then all zeros* $\alpha_{k,n}$ *of* $p_n(z)$ *satisfy the inequalities* $a_1 \le |\alpha_{k,n}| \le a_2$.

In our case, for n large enough, by using the asymptotic formulas (57) we find

$$B_n^{\alpha,M_0,M_1}(x) = x^n + \frac{2^n}{(\alpha+n+1)_n} \sum_{k=0}^{n-1} \binom{n}{k} (n+\alpha+1)_k \times$$

$$\times \left[1 + \frac{8M_1(-1)^n}{2^{3(\alpha+1)} n^{\alpha-6}} \left(\frac{e}{2n} \right)^{4n} + o\left(\frac{e^n}{2^n n^{4n} + \alpha - 6} \right) \right] \left(\frac{x}{2} \right)^k.$$

or, equivalently, using (44) and (57) one deduces that the product $(k+\beta_0)(k+\beta_1)$ for n large enough is positive.

Then, for n large enough, the ratio satisfies the condition

$$\frac{b_j}{b_{j+1}} = \frac{2(k+1)}{(n-k)(n+\alpha+k+1)},$$

as well as

$$\frac{2}{n(n+\alpha+1)} < \frac{b_j}{b_{j+1}} < \frac{2n}{2n+\alpha+1}, \quad \alpha > -1,$$

which yields $|\alpha_{k,n}| \le 1$. $\qquad \blacksquare$

Then, the following proposition holds.

Proposition 4 *If* $\mathcal{L}_1$ *is assumed to be a quasi-definite moment functional all the zeros of* $B_n^{\alpha,M_0,M_1}(x)$ *for* n *large enough are located inside the unit circle.*

Proof: The statement is a consequence of the previous Lemma and the asymptotic relation (57). In fact, by using the representation (60). $\qquad \blacksquare$

9 Perturbation matrix and eigenvalue problem.

From

$$x^2 \mathcal{L}_1 = x^2 \mathcal{L}_0, \tag{78}$$

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if $\{B_n^\alpha(x)\}_0^\infty$, $\{B_n^{\alpha,M_0,M_1}(x)\}_0^\infty$, and $\{B_n^{\alpha+2}(x)\}_0^\infty$ denote the orthogonal polynomial sequences with respect to $\mathcal{L}_0$, $\mathcal{L}_1$, and $x^2\mathcal{L}_0$, respectively, we can write $B_n^{\alpha,M_0,M_1}(x)$ in terms of $B_n^{\alpha+2}(x)$ as in (44). In matrix form

$$
\begin{pmatrix}
B_0^{\alpha,M_0,M_1}(x) \\
B_1^{\alpha,M_0,M_1}(x) \\
B_2^{\alpha,M_0,M_1}(x) \\
\vdots \\
B_n^{\alpha,M_0,M_1}(x)
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
a_1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
b_2 & a_2 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & b_n & a_n & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
B_0^{\alpha+2}(x) \\
B_1^{\alpha+2}(x) \\
B_2^{\alpha+2}(x) \\
\vdots \\
B_n^{\alpha+2}(x)
\end{pmatrix},
\tag{79}
$$

or, equivalently,

$$
\mathbb{Q}_n = \mathsf{T}_n \cdot \bar{\mathbb{P}}_n,
\tag{80}
$$

where $\mathbb{Q}_n$ and $\bar{\mathbb{P}}_n$ denote the column vectors of (79), respectively.

On the other hand, from *the three-term recurrence relation* that $\{B_n^{\alpha,M_0,M_1}(x)\}_0^\infty$ and $\{B_n^{\alpha+2}(x)\}_0^\infty$ satisfy, one gets the matrix representation

$$
x\mathbb{Q}_n =
\underbrace{
\begin{pmatrix}
\tilde{\beta}_0 & 1 & 0 & \ldots & 0 \\
\tilde{\gamma}_1 & \tilde{\beta}_1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \tilde{\beta}_n
\end{pmatrix}
}_{\tilde{\mathbf{J}}_n}
\cdot \mathbb{Q}_n + B_{n+1}^{\alpha,M_0,M_1}(x)
\begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\tag{81}
$$

$$
x\bar{\mathbb{P}}_n =
\underbrace{
\begin{pmatrix}
\bar{\beta}_0 & 1 & 0 & \ldots & 0 \\
\bar{\gamma}_1 & \bar{\beta}_1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \bar{\beta}_n
\end{pmatrix}
}_{\bar{\mathbf{J}}_n}
\cdot \bar{\mathbb{P}}_n + B_{n+1}^{\alpha+2}(x)
\begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}.
\tag{82}
$$

Substituting (80) into (81) we have

$$
\begin{aligned}
x\mathsf{T}_n \cdot \bar{\mathbb{P}}_n &= \tilde{\mathbf{J}}_n \cdot \mathsf{T}_n \cdot \bar{\mathbb{P}}_n +
\begin{pmatrix}
& & \mathsf{O} & & \\
& & \ldots & & \\
0 & \ldots & 0 & b_{n+1} & a_{n+1} & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
\bar{\mathbb{P}}_n \\
\ldots \\
B_{n+1}^{\alpha,M_0,M_1}(x)
\end{pmatrix}
= \\
&= \left[\tilde{\mathbf{J}}_n \cdot \mathsf{T}_n +
\begin{pmatrix}
& & \mathsf{O} & \\
& & \ldots & \\
0 & \ldots & 0 & b_{n+1} & a_{n+1}
\end{pmatrix}
\right] \cdot \bar{\mathbb{P}}_n + B_{n+1}^{\alpha,M_0,M_1}(x)
\begin{pmatrix}
0 \\
\vdots \\
1
\end{pmatrix},
\end{aligned}
\tag{83}
$$

where the matrices $\tilde{\mathbf{J}}_n$ and $\bar{\mathbf{J}}_n$ are defined in (81)-(82) and O is the null matrix. Therefore

$$
\begin{aligned}
x\bar{\mathbb{P}}_n &= \mathsf{T}_n^{-1} \cdot \left[\tilde{\mathbf{J}}_n \cdot \mathsf{T}_n +
\begin{pmatrix}
& & \mathsf{O} & \\
& & \ldots & \\
0 & \ldots & 0 & b_{n+1} & a_{n+1}
\end{pmatrix}
\right] \cdot \bar{\mathbb{P}}_n + B_{n+1}^{\alpha,M_0,M_1}(x)\mathsf{T}_n^{-1} \cdot
\begin{pmatrix}
0 \\
\vdots \\
1
\end{pmatrix}
= \\
&= \left[\mathsf{T}_n^{-1} \cdot \tilde{\mathbf{J}}_n \cdot \mathsf{T}_n + \mathsf{T}_n^{-1} \cdot
\begin{pmatrix}
& & \mathsf{O} & \\
& & \ldots & \\
0 & \ldots & 0 & b_{n+1} & a_{n+1}
\end{pmatrix}
\right] \cdot \bar{\mathbb{P}}_n + B_{n+1}^{\alpha,M_0,M_1}(x)
\begin{pmatrix}
0 \\
\vdots \\
1
\end{pmatrix}.
\end{aligned}
\tag{84}
$$

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Taking into account (82) we deduce that

$$\bar{\mathbf{J}}_n = \mathsf{T}_n^{-1} \cdot \left[\tilde{\mathbf{J}}_n + \begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ 0 & \ldots & 0 & b_{n+1} & a_{n+1} \end{pmatrix} \cdot \mathsf{T}_n^{-1} \right] \cdot \mathsf{T}_n =$$

$$= \mathsf{T}_n^{-1} \cdot \left[\tilde{\mathbf{J}}_n + \begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ a_{11} & a_{12} & \ldots & a_{1,n+1} \end{pmatrix} \right] \cdot \mathsf{T}_n .$$

(85)

Thus, $\bar{\mathbf{J}}_n$ is, essentially, a rank one perturbation of the matrix $\tilde{\mathbf{J}}_n$. From (85) notice that

$$\begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ 0 & \ldots & 0 & b_{n+1} & a_{n+1} \end{pmatrix} = \begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ a_{11} & a_{12} & \ldots & a_{1,n+1} \end{pmatrix} \cdot \mathsf{T}_n =$$

$$= \begin{pmatrix} & & \mathsf{O} & \\ & \ldots & \\ a_{11} & a_{12} & \ldots & a_{1,n+1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_1 & \ddots & 0 & \ldots & 0 \\ b_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & b_n & a_n & 1 \end{pmatrix} .$$

(86)

This means

$$\begin{cases} a_{1,n+1} = a_{n+1}, \\ a_{1,n} + a_{1,n+1} a_n = b_{n+1}, \\ a_{1,n-1} + a_{1,n} a_{n-1} + a_{1,n+1} b_n = 0, \\ a_{1,n-2} + a_{1,n-1} a_{n-2} + a_{1,n} b_{n-1} = 0, \\ \qquad \ldots \\ a_{1,1} + a_{1,2} a_1 + a_{1,3} b_2 = 0. \end{cases} \Leftrightarrow \begin{cases} a_{1,n+1} = a_{n+1}, \\ a_{1,n} = b_{n+1} - a_{n+1} a_n, \\ a_{1,n-1} = -a_{n+1} b_n - a_{n-1}[b_{n+1} - a_{n+1} a_n], \\ \qquad = -b_n a_{n+1} - b_{n+1} a_{n-1} + a_{n-1} a_n a_{n+1}, \\ \qquad \ldots \end{cases} .$$

(87)

Then, for the rank one matrix, the entries of the last row can be generated in a straightforward way. Let

$$\mathsf{A}_n = \mathsf{T}_n^{-1} \cdot \begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ 0 & \ldots & 0 & b_{n+1} & a_{n+1} \end{pmatrix} .$$

(88)

Taking into account that T_n is a lower triangular matrix, we deduce that the product matrix has the same structure as the matrix of the second factor on the previous expression, i.e,

$$\mathsf{A}_n = \begin{pmatrix} & & \mathsf{O} & & \\ & & \ldots & & \\ 0 & \ldots & 0 & b_{n+1} & a_{n+1} \end{pmatrix} ,$$

(89)

Hence

$$\bar{\mathbf{J}}_n = \mathsf{T}_n^{-1} \cdot \tilde{\mathbf{J}}_n \cdot \mathsf{T}_n + \mathsf{A}_n .$$

(90)

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