Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions

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Abstract

We consider the general theory of the modifications of quasi-definite linear functionals by adding discrete measures. We analyze the existence of the corresponding orthogonal polynomial sequences with respect to such linear functionals. The three-term recurrence relation, lowering and raising operators as well as the second order linear differential equation that the sequences of monic orthogonal polynomials satisfy when the linear functional is semiclassical are also established. A relevant example is considered in details.

1 Introduction.

Let $\mathfrak{u}$ be a linear functional in the vector space $\mathbb{P}$ of polynomials with complex coefficients. This functional is said to be quasi-definite [8] if the principal submatrices of the infinite Hankel matrix associated with the sequence of the moments $(u_n)_n$ of the linear functional $\mathfrak{u}$ are non-singular. Notice that $u_n = \langle \mathfrak{u}, x^n \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the *duality bracket.*

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Assuming $\mathfrak{u}$ is a quasi-definite linear functional, there exists a sequence of monic polynomials $(P_n)_n$ with $\deg P_n = n$, such that

$$\langle \mathfrak{u}, P_n\, P_m\rangle = k_n \delta_{n,m}, \qquad k_n \neq 0.$$

We introduce a new linear functional

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{i=1}^{M} A_i \delta(x - a_i) - \sum_{j=1}^{N} B_j \delta'(x - b_j), \tag{1.1}$$

where $(A_i)_{i=1}^{M}$ and $(B_j)_{j=1}^{N}$ are real numbers, $\delta(x-y)$ and $\delta'(x-y)$ mean the Dirac linear functional and its derivative, respectively, defined by

$$\langle \delta(x-y), p(x)\rangle = p(y), \qquad \langle \delta'(x-y), p(x)\rangle = -p'(y), \qquad \forall p \in \mathbb{P}.$$

Such a kind of linear functionals have been studied intensively in the last ten years. For instance, if $B_i = 0$, $i = 1, 2, \ldots, N$ and $\mathfrak{u}$ is a classical linear functional (Hermite, Laguerre, Jacobi, and Bessel) necessary and sufficient conditions in order to $\widetilde{\mathfrak{u}}$ be quasi-definite are given in [3]. Notice that in the Laguerre case (with $M = 1$ and $a_1 = 0$), Jacobi (with $M = 2$ and $a_1 = 1$, $a_2 = -1$), Bessel case (with $M = 1$ and $a_1 = 0$) we get the so-called Krall-type orthogonal polynomials which are related to the spectral analysis of linear differential operators (see e.g. [10, 15]) and also appear in the Darboux transformation on the Jacobi matrix associated with the three-term recurrence relation which such polynomials satisfy (see [12, 13], as well as [14] for some iteration of the Darboux transformation).

If some B_i is nonzero, then a general approach to the quasi-definiteness of the linear functional $\widetilde{\mathfrak{u}}$ is presented in [6]. For the case of classical functionals we have studied the corresponding sequences of orthogonal polynomials in several papers: For the Laguerre functional see [1, 2], for the Bessel linear functional see [4] and, finally, for the Jacobi linear functional see [5]. Notice that in these cases the points where the Dirac linear functionals are supported are intimately related to the end points of the support of $\mathfrak{u}$.

Very recently and following a multiple integral representation of Heine for Hankel matrices as well as the corresponding sequences of orthogonal polynomials, in [9] a similar study has been done. Surprisingly, the authors do not refer the above contributions and even, they do not discuss the quasi-definite character of the linear functional $\widetilde{\mathfrak{u}}$. Basically they obtain our original results using a different approach.

The aim of this contribution is to analyze the linear functional $\widetilde{\mathfrak{u}}$ introduced in (1.1) and to find necessary and sufficient conditions for its quasi-definiteness. Next, we deduce an explicit representation of the corresponding sequence of monic orthogonal polynomials $(\widetilde{P}_n)_n$. This constitutes the contents of the section 2. In section 3 we obtain the coefficients of the three-term recurrence relation. In section 4 we assume $\mathfrak{u}$ is a semiclassical linear functional and then we deduce a raising as well as a lowering operator associated with $\widetilde{\mathfrak{u}}$. In section 5 we deduce a second order linear differential equation from the above results. Finally, in section 6 an example related to the Hermite polynomials is worked out.

2 Definition of the functional $\widetilde{\mathfrak{u}}$.

Consider the linear functional $\widetilde{\mathfrak{u}}$ given in (1.1), i.e.,

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{i=1}^{M} A_i \delta(x-a_i) - \sum_{j=1}^{N} B_j \delta'(x-b_j).$$

If $\widetilde{\mathfrak{u}}$ is quasi-definite then there exists a sequence of monic polynomials $(\widetilde{P}_n)_n$ orthogonal with respect to $\widetilde{\mathfrak{u}}$ and therefore we can consider the Fourier expansion

$$\widetilde{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} \lambda_{n,k} P_k(x). \tag{2.1}$$

Then, for $0 \le k \le n-1$,

$$\lambda_{n,k} = \frac{\langle \mathfrak{u}, \widetilde{P}_n(x) P_k(x) \rangle}{\langle \mathfrak{u}, P_k^2(x) \rangle} = -\frac{\sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) P_k(a_i) + \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) P_k'(b_j) + \sum_{j=1}^{N} B_j \widetilde{P}_n'(b_j) P_k(b_j)}{\langle \mathfrak{u}, P_k^2(x) \rangle}$$

$$= -\sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \frac{P_k(a_i)}{\langle \mathfrak{u}, P_k^2(x) \rangle} - \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) \frac{P_k'(b_j)}{\langle \mathfrak{u}, P_k^2(x) \rangle} - \sum_{j=1}^{N} B_j \widetilde{P}_n'(b_j) \frac{P_k(b_j)}{\langle \mathfrak{u}, P_k^2(x) \rangle}.$$

Thus, (2.1) becomes

$$\widetilde{P}_n(x) = P_n(x) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}(x,a_i) - \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) \mathrm{K}_{n-1}^{(0,1)}(x,b_j) - \sum_{j=1}^{N} B_j \widetilde{P}_n'(b_j) \mathrm{K}_{n-1}(x,b_j), \tag{2.2}$$

where, as usual, we denote

$$\mathrm{K}_n^{(i,j)}(x,y) = \sum_{l=0}^{n} \frac{P_l^{(i)}(x) P_l^{(j)}(y)}{\langle \mathfrak{u}, P_l^2(x) \rangle}, \qquad P_l^{(i)}(x) := \frac{d^i}{dx^i} P_l(x), \quad i,j \in \mathbb{N}.$$

For a sake of simplicity $\mathrm{K}_n^{(0,0)}(x,y) := \mathrm{K}_n(x,y)$ denotes the reproducing kernel associated with the linear functional $\mathfrak{u}$. It is very well known that $\langle \mathfrak{u}, \mathrm{K}_n(x,y) p(x) \rangle = p(y)$ for every polynomial $p(x)$ of degree less than or equal to n.

If we evaluate (2.2) for $x = a_k$, $k = 1, 2, \ldots, M$ we get

$$\begin{aligned} \widetilde{P}_n(a_k) = \quad & P_n(a_k) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}(a_k,a_i) - \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) \mathrm{K}_{n-1}^{(0,1)}(a_k,b_j) \\ & - \sum_{j=1}^{N} B_j \widetilde{P}_n'(b_j) \mathrm{K}_{n-1}(a_k,b_j). \end{aligned} \tag{2.3}$$

A similar evaluation in (2.2) for $x = b_k$, $k = 1, 2, \dots, N$, yields

$$\begin{aligned} \widetilde{P}_n(b_k) = \quad & P_n(b_k) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}(b_k, a_i) - \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) \mathrm{K}_{n-1}^{(0,1)}(b_k, b_j) \\ & - \sum_{j=1}^{N} B_j \widetilde{P}'_n(b_j) \mathrm{K}_{n-1}(b_k, b_j). \end{aligned} \tag{2.4}$$

Finally, taking derivatives in (2.2) and evaluating the resulting expression for $x = b_k$, $k = 1, 2, \dots, N$, we obtain

$$\begin{aligned} \widetilde{P}'_n(b_k) = \quad & P'_n(b_k) - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i) \mathrm{K}_{n-1}^{(1,0)}(b_k, a_i) - \sum_{j=1}^{N} B_j \widetilde{P}_n(b_j) \mathrm{K}_{n-1}^{(1,1)}(b_k, b_j) \\ & - \sum_{j=1}^{N} B_j \widetilde{P}'_n(b_j) \mathrm{K}_{n-1}^{(1,0)}(b_k, b_j). \end{aligned} \tag{2.5}$$

Thus we get a system of $M + 2N$ linear equations in the variables $(\widetilde{P}_n(a_k))_{k=1}^{M}$, $(\widetilde{P}_n(b_k))_{k=1}^{N}$ and $(\widetilde{P}'_n(b_k))_{k=1}^{N}$.

In order to simplify the above expressions we introduce the following notations (A^T is the transpose of A):

$$\mathcal{P}_n(\vec{z}) = (P_n(z_1), P_n(z_2), \dots, P_n(z_k))^T, \qquad \mathcal{P}'_n(\vec{z}) = (P'_n(z_1), P'_n(z_2), \dots, P'_n(z_k))^T,$$

where $\vec{z} = (z_1, z_2, \cdots, z_k)^T$. Also we introduce the matrices $\mathcal{K}_{n-1}^{(i,j)}(\vec{z}, \vec{y}) \in \mathbb{C}^{p \times q}$ whose (m, n) entry is $\mathrm{K}_{n-1}^{(i,j)}(z_m, y_n)$. Here $\vec{z} = (z_1, z_2, \dots, z_p)$ and $\vec{y} = (y_1, y_2, \dots, y_q)$. Finally, we introduce the matrices associated with the mass points $A = \operatorname{diag}(A_1, A_2, \dots, A_M)$ and $B = \operatorname{diag}(B_1, B_2, \dots, B_N)$.

With these notations the linear system of equations (2.3), (2.4) and (2.5) becomes

$$\begin{pmatrix} \widetilde{\mathcal{P}}_n(\vec{a}) \\ \widetilde{\mathcal{P}}_n(\vec{b}) \\ \widetilde{\mathcal{P}}'_n(\vec{b}) \end{pmatrix} = \begin{pmatrix} \mathcal{P}_n(\vec{a}) \\ \mathcal{P}_n(\vec{b}) \\ \mathcal{P}'_n(\vec{b}) \end{pmatrix} - \mathbb{K}_{n-1} \mathrm{D} \begin{pmatrix} \widetilde{\mathcal{P}}_n(\vec{a}) \\ \widetilde{\mathcal{P}}_n(\vec{b}) \\ \widetilde{\mathcal{P}}'_n(\vec{b}) \end{pmatrix}, \tag{2.6}$$

where $\vec{a} = (a_1, a_2, \dots, a_M)^T$, $\vec{b} = (b_1, b_2, \dots, b_N)^T$, and

$$\mathrm{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix}, \qquad \mathbb{K}_{n-1} = \begin{pmatrix} \mathcal{K}_{n-1}(\vec{a}, \vec{a}) & \mathcal{K}_{n-1}(\vec{a}, \vec{b}) & \mathcal{K}_{n-1}^{(0,1)}(\vec{a}, \vec{b}) \\ \mathcal{K}_{n-1}(\vec{b}, \vec{a}) & \mathcal{K}_{n-1}(\vec{b}, \vec{b}) & \mathcal{K}_{n-1}^{(0,1)}(\vec{b}, \vec{b}) \\ \mathcal{K}_{n-1}^{(1,0)}(\vec{b}, \vec{a}) & \mathcal{K}_{n-1}^{(0,1)}(\vec{b}, \vec{b}) & \mathcal{K}_{n-1}^{(1,1)}(\vec{b}, \vec{b}) \end{pmatrix},$$

or, equivalently,

$$\begin{pmatrix} \widetilde{\mathcal{P}}_n(\vec{a}) \\ \widetilde{\mathcal{P}}_n(\vec{b}) \\ \widetilde{\mathcal{P}}'_n(\vec{b}) \end{pmatrix} = \begin{pmatrix} \mathcal{P}_n(\vec{a}) \\ \mathcal{P}_n(\vec{b}) \\ \mathcal{P}'_n(\vec{b}) \end{pmatrix} - \begin{pmatrix} \mathcal{K}_{n-1}(\vec{a}, \vec{a}) A \widetilde{\mathcal{P}}_n(\vec{a}) + \mathcal{K}_{n-1}^{(0,1)}(\vec{a}, \vec{b}) B \widetilde{\mathcal{P}}_n(\vec{b}) + \mathcal{K}_{n-1}(\vec{a}, \vec{b}) B \widetilde{\mathcal{P}}'_n(\vec{b}) \\ \mathcal{K}_{n-1}(\vec{b}, \vec{a}) A \widetilde{\mathcal{P}}_n(\vec{a}) + \mathcal{K}_{n-1}^{(0,1)}(\vec{b}, \vec{b}) B \widetilde{\mathcal{P}}_n(\vec{b}) + \mathcal{K}_{n-1}(\vec{b}, \vec{b}) B \widetilde{\mathcal{P}}'_n(\vec{b}) \\ \mathcal{K}_{n-1}^{(1,0)}(\vec{b}, \vec{a}) A \widetilde{\mathcal{P}}_n(\vec{a}) + \mathcal{K}_{n-1}^{(1,1)}(\vec{b}, \vec{b}) B \widetilde{\mathcal{P}}_n(\vec{b}) + \mathcal{K}_{n-1}^{(0,1)}(\vec{b}, \vec{b}) B \widetilde{\mathcal{P}}'_n(\vec{b}) \end{pmatrix}.$$

Notice that all the involved block matrices have the appropriate dimensions. Thus, if the matrix $\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D}$, where I is the identity matrix, is non singular, then we get the existence and uniqueness for the solution of (2.6).

In such a case (2.2) becomes

$$\widetilde{P}_n(x) = P_n(x) - (\,\mathcal{K}^T_{n-1}(x,\vec{a}), \mathcal{K}^T_{n-1}(x,\vec{b}), \mathcal{K}^{(0,1)^T}_{n-1}(x,\vec{a})\,)\,\mathrm{D}(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}'_n(\vec{b})\end{pmatrix}. \qquad (2.7)$$

On the other hand, taking into account the quasi-definite character of $\widetilde{\mathbf{u}}$, we get

$$0 \neq \langle \widetilde{\mathbf{u}}, \widetilde{P}^2_n(x)\rangle = \langle \widetilde{\mathbf{u}}, \widetilde{P}_n(x)P_n(x)\rangle = \langle \mathbf{u}, \widetilde{P}_n(x)P_n(x)\rangle + \sum_{i=1}^{M} A_i\widetilde{P}_n(a_i)P_n(a_i)$$
$$+\sum_{j=1}^{N} B_j\left(\widetilde{P}'_n(b_j)P_n(b_j)+\widetilde{P}_n(b_j)P'_n(b_j)\right).$$

Using the previous notations, as well as $\langle \mathbf{u}, \widetilde{P}_n(x)P_n(x)\rangle = \langle \mathbf{u}, P^2_n(x)\rangle$, the above expression becomes

$$\langle \widetilde{\mathbf{u}}, \widetilde{P}^2_n(x)\rangle = \langle \mathbf{u}, P^2_n(x)\rangle + (\,\mathcal{P}^T_n(\vec{a}), \mathcal{P}^T_n(\vec{b}), \mathcal{P}'^T_n(\vec{b})\,)\,\mathrm{D}\begin{pmatrix}\widetilde{\mathcal{P}}_n(\vec{a})\\ \widetilde{\mathcal{P}}_n(\vec{b})\\ \widetilde{\mathcal{P}}'_n(\vec{b})\end{pmatrix} \qquad (2.8)$$

$$= \langle \mathbf{u}, P^2_n(x)\rangle + (\,\mathcal{P}^T_n(\vec{a}), \mathcal{P}^T_n(\vec{b}), \mathcal{P}'^T_n(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}'_n(\vec{b})\end{pmatrix}.$$

Thus, as a conclusion, we have proved that if $\widetilde{\mathbf{u}}$ is a quasi-definite linear functional then the following conditions hold:

1. The matrix $\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D}$ is nonsingular for every $n\in\mathbb{N}$, i.e.

$$\det(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D}) \neq 0, \qquad \forall n\in\mathbb{N}. \qquad (2.9)$$

2. For all $n\in\mathbb{N}$

$$\langle \widetilde{\mathbf{u}}, \widetilde{P}^2_n(x)\rangle = \langle \mathbf{u}, P^2_n(x)\rangle + (\,\mathcal{P}^T_n(\vec{a}), \mathcal{P}^T_n(\vec{b}), \mathcal{P}'^T_n(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}'_n(\vec{b})\end{pmatrix} \neq 0. \qquad (2.10)$$

Conversely, (2.9) and (2.10) are also sufficient conditions for the quasi-definite character of $\widetilde{\mathbf{u}}$. In order to prove it we proceed as follows. Let $\widetilde{P}_n$ be the polynomial given by (2.7). Then, for $0\leq j\leq n-1$

$$\langle \widetilde{\mathbf{u}}, \widetilde{P}_n(x)P_j(x)\rangle = \langle \mathbf{u}, \widetilde{P}_n(x)P_j(x)\rangle + (\,\mathcal{P}^T_j(\vec{a}), \mathcal{P}^T_j(\vec{b}), \mathcal{P}'^T_j(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}'_n(\vec{b})\end{pmatrix}.$$

Now, taking into account the reproducing property of the kernel and $\langle \mathbf{u}, P_j(x)\mathrm{K}_{n-1}^{(0,1)}(x,b)\rangle = P_j'(b)$, we get

$$\langle \mathbf{u}, \widetilde{P}_n(x)P_j(x)\rangle = \langle \mathbf{u}, P_n(x)P_j(x)\rangle - (\,\mathcal{P}_j^T(\vec{a}), \mathcal{P}_j^T(\vec{b}), \mathcal{P}_j'^{\,T}(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}_n'(\vec{b})\end{pmatrix}.$$

From the above two formulas one has $\langle \widetilde{\mathbf{u}}, \widetilde{P}_n(x)P_j(x)\rangle = \langle \mathbf{u}, P_n(x)P_j(x)\rangle = 0$, for $0 \le j \le n-1$. On the other hand,

$$\langle \widetilde{\mathbf{u}}, \widetilde{P}_n(x)P_n(x)\rangle = \langle \mathbf{u}, \widetilde{P}_n(x)P_n(x)\rangle + (\,\mathcal{P}_n^T(\vec{a}), \mathcal{P}_n^T(\vec{b}), \mathcal{P}_n'^{\,T}(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}_n'(\vec{b})\end{pmatrix}$$

$$= \langle \mathbf{u}, P_n^2(x)\rangle + (\,\mathcal{P}_n^T(\vec{a}), \mathcal{P}_n^T(\vec{b}), \mathcal{P}_n'^{\,T}(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}_n'(\vec{b})\end{pmatrix} \neq 0,$$

from (2.10). Thus, $\widetilde{P}_n$ is the n-th monic polynomial orthogonal with respect to the linear functional $\widetilde{\mathbf{u}}$. Then we have proved

Theorem 1 *The linear functional $\widetilde{\mathbf{u}}$ given by (1.1) is a quasi-definite linear functional if and only if*

(i) The matrix $\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D}$ is nonsingular for every $n \in \mathbb{N}$.

(ii) $\langle \mathbf{u}, P_n^2(x)\rangle + (\,\mathcal{P}_n^T(\vec{a}), \mathcal{P}_n^T(\vec{b}), \mathcal{P}_n'^{\,T}(\vec{b})\,)\,\mathrm{D}\,(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}_n'(\vec{b})\end{pmatrix} \neq 0$, *for every* $n \in \mathbb{N}$.

In such a case, the corresponding sequence $(\widetilde{P}_n)_n$ of monic orthogonal polynomials is given by

$$\widetilde{P}_n(x) = P_n(x) - (\,\mathcal{K}_{n-1}^T(x,\vec{a}), \mathcal{K}_{n-1}^T(x,\vec{b}), \mathcal{K}_{n-1}^{(0,1)^T}(x,\vec{a})\,)\,\mathrm{D}(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1}\begin{pmatrix}\mathcal{P}_n(\vec{a})\\ \mathcal{P}_n(\vec{b})\\ \mathcal{P}_n'(\vec{b})\end{pmatrix}.$$

Remarks:

1. If the entries of the matrix D are nonzero, then D is an hermitian matrix which is nonsingular. Then
$$\mathrm{D}(\mathrm{I}+\mathbb{K}_{n-1}\mathrm{D})^{-1} = (\mathrm{D}^{-1}+\mathbb{K}_{n-1})^{-1} = \mathrm{M}_{n-1},$$
where $\mathrm{D}^{-1}+\mathbb{K}_{n-1}$ is an hermitian matrix. Thus (2.10) means that
$$1+\varepsilon_n\,(\,\widehat{\mathcal{P}}_n^T(\vec{a}), \widehat{\mathcal{P}}_n^T(\vec{b}), \widehat{\mathcal{P}}_n'^{\,T}(\vec{b})\,)\,\mathrm{M}_{n-1}\begin{pmatrix}\widehat{\mathcal{P}}_n(\vec{a})\\ \widehat{\mathcal{P}}_n(\vec{b})\\ \widehat{\mathcal{P}}_n'(\vec{b})\end{pmatrix} \neq 0, \qquad \widehat{P}_n(a_i) = \frac{P_n(a_i)}{\sqrt{|\langle \mathbf{u}, P_n^2(x)\rangle|}},$$
and $\varepsilon_n = \operatorname{sign}(\langle \mathbf{u}, P_n^2(x)\rangle)$.

2. If some entries of D are zero, then we can reduce the size of our system. The same situation holds when $(a_i)_{i=1}^M \bigcap (b_j)_{j=1}^N \neq \emptyset$. In such a case, the corresponding equations in (2.3) are reduced because of the repetition in (2.4).

3. Taking into account the Christoffel-Darboux formula we have

$$\mathrm{K}_{n-1}(x,y) = \frac{1}{k_n}\left[\frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x-y}\right], \qquad k_n = \langle \mathbf{u}, P_n^2(x)\rangle,$$

as well as

$$\mathrm{K}_{n-1}^{(0,1)}(x,y) = \frac{1}{k_n}\left[\frac{P_n(x)P'_{n-1}(y) - P'_n(y)P_{n-1}(x)}{x-y} + \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x-y)^2}\right].$$

Inserting this two expressions in (2.7) and denoting $\phi(x) = \prod_{i=1}^M (x-a_i)\prod_{j=1}^N (x-b_j)^2$, we deduce

$$\phi(x)\widetilde{P}_n(x) = A(x;n)P_n(x) + B(x;n)P_{n-1}(x), \tag{2.11}$$

where $A(x;n)$ and $B(x;n)$ are polynomials of degree independent of n and at most $2N+M$ and $2N+M-1$, respectively.

On the other hand, from the three-term recurrence relation that the sequence $(P_n)_n$ satisfies

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \tag{2.12}$$

and taking into account (2.11) we get, for $n \geq 1$

$$\phi(x)\widetilde{P}_{n-1}(x) = \quad C(x;n)P_n(x) + D(x;n)P_{n-1}(x), \tag{2.13}$$

where

$$C(x;n) = -\frac{B(x;n-1)}{\gamma_{n-1}}, \qquad D(x;n) = A(x;n-1) + \frac{x-\beta_{n-1}}{\gamma_{n-1}}B(x;n-1).$$

4. An inverse process can be done in order to recover the linear functional $\mathbf{u}$ in terms of $\widetilde{\mathbf{u}}$ (we need to add to $\widetilde{\mathbf{u}}$ the same masses but with opposite sign). In such a way we can deduce the existence of polynomials $\overline{A}(x;n)$ and $\overline{B}(x;n)$ with degrees independent of n such that

$$\phi(x)P_n(x) = \overline{A}(x;n)\widetilde{P}_n(x) + \overline{B}(x;n)\widetilde{P}_{n-1}(x), \tag{2.14}$$

as well as the counterpart

$$\phi(x)P_{n-1}(x) = \overline{C}(x;n)\widetilde{P}_n(x) + \overline{D}(x;n)\widetilde{P}_{n-1}(x). \tag{2.15}$$

3 A Three-Term Recurrence Relation for $(\widetilde{P}_n)_n$.

In the following we assume that $\widetilde{\mathbf{u}}$ is quasi-definite. Then, the sequence $(\widetilde{P}_n)_n$ of monic polynomials orthogonal with respect to $\widetilde{\mathbf{u}}$ satisfies a three-term recurrence relation (TTRR)

$$x\widetilde{P}_n(x) = \widetilde{P}_{n+1}(x) + \widetilde{\beta}_n\widetilde{P}_n(x) + \widetilde{\gamma}_n\widetilde{P}_{n-1}(x), \qquad n \in \mathbb{N}, \tag{3.1}$$

with the initial conditions $\widetilde{P}_{-1}(x) = 0$, $\widetilde{P}_0(x) = 1$.

Our aim is to obtain the coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ of the TTRR (3.1) for the polynomials $\widetilde{P}_n$ orthogonal with respect to $\widetilde{\mathbf{u}}$, in terms of the coefficients β_n and γ_n of the TTRR (2.12) of the monic polynomials orthogonal with respect to $\mathbf{u}$.

To do it we proceed as follows. By definition

$$\widetilde{\gamma}_n = \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x)\rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_{n-1}^2(x)\rangle}. \tag{3.2}$$

Taking into account (2.10) as well as remark 1 of theorem 1, we get, for $n > 1$

$$\widetilde{\gamma}_n = \gamma_n \frac{1 + \varepsilon_n\,(\,\widehat{\mathcal{P}}_n^T(\vec{a}), \widehat{\mathcal{P}}_n^T(\vec{b}), \widehat{\mathcal{P}}_n'^{\,T}(\vec{b})\,)\,\mathrm{M}_{n-1}\,(\,\widehat{\mathcal{P}}_n(\vec{a}), \widehat{\mathcal{P}}_n(\vec{b}), \widehat{\mathcal{P}}'_n(\vec{b})\,)^T}{1 + \varepsilon_{n-1}\,(\,\widehat{\mathcal{P}}_{n-1}^T(\vec{a}), \widehat{\mathcal{P}}_{n-1}^T(\vec{b}), \widehat{\mathcal{P}}_{n-1}'^{\,T}(\vec{b})\,)\,\mathrm{M}_{n-2}\,(\,\widehat{\mathcal{P}}_{n-1}(\vec{a}), \widehat{\mathcal{P}}_{n-1}(\vec{b}), \widehat{\mathcal{P}}'_{n-1}(\vec{b})\,)^T},$$

as well as, for $n = 1$

$$\begin{aligned}\widetilde{\gamma}_1 = & \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_1^2(x)\rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_0^2(x)\rangle} = \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_1^2(x)\rangle}{\langle \mathbf{u}, \widetilde{P}_0^2(x)\rangle + \sum_{i=1}^M A_i} \\ = & \ \gamma_1 \frac{1 + \varepsilon_1\,(\,\widehat{\mathcal{P}}_1^T(\vec{a}), \widehat{\mathcal{P}}_1^T(\vec{b}), \widehat{\mathcal{P}}_1'^{\,T}(\vec{b})\,)\,\mathrm{M}_0\,(\,\widehat{\mathcal{P}}_1(\vec{a}), \widehat{\mathcal{P}}_1(\vec{b}), \widehat{\mathcal{P}}'_1(\vec{b})\,)^T}{1 + \sum_{i=1}^M A_i/u_0},\end{aligned}$$

where $u_0 = \langle \mathbf{u}, 1\rangle$ is the first moment of the functional $\mathbf{u}$.

On the other hand, $\widetilde{\beta}_n = \langle \widetilde{\mathbf{u}}, x\widetilde{P}_n^2(x)\rangle / \langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x)\rangle$. Nevertheless, it is better to compute $\widetilde{\beta}_n$ in a different way. If $\widetilde{b}_n$ denote the coefficient of x^{n-1} for $\widetilde{P}_n$ and b_n the corresponding coefficient of x^{n-1} for P_n we have, $\widetilde{\beta}_n = \widetilde{b}_n - \widetilde{b}_{n+1}$. To obtain $\widetilde{b}_n$ we use the Eq. (2.7) which yields

$$\widetilde{b}_n = b_n - \varepsilon_n\varepsilon_{n-1}|\gamma_n|^{1/2}\,(\,\widehat{\mathcal{P}}_{n-1}^T(\vec{a}), \widehat{\mathcal{P}}_{n-1}^T(\vec{b}), \widehat{\mathcal{P}}_{n-1}'^{\,T}(\vec{b})\,)\,\mathrm{M}_{n-1}\begin{pmatrix}\widehat{\mathcal{P}}_n^T(\vec{a})\\ \widehat{\mathcal{P}}_n^T(\vec{b})\\ \widehat{\mathcal{P}}_n'^{\,T}(\vec{b})\end{pmatrix}.$$

Thus, for $n \geq 1$,

$$\begin{aligned}\widetilde{\beta}_n = \beta_n + & \ \varepsilon_n\varepsilon_{n+1}|\gamma_{n+1}|^{1/2}\,(\,\widehat{\mathcal{P}}_n^T(\vec{a}), \widehat{\mathcal{P}}_n^T(\vec{b}), \widehat{\mathcal{P}}_n'^{\,T}(\vec{b})\,)\,\mathrm{M}_n\begin{pmatrix}\widehat{\mathcal{P}}_{n+1}^T(\vec{a})\\ \widehat{\mathcal{P}}_{n+1}^T(\vec{b})\\ \widehat{\mathcal{P}}_{n+1}'^{\,T}(\vec{b})\end{pmatrix} \\ & - \varepsilon_n\varepsilon_{n-1}|\gamma_n|^{1/2}\,(\,\widehat{\mathcal{P}}_{n-1}^T(\vec{a}), \widehat{\mathcal{P}}_{n-1}^T(\vec{b}), \widehat{\mathcal{P}}_{n-1}'^{\,T}(\vec{b})\,)\,\mathrm{M}_{n-1}\begin{pmatrix}\widehat{\mathcal{P}}_n^T(\vec{a})\\ \widehat{\mathcal{P}}_n^T(\vec{b})\\ \widehat{\mathcal{P}}_n'^{\,T}(\vec{b})\end{pmatrix}.\end{aligned}$$

Finally, for $n = 0$ we have

$$\widetilde{\beta}_0 = \frac{\langle \widetilde{\mathbf{u}}, x \rangle}{\langle \widetilde{\mathbf{u}}, 1 \rangle} = \frac{u_1 + \sum_{i=1}^{M} a_i A_i + \sum_{j=1}^{N} B_j}{u_0 + \sum_{i=1}^{M} A_i}.$$

4 Raising and lowering operators for $(\widetilde{P}_n)_n$.

In the following we assume that the linear functional $\mathbf{u}$ is semiclassical, i.e., there exist polynomials ψ and ν, with $\deg \nu \geq 1$, such that

$$D(\psi \mathbf{u}) = \nu \mathbf{u}, \qquad D = \frac{d}{dx}. \tag{4.1}$$

Here we use the distributional notation in the sense that for a polynomial π we define the linear functional $\pi \mathbf{u}$ in such a way that

$$\langle \pi \mathbf{u}, p \rangle := \langle \mathbf{u}, \pi p \rangle, \qquad \langle D\mathbf{u}, p \rangle := -\langle \mathbf{u}, p' \rangle, \qquad \forall p \in \mathbb{P}.$$

Proposition 1 *If $\mathbf{u}$ is a semiclassical linear functional, then the linear functional $\widetilde{\mathbf{u}}$ introduced in (1.1) is also a semiclassical functional.*

Proof: Taking into account the fact that for the polynomial $\phi(x) = \prod_{i=1}^{M}(x - a_i) \prod_{j=1}^{N}(x - b_j)^2$ we get $\phi \mathbf{u} = \phi \widetilde{\mathbf{u}}$, we can consider

$$D(\phi^2 \psi \widetilde{\mathbf{u}}) = D(\phi^2 \psi \mathbf{u}) = \phi^2 D(\psi \mathbf{u}) + 2\phi\phi'\psi \mathbf{u} = \phi^2 \nu \mathbf{u} + 2\phi'\psi\phi \mathbf{u} = (\nu\phi + 2\phi'\psi)\phi \mathbf{u} = (\nu\phi + 2\phi'\psi)\phi \widetilde{\mathbf{u}},$$

i.e., there exist polynomials $\widetilde{\psi} = \phi^2 \psi$ and $\widetilde{\nu} = (\nu\phi + 2\phi'\psi)\phi$ such that $D(\widetilde{\psi}\widetilde{\mathbf{u}}) = \widetilde{\nu}\widetilde{\mathbf{u}}$. ■

Notice that the choice of $\widetilde{\psi}$ and $\widetilde{\nu}$ is not, in general, optimal. For instance, if

$$D(\phi\psi \widetilde{\mathbf{u}}) = D(\phi\psi \mathbf{u}) = \phi D(\psi \mathbf{u}) + \phi'\psi \mathbf{u} = (\phi\nu + \phi'\psi)\mathbf{u},$$

and we assume $\phi'\psi$ is a multiple of ϕ, i.e., $\phi'\psi = \eta\phi$, where η is a polynomial then the above equation yields

$$D(\phi\psi \widetilde{\mathbf{u}}) = \phi(\nu + \eta)\mathbf{u} = \phi(\nu + \eta)\widetilde{\mathbf{u}},$$

and thus $\widetilde{\psi} = \phi\psi$ and $\widetilde{\nu} = \phi(\nu + \eta)$. This is the reason why the study of the cases when the set $(a_i)_{i=1}^{M} \bigcup (b_j)_{j=1}^{N}$ is reduced to the set of zeros of ψ allows to reduce substantially the computation.

Proposition 2 *(Maroni [16]) If $\mathbf{u}$ is a semiclassical linear functional, then there exist polynomials $M_1(x;n)$, $N_1(x;n)$ with degree independent of n, such that*

$$\psi(x)P_n'(x) = M_1(x;n)P_n(x) + N_1(x;n)P_{n-1}(x). \tag{4.2}$$

In such a sense the operator $\mathcal{L}_n := \psi D - M_1(x;n)I$ reads as a lowering operator. On the other hand, using the TTRR (2.12) we get

$$\psi(x)P_n'(x) = M_2(x;n)P_n(x) + N_2(x;n)P_{n+1}(x). \tag{4.3}$$

Thus the operator $\mathcal{R}_n := \psi D - M_2(x;n)I$ reads as a raising operator. As a consequence of the existence of lowering and raising operators associated with a semiclassical linear functional, there exist polynomials with degrees independent of n, $R(x;n)$, $S(x;n)$, and $T(x;n)$ such that (see [16, 17])

$$R(x;n)P_n''(x) + S(x;n)P_n'(x) + T(x;n)P_n(x) = 0.$$

The interest of such a differential equation for the electrostatic interpretation of the location of the zeros of P_n was increased in the last years. For more details in the case when the linear functional $\mathfrak{u}$ is associated with an external field with eventually mass points outside its support see [18, 19] (see also [11] and references therein).

With the above notation we have

Proposition 3 *The lowering operator $\widetilde{\mathcal{L}}_n$ associated with the semiclassical linear functional $\widetilde{\mathfrak{u}}$ is $\widetilde{\mathcal{L}}_n = \widetilde{\psi}(x)D - \widetilde{M_1}(x;n)\mathrm{I}$, i.e.,*

$$\widetilde{\psi}(x)\widetilde{P}_n'(x) = \widetilde{M_1}(x;n)\widetilde{P}_n(x) + \widetilde{N}_1(x;n)\widetilde{P}_{n-1}(x), \tag{4.4}$$

where $\widetilde{\psi}(x) = \phi^2(x)\psi(x)$ and $\widetilde{M_1}(x;n)$ and $\widetilde{N}_1(x;n)$ are given by formulas (4.5) and (4.6), respectively.

Proof: Using (2.11) we have

$$\begin{aligned}
\widetilde{\psi}(x)\widetilde{P}_n'(x) = &\quad \phi(x)\psi(x)[(\phi(x)\widetilde{P}_n(x))' - \phi'(x)\widetilde{P}_n(x)] \\
= &\quad \psi(x)[A'(x;n)\phi(x)P_n(x) + B'(x;n)\phi(x)P_{n-1}(x)] \\
&\quad +\phi(x)[A(x;n)\psi(x)P_n'(x) + B(x;n)\psi(x)P_{n-1}'(x)] - \psi(x)\phi'(x)\phi(x)\widetilde{P}_n(x).
\end{aligned}$$

But from (4.2) and (4.3), the above expression becomes

$$\begin{aligned}
\widetilde{\psi}(x)\widetilde{P}_n'(x) = &\quad \psi(x)[A'(x;n)\phi(x)P_n(x) + B'(x;n)\phi(x)P_{n-1}(x)] \\
&\quad +\phi(x)\{A(x;n)[M_1(x;n)P_n(x) + N_1(x;n)P_{n-1}(x)] \\
&\quad +B(x;n)[M_2(x;n-1)P_{n-1}(x) + N_2(x;n-1)P_n(x)]\} - \psi(x)\phi'(x)\phi(x)\widetilde{P}_n(x),
\end{aligned}$$

i. e.,

$$\begin{aligned}
\widetilde{\psi}(x)\widetilde{P}_n' = &\quad \phi(x)P_n(x)[\psi(x)A'(x;n) + A(x;n)M_1(x;n) + B(x;n)N_2(x;n-1)] \\
&\quad +\phi(x)P_{n-1}(x)[\psi(x)B'(x;n) + A(x;n)N_1(x;n) + B(x;n)M_2(x;n-1)] \\
&\quad -\psi(x)\phi'(x)\phi(x)\widetilde{P}_n(x).
\end{aligned}$$

Finally, from (2.14) and (2.15) we get

$$\widetilde{\psi}(x)\widetilde{P}_n'(x) = \widetilde{M}_1(x;n)\widetilde{P}_n(x) + \widetilde{N}_1(x;n)\widetilde{P}_{n-1}(x),$$

where

$$\begin{aligned}\widetilde{M}_1(x;n) = \;& \psi(x)[\overline{A}(x;n)A'(x;n) + \overline{C}(x;n)B'(x;n) - \phi(x)\phi'(x) \\ & +A(x;n)[\overline{A}(x;n)M_1(x;n) + \overline{C}(x;n)N_1(x;n)] \\ & +B(x;n)[\overline{A}(x;n)N_2(x;n-1) + \overline{C}(x;n)M_2(x;n-1)],\end{aligned} \tag{4.5}$$

as well as

$$\begin{aligned}\widetilde{N}_1(x;n) = \;& \psi(x)[\overline{B}(x;n)A'(x;n) + \overline{D}(x;n)B'(x;n) \\ & +A(x;n)[\overline{B}(x;n)M_1(x;n) + \overline{D}(x;n)N_1(x;n)] \\ & +B(x;n)[\overline{B}(x;n)N_2(x;n-1) + \overline{D}(x;n)M_2(x;n-1)].\end{aligned} \tag{4.6}$$

In a similar way we can deduce the representation for the raising operator. Taking into account (4.4) and the TTRR (3.1) we obtain

$$\widetilde{\psi}(x)\widetilde{P}_n'(x) = \widetilde{M}_2(x;n)\widetilde{P}_n(x) + \widetilde{N}_2(x;n)\widetilde{P}_{n+1}(x), \tag{4.7}$$

where

$$\widetilde{M}_2(x;n) = \widetilde{M}_1(x;n) + \frac{x-\widetilde{\beta}_n}{\widetilde{\gamma}_n}\widetilde{N}_1(x;n), \qquad \widetilde{N}_2(x;n) = -\frac{\widetilde{N}_1(x;n)}{\widetilde{\gamma}_n}.$$

Remark: Notice that is the degree of $\widetilde{\psi}$ and $\tilde{\eta}$ is not optimal then we can reduce the degrees of the polynomials $\widetilde{M}_1$ and $\widetilde{N}_1$ in (4.4).

5 The second order differential equation for $(\widetilde{P}_n)_n$.

From (4.4) and (4.7) a second order linear differential equation immediately follows. In fact, from (4.4) we have, formally,

$$\widetilde{P}_{n-1}(x) = \frac{\widetilde{\psi}(x)\widetilde{P}_n'(x) - \widetilde{M}_1(x;n)\widetilde{P}_n(x)}{\widetilde{N}_1(x;n)}. \tag{5.1}$$

But if we consider (4.7) for polynomials of degree $n-1$ instead of n then

$$\widetilde{\psi}(x)\widetilde{P}_{n-1}'(x) = \widetilde{M}_2(x;n-1)\widetilde{P}_{n-1}(x) + \widetilde{N}_2(x;n-1)\widetilde{P}_n(x). \tag{5.2}$$

Thus the substitution of (5.1) in (5.2) yields

$$\begin{aligned}\widetilde{\psi}(x)\Bigg\{ & \frac{1}{\widetilde{N}_1(x;n)}[\widetilde{\psi}(x)\widetilde{P}_n''(x) + \widetilde{\psi}'(x)\widetilde{P}_n'(x) - \widetilde{M}_1'(x;n)\widetilde{P}_n(x) - \widetilde{M}_1(x;n)\widetilde{P}_n'(x)] \\ & - \left(\frac{\widetilde{N}_1'(x;n)}{\widetilde{N}_1^2(x;n)} + \frac{\widetilde{M}_2(x;n-1)}{\widetilde{N}_1(x;n)}\right)[\widetilde{\psi}(x)\widetilde{P}_n'(x) - \widetilde{M}_1(x;n)\widetilde{P}_n(x)]\Bigg\} = \widetilde{N}_2(x;n-1)\widetilde{P}_n(x).\end{aligned}$$

If we now multiply the above equation by $\widetilde{N}_1^2(x;n)$, then we deduce that there exist polynomials $\widetilde{R}(x;n)$, $\widetilde{S}(x;n)$ and $\widetilde{T}(x;n)$ of degree independent of n such that

$$\widetilde{R}(x;n)\widetilde{P}_n''(x) + \widetilde{S}(x;n)\widetilde{P}_n'(x) + \widetilde{T}(x;n)\widetilde{P}_n(x) = 0, \tag{5.3}$$

with

$$\widetilde{R}(x;n) = \widetilde{\psi}^2(x)\widetilde{N}_1(x;n),$$
$$\widetilde{S}(x;n) = \widetilde{\psi}(x)\widetilde{N}_1(x;n)[\widetilde{\psi}'(x) - \widetilde{M}_1(x;n) - \widetilde{\psi}(x)\widetilde{M}_2(x;n-1)] - \widetilde{N}_1'(x;n)\widetilde{\psi}^2(x) -$$
$$\widetilde{T}(x;n) = [\widetilde{N}_1'(x;n)\widetilde{M}_1(x;n) - \widetilde{N}_1(x;n)\widetilde{M}_1'(x;n)]\widetilde{\psi}(x) - \widetilde{N}_2(x;n-1)\widetilde{N}_1^2(x;n).$$

6 Example: The modified Hermite polynomials

In this section we will consider an example related to Hermite polynomials. More precisely, we will consider the polynomials $\widetilde{H}_n(x)$ orthogonal with respect to the functional $\widetilde{\mathbf{u}}$ such that

$$\langle \widetilde{\mathbf{u}}, p\rangle = \langle \mathbf{u}, p\rangle + Ap(0) + Bp'(0) = \int_{\mathbb{R}} p(x)e^{-x^2}dx + Ap(0) + Bp'(0). \tag{6.1}$$

In this case we have

$$\mathbb{K}_{n-1} = \begin{pmatrix} \mathrm{K}_{n-1}(0,0) & \mathrm{K}_{n-1}(0,0) & \mathrm{K}_{n-1}^{(0,1)}(0,0) \\ \mathrm{K}_{n-1}(0,0) & \mathrm{K}_{n-1}(0,0) & \mathrm{K}_{n-1}^{(0,1)}(0,0) \\ \mathrm{K}_{n-1}^{(1,0)}(0,0) & \mathrm{K}_{n-1}^{(0,1)}(0,0) & \mathrm{K}_{n-1}^{(1,1)}(0,0) \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix},$$

thus (2.7) yields

$$\widetilde{H}_n(x) = H_n(x) - (\,\mathrm{K}_{n-1}(x,0), \mathrm{K}_{n-1}(x,0), \mathrm{K}_{n-1}^{(0,1)}(x,0)\,)\,\mathrm{D}(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1} \begin{pmatrix} H_n(0) \\ H_n(0) \\ H_n'(0) \end{pmatrix}.$$

As we already pointed out in the case when some point masses coincide with some points for the derivatives of the delta Dirac masses, as in this case, the formulas can be reduced. In fact since in this case (2.3) and (2.4) are the same, we can reduce the dimensions of the matrices $\mathbb{K}_{n-1}$ and D

$$\mathbb{K}_{n-1} = \begin{pmatrix} \mathrm{K}_{n-1}(0,0) & \mathrm{K}_{n-1}^{(0,1)}(0,0) \\ \mathrm{K}_{n-1}^{(1,0)}(0,0) & \mathrm{K}_{n-1}^{(1,1)}(0,0) \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}. \tag{6.2}$$

Thus (2.7) yields

$$\widetilde{H}_n(x) = H_n(x) - (\,\mathrm{K}_{n-1}(x,0), \mathrm{K}_{n-1}^{(0,1)}(x,0)\,)\,\mathrm{D}(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1} \begin{pmatrix} H_n(0) \\ H_n'(0) \end{pmatrix},$$

or, equivalently,

$$\widetilde{H}_n(x) = H_n(x) - A\widetilde{H}_n(0)\mathrm{K}_{n-1}(x,0) - B\widetilde{H}_n(0)\mathrm{K}_{n-1}^{(0,1)}(x,0) - B\widetilde{H}_n'(0)\mathrm{K}_{n-1}(x,0), \tag{6.3}$$

where, using the notation (6.2) we have

$$\begin{pmatrix} \widetilde{H}_n(0) \\ \widetilde{H}_n'(0) \end{pmatrix} = (\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1} \begin{pmatrix} H_n(0) \\ H_n'(0) \end{pmatrix}.$$

Here (H_n) denotes the classical monic Hermite polynomials.

6.1 An explicit representation formula

In order to give explicit formulas for these new polynomials $\widetilde{H}_n$ we should compute the corresponding kernels for the Hermite polynomials. We will distinguish the even and odd polynomials. To compute the kernels $\mathrm{K}_{n-1}(x,0)$ we use the Christoffel-Darboux formula

$$\mathrm{K}_{n-1}(x,y) = \frac{1}{\langle \mathbf{u}, H_{n-1}^2\rangle}\frac{H_n(x)H_{n-1}(y)-H_n(y)H_{n-1}(x)}{x-y}, \quad n \geq 1\,. \tag{6.4}$$

For the Hermite polynomials H_n we have $\langle \mathbf{u}, H_n^2\rangle = \dfrac{n!\sqrt{\pi}}{2^n}$ and

$$H_{2m}(0) = \frac{(-1)^m(2m-1)!}{2^{2m-1}(m-1)!}, \quad H_{2m-1}(0)=0, \qquad m=1,2,3,\ldots\,.$$

Then, using (6.4) we obtain

$$\mathrm{K}_{2m-1}(x,0) = \frac{(-1)^{m-1}}{2\sqrt{\pi}m!}\frac{H_{2m}'(x)}{x}, \qquad \mathrm{K}_{2m}(x,0) = \frac{(-1)^m}{\sqrt{\pi}m!}\frac{H_{2m+1}(x)}{x}, \tag{6.5}$$

from where, taking the limit $x \to 0$ and using the property $H_n'(x) = nH_{n-1}(x)$, we get

$$\mathrm{K}_{2m-1}(0,0) = \frac{(2m-1)!}{2^{2m-2}\sqrt{\pi}(m-1)!^2}, \qquad \mathrm{K}_{2m}(0,0) = \frac{(2m+1)!}{2^{2m}\sqrt{\pi}\,m!^2}. \tag{6.6}$$

Next we compute the kernel $\mathrm{K}_{n-1}^{(0,1)}(x,0)$. For doing that we take the derivative of (6.4) in y and put $y=0$. With the help of the aforesaid properties of the Hermite polynomials this yields

$$\mathrm{K}_{2m-1}^{(0,1)}(x,0) = \frac{(-1)^{m-1}}{2\sqrt{\pi}m!}\frac{4mxH_{2m}(x)+H_{2m}'(x)}{x^2}, \quad \mathrm{K}_{2m}^{(0,1)}(x,0) = \frac{(-1)^m}{\sqrt{\pi}m!}\frac{H_{2m+1}(x)-xH_{2m+1}'(x)}{x^2}, \tag{6.7}$$

from where easily follows the values $\mathrm{K}_{2m-1}^{(0,1)}(0,0) = \mathrm{K}_{2m}^{(0,1)}(0,0) = 0$. Thus $\mathrm{K}_{n-1}^{(0,1)}(0,0) = \mathrm{K}_{n-1}^{(1,0)}(0,0)$. Finally, we need to compute the value $\mathrm{K}_{n-1}^{(1,1)}(0,0)$. For doing this we take derivatives of (6.4) in x and y, respectively and put $x=y=0$. This yields the values

$$\mathrm{K}_{2m}^{(1,1)}(0,0) = \mathrm{K}_{2m-1}^{(1,1)}(0,0) = \frac{(2m+1)!}{3\cdot 2^{2m-2}\sqrt{\pi}(m-1)!m!}. \tag{6.8}$$

Using all the above results we obtain

$$\begin{aligned}
\widetilde{H}_{2m}(0) &= \frac{H_{2m}(0)}{1+A\mathrm{K}_{2m-1}(0,0)-B^2\mathrm{K}_{2m-1}(0,0)\mathrm{K}_{2m-1}^{(1,1)}(0,0)},\\
\widetilde{H}_{2m}'(0) &= \frac{-B\mathrm{K}_{2m-1}^{(1,1)}(0,0)H_{2m}(0)}{1+A\mathrm{K}_{2m-1}(0,0)-B^2\mathrm{K}_{2m-1}(0,0)\mathrm{K}_{2m-1}^{(1,1)}(0,0)},\\
\widetilde{H}_{2m+1}(0) &= \frac{B(2m+1)\mathrm{K}_{2m}(0,0)H_{2m}(0)}{1+A\mathrm{K}_{2m}(0,0)-B^2\mathrm{K}_{2m}(0,0)\mathrm{K}_{2m}^{(1,1)}(0,0)},\\
\widetilde{H}_{2m+1}'(0) &= \frac{(2m+1)(1+A\mathrm{K}_{2m}(0,0))H_{2m}(0)}{1+A\mathrm{K}_{2m}(0,0)-B^2\mathrm{K}_{2m}(0,0)\mathrm{K}_{2m}^{(1,1)}(0,0)}.
\end{aligned} \tag{6.9}$$

Then, (6.3) transforms into the following explicit representation formula in terms of the classical monic Hermite polynomials

$$
\begin{aligned}
x^2\widetilde{H}_{2m}(x) &= [x^2 + a(m)x]H_{2m}(x) + [b(m)x + c(m)]H'_{2m}(x), \qquad m \geq 1 \\
x^2\widetilde{H}_{2m+1}(x) &= [x^2 + d(m)x + e(m)]H_{2m+1}(x) + f(m)xH'_{2m+1}(x), \qquad m \geq 0,
\end{aligned}
\tag{6.10}
$$

where

$$
\begin{aligned}
a(m) &= \frac{2B(-1)^m\widetilde{H}_{2m}(0)}{\sqrt{\pi}(m-1)!}, & b(m) &= \frac{(-1)^m}{2\sqrt{\pi}\,m!}[A\widetilde{H}_{2m}(0) + B\widetilde{H}'_{2m}(0)], \\
c(m) &= \frac{B(-1)^m\widetilde{H}_{2m}(0)}{2\sqrt{\pi}\,m!}, & d(m) &= \frac{(-1)^{m+1}}{\sqrt{\pi}\,m!}[A\widetilde{H}_{2m+1}(0) + B\widetilde{H}'_{2m+1}(0)], \\
e(m) &= \frac{B(-1)^{m+1}\widetilde{H}_{2m+1}(0)}{\sqrt{\pi}\,m!}, & f(m) &= \frac{B(-1)^m\widetilde{H}_{2m+1}(0)}{\sqrt{\pi}\,m!}.
\end{aligned}
$$

6.2 The TTRR relation

To compute the coefficient $\widetilde{\gamma}_n$ of the TTRR we use (3.2) $\widetilde{\gamma}_n = \langle \widetilde{\mathfrak{u}}, \widetilde{H}_n^2(x)\rangle / \langle \widetilde{\mathfrak{u}}, \widetilde{H}_{n-1}^2(x)\rangle$, where $\langle \widetilde{\mathfrak{u}}, \widetilde{H}_n^2(x)\rangle$ is given by (2.8), i.e.,

$$
\begin{aligned}
\langle \widetilde{\mathfrak{u}}, \widetilde{H}_n^2(x)\rangle &= \langle \mathfrak{u}, H_n^2(x)\rangle + (\, H_n(0), H'_n(0)\,) \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} \begin{pmatrix} \widetilde{H}_n(0) \\ \widetilde{H}'_n(0) \end{pmatrix} \\
&= \langle \mathfrak{u}, H_n^2(x)\rangle + AH_n(0)\widetilde{H}_n(0) + BH_n(0)\widetilde{H}'_n(0) + BH'_n(0)\widetilde{H}_n(0).
\end{aligned}
$$

Thus

$$
\widetilde{\gamma}_{2m} = m\frac{1 + \frac{(-1)^m}{\sqrt{\pi}m!}[A\widetilde{H}_{2m}(0) + B\widetilde{H}'_{2m}(0)]}{1 + \frac{2(-1)^{m+1}}{\sqrt{\pi}(m-1)!}B\widetilde{H}_{2m-1}(0)}, \quad \widetilde{\gamma}_{2m+1} = \frac{2m+1}{2}\frac{1 + \frac{2(-1)^m}{\sqrt{\pi}m!}B\widetilde{H}_{2m+1}(0)}{1 + \frac{(-1)^m}{\sqrt{\pi}m!}[A\widetilde{H}_{2m}(0) + B\widetilde{H}'_{2m}(0)]}.
$$

To compute $\widetilde{\beta}_n$ we use the formula $\widetilde{\beta}_n = \widetilde{b}_n - \widetilde{b}_{n-1}$ where $\widetilde{b}_n$ denotes the coefficient of x^{n-1} of $\widetilde{H}_n$. Then using (6.10) we find $\widetilde{b}_{2m} = a(m)$, $\widetilde{b}_{2m+1} = d(m)$, thus

$$
\begin{aligned}
\widetilde{\beta}_{2m} &= \frac{(-1)^m}{\sqrt{\pi}(m-1)!}[2B\widetilde{H}_{2m}(0) - A\widetilde{H}_{2m-1}(0) - B\widetilde{H}'_{2m-1}(0)], \\
\widetilde{\beta}_{2m+1} &= \frac{(-1)^{m+1}}{\sqrt{\pi}m!}[2B\widetilde{H}_{2m}(0) + A\widetilde{H}_{2m+1}(0) + B\widetilde{H}'_{2m+1}(0)].
\end{aligned}
$$

6.3 The second order differential equation

Let now obtain the SODE. We will follow the same algorithm as in [3]. First of all, notice that (6.10) can be written in the unified form

$$
x^2\widetilde{H}_n(x) = \alpha(x,n)H_n(x) + \beta(x,n)H'_n(x). \tag{6.11}
$$

Taking the derivatives of the above expression we have

$$x^2\widetilde{H}_n'(x) + 2x\widetilde{H}_n(x) = \alpha'(x,n)H_n(x) + \beta'(x,n)H_n'(x) + \alpha(x,n)H_n'(x) + \beta(x,n)H_n''(x).$$

Now we multiply by x and use (6.11) to eliminate the function $\widetilde{H}_n(x)$ as well as the SODE for the classical Hermite polynomials $H_n''(x) - 2xH_n'(x) + \lambda_n H_n(x) = 0$ $(\lambda_n = n/2)$ to obtain

$$x^3\widetilde{H}_n'(x) = \gamma(x,n)H_n(x) + \delta(x,n)H_n'(x), \tag{6.12}$$

where

$$\gamma(x,n) = x\alpha'(x,n) - x\lambda_n\beta(x,n) - 2\alpha(x,n), \quad \delta(x,n) = x\alpha(x,n) + x\beta'(x,n) + 2(x^2-1)\beta(x,n).$$

Applying the same procedure but starting with (6.12) we obtain

$$x^4\widetilde{H}_n''(x) = \epsilon(x,n)H_n(x) + \xi(x,n)H_n'(x), \tag{6.13}$$

where

$$\epsilon(x,n) = x\gamma'(x,n) - x\lambda_n\delta(x,n) - 3\gamma(x,n), \quad \xi(x,n) = x\gamma(x,n) + x\delta'(x,n) + (2x^2-3)\delta(x,n).$$

From (6.11), (6.12) and (6.13) it follows that

$$\begin{vmatrix} x^2\widetilde{H}_n(x) & \alpha(x,n) & \beta(x,n) \\ x^3\widetilde{H}_n'(x) & \gamma(x,n) & \delta(x,n) \\ x^4\widetilde{H}_n''(x) & \epsilon(x,n) & \xi(x,n) \end{vmatrix} = 0,$$

or, equivalently,

$$x^2[\alpha(x,n)\delta(x,n) - \gamma(x,n)\beta(x,n)]\widetilde{H}_n''(x) + x[\epsilon(x,n)\beta(x,n) - \alpha(x,n)\xi(x,n)]\widetilde{H}_n'(x)$$
$$+[\gamma(x,n)\xi(x,n) - \epsilon(x,n)\delta(x,n)]\widetilde{H}_n(x) = 0.$$

For the even case, we have $\alpha(x,2m) = x^2 + a(m)x$, $\beta(x,2m) = b(m)x + c(m)$ and $\lambda_{2m} = 4m$, then, after some straightforward but cumbersome calculations (we use Mathematica in order to simplify the expressions) we find

$$\widetilde{\sigma}(x,2m)\widetilde{H}_{2m}''(x) + \widetilde{\tau}(x,2m)\widetilde{H}_{2m}'(x) + \widetilde{\lambda}(x,2m)\widetilde{H}_{2m}(x) = 0,$$

where

$$\begin{aligned}\widetilde{\sigma}(x,2m) = \; & x^2\Big[x^4 + x^2a^2(m) + 4mx^2b^2(m) - 2xc(m) + 2x^3c(m) + 4mc^2(m) \\ & +xb(m)\left(-x + 2x^3 + 8mc(m)\right) + a(m)\left(2x^3 + 2x^3b(m) - c(m) + 2x^2c(m)\right)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}(x,2m) = -x\Big[&2x^6 + 2x^2\left(x^2-1\right)a^2(m) + 4mx^2\left(1+2x^2\right)b^2(m) + 9xc(m) \\ & -6x^3c(m) + 4x^5c(m) - 4mc^2(m) + 8mx^2c^2(m) + x^2b(m)\left(5 - 2x^2 + 4x^4 + 16mxc(m)\right) \\ & +2a(m)\left(2x^5 - x^3 + x\left(3 - x^2 + 2x^4\right)b(m) + 5c(m) - 3x^2c(m) + 2x^4c(m)\right)\Big],\end{aligned}$$

and

$$\begin{aligned}\widetilde{\lambda}(x,2m) = & \, 3x^4+4mx^6+\left(6x^2+(-2+4m)\,x^4\right)a^2(m)+4mx^2\left(-3+2x^2+4mx^2\right)b^2(m)\\ & -6xc(m)+6x^3c(m)-12mx^3c(m)+8mx^5c(m)-16mc^2(m)+16m^2x^2c^2(m)\\ & +xb(m)\left(-3x+6x^3+8mx^5-36mc(m)+8mx^2c(m)+32m^2x^2c(m)\right)\\ & +a(m)[7x^3-2x^5+8mx^5+2x\left(5x^2+6mx^2-2x^4+4mx^4-3\right)b(m)-14c(m)\\ & +14x^2c(m)-4x^4c(m)+8mx^4c(m)].\end{aligned}$$

For the odd case, $\alpha(x,2m+1)=x^2+d(m)x+e(m)$, $\beta(x,2m+1)=f(m)x$, and $\lambda_{2m+1}=4m+2$, then

$$\widetilde{\sigma}(x,2m+1)\widetilde{H}''_{2m+1}(x)+\widetilde{\tau}(x,2m+1)\widetilde{H}'_{2m+1}(x)+\widetilde{\lambda}(x,2m+1)\widetilde{H}_{2m+1}(x)=0,$$

where

$$\begin{aligned}\widetilde{\sigma}(x,2m+1)= \; & x^2\Big[x^2d^2(m)+e^2(m)+e(m)\left(2x^2+f(m)+2x^2f(m)\right)+x^2\left(x^2-f(m)\right)\\ & +x^2\left(2x^2f(m)+2f^2(m)+4mf^2(m)\right)+2xd(m)\left(e(m)+x^2\left(1+f(m)\right)\right)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}(x,2m+1)= & -x\Big[2x^2\left(x^2-1\right)d^2(m)+2\left(x^2-2\right)e^2(m)+e(m)\left(4x^4-4x^2+5f(m)-2x^2f(m)\right.\\ & \left.+4x^4f(m)\right)+d(m)\left(4x^5-2x^3+\left(4x^3-6x\right)e(m)+6xf(m)-2x^3f(m)+4x^5f(m)\right)\\ & +x^2\left(2x^4+5f(m)-2x^2f(m)+4x^4f(m)+2f^2(m)+4mf^2(m)+4x^2f^2(m)+8mx^2f^2(m)\right)\Big],\end{aligned}$$

and

$$\begin{aligned}\widetilde{\lambda}(x,2m+1)= & \left(6x^2+4mx^4\right)d^2(m)+\left(11-2x^2+4mx^2\right)e^2(m)+xd(m)\left(7x^2+2x^4+8mx^4\right.\\ & \left.+\left(15-2x^2+8mx^2\right)e(m)-6f(m)+16x^2f(m)+12mx^2f(m)+8mx^4f(m)\right)\\ & +e(m)\left(6x^2+8mx^4-7f(m)+22x^2f(m)+24mx^2f(m)-4x^4f(m)+8mx^4f(m)\right)\\ & +x^2\left(3x^2+2x^4+4mx^4-3f(m)+6x^2f(m)+4x^4f(m)+8mx^4f(m)-6f^2(m)\right.\\ & \left.-12mf^2(m)+8x^2f^2(m)+24mx^2f^2(m)+16m^2x^2f^2(m)\right)\end{aligned}$$

Using the above SODE we can study the distribution of zeros of the $\widetilde{H}_n$ polynomials by means of the central moments [7] and the WKB algorithm [20] in a similar way as it was done in [3].

6.4 The raising and lowering operators

For the raising and lowering operators we follow the algorithm described in section 4. We need to obtain the relations (2.14) and (2.15). Since (6.1) we have $\langle \mathbf{u}, p\rangle = \langle \widetilde{\mathbf{u}}, p\rangle - Ap(0) - Bp'(0)$, i.e., the classic Hermite functional can be obtained from the functional $\widetilde{\mathbf{u}}$ via the addition of the same masses A and B but with a different sign. Then we have the following representation (compare with (6.3))

$$H_n(x)=\widetilde{H}_n(x)+AH_n(0)\widetilde{\mathrm{K}}_{n-1}(x,0)+BH_n(0)\widetilde{\mathrm{K}}^{(0,1)}_{n-1}(x,0)+BH'_n(0)\widetilde{\mathrm{K}}_{n-1}(x,0), \qquad (6.14)$$

where $\widetilde{\mathrm{K}}_n$ and $\widetilde{\mathrm{K}}_n^{(0,1)}$ are the kernel polynomials corresponding to the family $\widetilde{H}_n$. Then using the Christoffel-Darboux formula for the $\widetilde{H}_n$ polynomials we obtain

$$\begin{aligned} x^2 H_{2m}(x) &= [x^2 + \widetilde{a}(m)x + \widetilde{b}(m)]\widetilde{H}_{2m}(x) + [\widetilde{c}(m)x + \widetilde{d}(m)]\widetilde{H}_{2m-1}(x), \quad m \geq 1 \\ x^2 H_{2m+1}(x) &= [x^2 + \widetilde{e}(m)x]\widetilde{H}_{2m+1}(x) + \widetilde{f}(m)x\widetilde{H}_{2m}(x), \quad m \geq 0, \end{aligned} \tag{6.15}$$

where

$$\widetilde{a}(m) = \frac{H_{2m}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m-1}\rangle}[A\widetilde{H}_{2m-1}(0) + B\widetilde{H}'_{2m-1}(0)], \qquad \widetilde{b}(m) = \frac{BH_{2m}(0)\widetilde{H}_{2m-1}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m-1}\rangle},$$

$$\widetilde{c}(m) = -\frac{H_{2m}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m-1}\rangle}[A\widetilde{H}_{2m}(0) + B\widetilde{H}'_{2m}(0)], \qquad \widetilde{d}(m) = -\frac{BH_{2m}(0)\widetilde{H}_{2m}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m-1}\rangle},$$

$$\widetilde{e}(m) = \frac{B(2m+1)H_{2m}(0)\widetilde{H}_{2m}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m}\rangle}, \qquad \widetilde{f}(m) = -\frac{B(2m+1)H_{2m}(0)\widetilde{H}_{2m+1}(0)}{\langle \widetilde{\mathbf{u}}, \widetilde{H}_{2m}\rangle}.$$

From the last equation in (6.15) if we change m by $m-1$ and use the TTRR for the polynomials $\widetilde{H}_n$ we obtain the following expression for the polynomials $H_{2m-1}(x)$

$$x^2 H_{2m-1}(x) = \left[x^2 + \widetilde{e}(m-1)x + \frac{x - \widetilde{\beta}_{2m-1}}{\widetilde{\gamma}_{2m-1}} x\widetilde{f}(m-1)\right]\widetilde{H}_{2m-1}(x) - \frac{x\widetilde{f}(m)}{\widetilde{\gamma}_{2m-1}}\widetilde{H}_{2m}(x). \tag{6.16}$$

Next, we rewrite the first equation in (6.10)

$$x^2 \widetilde{H}_{2m}(x) = \alpha(x, 2m)H_{2m}(x) + 2m\beta(x, 2m)H_{2m-1}(x). \tag{6.17}$$

Thus $A(x, 2m) = \alpha(x, 2m)$ and $B(x, 2m) = 2m\beta(x, 2m)$. Using the TTRR for the Hermite polynomials the second equation can be easily transformed as follows

$$x^2 \widetilde{H}_{2m-1}(x) = -2\beta(x, 2m-1)H_{2m}(x) + [\alpha(x, 2m-1) + 2x\beta(x, 2m-1)]H_{2m-1}(x). \tag{6.18}$$

Thus $C(x, 2m) = -2\beta(x, 2m-1)$ and $D(x, 2m) = \alpha(x, 2m-1) + 2x\beta(x, 2m-1)$.

From (6.15) and (6.16) we have

$$\overline{A}(x, 2m) = x^2 + \widetilde{a}(m)x + \widetilde{b}(m), \quad \overline{B}(x, 2m) = \widetilde{c}(m)x + \widetilde{d}(m),$$

$$\overline{C}(x, 2m) = -\frac{x\widetilde{f}(m)}{\widetilde{\gamma}_{2m-1}}, \quad \overline{D}(x, 2m) = x^2 + \widetilde{e}(m-1)x + \frac{x - \widetilde{\beta}_{2m-1}}{\widetilde{\gamma}_{2m-1}} x\widetilde{f}(m-1).$$

Finally, from the properties of the Hermite polynomials in formulas (2.11) and (2.13) we get

$$M_1(x, 2m) = 0, \quad N_1(x, 2m) = 2m, \quad M_2(x, 2m-1) = 2x, \quad N_2(x, 2m-1) = -2.$$

Substituting all the above formulas in (4.5) and (4.6) we obtain the lowering operator, and in (4.7) the raising operator. The same can be performed for the odd case.

Before concluding this section let us point out that there exist the lowering-like and raising-like operators. For the sake of completeness we will show how they can be obtained for the even case. The odd case in completely similar. If we rewrite (6.12) in the form

$$x^3\widetilde{H}'_{2m}(x) = \gamma(x,2m)H_{2m}(x) + 2m\delta(x,2m)H_{2m-1}(x),$$

and use the equations (6.17) and (6.18) we get the lowering-like operator

$$\widetilde{\psi}(x;2m)\widetilde{H}'_{2m}(x) = \overline{M}_1(x;2m)\widetilde{H}_{2m}(x) + \overline{N}_1(x;2m)\widetilde{H}_{2m-1}(x), \tag{6.19}$$

where

$$\begin{aligned}
\widetilde{\psi}(x,2m) = &\quad x[\alpha(x,2m)(\alpha(x,2m-1)+2x\beta(x,2m-1))+4m\beta(x,2m-1)\beta(x,2m)],\\
\overline{M}_1(x;2m) = &\quad \gamma(x,2m)[\alpha(x,2m-1)+2x\beta(x,2m-1)]+4m\beta(x,2m-1)\delta(x,2m),\\
\overline{N}_1(x;2m) = &\quad 2m[\alpha(x,2m)\delta(x,2m)-\beta(x,2m)\gamma(x,2m)].
\end{aligned}$$

To obtain the raising-like operator we can use the TTRR for the polynomials $\widetilde{H}_n$ to substitute $\widetilde{H}_{2m-1}(x)$ in (6.19) that leads to

$$\widetilde{\psi}(x;2m)\widetilde{H}'_{2m}(x) = \overline{M}_2(x;2m)\widetilde{H}_{2m}(x) + \overline{N}_2(x;2m)\widetilde{H}_{2m+1}(x), \tag{6.20}$$

$$\overline{M}_2(x;n) = \overline{M}_1(x;n) + \frac{x-\widetilde{\beta}_n}{\widetilde{\gamma}_n}\overline{N}_1(x;n), \qquad \overline{N}_2(x;n) = -\frac{\overline{N}_1(x;n)}{\widetilde{\gamma}_n}.$$

The odd case can be obtained in a similar way.

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