THE q-RACAH-KRALL-TYPE POLYNOMIALS

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Abstract. In this paper the Krall-type polynomials obtained via the addition of two mass points to the weight function of the *standard* q-Racah polynomials are introduced. Several algebraic properties of these polynomials are obtained and some of their limit cases are discussed.

1. Introduction

In the last decades the study of the discrete analogues of the classical special functions and, in particular, the orthogonal polynomials, has received an increasing interest (for a review see [2, 16, 14]). Special emphasis was given to the q-analogues of the orthogonal polynomials or q-polynomials, which are closely related with different topics in other fields of actual science: Mathematics and Physics (see e.g., [3, 4, 14, 17, 18] and references therein). One of the possible extensions of the classical polynomials are the so-called Krall-type polynomials.

The Krall-type polynomials are polynomials which are orthogonal with respect to a linear functional $\widetilde{\mathfrak{u}}$ obtained from a quasi-definite functional $\mathfrak{u} : \mathbb{P} \mapsto \mathbb{C}$ ($\mathbb{P}$, denotes the space of complex polynomials with complex coefficients) via the addition of delta Dirac measures. In the last years the study of such polynomials have attracted an increasing interest with a special emphasis on the case when the starting functional $\mathfrak{u}$ is a classical *continuous*, discrete or q linear functional (for more details see [7] and references therein). These kind of polynomials appear as eigenfunctions of a fourth order linear differential operator with polynomial coefficients that do not depend on the degree of the polynomials. They were firstly considered by Krall in 1940 (see e.g. [13, chapter XV]) and further studied by several authors (for more details see [7] and references therein). For the case of the discrete lattice A. Durán has discovered very recently [10, 11] a method for obtaining the orthogonal polynomials satisfying higher order differential and difference equations.

In two recent papers [7, 8] a general theory of the Krall-type polynomials on non-uniform lattices was developed. In fact, in [7, 8] the authors studied the polynomials $\widetilde{P}_n(s)_q$ which are orthogonal with respect to the linear functionals $\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{k=1}^{N} A_k \delta_{x_k}$ defined on the q-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. For these polynomials the following expression have been found [7, Eq. (13)]

$$\widetilde{P}_n(s)_q = P_n(s)_q - \sum_{i=1}^{M} A_i \widetilde{P}_n(a_i)_q \mathrm{K}_{n-1}(s, a_i), \tag{1}$$

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provided that the sequence of monic polynomials $(\widetilde{P}_n)_n$ exist. Here, $(P_n)_n$ are the polynomials orthogonal with respect to $\mathfrak{u}$, $A_i \in \mathbb{C}$ and $K_n(x,y)$ are the n-th reproducing kernels given by the Christoffel-Darboux formula

$$\mathrm{K}_n(s_1,s_2) := \sum_{k=0}^{n} \frac{P_k(s_1)_q P_k(s_2)_q}{d_k^2} = \frac{\alpha_n}{d_n^2} \frac{P_{n+1}(s_1)_q P_n(s_2)_q - P_{n+1}(s_2)_q P_n(s_1)_q}{x(s_1)-x(s_2)}.$$

Moreover, in [7], we have studied the modifications of the *non-standard* q-Racah polynomials defined on the lattice $x(s)=[s]_q[s+1]_q$ [6, 14], where $[s]_q$ denotes the symmetric q-numbers $[s]_q=(q^{s/2}-q^{-s/2})/(q^{1/2}-q^{-1/2})$, whereas in [8] we developed the theory for the Askey-Wilson-Krall polynomials. Here, in this paper we will consider the "*standard*" (or classical) q-Racah-Krall polynomials defined on the lattice $x(s)=q^{-s}+\delta q^{-N}q^s$ introduced in [9] (see also [16, page 422]). For doing that we will follow the general results obtained in [7, section 2, 3].

The structure of the paper is as follows. In Section 2, the *standard* q-Racah-Krall polynomials are introduced and all their main characteristics are studied in detail. Finally in Section 3, some important limit cases are considered.

2. Standard q-Racah-Krall polynomials

In this section, we study the *standard* q-Racah-Krall polynomials, i.e., the polynomials obtained from the *standard* q-Racah polynomials $R_n^{\alpha,\beta}(s)_q$ via the addition of two mass points. In the following we follow the notations introduced in [7].

2.1. **Preliminaries.** The *standard* q-Racah $R_n^{\alpha,\beta}(s)_q := R_n(x(s),\alpha,\beta,\gamma,\delta|q)$ are polynomials on the q-quadratic lattice $x(s)=q^{-s}+\delta\gamma q^{s+1}$ introduced in [9] (see also the more recent book [16, page 422]). They are defined by the following basic series (for properties of basic series see [12])

$$R_n(x(s),\alpha,\beta,\gamma,\delta|q) = \frac{(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}\,{}_4\varphi_3\left(\begin{matrix} q^{-n},\alpha\beta q^{n+1},q^{-s},\delta\gamma q^{s+1} \\ \alpha q,\beta\delta q,\gamma q\end{matrix}\,\middle|\, q;q\right), \tag{2}$$

where $x(s)=q^{-s}+\delta\gamma q^{s+1}$, and $\alpha q = q^{-N}$, $\beta\delta q = q^{-N}$, $\gamma q = q^{-N}$, N, a nonnegative integer. Their main data can be found in table 1. Notice from (2) that we have

$$R_n^{\alpha,\beta}(0)_q := R_n(x(0),\alpha,\beta,\gamma,\delta|q) = \frac{(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n},$$

and $R_n^{\alpha,\beta}(N)_q := R_n(x(N),\alpha,\beta,\gamma,\delta|q)$

$$R_n^{\alpha,\beta}(N)_q = (\delta\gamma q^{N+1})^n \frac{(q^{-N},\beta\gamma^{-1}q^{-N},\delta^{-1}q^{-N};q)_n}{(\beta q^{n-N};q)_n}, \quad \text{if } \alpha q = q^{-N},$$

$$R_n^{\alpha,\beta}(N)_q = \delta^n \frac{(q^{-N},\beta q,\delta^{-1}\alpha q;q)_n}{(\alpha\beta q^{n+1};q)_n}, \quad \text{if } \gamma q = q^{-N},$$

$$R_n^{\alpha,\beta}(N)_q = (\beta^{-1}\gamma)^n \frac{(q^{-N},\beta q,\delta^{-1}\gamma^{-1}\alpha q^{-N};q)_n}{(\alpha\beta q^{n+1};q)_n}, \quad \text{if } \beta\delta q = q^{-N}.$$

We also need the following identity for the *standard* q-Racah polynomials which follows from [7, Eq. (7)]

$$R_{n-1}^{\alpha,\beta}(s)_q = \Theta(s,n)R_n^{\alpha,\beta}(s)_q + \Xi(s,n)R_n^{\alpha,\beta}(s+1)_q, \tag{3}$$

TABLE 1. Main data of the monic q-Racah polynomials

$P_n(s)$	$R_n(x(s),\alpha,\beta,\gamma,\delta\vert q)\,,\quad x(s)=q^{-s}+\delta\gamma q^{s+1}\,,\quad \Delta x(s)=q^{-s}(1-\delta\gamma q^{2s+2})(q^{-1}-1)$
(a,b)	$[0,N]$
$\rho(s)$	$\dfrac{(\alpha\beta)^{-s}}{(q^{-1/2}-q^{1/2})}\dfrac{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}{(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q)_\infty}\dfrac{(\delta\gamma q,\alpha q,\beta\delta q,\gamma q;q)_s}{(q,\alpha^{-1}\delta\gamma q,\beta^{-1}\gamma q,\delta q;q)_s}$
$\sigma(s)$	$\delta^2q^{-2N}(q^{1/2}-q^{-1/2})^2q^{-2s}(q^s-1)(q^s-\delta^{-1})(q^s-\beta\gamma^{-1})(q^s-\alpha\delta^{-1}\gamma^{-1})$
$\Phi(s)$	$(q^{1/2}-q^{-1/2})^2q^{-2s}(1-\alpha q^{s+1})(1-\beta\delta q^{s+1})(1-\gamma q^{s+1})(1-\delta\gamma q^{s+1})$
$\tau(s)$	$\frac{(q^{-1/2}-q^{1/2})q^{-s}}{(1-\gamma\delta q^{2s+1})}\Big[(1-\alpha q^{s+1})(1-\beta\delta q^{s+1})(1-\gamma q^{s+1})(1-\delta\gamma q^{s+1})-(\delta\gamma q)^2(q^s-1)$ $\times(q^s-\delta^{-1})(q^s-\beta\gamma^{-1})(q^s-\alpha\delta^{-1}\gamma^{-1})\Big]$
$\tau_n(s)$	$-q^{-n}(q^{1/2}-q^{-1/2})\Big\{(1-\alpha\beta q^{2n+2})x(s+\frac{n}{2})+q^{-n/2}\Big[(1-\alpha q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})$ $-(1+\delta\gamma q^{n+1})(1-\alpha\beta q^{2n+2})\Big]\Big\}$
λ_n	$-q^{-n+\frac{1}{2}}(1-q^n)(1-\alpha\beta q^{n+1})$
d_n^2	$\dfrac{(1-\alpha\beta q)(\delta\gamma q)^n}{(1-\alpha\beta q^{2n+1})}\dfrac{(\alpha q,\beta\delta q,\gamma q,q,q\alpha\beta\gamma^{-1},\alpha\delta^{-1}q,\beta q;q)_n}{(\alpha\beta q,\alpha\beta q^{n+1},\alpha\beta q^{n+1};q)_n}$
β_n	$1+\delta\gamma q-\dfrac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}$ $-\dfrac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}$
γ_n	$\dfrac{(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})}\,\dfrac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}$
$\overline{\alpha}_n=\widehat{\alpha}_n$	$q^{-n+\frac{1}{2}}(q^{-1/2}-q^{1/2})(1-\alpha\beta q^{2n+1})$
$\overline{\beta}_n(s)$	$q^{-n/2+1/2}(q^{1/2}-q^{-1/2})\dfrac{(1-\alpha\beta q^{n+1})}{(1-\alpha\beta q^{2n+2})}\Big\{(1-\alpha\beta q^{2n+2})x(s+\dfrac{n}{2})+q^{-n/2}\Big[(1-\alpha q^{n+1})$ $\times(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})-(1+\delta\gamma q^{n+1})(1-\alpha\beta q^{2n+2})\Big]\Big\}$
$\widehat{\beta}_n(s)$	$\overline{\beta}_n(s)-q^{-s-n+\frac{1}{2}}(q^{1/2}-q^{-1/2})(1-q^n)(1-\alpha\beta q^{n+1})(1-\delta\gamma q^{2s+1})$

where

$$
\begin{aligned}
\Theta(s,n) &= -\Xi(s,n)+\frac{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})^2}{(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^n)(1-\beta q^n)}\\
&\times\frac{1}{(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}\Big[(q^{-s}-1)(1-\delta\gamma q^{s+1})+\frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}\\
&\times(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})+\frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}\Big]\\
&-\frac{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})^2}{(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)}\\
&\times\frac{1}{(\delta-\alpha q^n)}\Bigg\{q^{n/2}\frac{(1-\alpha\beta q^{n+1})}{(1-\alpha\beta q^{2n+2})}\Big\{(1-\alpha\beta q^{2n+2})x_n(s)+q^{-n/2}\Big[(1-\alpha q^{n+1})\\
&\times(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})-(1+\delta\gamma q^{n+1})(1-\alpha\beta q^{2n+2})\Big]\Big\}-q^{-s}(1-q^n)\\
&\times(1-\alpha\beta q^{n+1})(1-\delta\gamma q^{2s+1})\Bigg\},
\end{aligned}
$$

$$\Xi(s,n) = -\frac{q^{-s-1+n}(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})^2(1-\alpha q^{s+1})}{(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^n)(1-\beta q^n)} \times \frac{(1-\beta\delta q^{s+1})(1-q^{-N+s})(1-\delta\gamma q^{s+1})}{(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)(1-\delta\gamma q^{2s+2})}.$$

2.2. **The case of two mass point.** Let us now consider the modification of the *standard* q-Racah polynomials defined in (2). Let $\mathfrak{u}$ be the functional with respect to which the *standard* q-Racah polynomials (2) are orthogonal. Then we study the orthogonal polynomials that are orthogonal with respect to the following linear functional $\widetilde{\mathfrak{u}} = \mathfrak{u} + A\delta(x(s)-x(0)) + B\delta(x(s)-x(N))$. In other words, the q-Racah-Krall polynomials, $R_n^{\alpha,\beta,A,B}(s)_q := R_n^{A,B}(x(s),\alpha,\beta,\gamma,\delta|q)$ satisfying the following orthogonality relation

$$\sum_{s=0}^{N} R_n^{\alpha,\beta,A,B}(s)_q R_m^{\alpha,\beta,A,B}(s)_q \rho(s)\Delta x(s-1/2) + A R_n^{\alpha,\beta,A,B}(0)_q R_m^{\alpha,\beta,A,B}(0)_q + B R_n^{\alpha,\beta,A,B}(N)_q R_m^{\alpha,\beta,A,B}(N)_q = \delta_{n,m}\widetilde{d}_n^2, \tag{4}$$

where ρ is the *standard* q-Racah weight function (see table 1 (as in [7] we have chosen $\rho(s)$ to be a probability measure). From [7, Eqs. (31)-(32)] it follows that

$$\begin{aligned} R_n^{\alpha,\beta,A,B}(0)_q &= \frac{(1+B\mathrm{K}_{n-1}^{\alpha,\beta}(N,N))R_n^{\alpha,\beta}(0)_q - B\mathrm{K}_{n-1}^{\alpha,\beta}(0,N)R_n^{\alpha,\beta}(N)_q}{\kappa_{n-1}^{\alpha,\beta}(0,N)}, \\ R_n^{\alpha,\beta,A,B}(N)_q &= \frac{-A\mathrm{K}_{n-1}^{\alpha,\beta}(N,0)R_n^{\alpha,\beta}(0)_q + (1+A\mathrm{K}_{n-1}^{\alpha,\beta}(0,0))R_n^{\alpha,\beta}(N)_q}{\kappa_{n-1}^{\alpha,\beta}(0,N)}, \end{aligned} \tag{5}$$

$$\widetilde{d}_n^2 = d_n^2 + \frac{A(R_n^{\alpha,\beta}(0)_q)^2\{1+B\mathrm{K}_{n-1}^{\alpha,\beta}(N,N)\} + B(R_n^{\alpha,\beta}(N)_q)^2\{1+A\mathrm{K}_{n-1}^{\alpha,\beta}(0,0)\}}{\kappa_{n-1}^{\alpha,\beta}(0,N)} - \frac{2ABR_n^{\alpha,\beta}(0)_q R_n^{\alpha,\beta}(N)_q \mathrm{K}_{n-1}^{\alpha,\beta}(0,N)}{\kappa_{n-1}^{\alpha,\beta}(0,N)},$$

where

$$\begin{aligned} \kappa_m^{\alpha,\beta}(s,t) &= 1 + A\mathrm{K}_m^{\alpha,\beta}(s,s) + B\mathrm{K}_m^{\alpha,\beta}(t,t) \\ &\quad + AB\left\{\mathrm{K}_m^{\alpha,\beta}(s,s)\mathrm{K}_m^{\alpha,\beta}(t,t) - (\mathrm{K}_m^{\alpha,\beta}(s,t))^2\right\}, \end{aligned} \tag{6}$$

where $\mathrm{K}_m^{\alpha,\beta}(s,t)$ are the kernels $\mathrm{K}_m^{\alpha,\beta}(s,t) = \sum_{k=0}^{m} R_k^{\alpha,\beta}(s)_q R_k^{\alpha,\beta}(t)_q / d_k^2$, and d_n^2 denotes the squared norm of the n-th *standard* q-Racah polynomials (see table 1). Notice that $\forall A,B>0$ and $a\neq b$, $\mathrm{K}_m^{\alpha,\beta}(a,b)>0$. Thus, by the Proposition 1 in [7] the polynomials $R_n^{\alpha,\beta,A,B}(s)_q$ are well defined for all values $A,B>0$.

Remark 1. *If A,B are in general complex numbers then, according to Proposition 1 in* [7], *in order that there exists a sequence of orthogonal polynomials $(R_n^{\alpha,\beta,A,B}(s)_q)$ the condition $\kappa_{n-1}^{\alpha,\beta}(0,N)\neq 0$, where $\kappa_{n-1}^{\alpha,\beta}(0,N)$ is defined in (6), should be hold for all $n\in\mathbb{N}$, $A,B\in\mathbb{C}$.*

Representation formulas for $R_n^{\alpha,\beta,A,B}(s)_q$. Let us now obtain some explicit formulas for the q-Racah-Krall polynomials. The first formula follows from [7, Eq. (29)] (see also (1) from above)

$$R_n^{\alpha,\beta,A,B}(s)_q = R_n^{\alpha,\beta}(s)_q - AR_n^{\alpha,\beta,A,B}(0)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,0) - BR_n^{\alpha,\beta,A,B}(N)_q \mathrm{K}_{n-1}^{\alpha,\beta}(s,N) \tag{7}$$

where

$$\mathrm{K}_{n-1}^{\alpha,\beta}(s,0) = \varkappa_0^{\alpha,\beta}(s,n) R_{n-1}^{\alpha,\beta}(s)_q + \overline{\varkappa}_0^{\alpha,\beta}(s,n) \frac{\nabla R_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)}, \tag{8}$$

$$\begin{aligned}
\varkappa_0^{\alpha,\beta}(s,n) =& \frac{(\delta\gamma q)^{-n+1}(1-\alpha\beta q^n)(1-\delta\gamma q^{s+n})(\alpha\beta q, \alpha\beta q^{n+1}; q)_{n-1}}{(1-\alpha\beta q)(1-\delta\gamma q^{s+1})(q, \alpha\beta\gamma^{-1}q, \alpha\delta^{-1}q, \beta q; q)_{n-1}}, \\
\overline{\varkappa}_0^{\alpha,\beta}(s,n) =& \frac{q^{2-s}(\delta\gamma)^{-n+3}(\alpha\beta q, \alpha\beta q^{n+1}; q)_{n-1}(q^s-1)(q^s-\delta^{-1})}{(1-\alpha\beta q)(1-\delta\gamma q^{2s+2})} \\
& \times \frac{(q^s-\beta\gamma^{-1})(q^s-\alpha\delta^{-1}\gamma^{-1})}{(q, \alpha\beta\gamma^{-1}q, \alpha\delta^{-1}q, \beta q; q)_{n-1}},
\end{aligned} \tag{9}$$

and

$$\mathrm{K}_{n-1}^{\alpha,\beta}(s,N) = \varkappa_N^{\alpha,\beta}(s,n) R_{n-1}^{\alpha,\beta}(s)_q + \overline{\varkappa}_N^{\alpha,\beta}(s,n) \frac{\Delta R_{n-1}^{\alpha,\beta}(s)_q}{\Delta x(s)}, \tag{10}$$

$$\begin{aligned}
\varkappa_N^{\alpha,\beta}(s,n) =& \frac{(\gamma)^{-n+1}(1-\alpha\beta q^n)(1-\delta q^{s-n+1})(\alpha\beta q, \alpha\beta q^{n+1}; q)_{n-1}}{(1-\alpha\beta)(1-\delta q^s)(\beta^{-1}\delta^{-1}q^{-n}, \alpha q, \beta\delta q, q, \alpha\beta\gamma^{-1}q; q)_{n-1}}, \\
\overline{\varkappa}_N^{\alpha,\beta}(s,n) =& \frac{(\gamma)^{-n+1}q^{-s}(1-\alpha q^{s+1})(1-\beta\delta q^{s+1})(1-\gamma q^{s+1})(1-\delta\gamma q^{s+1})}{(1-\alpha\beta q)(1-\delta\gamma q^{2s+2})} \\
& \times \frac{(\alpha\beta q, \alpha\beta q^{n+1}; q)_{n-1}}{(\beta^{-1}\delta^{-1}q^{-n}, \alpha q, \beta\delta q, q, \alpha\beta\gamma^{-1}q; q)_{n-1}}.
\end{aligned}$$

Inserting (8) and (10) into the formula (7) one finds the $1st$ representation formula

$$\begin{aligned}
R_n^{\alpha,\beta,A,B}(s)_q = R_n^{\alpha,\beta}(s)_q + \overline{A}(s,n) R_{n-1}^{\alpha,\beta}(s)_q + \overline{B}(s,n) \frac{\nabla R_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)} \\
+ \overline{C}(s,n) \frac{\Delta R_{n-1}^{\alpha,\beta}(s)_q}{\Delta x(s)}
\end{aligned} \tag{11}$$

with

$$\overline{A}(s,n) = -AR_n^{\alpha,\beta,A,B}(0)_q \varkappa_0^{\alpha,\beta}(s,n) - BR_n^{\alpha,\beta,A,B}(N)_q \varkappa_N^{\alpha,\beta}(s,n),$$

$$\overline{B}(s,n) = -AR_n^{\alpha,\beta,A,B}(0)_q \overline{\varkappa}_0^{\alpha,\beta}(s,n), \quad \overline{C}(s,n) = -BR_n^{\alpha,\beta,A,B}(N)_q \overline{\varkappa}_N^{\alpha,\beta}(s,n),$$

where $R_n^{\alpha,\beta,A,B}(0)_q$ and $R_n^{\alpha,\beta,A,B}(N)_q$ are given in (5). Notice that $R_n^{\alpha,\beta,A,B}(s)_q$ defined in (7) is a polynomial of degree n in $x(s)$. However, it is not easy to see that $R_n^{\alpha,\beta,A,B}(s)_q$ is a polynomial of degree n in $x(s)$ by the formula (11) since the functions $\overline{A}$, $\overline{B}$ and $\overline{C}$ as well as $\nabla R_{n-1}^{\alpha,\beta}(s)_q/\nabla x(s)$ and $\Delta R_{n-1}^{\alpha,\beta}(s)_q/\Delta x(s)$ in (11) are not, in general, polynomials in $x(s)$.

The $2nd$ representation formula of the q-Racah-Krall polynomials follows from [7, section 3, (17)] and has the following form

$$\phi(s) R_n^{\alpha,\beta,A,B}(s)_q = A(s,n) R_n^{\alpha,\beta}(s)_q + B(s,n) R_{n-1}^{\alpha,\beta}(s)_q, \tag{12}$$

$$\phi(s) = (1-\delta\gamma q^{N+1}q^s)(1-\delta\gamma q^{s+1})(q^{-s}-1)(q^{-s}-q^{-N}),$$

$$\begin{aligned} A(s,n) &= \phi(s) - \frac{1}{d_{n-1}^2}\Big\{AR_n^{\alpha,\beta,A,B}(0)_q R_{n-1}^{\alpha,\beta}(0)_q (q^{-s}-q^{-N})(1-\delta\gamma q^{N+1}q^s) \\ &\qquad + BR_n^{\alpha,\beta,A,B}(N)_q R_{n-1}^{\alpha,\beta}(N)_q (1-\delta\gamma q^{s+1})(q^{-s}-1)\Big\}, \\ B(s,n) &= \frac{1}{d_{n-1}^2}\Big\{AR_n^{\alpha,\beta,A,B}(0)_q R_n^{\alpha,\beta}(0)_q (q^{-s}-q^{-N})(1-\delta\gamma q^{N+1}q^s) \\ &\qquad + BR_n^{\alpha,\beta,A,B}(N)_q R_n^{\alpha,\beta}(N)_q (1-\delta\gamma q^{s+1})(q^{-s}-1)\Big\}, \end{aligned} \tag{13}$$

where $R_n^{\alpha,\beta,A,B}(0)_q$ and $R_n^{\alpha,\beta,A,B}(N)_q$ are given in (5). Note that the functions $\phi(s)$ and $A(s,n)$ are polynomials of degree 2 in $x(s)$ and $B(s,n)$ is 1st degree polynomial in $x(s)$. Therefore it is obvious from the right hand side of the formula (12) that $\phi(s)R_n^{\alpha,\beta,A,B}(s)_q$ is a polynomial of degree $n+2$ in $x(s)$.

Another representation formula for the q-Racah-Krall polynomials may be obtained by putting the relation (3) into (12)

$$\phi(s)R_n^{\alpha,\beta,A,B}(s)_q = a(s;n)R_n^{\alpha,\beta}(s)_q + b(s;n)R_n^{\alpha,\beta}(s+1)_q,$$

where $a(s;n) = A(s;n) + B(s;n)\Theta(s;n)$, $b(s;n) = B(s;n)\Xi(s;n)$, and A, B and Θ, Ξ are given by (13) and (3), respectively.

We note that from Proposition 3 in [7] it follows that the polynomials $R_n^{\alpha,\beta,A,B}(s)_q$ satisfy a second order linear difference equation where the coefficients can be computed explicitly. Since the expression is large enough we will omit them here.

Finally, by use of [7, Eq (33), section 3] one can obtain the TTRR for $R_n^{\alpha,\beta,A,B}(s)_q$

$$x(s)R_n^{\alpha,\beta,A,B}(s)_q = R_{n+1}^{\alpha,\beta,A,B}(s)_q + \widetilde{\beta}_n R_n^{\alpha,\beta,A,B}(s)_q + \widetilde{\gamma}_n R_{n-1}^{\alpha,\beta,A,B}(s)_q$$

with the following coefficients

$$\begin{aligned} \widetilde{\beta}_n &= \beta_n - A\left(\frac{R_n^{\alpha,\beta,A,B}(0)_q R_{n-1}^{\alpha,\beta}(0)_q}{d_{n-1}^2} - \frac{R_{n+1}^{\alpha,\beta,A,B}(0)_q R_n^{\alpha,\beta}(0)_q}{d_n^2}\right) \\ &\quad - B\left(\frac{R_n^{\alpha,\beta,A,B}(N)_q R_{n-1}^{\alpha,\beta}(N)_q}{d_{n-1}^2} - \frac{R_{n+1}^{\alpha,\beta,A,B}(N)_q R_n^{\alpha,\beta}(N)_q}{d_n^2}\right), \\ \widetilde{\gamma}_n &= \gamma_n \frac{1+\Delta_n^{\alpha,\beta,A,B}}{1+\Delta_{n-1}^{\alpha,\beta,A,B}},\ \Delta_n^{\alpha,\beta,A,B} = \frac{AR_n^{\alpha,\beta,A,B}(0)_q R_n^{\alpha,\beta}(0)_q}{d_n^2} + \frac{BR_n^{\alpha,\beta,A,B}(N)_q R_n^{\alpha,\beta}(N)_q}{d_n^2} \end{aligned} \tag{14}$$

where we use the notations defined in Eqs. (5).

Representation of $R_n^{\alpha,\beta,A,B}(s)_q$ in terms of basic series. Let us now introduce the representation of $R_n^{\alpha,\beta,A,B}(s)_q$ in terms of the basic hypergeometric series. To this end, we substitute (2) into the formula (12) which leads to

$$\phi(s)R_n^{\alpha,\beta,A,B}(s)_q = \frac{(\alpha q;q)_{n-1}(\beta\delta q;q)_{n-1}(\gamma q;q)_{n-1}}{(\alpha\beta q^n;q)_{n-1}} \times \sum_{k=0}^{\infty} \frac{(q^{-n},\alpha\beta q^n, q^{-s}, \delta\gamma q^{s+1};q)_k}{(\alpha q, \beta\delta q, \gamma q, q;q)_k} q^k \Pi_1(q^k),$$

where $\phi(s)$, $A(s,n)$ and $B(s,n)$ are given in (13) and

$$\begin{aligned}\Pi_1(q^k) &= A(s,n)\frac{(1-\alpha q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-\alpha\beta q^{n+k})}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})} + B(s,n)\frac{(1-q^{-n+k})}{(1-q^{-n})}\\ &= -\frac{1}{1-q^{-n}}\Big\{\alpha\beta q^n A(s,n)\vartheta_n^{\alpha,\beta,\delta,\gamma} + B(s,n)q^{-n}\Big\}(q^k - q^{\beta_1}),\end{aligned} \tag{15}$$

where

$$q^{\beta_1} = \frac{A(s,n)\vartheta_n^{\alpha,\beta,\delta,\gamma} + B(s,n)}{A(s,n)\alpha\beta q^n\vartheta_n^{\alpha,\beta,\delta,\gamma} + B(s,n)q^{-n}},$$

$$\vartheta_n^{\alpha,\beta,\delta,\gamma} = \frac{(1-\alpha q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^{-n})}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})}.$$

Moreover, by use of the identity $(q^k - q^z)(q^{-z};q)_k = (1-q^z)(q^{1-z};q)_k$ we arrive at the following representation

$$\phi(s)R_n^{\alpha,\beta,A,B}(s)_q = D_n^{\alpha,\beta,\beta_1}(s)\,{}_5\varphi_4\left(\begin{matrix} q^{-n}, \alpha\beta q^n, q^{-s}, \delta\gamma q^{s+1}, q^{1-\beta_1}\\ \alpha q, \beta\delta q, \gamma q, q^{-\beta_1}\end{matrix}\,\middle|\, q, q\right), \tag{16}$$

where

$$\begin{aligned}D_n^{\alpha,\beta,\beta_1}(s) &= -\frac{(\alpha q;q)_{n-1}(\beta\delta q;q)_{n-1}(\gamma q;q)_{n-1}}{(\alpha\beta q^n;q)_{n-1}}(1-q^{\beta_1})\\ &\quad\times\frac{1}{1-q^{-n}}\Big\{A(s,n)\alpha\beta q^n\vartheta_n^{\alpha,\beta,\delta,\gamma} + B(s,n)q^{-n}\Big\}.\end{aligned}$$

Remark 2. *Notice that $\phi(s)R_n^{\alpha,\beta,A,B}(s)_q$ in (16) is a polynomial of degree $n+2$ in $x(s)$. To see that formula (16) gives a polynomial of degree $n+2$ it is sufficient to notice that the function Π_1 defined in (15) is a polynomial of degree 2 in $x(s)$ since $A(s,n)$ and $B(s,n)$ defined in (13) are polynomials of degree 2 and 1 in $x(s)$, respectively.*

Finally let us mention that the direct substitution of (2) into (12) leads to the following representation formula

$$\begin{aligned}\phi(s)R_n^{\alpha,\beta,A,B}(s)_q &= A(s,n)\Lambda_n^{\alpha,\beta,\delta,\gamma}\,{}_4\varphi_3\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-s}, \delta\gamma q^{s+1}\\ \alpha q, \beta\delta q, \gamma q\end{matrix}\,\middle|\, q\,,\, q\right)\\ &\quad + B(s,n)\Lambda_{n-1}^{\alpha,\beta,\delta,\gamma}\,{}_4\varphi_3\left(\begin{matrix} q^{-n+1}, \alpha\beta q^{n}, q^{-s}, \delta\gamma q^{s+1}\\ \alpha q, \beta\delta q, \gamma q\end{matrix}\,\middle|\, q\,,\, q\right),\end{aligned}$$

where

$$\Lambda_n^{\alpha,\beta,\delta,\gamma} = \frac{(\alpha q, \beta\delta q, \gamma q;q)_n}{(\alpha\beta q^n;q)_n}.$$

2.3. **The case of one mass point.** In this section we introduce the *standard* q-Racah polynomials but with one mass point at the value $s=0$. All the formulas follow by replacing $B=0$ into the ones in section 2.2. For example the first representation formula of $R_n^{\alpha,\beta,A}(s)_q$ is produced by inserting $B=0$ into (11),

$$R_n^{\alpha,\beta,A}(s)_q = R_n^{\alpha,\beta}(s)_q + \overline{A}(s,n)R_{n-1}^{\alpha,\beta}(s)_q + \overline{B}(s,n)\frac{\nabla R_{n-1}^{\alpha,\beta}(s)_q}{\nabla x(s)},$$

where

$$\overline{A}(s,n) = -AR_n^{\alpha,\beta,A}(0)_q\varkappa_0^{\alpha,\beta}(s,n), \quad \overline{B}(s,n) = -AR_n^{\alpha,\beta,A}(0)_q\overline{\varkappa}_0^{\alpha,\beta}(s,n),$$

and

$$R_n^{\alpha,\beta,A}(0)_q = \frac{R_n^{\alpha,\beta}(0)_q}{1 + A\mathrm{K}_{n-1}^{\alpha,\beta}(0,0)} \tag{17}$$

in which $\varkappa_0^{\alpha,\beta}(s,n)$ and $\overline{\varkappa}_0^{\alpha,\beta}(s,n)$ are defined in (9), respectively. Moreover, the second representation formula follows by evaluating (12) for $B = 0$

$$\phi(s)R_n^{\alpha,\beta,A}(s)_q = A(s;n)R_n^{\alpha,\beta}(s)_q + B(s;n)R_{n-1}^{\alpha,\beta}(s)_q,$$

where $\phi(s) = (q^{-s} - 1)(1 - \delta\gamma q^{s+1})$,

$$A(s,n) = \phi(s) - \frac{A}{d_{n-1}^2} R_n^{\alpha,\beta,A}(0)_q R_{n-1}^{\alpha,\beta}(0)_q, \quad B(s,n) = \frac{A}{d_{n-1}^2} R_n^{\alpha,\beta,A}(0)_q R_n^{\alpha,\beta}(0)_q, \tag{18}$$

and $R_n^{\alpha,\beta,A}(0)_q$ is given in (17). Finally, the third representation formula follows by use of the same idea

$$\phi(s)R_n^{\alpha,\beta,A}(s)_q = a(s;n)R_n^{\alpha,\beta}(s)_q + b(s;n)R_n^{\alpha,\beta}(s+1)_q, \tag{19}$$

where $a(s;n) = A(s;n) + B(s;n)\Theta(s;n)$ and $b(s;n) = B(s;n)\Xi(s;n)$, and A, B and Θ, Ξ are given by (18) and (3), respectively. Notice that, as for two mass points case, from the above representation formula (19) the second order difference equation for $R_n^{\alpha,\beta,A}(s)_q$ follows [1, 7].

Furthermore, one can obtain the coefficients of the TTRR by replacing $B = 0$ into (14) as the following

$$\widetilde{\beta}_n = \beta_n - A\left(\frac{R_n^{\alpha,\beta,A}(0)_q R_{n-1}^{\alpha,\beta}(0)_q}{d_{n-1}^2} - \frac{R_{n+1}^{\alpha,\beta,A}(0)_q R_n^{\alpha,\beta}(0)_q}{d_n^2}\right),$$

$$\widetilde{\gamma}_n = \gamma_n \frac{1 + \Delta_n^{\alpha,\beta,A}}{1 + \Delta_{n-1}^{\alpha,\beta,A}}, \quad \Delta_n^{\alpha,\beta,A} = \frac{A R_n^{\alpha,\beta,A}(0)_q R_n^{\alpha,\beta}(0)_q}{d_n^2}.$$

Notice that putting $B = 0$ in the basic series representation formulas (16) we obtain the corresponding basic series representations for the q-Racah-Krall polynomials $R_n^{\alpha,\beta,A}(s)_q$.

2.4. **Some limit cases.** We first consider the modification of *standard* dual q-Hahn polynomials defined on the lattice $x(s) = q^{-s} + \gamma\delta q^{s+1}$ by

$$R_n(x(s), \gamma, \delta, N|q) = (q^{-N}, \gamma q; q)_n {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-s}, \delta\gamma q^{s+1} \\ q^{-N}, \gamma q \end{matrix}\,\middle|\, q; q\right),$$

that are related with the q-Racah polynomials by the expression [16]

$$\lim_{\beta \to 0} R_n^{\alpha,\beta}(s)_q = R_n(s)_q. \tag{20}$$

In order to obtain the modification of *standard* dual q-Hahn polynomials by adding two mass points at the end of the interval of orthogonality, i.e., $R_n^{A,B}(s)_q$ satisfying the orthogonality relation

$$\begin{aligned} \sum_{s=0}^{N} R_n^{A,B}(s)_q R_m^{A,B}(s)_q \rho(s) \Delta x(s - 1/2) + A R_n^{A,B}(0)_q R_m^{A,B}(0)_q \\ + B R_n^{A,B}(N)_q R_m^{A,B}(N)_q = \delta_{n,m} \widetilde{d}_n^2. \end{aligned} \tag{21}$$

where

$$\rho(s)=\frac{q^{Ns-\binom{s}{2}}(\gamma q)^N}{(-\gamma)^s(1-\gamma\delta q)(q^{-1/2}-q^{1/2})}\frac{(\delta q;q)_N}{(\gamma\delta q^2;q)_N}\frac{(\gamma q,\gamma\delta q,q^{-N};q)_s}{(q,\delta\gamma q^{N+2},\delta q;q)_s}.$$

Using the same procedure as before it follows that

$$R_n^{A,B}(0)_q=\frac{(1+B\mathrm{K}_{n-1}(N,N))R_n(0)_q-B\mathrm{K}_{n-1}(0,N)R_n(N)_q}{\kappa_{n-1}(0,N)},$$

$$R_n^{A,B}(N)_q=\frac{-A\mathrm{K}_{n-1}(N,0)R_n(0)_q+(1+A\mathrm{K}_{n-1}(0,0))R_n(N)_q}{\kappa_{n-1}(0,N)},$$

$$\widetilde{d}_n^2=d_n^2+\frac{A(R_n(0)_q)^2\{1+B\mathrm{K}_{n-1}(N,N)\}+B(R_n(N)_q)^2\{1+A\mathrm{K}_{n-1}(0,0)\}}{\kappa_{n-1}(0,N)}-\frac{2ABR_n(0)_qR_n(N)_q\mathrm{K}_{n-1}(0,N)}{\kappa_{n-1}(0,N)}.$$

In the above formulas

$$\kappa_m(s,t)=1+A\mathrm{K}_m(s,s)+B\mathrm{K}_m(t,t)+AB\left\{\mathrm{K}_m(s,s)\mathrm{K}_m(t,t)-(\mathrm{K}_m(s,t))^2\right\},$$

$\mathrm{K}_m(s,t)=\sum_{k=0}^m R_n(s)_qR_n(t)_q/d_k^2$, and d_n^2 denote m-th Kernel and the norm of the *standard* dual q-Hahn polynomials

$$d_n^2=(\gamma\delta q)^n(q,\delta^{-1}q^{-N},\gamma q,q^{-N};q)_n.$$

If we now fix $\alpha q=q^{-N}$ in the orthogonality relation for the *standard* q-Racah polynomials (4) and take the limit $\beta\to 0$, then using (20) we obtain the orthogonality relation (21), and therefore, it is straightforward to see that

$$\lim_{\beta\to 0}R_n^{\alpha,\beta,A,B}(s)_q=R_n^{A,B}(s)_q.$$

Thus, all properties of the modified dual q-Hahn polynomials $R_n^{A,B}(s)_q$ can be obtained from the corresponding properties of the modified q-Racah polynomials $R_n^{\alpha,\beta,A,B}(s)_q$ by taking the appropriate limit.

To conclude this paper we discuss two other important limit cases of the q-Racah-Krall polynomials $R_n^{\alpha,\beta,A,B}(s)_q$.

The first one is when we take the limit $q\to 1$. In fact, as $q\to 1$ in (2) we recover the Racah polynomials in the quadratic lattice $x(s)=s(s+\gamma+\delta+1)$ [16], i.e.,

$$\lim_{q\to 1}R_n(q^{-s}+\delta\gamma q^{s+1},q^\alpha,q^\beta,q^\gamma,q^\delta)_q=R_n(s(s+\gamma+\delta+1),\alpha,\beta,\gamma,\delta),$$

where

$$R_n^{\alpha,\beta}(s)=\frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+n+1)_n}\times {}_4\mathrm{F}_3\left(\begin{matrix}-n,\alpha+\beta+n+1,-s,s+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\,\middle|\,1\right).$$

Straightforward calculations show that all the properties of the *standard* q-Racah polynomials $R_n^{\alpha,\beta}(s)_q$ becomes into the properties of the *standard* Racah ones. In particular, we have the following limit relation

$$\lim_{q\to 1}R_n^{\alpha,\beta,A,B}(q^{-s}+\delta\gamma q^{s+1})_q=R_n^{\alpha,\beta,A,B}(s(s+\delta+\gamma+1)),$$

where $R_n^{\alpha,\beta,A,B}(s(s+\delta+\gamma+1))$ denotes the modification of the Racah polynomials by adding two delta Dirac masses at the points 0 and N. Thus, taking appropriate limits one can construct the analogue of the Askey Tableau but for the Krall type polynomials.

Another important family of Krall-type polynomials are the so called q-Hahn-Krall tableau of orthogonal polynomials considered in [5, 7]. In order to obtain it first of all notice that the *standard* q-Racah polynomials are defined on the lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ where $c_1 = \gamma\delta q$, $c_2 = 1$ and $c_3 = 0$. Then, making the transformation $q^{\delta-N}x(s) = c_1\mu(s)$, $\alpha = \nu$, $\beta = \mu$, $\gamma q \to q^{-N}$ and $\delta \to q^\delta$ in (2) and then taking the limit $q^\delta \to 0$ by use of the identity [2]

$$(q^{-s};q)_k(q^{s+\zeta};q)_k = (-1)^k q^{k(\zeta+\frac{k-1}{2})}\prod_{i=0}^{k-1}\Big[\frac{x(s)-c_3}{c_1} - q^{-\frac{\zeta}{2}}(q^{i+\frac{\zeta}{2}} + q^{-i-\frac{\zeta}{2}})\Big],$$

we obtain

$$C_n R_n(x(s),\alpha,\beta,\gamma,\delta|q) \stackrel{q^\delta\to 0}{\longrightarrow} h_n^{\mu,\nu}(\mu(s);N|q), \tag{22}$$

where $h_n^{\mu,\nu}(\mu(s);N|q)$ are the q-Hahn polynomials on the lattice $\mu(s) = q^{-s}$

$$h_n^{\mu,\nu}(\mu(s);N|q) := \frac{(\nu q;q)_n(q^{-N};q)_n}{(\mu\nu q^{n+1};q)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, \mu\nu q^{n+1}, \mu(s) \\ \nu q, q^{-N}\end{matrix}\,\Bigg|\, q\,,\,q\right),$$

and $C_n := 1$. Then we get the q-Hahn-Krall polynomials by use of the above limit relation and the property that

$$C_n R_n^{\alpha,\beta}(0) \stackrel{q^\delta\to 0}{\longrightarrow} h_n^{\mu,\nu}(\mu(0);N|q) \quad C_n R_n^{\alpha,\beta}(N) \stackrel{q^\delta\to 0}{\longrightarrow} h_n^{\mu,\nu}(\mu(N);N|q), \tag{23}$$

and

$$C_k d_k^2 \stackrel{q^\delta\to 0}{\longrightarrow} \overline{d}_k^2 = (-\nu q)^n q^{\binom{n}{2}-Nn}\frac{(q,\mu q,\nu q,q^{-N},\mu\nu q^{N+2};q)_n}{(\mu\nu q^2;q)_{2n}(\mu\nu q^{n+1};q)_n},$$

where d_k and $\overline{d}_k^2$ denote the norms for the *standard* q-Racah and the q-Hahn polynomials, respectively. The following limit relation for the kernels of the *standard* q-Racah and q-Hahn polynomials are introduced by applying the aforesaid transformation

$$\begin{aligned}\mathrm{K}_n^{\alpha,\beta}(s_1,s_2) := \sum_{k=0}^{n}\frac{C_k R_k^{\alpha,\beta}(s_1)_q C_k R_k^{\alpha,\beta}(s_2)_q}{C_k^2 d_k^2} \stackrel{q^\delta\to 0}{\longrightarrow} \\ \sum_{k=0}^{n}\frac{h_k^{\mu,\nu}(x(\overline{s}_1);N|q)h_k^{\mu,\nu}(x(\overline{s}_2);N|q)}{\overline{d}_k^2} := \mathrm{K}_n^{\mu,\nu}(\overline{s}_1,\overline{s}_2).\end{aligned} \tag{24}$$

Therefore, as a result of the limit relations defined by (22), (23), and (24) and the formula by (7), we obtain that

$$\lim_{q^a\to 0} C_n R_n^{\alpha,\beta,A,B}(s)_q = h_n^{\mu,\nu,A,B}(\mu(s);N|q) := h_n^{\mu,\nu,A,B}(s)_q,$$

where $C_n = 1$. In other words, we obtain the q-Hahn-Krall polynomials on the lattice $\mu(s) = q^{-s}$ which satisfy the orthogonality relation

$$\begin{aligned}\sum_{s=0}^{N} h_n^{\mu,\nu,A,B}(s)_q h_m^{\mu,\nu,A,B}(s)_q \rho(s)\Delta x(s-\tfrac{1}{2}) + A h_n^{\mu,\nu,A,B}(0)_q h_m^{\mu,\nu,A,B}(0)_q \\ + B h_n^{\mu,\nu,A,B}(N)_q h_m^{\mu,\nu,A,B}(N)_q = \delta_{n,m}\overline{d}_k^2, \quad \mu(s) = q^{-s},\end{aligned}$$

where ρ is the weight function of the q-Hahn polynomials [16, page 445].

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References

[1] R. Álvarez-Nodarse, Second order difference equations for certain families of 'discrete' polynomials, *J. Comput. Appl. Math.* **99** (1998), 37-45.

[2] R. Álvarez-Nodarse, *Polinomios hipergemétricos y q-polinomios.* Monografías del Seminario García Galdeano. Universidad de Zaragoza. Vol. **26**. Prensas Universitarias de Zaragoza, Zaragoza, Spain, 2003. (In Spanish).

[3] R. Álvarez-Nodarse N. M. Atakishiyev, and R.S. Costas-Santos: Factorization of the hypergeometric-type difference equation on the non-uniform lattices: dynamical algebra J. Phys. A: Math. Gen. 38 (2005) 153-174.

[4] R. Álvarez-Nodarse N. M. Atakishiyev, and R.S. Costas-Santos: Factorization of the hypergeometric-type difference equation on the uniform lattice. ETNA 27 (2007) 34-50

[5] R. Álvarez-Nodarse, J. Petronilho, On the Krall-type discrete polynomials, *J. Math. Anal. Appl.* **295** (2004), 55-69.

[6] R. Álvarez-Nodarse, Yu. F. Smirnov, R. S. Costas-Santos, A q-Analog of Racah Polynomials and q-Algebra $SU_q(2)$ in Quantum Optics, *J.Russian Laser Research* **27** (2006), 1-32.

[7] R. Álvarez-Nodarse, R. Sevinik Adıgüzel, On the Krall type polynomials on q-quadratic lattices, *Indagationes Mathematicae* **21** (2011), 181-203.

[8] R. Álvarez-Nodarse, R. Sevinik Adıgüzel, On the Krall-type Askey-Wilson Polynomials. (2011) Submitted.

[9] R. Askey and J. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, *SIAM J. Math. Anal.* **10** (1979), 1008-1016.

[10] A.J. Durán, Orthogonal polynomials satisfying higher order difference equations, (2011) Submitted.

[11] A.J. Durán, Using $\mathcal{D}-$operators to construct orthogonal polynomials satisfying higher order difference equations, (2011) Submitted.

[12] M. Gasper and G. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications (No. 96), Cambridge University Press (2nd edition), Cambridge, 2004.

[13] A. M. Krall, *Hilbert space, boundary value problems and orthogonal polynomials,* Operator Theory: Advances and Applications, **133**. Birkhuser Verlag, Basel, 2002.

[14] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable.* Springer Ser. Comput. Phys., Springer-Verlag, Berlin, 1991.

[15] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics.* Birkhäuser Verlag, Basel, 1988.

[16] R. Koekoek, Peter A. Lesky, and R.F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues,* Springer-Verlag, Berlin-Heidelberg, 2010.

[17] T. H. Koornwinder, Compact quantum groups and q-special functions. In *Representations of Lie groups and quantum groups.* V. Baldoni & M.A. Picardello (Eds.) Pitman Research Notes in Mathematics series 311, Longman Scientific & Technical, (1994), 46-128.

[18] N. Ja. Vilenkin and A.U. Klimyk, *Representations of Lie Groups and Special Functions.* **Vol. I,II,III**. Kluwer Academic Publishers. Dordrecht, 1992.

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