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ON THE TOPOLOGY OF LOCALLY 2-CONNECTED PEANO CONTINUA

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ABSTRACT. Several recent results by Thomassen ([**23, 24**]) concerning locally (2-)connected, compact, connected metrizable spaces are considered in the setting of continuum theory. By doing that we find out that Thomassen's theorems are closely related to classical powerful theorems, due to Kuratowski, Claytor and Borsuk, among others, which allow an alternative approach to them. This way we are able to generalize Thomassen's results to locally compact spaces.

1. Introduction. This paper provides an alternative approach to the study of locally 2-connected compact metric spaces carried out by Thomassen [**24**]. The goal is to show that a good deal of the results in [**24**] and its companion [**23**] are essentially consequences of well-established theorems of continuum theory. In addition we are able to generalize Thomassen's results to locally compact spaces. More precisely, we give purely topological proofs of the two following theorems generalizing results in [**24**].

Theorem A. *Let X be a locally 2-connected, locally planar, locally compact metric space. Then X is a closed subset of a surface M_X whose boundary $\partial M_X = \sqcup_{i\in I}\mathbf{R}$ consists of a sequence (possibly empty or finite) of copies of the Euclidean line. Moreover, the inclusion $i : X \subset M_X$ induces a homeomorphism $i_* : \mathcal{F}(X) \cong \mathcal{F}(M_X)$ between the Freudenthal end spaces.*

Furthermore, M_X is determined by X in the following sense

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Theorem B. *A* 1*-dimensional metric space* Y *embeds as a closed set of* X *if and only if* Y *does so in* M_X*. Moreover, given a (closed) embedding* $\psi : Y \to M_X$*, the embedding* $\varphi : Y \to X$ *can be chosen in such a way that the homeomorphism* $\overline{i_* : \mathcal{F}(X) \cong \mathcal{F}(M_X)}$ *restricts to a homeomorphism* $\overline{\varphi(Y)} \cap \mathcal{F}(X) \cong \overline{\psi(Y)} \cap \mathcal{F}(M_X)$*. Here the closures are taken in the corresponding Freudenthal compactifications.*

As a consequence of Theorem B, the local planarity of locally 2-connected generalized Peano continua is characterized by any of the two curves L_1 and L_2 added by Claytor [**7**] to the two Kuratowski's forbidden graphs $K_{3,3}$ and K_5 in order to characterize planar Peano continua; see Figure 1. Although a curve is usually assumed to be compact, in this paper we will extend this term to the non-compact setting by calling a *curve* any 1-dimensional generalized Peano continuum (see Section 2 for definitions). Namely, we prove

Theorem C. *Let* X *be a locally* 2*-connected generalized Peano continuum. The following statements are equivalent*:

(a) X *is not locally planar at* $p \in X$*.*

(b) *There is an embedding* $\varphi : L_1 \to X$ *such that* $\varphi(p_1) = p$*.*

(c) *There is an embedding* $\varphi : L_2 \to X$ *with* $\varphi(p_2) = p$*.*

This theorem provides an alternative characterization to the one given by Thomassen in [**24**, Theorem 4.6] by using complete infinite graphs. Although Thomassen's theorem is stronger than the previous theorem, in the sense that any infinite complete graph contains the curves L_1 and L_2 (Remark 5.7), Theorem C highlights Claytor's curves as the minimal forbidden configurations characterizing the local planarity of a locally 2-connected Peano continuum. We will also use Claytor's curves to disprove a conjecture by Thomassen on the characterization of planarity for Peano continua, see subsection 3A.

2. Locally 2**-connected Peano continua.** We recall that a *Peano continuum* X is a compact, connected, locally connected metrizable space. When compactness is replaced by local compactness the space X is called a *generalized Peano continuum*. Any connected open set of a generalized Peano continuum is arcwise connected [**22**, Theorem

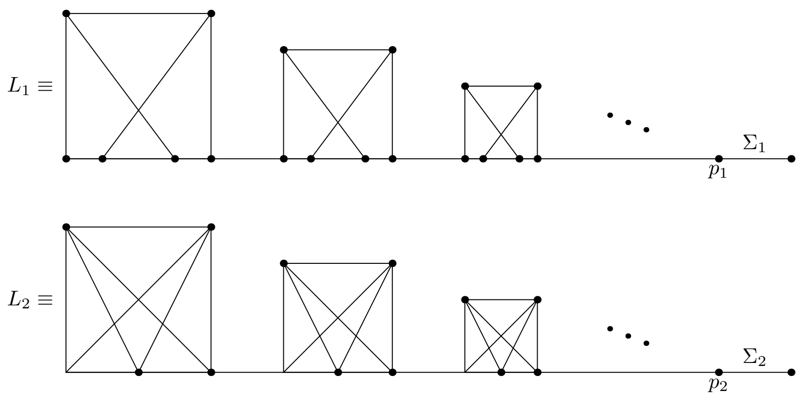


FIGURE 1. Claytor's curves L_1 and L_2.

4.2.5]. Moreover, it follows from [**10**, Theorem 4.4 F.(c)] that any generalized Peano continuum is separable and hence second countable and σ-compact ([**10**, Theorem 4.1.16] and [**10**, Theorem 3.8.C(b)]). The local compactness together with the σ-compactness yield that X is a countable union $\cup_{n=1}^{\infty} K_n$ of compact subsets $K_n \subset X$ with $K_n \subset \operatorname{int} K_{n+1}$. Given such a sequence $\{K_n\}_{n\geq 1}$, a *Freudenthal end* of X is a sequence $\varepsilon = (C_n)_{n\geq 1}$ of components $C_n \subset X - K_n$ with $C_{n+1} \subset C_n$. Let $\mathcal{F}(X)$ denote the set of Freudenthal ends of X. The set $\widehat{X} = X \cup \mathcal{F}(X)$ admits a compact topology whose base consists of the open sets of X together with the sets $\widehat{C_n} = C_n \cup \{\varepsilon \in \mathcal{F}(X); C_n \text{ appears in } \varepsilon\}$ $(n \geq 1)$. This topology is called the Freudenthal topology and $\widehat{X}$ is called the *Freudenthal compactification* of X. Moreover the subspace $\mathcal{F}(X)$ turns to be homeomorphic to a closed subset of the Cantor set (see [**12**] for details). It is well known that any *proper map* $f : X \to Y$ (i.e., a continuous map such that $f^{-1}(K)$ is compact for any compact subset K) between generalized continua extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ which restricts to a map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Namely if $\varepsilon = (C_n)_{n\geq 1}$, then $\widehat{f}(\varepsilon) = f_*(\varepsilon) = (D_k)_{k\geq 1}$ where $f(C_{n_k}) \subset D_k$ for some increasing subsequence $(C_{n_k})_{k\geq 1}$ of ε.

A space X is called 2-*connected* if no point separates X. More generally, a space X is said to be *n-connected*[1] if no set with fewer or equal than $n-1$ points separates X. By an *ω-connected space* we mean a space which is n-connected for all $n \geq 1$. The corresponding local version of 2-connectedness is the following. A space X is termed *locally* 2-*connected* if for any $x \in X$ and any neighborhood of x, U, there is another neighborhood $V \subset U$ such that $V - \{x\}$ is connected. The following lemma shows that local 2-connectedness is in fact a strong connectivity property.

Lemma 2.1. *Any connected open set $U \subset X$ in a locally 2-connected generalized Peano continuum X is ω-connected. Moreover, if the closure $\overline{U}$ is locally connected (i.e., $\overline{U}$ is a generalized Peano continuum), then $\overline{U}$ is also ω-connected.*

Proof. As quoted above, connected open subsets of Peano continua are arcwise connected; that is, U is 1-connected. Assume inductively that U is n-connected and, given x, $y \in U$, let $A = \{a_1, \ldots, a_n\} \subset U - \{x,y\}$ be any set with n points. By the induction hypothesis there is at least an arc $\gamma \subset U$ joining x to y such that $\gamma \cap A$ contains at most $\{a_n\}$. If $a_n \in \gamma$, then we choose a 2-connected neighborhood of a_n, $V \subset U$ such that $(A \cup \{x,y\}) \cap V = \{a_n\}$ and we modify γ inside V to get a new arc γ' with $\gamma' \cap A = \varnothing$.

Assume that $\overline{U}$ is locally connected. In order to show that $\overline{U}$ is n-connected ($n \geq 1$), take $x_1, x_2 \in \overline{U}$ and a set $S \subset \overline{U} - \{x_1, x_2\}$ consisting of $n-1$ points we use local connectedness to find small arcwise connected open neighborhoods W_1 and W_2 of x_1 and x_2, respectively, in $\overline{U}$ with $(W_1 \cup W_2) \cap S = \varnothing$. Then we choose $y_1 \in U \cap W_1$ and $y_2 \in U \cap W_2$ and apply the first part of the lemma to get an arc $\gamma \subset U - S$ joining y_1 to y_2. We easily find an arc in $\gamma \cup W_1 \cup W_2 \subset \overline{U}$ joining x_1 to x_2 and missing S. □

The following lemma uses the so-called property $\mathcal{S}$, due to Sierpiński, which characterizes local connectedness. Namely, a non-empty subset $Y \subset X$ of a metric space X is said to have *property* $\mathcal{S}$ if, for each $\epsilon > 0$, there are finitely many connected subsets $A_1, \ldots, A_n$ of Y such that $Y = \cup_{i=1}^{n} A_i$ with diameter $\operatorname{diam}(A_i) < \epsilon$ for each $i = 1, \ldots, n$.

Lemma 2.2. *Let X be a locally 2-connected generalized Peano continuum, and let U be any open neighborhood of a continuum $C \subset X$. Then there is a neighborhood $V \subset U$ of C which is an ω-connected*

Peano continuum, whose interior is connected, and hence a locally 2*-connected generalized Peano continuum.*

Proof. It is known that X decomposes in an increasing union of Peano subcontinua $X = \cup_{i=1}^{\infty} X_i$ with $X_i \subset \operatorname{int} X_{i+1}$, see [**4**]. In particular, there is a Peano subcontinuum $Y \subset X$ with $C \subset \operatorname{int} Y$. As Y has property $\mathcal{S}$ [**17**, 8.4], we use [**17**, 8.8] to find a connected open neighborhood W of C in $\operatorname{int} Y$, and hence in X, satisfying property $\mathcal{S}$ and such that $C \subset W \subset \overline{W} \subset U$. The set $W = S(C, \epsilon)$ is constructed by adding to C all ϵ-chains starting from C with ϵ small enough to keep all those chains inside $U \cap \operatorname{int} Y$, see ([**17**, 8.6]). Moreover, by [**17**, 8.5], the closure $\overline{W}$ also has property $\mathcal{S}$ and so $V = \overline{W}$ is locally connected (i.e., a Peano continuum) by [**17**, 8.4] and so it is ω-connected by Lemma 2.1.

Clearly, an open set of X is a locally 2-connected generalized Peano continuum if and only if it is connected. In particular, for $\operatorname{int} V$ above we observe that, as W is an open set, we have $W \subset \operatorname{int} V \subset V = \overline{W}$ and hence $\operatorname{int} V$ is connected since W is so. □

Henceforth, by an *ω-neighborhood* of a subset A we mean an ω-connected Peano continuum V whose interior is a connected open set (i.e., a locally 2-connected generalized Peano continuum) containing A. The previous lemma yields

Corollary 2.3. *Any subcontinuum of a locally* 2*-connected generalized Peano continuum admits a neighborhood base consisting of ω-neighborhoods.*

Concerning n-connectedness, Nöbeling and Zippin extended in [**18, 26**] the classical Menger theorem for graphs to (generalized) continua. Namely,

Theorem 2.4 [**26**, Principal theorem, page 96]. *Let X be an n-connected generalized Peano continuum. Given any two points $p, q \in X$, there exist n independent arcs in X running from p to q (that is, the arcs are pairwise disjoint except in $\{p, q\}$).*

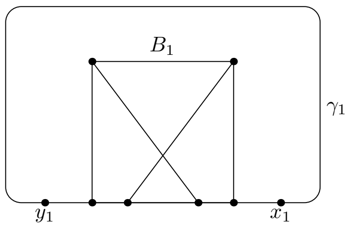


FIGURE 2. $K_{3,3} = B_1 \cup [y_1, x_1] \cup \gamma_1$.

We finish this section with the statement of the following lemma, a consequence of Nöbeling-Zippin's theorem, proved by Zippin.

Lemma 2.5 [**26**, page 112]. *For any $n \geq 1$, let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be two disjoint sets in an ω-connected generalized Peano continuum X. Then there exist n independent arcs $\gamma_1, \dots, \gamma_n$ connecting all points in A with all points in B; that is, the arcs are pairwise disjoint and their extremes lie in $A \cup B$. Moreover, given any $x \in X - B$ there are n independent arcs from x to B.*

3. Planarity of 2-connected Peano continua. In this section we collect some remarks and observations concerning the planarity of (generalized) Peano continua. We start by pointing out that a recent result due to Thomassen [**23**] characterizing the planarity of locally 2-connected Peano continua is an immediate consequence of an old theorem due to Claytor [**7**]; compare with [**19**, 1.2]. Here we show this fact directly for generalized Peano continua. For this we use the following extension of Claytor's theorem.

Theorem 3.1 [**3**, 1.1]. *Let X be a generalized Peano continuum. Then the following statements are equivalent*:

(1) *X is embeddable in S^2 (or equivalently in $\mathbf{R}^2$ if $X \neq S^2$).*

(2) *Any subcontinuum $K \subset X$ embeds in S^2.*

(3) *X contains no set homeomorphic to K_5, $K_{3,3}$, L_1, L_2.*

(4) *The Freudenthal compactification $\widehat{X}$ of X is embeddable in S^2.*

As remarked upon by Claytor [**7**, Theorem C], the curves L_1 and L_2 are redundant in Claytor's theorem if X is 2-connected. More precisely, in that case the limit point $p_i \in L_i$ in Figure 1 does not separate L_i in X and one finds an arc $\gamma_i \subset X - \{p_i\}$ from a point $x_i \in \Sigma_i$ to a point $y_i \in L_i - \Sigma_i$. The unions $L_1 \cup \gamma_1$ and $L_2 \cup \gamma_2$ contain a copy of $K_{3,3}$ and K_5, respectively. Indeed, since $p_i \notin \gamma_i$ there is a neighborhood $U_i \subset L_i$ of p_i with $U_i \cap \gamma_i = \varnothing$. Then the union of a small "block" $B_i \subset U_i$ with the arcs $[y_i, x_i] \subset L_i$ and γ_i is either $K_{3,3}$ or K_5, see Figure 2.

We have proved as a straightforward consequence of Theorem 3.1 the following extension of [**23**, Theorem 4.3]:

Theorem 3.2. *A connected, locally* 2*-connected, locally compact metrizable space* X *is embeddable in the* 2*-sphere if and only if* X *contains none of the Kuratowski graphs* K_5 *and* $K_{3,3}$.

It is worth mentioning that, prior to his general theorem, Claytor gave in [**8**] a long and somewhat cumbersome proof of the special case when X is 2-connected. See [**16**] for a short proof of this special case, as well as a further alternative proof of Thomassen's theorem [**23**, 4.3].

It is well known that the graphs $K_{3,3}$ and K_5 do not play the same role in the Kuratowski-Claytor planarity criterion for Peano continua. Actually any embedding $K_5 \subset X$ in a 3-connected (generalized) Peano continuum yields an embedding $K_{3,3} \subset X$ [**13**] and so $K_{3,3}$ suffices to characterize planarity in the realm of 3-connected Peano continua. Next we prove that the roles of $K_{3,3}$ and K_5 turn out to be equivalent under the assumption of local 2-connectedness.

Proposition 3.3. *Let* X *be a locally* 2*-connected generalized Peano continuum. The following statements are equivalent:*

(a) X *is not planar.*

(b) *There is an embedding* $K_{3,3} \subset X$.

(c) *There is an embedding* $K_5 \subset X$.

Proof. (a) $\Rightarrow$ (b) was observed above.

Moreover, (c) $\Rightarrow$ (a) is obvious.

It remains to check (b) $\Rightarrow$ (c). Let $L \subset X$ be a bipartite graph with vertices $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. It will suffice to find two arcs α and β under the conditions of Lemma 3.4 below. For this we proceed as follows. Let Ω_{a_3} and Ω_{b_3} be disjoint ω-neighborhoods of a_3 and b_3, respectively, missing the union $\cup_{i,j=1,2}[a_i, b_i]$. Here we use Corollary 2.3. Let $a_3^i \in \Omega_{a_3}$ and $b_3^i \in \Omega_{b_3}$ $(i = 1, 2, 3)$ be the last points in the intersections $[a_3, b_i] \cap \Omega_{a_3}$ and $[a_i, b_3] \cap \Omega_{b_3}$, respectively. As $\operatorname{int}\Omega_{a_3}$ and $\operatorname{int}\Omega_{b_3}$ are locally 2-connected, and hence ω-connected (Lemma 2.1), Theorem 2.4 provides circles $\Sigma_{a_3} \subset \operatorname{int}\Omega_{a_3}$ and $\Sigma_{b_3} \subset \operatorname{int}\Omega_{b_3}$. Then we join the points a_3^i to Σ_{a_3} by three disjoint arcs $\gamma_{a_3}^i \subset \Omega_{a_3}$. Here we apply Lemma 2.5. Similarly the points b_3^i are joined to Σ_{b_3} by three disjoint arcs $\gamma_{b_3}^i \subset \Omega_{b_3}$. One readily constructs a new copy $\widetilde{L}$ of $K_{3,3}$ in $L \cup (\cup_{i=1,2}\gamma_{b_3}^i) \cup (\cup_{i=1,2}\gamma_{a_3}^i) \cup \Sigma_{a_3} \cup \Sigma_{b_3}$ such that $\widetilde{L}$ admits two arcs $\alpha \subset \Sigma_{b_3}$ and $\beta \subset \Sigma_{a_3}$ under conditions of Lemma 3.4, and the proof is complete. □

Lemma 3.4. *Let $K_{3,3}$ be a bipartite graph with vertices $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ in a locally 2-connected generalized Peano continuum X. Assume that there are two disjoint arcs $\alpha, \beta \subset X$ such that α runs between $p_1 \in (a_1, b_3)$ and $p_2 \in (a_2, b_3)$ and β runs between $q_1 \in (b_1, a_3)$ and $q_2 \in (b_2, a_3)$ and $(\alpha \cup \beta) \cap K_{3,3} = \{p_1, p_2, q_1, q_2\}$. Then there is an embedding $K_5 \subset X$.*

Proof. We apply Corollary 2.3 to obtain disjoint ω-neighborhoods A_1, A_2, , B_1, B_2 of the arcs $[a_1, p_1]$, $[a_2, p_2]$, $[b_1, q_1]$ and $[b_2, q_2]$, respectively, in X. Let s_0^i, $s_j^i \in A_i$ $(1 \le j \le 3)$ be the last points in the intersections $\alpha \cap A_i$, and $[a_i, b_j] \cap A_i$, respectively. Similarly, let t_0^i, $t_j^i \in B_i$ $(1 \le j \le 3)$ be the last points in $\beta \cap B_i$, and $[a_i, b_j] \cap B_i$, respectively. As A_i and B_i are ω-connected we can find by Lemma 2.5 four independent arcs $\sigma_j^i \subset A_i$ $(0 \le j \le 3$ and $i = 1, 2)$ connecting a point $\widetilde{a}_i \in \operatorname{int} A_i$ to s_j^i. Similarly we choose four independent arcs $\tau_j^i \subset B_i$ joining a point $\widetilde{b}_i \in \operatorname{int} B_i$ to t_j^i.

Again we apply Corollary 2.3 to get an ω-neighborhood $\Omega \subset X$ of the edge $[a_3, b_3]$ avoiding the union $\alpha \cup \beta \cup (\cup_{i,j=1,2}[a_i, b_j])$. Let ρ_{a_i}, $\rho_{b_i} \subset \Omega$ $(i = 1, 2)$ be four independent arcs joining a point $v \in \operatorname{int}\Omega$ to the last points in the intersections $\Omega \cap [a_i, b_j]$ and $\Omega \cap [b_i, a_3]$, respectively. Now it is easy to find a copy of K_5, with vertices $\widetilde{a}_i$, $\widetilde{b}_i$ $(i = 1, 2)$ and v, in the union $\Omega \cup (\cup_{i=1}^2 (A_i \cup B_i)) \cup K_{3,3}$. □

Remark 3.5. A characterization of the generalized Peano continua which are embeddable as closed sets in $\mathbf{R}^2$ (i.e., without accumulation points) is given in [**2**] by using the Halin graphs. It seems to be feasible that a result similar to Proposition 3.3 should hold and locally 2-connected generalized Peano continua which admit a closed planar embedding may eventually be characterized by any pair of forbidden graphs consisting of one Kuratowski's graph together with one Halin's graph.

3A. On a statement by Thomassen. Next we point out that the non-planar curves L_1 and L_2 provided by Claytor's theorem are counterexamples to the following theorem suggested in [**23**] as a by-product of the proof of Theorem 3.2 given there.

Theorem 3A.1 [**23**, 4.5]. *A connected locally connected compact metrizable space* (*i.e., a Peano continuum*) *X is embeddable in the* 2-*sphere if and only if X contains none of the Kuratowski graphs* $K_{3,3}$ *and* K_5 *and no thumbtack with holes.*

In [**23**] a *thumbtack with holes* is defined as the union $D' = D \cup \Gamma$ of a compact, locally connected, essentially 3-connected subspace of the 2-sphere $D \subset S^2$ together with an arc Γ. Moreover, the intersection $D \cap \Gamma = \{q\}$ reduces to one extreme of Γ and q misses the faces boundary of D in the sense of [**23**]. Recall that a 2-connected space D is said to be *essentially* 3-*connected* if for any elements $x, y \in D$ the difference $D - \{x, y\}$ is either connected or has precisely two components, one of which is an arc from x to y. Notice that the embedding of D in S^2 is unique by [**19**]; see Remark 4.3 below.

It is readily checked that Claytor's curves L_i contain neither a copy of $K_{3,3}$ or K_5 nor a thumbtack with holes, and so Theorem 3A.1, as stated, does not hold for L_i. Indeed, any essentially 3-connected subspace $D \subset L_i$ is necessarily 2-connected and so, it is contained in a "block" $B \subset L_i$. This readily follows from the fact that every point in the arc lying between two consecutive blocks of L_i is a cut-point of L_i. Therefore, D is a graph by [**17**, 9.10.1]. Hence any union $D' = D \cup \Gamma \subset B \cup \Gamma$ of D with an arc Γ such that $D \cap \Gamma$ is an extreme of Γ is a planar graph. Hence D' cannot be a thumbtack.

Remark 3A.2. Incidentally, we mention here that Mardešić and Segal showed in [**15**, Theorem 1] that a polyhedron P contains a Claytor's curve L_i if and only if it contains a copy of the subspace of $\mathbf{R}^3$ (called disk with feeler)

$$F_2 = \{(x, y, 0); x^2 + y^2 \leq 1\} \cup \{(0, 0, z); 0 \leq z \leq 1\}.$$

Hence, a polyhedron P is planar if and only if P contains none of the Kuratowski graphs $K_{3,3}$, K_5 nor disk with feeler. Notice that F_2 is a thumbtack without holes.

4. The structure of locally planar, locally 2-connected generalized Peano continua. This section provides an alternative proof of Thomassen's result in [**24**] stating that a typical neighborhood in a locally planar, locally 2-connected (generalized) Peano continuum X is topologically equivalent to a 2-sphere from which the interiors of a null sequence of pairwise disjoint disks has been removed. This will allow us to detect inside X a null sequence of pairwise disjoint circles which will be crucial in the proofs of the main results of the paper in Section 5.

Recall that a space X is said to be *locally planar* if each point $x \in X$ admits a planar neighborhood, and hence a neighborhood base of planar sets. The following lemma is an immediate consequence of Corollary 2.3.

Lemma 4.1. *Let X be a locally planar, locally 2-connected generalized Peano continuum. Each point $x \in X$ admits a countable neighborhood base $\mathcal{B}_x$ consisting of planar ω-neighborhoods.*

As a consequence of [**14**, Theorem 4(ii), subsection 61 II], given any embedding $\varphi : C_i^x \to S^2$ of $C_i^x \in \mathcal{B}_x$, each component $R \subset S^2 - \varphi(C_i^x)$ is an open disk whose frontier $\operatorname{Fr} R$ is its boundary circle. This observation leads to the following

Definition 4.2. A *chart* of X at $x \in X$ is a pair (C_i^x, φ) where C_i^x is a planar ω-neighborhood of x and $\varphi : C_i^x \to S^2$ is an embedding. A point $x \in X$ is said to be *terminal* if there is a chart (C_i^z, φ) at some z

such that $x \in \operatorname{int} C_i^z$ and $\varphi(x) \in \operatorname{Fr} R$ lies in the frontier of a component $R \subset S^2 - \varphi(C_i^z)$.

Remark 4.3. We may use a recent theorem by Thomassen [**19**, Theorem 2], showing that planar 3-connected Peano continua uniquely embed in S^2, to guarantee that the definition of a terminal point does not depend on the chart used in its definition. Notwithstanding, for the sake of completeness, we include an independent proof in Appendix A below. It is worth pointing out that Thomassen's result [**19**, Theorem 2) was already contained in an old paper by Adkisson [**1**, Theorem II].

As observed in Remark 4.3, we have a well-defined set $T \subset X$ consisting of all terminal points in X. We will call T the *terminal set* of X. By a *terminal (open) arc or circle* we mean an (open) arc or circle contained in T. We will use later the following basic properties of the charts.

Lemma 4.4. *Given a chart* (C_i^x, φ)*, the components* $R_n \subset S^2 - \varphi(C_i^x)$ *are countable and* $\operatorname{diam}(R_n) \to 0$*, and hence* $\operatorname{diam}(\operatorname{Fr} R_n) \to 0$*, if there are infinitely many distinct components.*

Proof. It is simply an application of [**14**, Theorem 6, subsection 49 II] and [**10**, Theorem 10, subsection 61 II]. □

Lemma 4.5. *Given a chart* (C_i^x, φ)*, if* $z \in C_i^x$ *is the limit point of a sequence* $z_n \in \varphi^{-1}(\operatorname{Fr} R_n) - T$ *of non-terminal points lying in the frontier of components* $R_n \subset S^2 - \varphi(C_i^x)$*, then* $z \notin \operatorname{int} C_i^x$*.*

Proof. Otherwise there exists an n_0 such that $z_n \in \operatorname{int} C_i^x$ if $n \geq n_0$. But this means that z_n is a terminal point since it is on the boundary of a component of $S^2 - \varphi(C_i^x)$. □

Lemma 4.6. *For any chart* (C_i^x, φ) *and components* $R_1, R_2 \subset S^2 - \varphi(C_i^x)$*, the intersection* $\varphi^{-1}(\operatorname{Fr} R_1 \cap \operatorname{Fr} R_2) \cap \operatorname{int} C_i^x$ *is empty if* $\operatorname{Fr} R_1 \neq \operatorname{Fr} R_2$*.*

Proof. Assume on the contrary that $z \in \operatorname{Fr} R_1 \cap \operatorname{Fr} R_2$ and $\varphi^{-1}(z)$ is an interior point in C_i^x. Let $L_i \subset R_i \cup \{z\}$ be an arc such that $z \in L_i$. Here we use that $R_i \cup \operatorname{Fr} R_i$ is a closed disk.

We consider a closed disk $D \subset S^2$ around z and a subarc $L \subset L_1 \cup L_2$ with $\partial L = \partial D \cap L$. Let E_1 and E_2 be the two components of $D - L$. In both components one readily gets sequences of elements of $\varphi(C_i^x)$ converging to z since otherwise L_1 and L_2 would be joined by arcs that do not cut the frontiers $\operatorname{Fr} K_1 \cup \operatorname{Fr} K_2$.

As the interior $U = \operatorname{int} C_i^x$ is a locally 2-connected neighborhood of $\varphi^{-1}(z)$, we can find a 2-connected neighborhood of z $W \subset U \cap \varphi^{-1}(D)$ and points $p_i \in W \cap \varphi^{-1}(E_i)$ which can be joined by an arc $\Gamma \subset W - \{z\}$. By the Jordan theorem $\varphi(U) \cap L \neq \varnothing$ which is a contradiction since $\varphi(U) \cap (L - \{z\}) = \varnothing$ by construction. □

Notice that Lemma 4.6 and the fact that the definition of a terminal point does not depend on the choice of charts (see Remark 4.3 and also Proposition A.3) show that given any planar ω-neighborhood C_i^x and any terminal point $t \in T \cap \operatorname{int} C_i^x$ there is a unique circle $S_t^{(x,i)} \subset C_i^x$ such that $t \in S_t^{(x,i)}$ and $\varphi(S_t^{(x,i)})$ is the frontier of a component of $S^2 - \varphi(C_i^x)$ for any embedding $\varphi : C_i^x \to S^2$. We call $S_t^{(x,i)}$ the *characteristic circle* of t in C_i^x. Observe that $S_t^{(x,i)}$ needs not be a terminal circle.

Lemma 4.7. *Any $z \in T \cap \operatorname{int} C_i^x$ lies in an open arc contained in $T \cap S_z^{(x,i)}$. Moreover any (open) arc $\Gamma \subset T \cap \operatorname{int} C_i^x$ is part of the characteristic circle $S_z^{(x,i)}$ of some $z \in T$.*

Proof. Each $z \in \operatorname{int} C_i^x \cap T$ is an interior point in the characteristic circle $S_z^{(x,i)}$, and hence there is an open arc $z \in \Gamma \subset \operatorname{int} C_i^x \cap S_z^{(x,i)}$, and all points in Γ are terminal by definition.

Any arc $\Gamma \subset T \cap \operatorname{int} C_i^x$ can be expressed as a disjoint union $\Gamma = \sqcup_{p \in \Gamma}(S_p^{(x,i)} \cap \Gamma)$. Then there is only one $S_{p_0}^{(x,i)}$ with $\Gamma \subset S_{p_0}^{(x.i)}$ since otherwise Γ is the disjoint union of a countable family of closed sets and we reach a contradiction with [**14**, Theorem 6, subsection 47 III]. If Γ is an open arc we argue as we did previously for each arc in an increasing sequence of arcs $\Gamma_1 \subset \cdots \subset \Gamma_n \subset \cdots$ covering Γ. □

Recall that a *triod* is the union of three arcs $[a,b]$, $[a,c]$, $[a,d]$ which have pairwise only the point a in common.

Proposition 4.8. *The terminal set T does not contain a triod.*

Proof. If $A = [a,b] \cup [a,c] \cup [a,d] \subset T$ is a triod, then any planar ω-neighborhood C_i^a contains a triod $A' \subset A$ in its interior and, by Lemma 4.7, A' is contained in the characteristic circle $S_a^{(x,i)}$ which is a contradiction. □

The previous proposition will allow us to determine the topological nature of the terminal set T. Namely,

Proposition 4.9. *Each arcwise component of $C \subset T$ is either a terminal circle or a terminal open arc which is a closed set of X.*

Proof. Let $C \subset T$ be an arcwise component of T. Lemma 4.7 implies that if $z \in C \cap \operatorname{int} C_i^x$, then C contains an open arc Γ_z with $z \in \Gamma_z$. In particular $C \neq \{z\}$. Moreover, the arc $\Gamma_z \subset S_z^{(x,i)}$ is part of the characteristic circle of z in C_i^x, and the intersection $I_z^{(x,i)} = C \cap S_z^{(x,i)}$ is an open subset of $S_z^{(x,i)}$. Indeed, any $y \in I_z^{(x,i)}$ is a terminal point in $S_z^{(x,i)} \subset C_i^x$, and then $y \in C \cap \operatorname{int} C_i^x$. As for z, there exists an open arc $\Gamma_y \subset I_z^{(x,i)}$ since $S_z^{(x,i)}$ is also the terminal circle of y in C_i^x. In fact, we will check below

$$I_z^{(x,i)} = C \cap \operatorname{int} C_i^x \tag{1}$$

which proves that $I_z^{(x,i)}$ is also an open subset of C; that is, C is locally the Euclidean line and, by the classification of 1-manifolds, C is a terminal circle or a terminal open arc.

Next we see equality (1). If $p \in I_z^{(x,i)}$, then p is a terminal point and $p \in C \cap \operatorname{int} C_i^x$. Conversely, any point $q \in C \cap \operatorname{int} C_i^x$ is in $S_z^{(x,i)}$; otherwise, there exists an arc from q to $\Gamma_z \subset S_z^{(x,i)}$ which only cuts Γ_z in its final point, and hence T contains a triod which is a contradiction to Proposition 4.8.

Finally we prove that any open terminal arc $C = (a, b)$ is a closed subset in X. Indeed, otherwise there exists a sequence $\{w_n\}_{n\geq 1} \subset (a, b)$ that converges to some $w \in X - C$. Then, given any planar ω-neighborhood C_i^w of w we find n_0 with $w_n \in \operatorname{int} C_i^w$ if $n \geq n_0$. In addition the characteristic circles $S_{w_n}^{(w,i)}$ in C_i^w are not terminal (otherwise, $S_{w_n}^{(w,i)} \subset (a, b)$ which is impossible), and we can find points $x_n \in S_{w_n}^{(w,i)} - T$.

If there exists an n_1 such that $S_{w_{n_1}}^{(w,i)}$ contains infinitely many w_n's, then $w \in S_{w_{n_1}}^{(w,i)}$ and necessarily $w \in T$ because $w \in \operatorname{int} C_i^w$. Moreover, by Lemma 4.7, there exists an open arc $\Gamma \subset S_{w_{n_1}}^{(w,i)} \cap T$ with $w \in \Gamma$ and so Γ contains infinitely many w_n. In particular $w \in \Gamma \subset C$ as $w_n \in C$; this is not the case and necessarily there exist infinitely many circles $S_{w_n}^{(w,i)}$. Hence $\operatorname{diam}(S_{w_n}^{(w,i)}) \to 0$ by the first part of Lemma 4.4 and the sequence $x_n \in S_{w_n}^{(w,i)} - T$ converges to w due to $w_n \to w$. This contradicts Lemma 4.5 and the proof concludes. □

Corollary 4.10. *If X is compact then all arcwise components of T are circles.*

Let $\mathscr{A}_T$ denote the family of all arcwise components of T. In addition to Proposition 4.9 we have that the family $\mathscr{A}_T^* \subset \mathscr{A}_T$ consisting of all terminal circles is a null sequence. Recall that a collection $\mathcal{A}$ of subsets of a locally compact, second countable metric space (X, d) with diameter $\operatorname{diam}(A) \neq 0$ for all $A \in \mathcal{A}$ is said to form a *null sequence* if for any compact K and any $\epsilon > 0$ only finitely many of the sets $A \in \mathcal{A}$ with $A \cap K \neq \varnothing$ have diameter $\operatorname{diam}(A) > \epsilon$ greater than ϵ.[2]

Proposition 4.11. *The family $\mathscr{A}_T^*$ is a null sequence of pairwise disjoint circles.*

Proof. Assume that there are ϵ_0, a compact set K_0 and a sequence $S_1 \cdots S_n \cdots$ of terminal circles such that $\operatorname{diam}(S_n) \geq \epsilon_0$ and $S_n \cap K_0 \neq \varnothing$ for all $n \geq 1$. The compactness of K_0 allows us to assume that a sequence $x_n \in S_n \cap K_0$ converges to a point $x_0 \in K_0$. Let $(C_i^{x_0}, \varphi)$ be a chart at x_0 such that $\operatorname{diam}(C_i^{x_0}) \leq \epsilon_0/4$, whence $S_n - C_i^{x_0} \neq \varnothing$ for all $n \geq 1$. The convergence of $\{x_n\}_{n\geq 1}$ implies the existence of some

n_0 that $x_n \in \operatorname{int} C_i^{x_0}$ if $n \geq n_0$. By Lemma 4.7 each characteristic circle $S_{x_n}^{(x_0,i)}$ must contain non-terminal points $z_n \in S_{x_n}^{(x_0,i)} - T$ since $S_n \neq S_{x_n}^{(x_0,i)}$ for all n because $\operatorname{diam}(S_{x_n}^{(x_0,i)}) \leq \operatorname{diam}(C_i^{x_0})$, and this contradicts Lemma 4.5 since $x_0 \in \operatorname{int} C_i^{x_0}$. □

Proposition 4.12. *The family* $\mathscr{A}_T - \mathscr{A}_T^* = \{A_\alpha\}$ *of terminal open arcs is locally finite in* X *and hence countable.*

Proof. If $\{A_\alpha\}$ is not locally finite at $x \in X$ then there exists a sequence $a_s \in A_{\alpha_s}$ $(s \geq 1)$ of points in distinct A_{α_s} such that $\{a_s\}_{s\geq 1}$ converges to x. Fixed a planar w-neighborhood of x, C_i^x, we have that there exists an s_0 with $a_s \in C_i^x$ for $s \geq s_0$ and each a_s $(s \geq s_0)$ determines a characteristic circle $S_{a_s}^{(x,i)} \subset C_i^x$. Moreover, in each one of the latter we can find $x_s \in S_{a_s}^{(x,i)} - T \neq \varnothing$, otherwise $S_{a_s}^{(x,i)} \subset A_{\alpha_s}$ which is impossible. In addition, there can only exist finitely many such circles because otherwise $\operatorname{diam}(S_{a_s}^{(x,i)}) \to 0$ Lemma 4.4 and as $\{a_s\}_{s\geq 1}$ converges to $x \in \operatorname{int} C_i^x$, the sequence $\{x_s\}_{s\geq 1}$ also converges to x and this is a contradiction with Lemma 4.5.

Since $\{S_{a_s}^{(x,i)}\}_{s\geq 1}$ is finite, then infinitely many a_s's lie in the same characteristic circle $S_{a_{s_0}}^{(x,i)}$. In particular, $x \in S_{a_{s_0}}^{(x,i)}$, and $x \in T$ is necessarily a terminal point. Now, Lemma 4.7 provides us an open arc $\Gamma \subset T \cap S_{a_{s_0}}^{(x,i)}$ containing x and hence almost all of the a_s's appearing in $S_{a_{s_0}}^{(x,i)}$. But then, $\Gamma \subset A_{\alpha_s}$ for infinitely many α_s's, and this contradicts that the A_α's are disjoint. □

5. Proofs of the main results. We are ready to prove Theorems A, B and C in the Introduction. We start with the null sequence $\mathscr{A}_T^*$ of terminal circles of X in Proposition 4.11, and we form the set $M_X = X \cup \{cS;\, S \in \mathscr{A}_T^*\}$ consisting of the union of X and pairwise disjoint cones $cS = S \times [0,1]/S \times \{1\}$, where $S \in \mathscr{A}_T^*$. We consider the topological space obtained by endowing the set M_X with the topology for which the family of sets $\{U_i^x\}_{i\geq 1}$ defined in (a)-(c) below form a neighborhood base of every $x \in M_X$.

(a) If $x \in cS - X$, then $\{U_i^x\}_{i\geq 1}$ is a neighborhood base of x in the open cone $cS - S$.

(b) If $x \in X - T$, then $U_i^x = C_i^x \cup \{cS;\ S \subset C_i^x\}$ where $\{C_i^x\}_{i\geq 1}$ is a neighborhood base of x consisting of planar ω-neighborhoods.

(c) If $x \in T$ and its arcwise component $E_x \subset T$ is an open arc, then $U_i^x = C_i^x \cup \{cS;\ S \subset C_i^x\}$. Otherwise, if E_x is a circle, then $U_i^x = C_i^x \cup \{cS;\ S \subset C_i^x,\ S \neq S_x^{(x,i)}\} \cup W_i^x \times [0,(1/i+1)]$ where $\{W_i^x\}_{i\geq 1}$ is a neighborhood base of x in the characteristic circle $S_x^{(x,i)}$ and $W_i^x \times [0,(1/i+1)]$ denotes the obvious subset of the cone $cS_x^{(x,i)}$.

Proof of Theorem A. Given any chart (C_i^x, φ) we can assume that the frontier of each component $R_n \subset S^2 - \varphi(C_i^x)$ contains at least one terminal point (and hence it is a characteristic circle); indeed, otherwise $x \in \operatorname{int} C_i^x$ is the limit of a sequence $z_k \in \varphi^{-1}(\operatorname{Fr} R_{n_k}) - T$ which contradicts Lemma 4.5.

Moreover, if $x \in \operatorname{int} C_i^x$ is terminal, then we can assume in addition that its characteristic circle $S_x^{(x,i)}$ is the only one in C_i^x that contains a terminal point; since, otherwise one can find a sequence of characteristic circles $S_{y_n}^{(x,i)} \subset C_i^x$ with points $x_n \in S_{y_n}^{(x,i)} - T$ and such that $\{y_n\}_{n\geq 1}$ converges to x. Then $\operatorname{diam}(S_{y_n}^{(x,i)}) \to 0$ by Lemma 4.4 and hence the sequence of non-terminal points $\{x_n\}_{n\geq 1}$ converges to $x \in \operatorname{int} C_i^x$. This contradicts Lemma4.5.

Under the above conditions we extend the embedding $\varphi : C_i^x \to S^2$ to an embedding $\widetilde{\varphi} : U_i^x \to S^2$ as follows. For each terminal circle $S \in \mathscr{A}_T^*$ with $S \subset C_i^x$ ($S \neq S_x^{(x,i)}$ if x is terminal) $\widetilde{\varphi}$ is the cone extension carrying cS ($S \neq S_x^{(x,i)}$) onto the component of $S^2 - \varphi(C_i^x)$ bounded by $\varphi(S)$. If, in addition, x is terminal, then $\widetilde{\varphi}$ is defined as the cylindrical extension of $\varphi|_{W_i^x}$ mapping $W_i^x \times [0,(1/i+1)]$ into the component bounded by $\varphi(S_x^{(x,i)})$; that is, $\widetilde{\varphi}(x,t) = (\varphi(x),t)$ if we identify that component with the disk.

This way, unless $x \in X$ is on an open terminal arc, U_i^x is homeomorphic by $\widetilde{\varphi}$ to an Euclidean neighborhood of $\varphi(x)$ in S^2. Otherwise, U_i^x is homeomorphic to a neighborhood of $\varphi(x)$ in the complement $S^2 - \operatorname{int} B^2$ of an open disk with $\varphi(x) \in \partial B^2$. Hence, all points in M_X have 2-dimensional Euclidean neighborhoods and then M_X is a surface whose boundary coincides with the locally finite family of terminal open arcs in $\mathscr{A}_T - \mathscr{A}_T^*$, see Proposition 4.12.

It is an immediate consequence of the definition of M_X that the family of cones $\{cS;\, S \in \mathscr{A}_T^*\}$ is a null sequence and so Corollary B.2 in Appendix B yields that the inclusion $i : X \subset M_X$ induces a homeomorphism $\mathcal{F}(X) \cong \mathcal{F}(M_X)$. □

Remark 5.1. Notice that M_X is a closed surface (i.e., compact and without boundary) whenever X is compact.

Next we proceed to prove Theorem B. For this, recall that an *S-curve* in a surface M is a curve X whose complement $M - X = \cup_{k=1}^{\infty} \operatorname{int} D_k$ is the union of the interiors of a sequence of pairwise disjoint closed disks $D_k \subset M - \partial M$ $(k \geq 1)$. Notice that $\partial M \subset X$. The following lemma is essentially well-known.

Lemma 5.2. *Let $\mathcal{D} = \{D_i\}_{i \geq 1}$ be a null sequence (possibly empty) of pairwise disjoint closed disks in the interior of the surface M. Then there is an S-curve $C \subset M - \cup_{i \geq 1} \operatorname{int} D_i$. In particular, any surface contains an S-curve.*

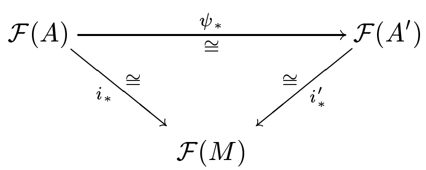
Proof. We will extend $\mathcal{D}$ to a null sequence of pairwise disjoint closed disks $\mathcal{B} = \{B_i\}_{i \geq 1}$ such that the union $\cup_{i=1}^{\infty} B_i$ is dense in M. This is equivalent to saying that $C = M - \cup_{i=1}^{\infty} \operatorname{int} B_i$ is an S-curve (see Appendix B).

In order to construct the sequence $\mathcal{B}$ we consider the closed set $F = \overline{\cup_{i \geq 1} D_i} \cup \partial M$. If $F = M$, there is nothing to proof. Otherwise, let $D = \{d_n\}_{n \geq 1}$ be a countable dense subset in the open subspace $M - F$. Then we define inductively a sequence $\mathcal{B}' = \{B_j'\}_{j \geq 1}$ such that B_j' is a closed disk of center d_{n_j} and radius ϵ_j where d_{n_j} is the first element in $\{d_n\}_{n \geq n_{j-1}}$ such that $d_n \notin \cup_{k=1}^{j-1} B_k$ and $\epsilon_j \leq (1/j)$ is chosen to get $B_j' \cap (\cup_{k=1}^{j-1} B_k' \cup F) = \varnothing$. We start with $n_1 = 1$, and choose $\epsilon_1 \leq 1$ with $B_1' \cap F = \varnothing$.

It is clear that all disks in $\mathcal{B}'$ are pairwise disjoint. Moreover $M - F \subset \overline{\cup_{j=1}^{\infty} B_j'}$; indeed, given $x \notin F$ and any open neighborhood Ω of x, let $d_n \in D \cap \Omega$. If $n \neq n_j$ for all j, then $n_{j_0} < n < n_{j_0+1}$ for some j_0 and $d_n \in B_k'$ for some $k \leq j_0$. Thus $x \in \overline{\cup_{j=1}^{\infty} B_j'}$. We set $\mathcal{B} = \mathcal{D} \cup \mathcal{B}'$ and so $M = (M - F) \cup F \subset \overline{\cup\{B;\, B \in \mathcal{B}\}}$. □

A theorem due to Borsuk [**5**, Theorem 7.4] states that any two S-curves in a closed surface M are homeomorphic. The same arguments, based on Moore's surface decomposition theorem, also work to show the same result for S-curves in an arbitrary surface with boundary, compact or not. Namely,

Theorem 5.3. *The inclusion $i : A \subset M$ of any S-curve in a surface M induces a homeomorphism $i_* : \mathcal{F}(A) \cong \mathcal{F}(M)$. Moreover, given another inclusion $i' : A' \subset M$ of an S-curve, there is a homeomorphism $\psi : A \to A'$ such that the diagram of homeomorphisms*



commutes.

In Appendix B we give a detailed proof of Theorem 5.3. Recall that a subspace $U \subset M$ is said to be *universal* for a class $\mathcal{P}$ of subspaces of M if any $P \in \mathcal{P}$ can be embedded as a subspace of U. The following proposition is an immediate consequence of Theorem 5.3.

Proposition 5.4. *Any S-curve U in a surface M is universal for the family $\mathcal{P}$ of* 1*-dimensional closed sets of M. Moreover, given a* 1*-dimensional closed set $C \subset M$, there is a* (*closed*) *embedding $\rho : C \to U$ for which the homeomorphism $i_* : \mathcal{F}(U) \cong \mathcal{F}(M)$ induced by the inclusion $U \subset M$ in Corollary* B.2 *restricts to a homeomorphism $\overline{\rho(C)} \cap \mathcal{F}(U) \cong \overline{C} \cap \mathcal{F}(M)$. Here the closures are taken in the corresponding Freudenthal compactifications.*

Proof. If $C \in \mathcal{P}$ and $F = C \cup \partial M$ then $\operatorname{int} F = \varnothing$ by [**11**, 1.8.12] and hence $\overline{M - F} = M$. Let $\{D_i\}_{i \geq 1}$ be a null sequence of pairwise disjoint closed disks contained in the open set $M - F$ and $M - F \subset \overline{\cup_{i \geq 1} D_i}$ as in the proof of Lemma 5.2. Then $M = \overline{\cup_{i \geq 1} D_i}$ since $M - F$ is dense in M. This suffices to show that $Z = M - \cup_{i \geq 1} \operatorname{int} D_i$ is an S-curve (see Appendix B). Theorem 5.3 yields a homeomorphism $\psi : Z \to U$ and hence a (closed) embedding $\rho : C \subset F \subset Z \xrightarrow{\psi} U$ for which one readily checks $i_*(\overline{\rho(C)} \cap \mathcal{F}(U)) = \overline{C} \cap \mathcal{F}(M)$ for the inclusion $i : U \subset M$. □

Proof of Theorem B. We use Lemma 5.2 to extend the null sequence of pairwise disjoint cones $cS \subset M_X - \partial M_X$ over terminal circles $S \in \mathscr{A}_T^*$ to a null sequence $\mathcal{B} = \{B_i\}_{i\geq 1}$ such that $M_X^1 = M_X - \cup_{i\geq 1} \mathrm{int}\, B_i$ is an S-curve in M_X. Moreover, given a closed embedding $\psi : Y \to M_X$ of a 1-dimensional metric space we use Proposition 5.4 to get a new embedding $\rho : \psi(Y) \to M_X^1 \subset X$, for which the inclusion $i : M_X^1 \to M$ induces a homeomorphism $\overline{\rho\psi(Y)} \cap \mathcal{F}(M_X^1) \cong \overline{\psi(Y)} \cap \mathcal{F}(M)$. It is easily checked that the embedding $\varphi = \rho\psi : Y \to X$ satisfies the equality $i_*(\overline{\varphi(Y)} \cap \mathcal{F}(X)) = \overline{\psi(Y)} \cap \mathcal{F}(M_X)$. Here we use the commutative diagram of homeomorphisms

$$\begin{array}{ccc} \mathcal{F}(M_X^1) & \xrightarrow{\;\cong\;} & \mathcal{F}(M_X) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & \mathcal{F}(X) & \end{array}$$

induced by the corresponding inclusions. □

Remark 5.5. It is clear that the surface M_X can be enlarged to a surface $\widetilde{M}_X$ without boundary by attaching a copy of the half-plane $\mathbf{R}_+^2$ to each component of $\partial M_X = \sqcup_{i\in I} \mathbf{R}$. However, easy examples show that not all curves A in $\widetilde{M}_X$ can be topologically embedded as a closed set in X. For instance, this happens for $X = M_X = \mathbf{R}_+^2$, $\widetilde{M}_X = \mathbf{R}^2$ and A the lattice $A = \mathbf{Z} \times \mathbf{R} \cup \mathbf{R} \times \mathbf{Z} \subset \widetilde{M}_X$.

Next we use Theorem B to prove Theorem C independently of Thomassen's theorem [**24**, Theorem 4.6]. For this we will need the following lemma; compare with [**26**, Theorem 4.2].

Lemma 5.6. *Let X be a locally 2-connected generalized Peano continuum. If $\{x_n\}_{n\geq 1} \subset X$ is a sequence converging to $x_0 \in X$, then there is an arc $[a, b] \subset X$ which contains x_0 in its interior and such that a subsequence of $\{x_n\}_{n\geq 1}$ lies in one of the subarcs determined by x_0.*

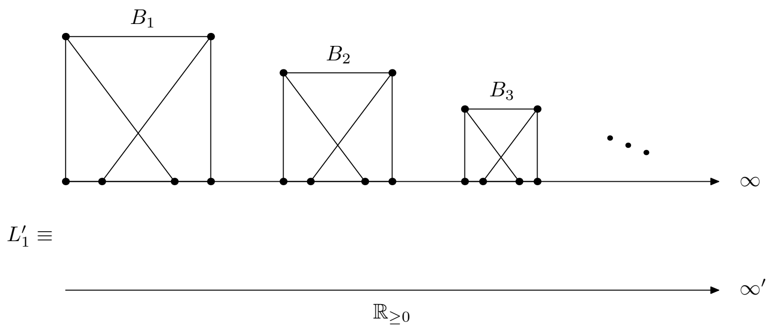


FIGURE 3. The graph L'_1.

Proof. As X is ω-connected (Lemma 2.1) given two points $a, b \in X - \{x_0\}$ there are two independent arcs from x_0 to a and b by Lemma 2.5, and hence there is an arc $\Gamma = [a, b] \subset X$ containing x_0 in its interior. Assume that $\Gamma \cap \{x_n\}_{n\geq 1}$ reduces to a finite number of points so that we can assume without loss of generality that $\Gamma \cap \{x_n\}_{n\geq 1} = \varnothing$. We use Corollary 2.3 to pick an ω-neighborhood W_1 of x_0. There is also no loss of generality in assuming that $\{x_n\}_{x\geq 1} \subset W_1$. We consider three points y_1, y_2, $y_3 \in W_1 \cap [x_0, b]$ in the interior of the subarc $[x_0, b]$. By using Lemma 2.5 there are three disjoint (except at x_1) arcs γ_1^1, γ_2^1, $\gamma_3^1 \subset W_1$ from x_1 to y_1, y_2, y_3, respectively, avoiding x_0. Let z_i be the first point in the intersection $\gamma_i^1 \cap \Gamma$. At least two out of the three points $\{z_1, z_2, z_3\}$ lie in one of the subarcs of $[a, b]$ determined by x_0. Again by Corollary 2.3 we choose a small ω-neighborhood $W_2 \subset W_1$ of x_0 avoiding the union $\widetilde{\gamma}_1^1 \cup \widetilde{\gamma}_2^1 \cup \widetilde{\gamma}_3^1$ where $\widetilde{\gamma}_i^1 \subset \gamma_i^1$ is the subarc running from x_1 to z_i. We pick $x_{n_2} \in W_2$ and obtain three arcs from x_{n_2} to the arc Γ such that two of them hit Γ at the same subarc. By proceeding inductively in this way we obtain an embedded sequence of ω-neighborhoods of x_0 $\{W_k\}_{k\geq 1}$ and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ with $x_{n_k} \in W_k$ such that for all $k \geq 1$ there are two arcs ρ_k^1 and ρ_k^2 from x_{n_k} to the same subarc of Γ, say $[x_0, b]$, verifying $\rho_k^1 \cap \rho_k^2 = \{x_{n_k}\}$ and $\rho_k^i \cap \rho_{k'}^j = \varnothing$ if $k \neq k'$. Then it is easy to change the subarc $[x_0, b]$ to get a new arc $\Gamma' \subset \Gamma \cup \{\rho^i\}_{k\geq 1}^{i=1,2}$ from a to b passing through x_0 and for which the subsequence $\{x_{n_k}\}_{k\geq 1}$ is part of the subarc of Γ' from x_0 to b. $\square$

Proof of Theorem C. We will prove (a) $\Longleftrightarrow$ (b). Similarly, by the use of Proposition 3.3 the same proof with the obvious changes proves (a) $\Longleftrightarrow$(c). Assume that there is a connected open neighborhood U of p such that $W = U - \{p\}$ is locally planar. Then W is a locally planar, locally 2-connected generalized Peano continuum, and hence there is a surface M_W satisfying Theorem A. Moreover the point p corresponds, via the homeomorphism $\mathcal{F}(W) \cong \mathcal{F}(M_W)$ in Theorem A, to an isolated Freudenthal end[3] $\varepsilon \in \mathcal{F}(M_W)$ which cannot be planar since otherwise U is locally planar at p. Hence, there is a sequence of handles or crosscaps in M_W converging to ε, and it is not hard to find in the surface M_W a closed embedding $\psi_0' : L_1' \to M_W$ of the non-connected infinite graph in Figure 3 in such a way that the ends ∞ and ∞' are mapped to ε, each block B_i lies in a handle or a crosscap, and $\mathbf{R}_+$ misses all handles and crosscaps. Then we apply Theorem B to obtain a closed embedding $\psi_1' : L_1' \to W$ for which the inclusion $i : W \to M_W$ satisfies $i_*(p) = \varepsilon$ if p is regarded as a Freudenthal end of W. Therefore ψ_1' induces an embedding $\psi_1 : L_1 = L_1' \cup \{p_1\} \to U = W \cup \{p\}$ with $\psi_1(p_1) = p$, and the theorem is proved if p is an isolated non-planar point.

Next we prove the theorem under the assumption that there is a sequence $\{p_n\}_{n\geq 1}$ converging to p such that X is not locally planar at each p_n. By use of Lemma 5.6 there is an arc $\Gamma = [a, b] \subset X$ with $p \in (a, b)$ and a subsequence of $\{p_n\}_{n\geq 1}$ in one of the subarcs of Γ defined by p, say $[p, b]$. In fact we can assume without loss of generality that $\{p_n\}_{n\geq 1} \subseteq [p, b]$ for the whole sequence. Now we use Corollary 2.3 to find a null sequence $C_1, \dots, C_i, \dots$ of ω-neighborhoods of $p_1, \dots, p_i, \dots$, respectively. As each $\operatorname{int} C_i$ is a non-planar locally 2-connected generalized Peano continuum, it contains a copy $K_i \subset \operatorname{int} C_i$ of the graph $K_{3,3}$, see Proposition 3.3. For each $i \geq 1$ we consider the points r_i, $s_i \in C_i$ defined as the first and last points in the intersection $C_i \cap \Gamma$, respectively. As the complement $C_i - V_i$ of the vertex set of K_i remains ω-connected, we get disjoint arcs ρ_i and σ_i in $C_i - V_i$ joining r_i and s_i to K_i, respectively. Here we use Lemma 2.5. Then one observes after the inspection of the cases originated by the (essentially three) possible positions of the intersections points $\rho_i \cap K_i$ and $\sigma_i \cap K_i$ that the graph G_i in Figure 4 is embedded in $K_i \cup \rho_i \cup \sigma_i \subset C_i$ with $G_i \cap \operatorname{Fr} C_i = \{s_i, r_i\}$. Now one readily finds a copy of the Claytor curve L_1 in the union $\Gamma \cup (\cup_{i\geq 1} G_i)$ with p in the place of p_1. This finishes the proof. □

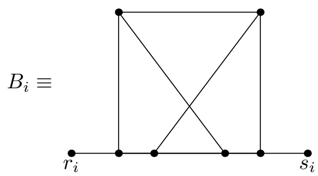


FIGURE 4. The graph G_i.

Remark 5.7. As pointed out at the end of Section 1, Thomassen [**24**, Theorem 4.6] characterized the failure of the local planarity of at $p \in X$ by the existence of an infinite complete graph $K_\infty \subset X$ containing p. It is readily checked that L_1 (analogously, L_2) can be embedded into the complete graph K_∞ whose vertices are the thick points of L_1 in Figure 1. Hence, Thomassen's characterization implies Theorem C. Nevertheless, it seems interesting to highlight, as Theorem C does, that the existence of an embedded copy of L_1 (or, equivalently, L_2) is enough to determine the non-planarity of X at p, avoiding the far more complicated graph K_∞.

APPENDIX

A. **The definition of terminal point.** We include here a proof of the fact that terminal points in Definition 4.2 are well defined. We start with the following:

Lemma A.1. *Let $U \subset S^2$ be a locally 2-connected subspace for which there is a circle Σ such that $U \cap \Sigma = (s,t)$ is an open arc and U misses one of the components of $S^2 - \Sigma$. Then $U - \Sigma$ is connected.*

Proof. Given a, $b \in U - \Sigma$ let $\rho \subset U$ be an arc running from a to b, and let a_0, $b_0 \in U$ be the first and last points in $\rho \cap \Sigma \neq \varnothing$ (if empty, there is nothing to prove). We choose an arc $[p,q] \subset (s,t)$ containing in its interior the arc $[a_0, b_0]$ (possibly $a_0 = b_0$). Let $W \subset U$ be a connected open set such that $[p,q] \subset W$ and $a, b \notin W$. By Lemma 2.1, W is ω-connected and so is $W - \{w\}$ for any point $w \in (a_0, b_0)$ (possibly $w = a_0 = b_0$ if $a_0 = b_0$). By Theorem 2.4 we find two independent arcs $\gamma_1, \gamma_2 \subset W$ from p to q (i.e., $\gamma_1 \cap \gamma_2 = \{p, q\}$) avoiding w. Let $x_1^i \in \gamma_i$

be the last point of γ_i in the open arc (s, w) and $x_2^i \in \gamma_i$ the first point after x_1^i in (w, t). A straightforward check shows that the Jordan theorem rules out the cases x_1^1, $x_1^2 \in (p, w)$ and x_2^1, $x_2^2 \in (w, q)$. Hence, $x_1^i \geq p$ or $x_2^i \geq q$ for some $i = 1, 2$. Let $\Gamma \subset \gamma_i \subset W$ be the subarc running from x_1^i to x_2^i. As $a, b \notin W$ one uses again the Jordan theorem to guarantee that $\rho \cap \Gamma \neq \varnothing$ and readily finds an arc in $\rho \cup \Gamma \subset U$ joining a to b outside Σ. □

Lemma A.2. *Let (C_i^x, φ) and (C_j^y, ψ) be two charts in a locally planar, locally 2-connected generalized Peano continuum. Assume that $z \in \operatorname{int} C_i^x \cap \operatorname{int} C_j^y$ is a point such that $\varphi(z) \in \operatorname{Fr} R$ for some component $R \subset S^2 - \varphi(C_i^x)$. Then there is a component $R' \subset S^2 - \varphi(C_j^y)$ with $\psi(z) \in \operatorname{Fr} R'$.*

Proof. Let $C_k^z \subset \operatorname{int} C_i^x \cap \operatorname{int} C_j^y$ be a planar ω-neighborhood of z. As $S^2 - \varphi(C_i^x) \subset S^2 - \varphi(C_k^z)$, let $R' \subset S^2 - \varphi(C_k^z)$ be the component containing R. Therefore $\varphi(z) \in \overline{R} \subset \overline{R}'$, and so $\varphi(z) \in \operatorname{Fr} R'$.

Next we pick a connected open neighborhood $U \subset C_k^z$ of z such that $U \cap S$ is an open arc where $S = \varphi^{-1}(\operatorname{Fr} R')$. By Lemma A.1 $U - S$ is connected, and hence $\psi(U) - \psi(S)$ is also connected and it is contained in a component of $S^2 - \psi(S) = R_1 \sqcup R_2$, say $\psi(U - S) \subset R_1$.

We next show that $\psi(z)$ belongs to the closure of a component $R_0 \subset S^2 - \psi(C_j^y)$ with $R_0 \subset R_2$; hence, $\psi(z) \in \operatorname{Fr} R_0$, and the lemma is proved. To find the component R_0 we observe that, as $\psi(z) \in \psi(S) = \operatorname{Fr} R_2$, there is a sequence $x_n \in R_2$ converging to $\psi(z)$. This sequence lies eventually outside $\psi(C_j^y)$ since otherwise, as U is a neighborhood of z, there is a subsequence of $\{x_n\}_{n \geq 1}$ contained in $\psi(U - S) \subset R_1$. Thus, $x_n \in R_2 - \psi(C_j^y)$ for n large enough as claimed. Then we consider the components $R'_n \subset S^2 - \psi(C_j^y) \subset S^2 - \psi(S)$ with $x_n \in R'_n$. Notice that $R'_n \subset R_2$ since $x_n \in R_2$. If the family $\mathcal{R} = \{R'_n\}_{n \geq 1}$ is infinite, we use Lemma 4.4 to get $\operatorname{diam}(R'_n) \to 0$ and so $\operatorname{diam}(\operatorname{Fr} R'_n) \to 0$. As U is a neighborhood of z there is some n_0 such that $\operatorname{Fr} R'_n \subset \psi(U)$ if $n \geq n_0$. Therefore $\operatorname{Fr} R'_n \subset \psi(U) \cap \overline{R}_2 \subset \psi(S)$ for all $n \geq n_0$ which is a contradiction since each $\operatorname{Fr} R'_n$ as well as $\psi(S)$ are circles. Thus $\mathcal{R}$ is necessarily finite and there is an $R_0 \in \mathcal{R}$ containing a subsequence of $\{x_n\}_{n \geq 1}$. Hence $\psi(z) \in \overline{R}_0$. □

As an immediate consequence of Lemma A.2, one gets

Proposition A.3. *The definition of a terminal point does not depend on the choice of charts.*

B. Proof of Theorem 5.3. In this appendix we prove Theorem 5.3 which extends a result due to Borsuk [**5**] stating the uniqueness of S-curves in a closed surface. Essentially the proof is the same as Borsuk's proof. We give it here for the sake of completeness. We start by recalling that, given a sequence $\mathcal{D} = \{D_k\}_{k\geq 1}$ of disjoint closed disks $D_k \subset M - \partial M$, the difference $A = M - \cup_{k\geq 1}\text{int}\, D_k$ is an S-curve in the surface M (possibly non-compact and with boundary) if and only if $\mathcal{D}$ is a null sequence and the union $\cup_{k=1}^{\infty} D_k$ is dense in M (see [**12**, 5.5A] for a detailed proof).

In the proof of Theorem 5.3 we apply some deep results on surface decompositions. Namely, given a null sequence of closed disks $\mathcal{D} = \{D_k\}_{k\geq 1}$ in a surface M, the quotient map $\pi : M \to M/\mathcal{D}$ induced by shrinking each D_k to a point $*_k$ is a cell-like upper semicontinuous decomposition ([**9**, Section 2, Proposition 9]) and so the celebrated Moore theorem on cell-like decomposition of surfaces yields:

Theorem B.1 [**9**, Section 25, Theorem 1]. *The quotient $\pi : M \to M/\mathcal{D}$ is a strongly shrinkable decomposition of M, and so for any open set $U \subset M$ with $\cup_{k\geq 1} D_k \subset U$ there is a homeomorphism $f : M \to M/\mathcal{D}$ such that $f = \pi$ outside U.*

Corollary B.2. *Let $\{D_i\}_{i\geq 1}$ be a null sequence of pairwise disjoint closed disks in a surface M. Then the inclusion $M - \cup_{i\geq 1}\text{int}\, D_i = \widetilde{M} \subset M$ induces a homeomorphism $\mathcal{F}(\widetilde{M}) \cong \mathcal{F}(M)$.*

Proof. By Theorem B.1, $M/\mathcal{D}$ is a surface and [**9**, Section 21, Theorem 4] applied to $\{*_k = \pi(D_k)\}_{k\geq 1}$ shows that there is a triangulation $M/\mathcal{D} \cong |K|$ such that the 1-skeleton K^1 misses the set $\{*_k\}_{k\geq 1}$. Hence, $\pi^{-1}(K^1)$ is a closed set in M avoiding $\cup_{k\geq 1} D_k$. Moreover, Theorem B.1 applied to the open set $U = M - \pi^{-1}(K^1)$ yields a homeomorphism $f : M \to M/\mathcal{D}$ such that $f = \pi$ on $\pi^{-1}(K^1)$ and so

$\pi : \pi^{-1}(K^1) \to K^1$ is a homeomorphism. In particular, $f^{-1} : |K| \to M$ is a triangulation of M such that each disk of $\mathcal{D}$ lies in the interior of a triangle of K. Therefore, for each triangle $\sigma \in K$ the family $\mathcal{D}_\sigma = \{D_k,\, D_k \subset \operatorname{int}\sigma\}$ is a null sequence in σ missing $\partial\sigma$. Hence each difference $\widetilde{\sigma} = \sigma - \cup\{\operatorname{int} D_k;\, D_k \in \mathcal{D}_\sigma\} = \cap_{D_k \in \mathcal{D}_k}(\sigma - \operatorname{int} D_k)$ is a compact connected set.

Let $\{N_j\}_{j\geq 1}$ be an increasing sequence of polyhedra $N_j = |K_j|$ triangulated by finite subcomplexes $K_j \subset K$ such that $N_j \subset \operatorname{int} N_{j+1}$ and the intersections $E_j = N_j \cap (\overline{M - N_j})$ are 1-dimensional. For each j, let $\widetilde{N}_j \subset N_j$ be the subset $\widetilde{N}_j = N_j - \cup\{\operatorname{int} D_k;\, D_k \in \mathcal{D}_\sigma,\, \sigma \in K_j\}$.

This way $\{\widetilde{N}_j\}_{j\geq 1}$ is an increasing sequence of compact sets in $\widetilde{M}$ and, moreover, since $K^1 \subset \widetilde{M}$, it is readily checked that there is a one-to-one correspondence between the components of $M - \operatorname{int} N_j$ and $\widetilde{M} - \operatorname{int} \widetilde{N}_j$ which carries the component $C \subset M - \operatorname{int} N_j$ to the component $\widetilde{C} = C - \cup\{D_k \in \mathcal{D}_\sigma;\, \sigma \subset C\}$. From this it easily follows that the inclusion $\widetilde{M} \subset M$ induces a homeomorphism $\mathcal{F}(\widetilde{M}) \cong \mathcal{F}(M)$. ◻

We are ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let $\mathcal{D} = \{D_k\}_{k\geq 1}$ and $\mathcal{D}' = \{D'_k\}_{k\geq 1}$ be two null sequences of pairwise disjoint closed disks in the interior of M defining the S-curves $A = M - \cup_{k\geq 1}\operatorname{int} D_k$ and $A' = M - \cup_{k\geq 1}\operatorname{int} D'_k$, respectively. As the decomposition spaces $M/\mathcal{D}$ and $M/\mathcal{D}'$ are surfaces (in fact homeomorphic to M) by Theorem B.1, the natural projections $\pi : M \to M/\mathcal{D}$ and $\pi' : M \to M/\mathcal{D}'$ are properly homotopic to homeomorphisms $f : M \to M/\mathcal{D}$ and $f' : M \to M/\mathcal{D}'$; see [**16**, Theorem 5.3]. In particular, for the composite $h = f' \circ f^{-1}$ we have a commutative diagram

(1)
$$\begin{array}{ccccc} & & \mathcal{F}(M) & & \\ & \swarrow^{f_*=\pi_*}_{\cong} & & \searrow^{f'_*=\pi'_*}_{\cong} & \\ \mathcal{F}(M/\mathcal{D}) & & \xrightarrow[h_*]{\cong} & & \mathcal{F}(M/\mathcal{D}') \end{array}$$

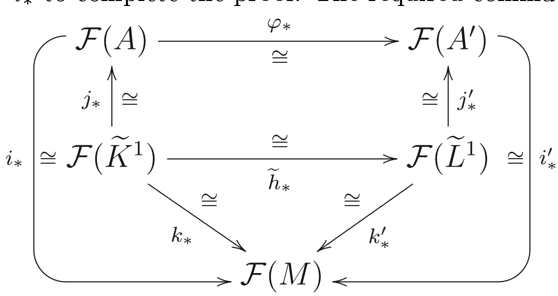
Recall that proper homotopic maps induce the same map between spaces of Freudenthal ends [**6**, Theorem 1.3].

Next, as was done in the proof of Corollary B.2, we can find a triangulation K of $M/\mathcal{D}$ such that its 1-skeleton K^1 misses the set $\{\pi(D_k)\}_{k\geq 1} \cup \{h^{-1}\pi'(D'_k)\}_{k\geq 1}$. Let L denote the triangulation of $M/\mathcal{D}'$ induced by K via h. Moreover, by Theorem B.1 there exist homeomorphisms $g : M \to M/\mathcal{D}$ and $g' : M \to M/\mathcal{D}'$ which coincide with π and π' outside the open sets $U = M - \pi^{-1}(K^1)$ and $U' = M - \pi'^{-1}(L^1)$, respectively. In particular, K and L lift to triangulations $\widetilde{K}$ and $\widetilde{L}$ of M with 1-skeletons $\widetilde{K}^1 = \pi^{-1}(K^1)$ and $\widetilde{L}^1 = \pi'^{-1}(L^1)$, respectively, and such that $\widetilde{h} = g'^{-1} \circ h \circ g : \widetilde{K} \to \widetilde{L}$ is a simplicial isomorphism.

Notice that for each triangle $\sigma \in \widetilde{K}$ the families $\mathcal{D}_\sigma = \{D_k; D_k \subset \operatorname{int}\sigma\}$ and $\mathcal{D}'_\sigma = \{D'_k; D'_k \subset \operatorname{int}\widetilde{h}(\sigma)\}$ are null sequences in σ and $\widetilde{h}(\sigma)$ missing $\partial\sigma$ and $\partial\widetilde{h}(\sigma) = \widetilde{h}(\partial\sigma)$, respectively. This way, $A \cap \sigma$ and $A' \cap \widetilde{h}(\sigma)$ are S-curves in σ and $\widetilde{h}(\sigma)$ containing $\partial\sigma$ and $\partial\widetilde{h}(\sigma)$. Then we apply to each σ a theorem due to Whyburn [**25**] to obtain a homeomorphism $\varphi_\sigma : A \cap \sigma \to A' \cap \widetilde{h}(\sigma)$ extending $\widetilde{h} : \partial\sigma \to \widetilde{h}(\partial\sigma)$. Therefore, $\varphi = \cup_{\sigma\in K}\varphi_\sigma : A \to A'$ is a homeomorphism extending $\widetilde{h}$ on $\widetilde{K}^1$.

Concerning the spaces of Freudenthal ends we have the following diagram where all arrows, except φ_* and $\widetilde{h}_*$, are induced by inclusions. Moreover, i_* and i'_* are homeomorphisms by Corollary B.2. In addition, we use the well-known fact that the Freudenthal ends of a polyhedron are determined by the 1-skeleton of any triangulation to get that k_* and k'_* are homeomorphisms.

Clearly the inner rectangle is commutative, and it remains to check that the inner triangle also is commutative to readily derive the equality $i'_* \circ \varphi_* = i_*$ to complete the proof. The required commutativity is



an immediate consequence of the fact that the π'_* in diagram (1) is a homeomorphism and the sequence of equalities

$$(\#) \qquad \pi'_* \circ k'_* \circ \widetilde{h}_* = \ell'_* \circ (g' \mid \widetilde{L}^1)_* \circ \widetilde{h}_* = h_* \circ \ell_* \circ (g \mid \widetilde{K}^1)_* = h_* \circ \pi_* \circ k_* = \pi'_* \circ k_*$$

where $\ell : K^1 \subset M/\mathcal{D}$ and $\ell' : L^1 \subset M/\mathcal{D}'$ are inclusions and (#) is the commutativity of diagram (1). Recall that $g \mid \widetilde{K}^1 = \pi \mid \widetilde{K}^1 : \widetilde{K}^1 \to K^1$ and $g' \mid \widetilde{L}^1 = \pi' \mid \widetilde{L}^1 : \widetilde{L}^1 \to L^1$. □

ENDNOTES

1. This terminology is used here as the natural extension of the well-established notion of n-connectivity in graph theory, far apart from its meaning in algebraic topology as the vanishing of homotopy groups in dimensions $\leq n$.

2. In case d is the restriction of a distance on the one-point compactification X^+, the role of the compact set K is irrelevant.

Notice also that $\mathcal{A} = \cup_{n=1}^{\infty} \cup_{j=1}^{\infty} \{A \in \mathcal{A};\ A \cap K_n \neq \varnothing \text{ and } \operatorname{diam}(A) \geq 1/j\}$ is indeed a sequence. Here $\{K_n\}_{n\geq 1}$ is an exhaustiveg sequence of X.

3. Indeed, if $\{K_n\}_{n\geq 1}$ is an exhaustive sequence of U and $\{U_n\}_{n\geq 1}$ is a nested neighborhood base of p consisting of 2-connected open sets, then $\{K_n - U_n\}_{n\geq 1}$ is an exhaustive sequence for $W = U - \{p\}$ for which p is identified with the end $\varepsilon \in \mathcal{F}(W)$ given by $\varepsilon = (U_n - \{p\})_{n\geq 1}$.

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