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ON THE FREUDENTHAL EXTENSIONS OF CONFLUENT PROPER MAPS

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Abstract. In this paper we study when Freudenthal extensions of proper maps preserve the (weak, semi) confluency. Also the extensions to the Alexandroff one-point compactificaton are considered.

1. Preliminaries

The interesting class of confluent maps (containing monotone as well as open maps) was introduced originally by J. J. Charatonik in [3]. Later on various generalizations have appeared in the literature. In this paper, besides confluent maps, we will consider semi-confluent and weakly confluent maps defined by T. Maćkowiak [14] and A. Lelek [11], respectively.

Some authors have used proper maps to extend results on confluent maps from the classical theory of *continua* (i.e., compact and connected metric spaces) to the non-compact setting. As a contribution to this effort, we study the behavior of confluent, semi-confluent and weakly confluent proper maps with respect to the Alexandroff and Freudenthal compactifications.

More precisely, we point out that none of the aforementioned types of confluent maps is preserved by the Freudenthal compactification (Example 3.4). Notwithstanding, positive results are attained for the three types of confluency for maps

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with locally connected codomain (Theorems 3.6, 5.5 and 6.4). Also the Freudenthal extension preserves confluency and weak confluency for maps inducing a homeomorphism between end spaces (Theorem 7.5). Similarly, the Alexandroff one-point compactification preserves confluent and weakly confluent maps (Theorem 7.7). In contrast, these results do not hold for semi-confluent maps (Example 7.8).

The paper contains two proofs of the preservation of confluency by the Freudenthal extension (Theorem 3.6), the first and shorter one (which does not work for the other two types of confluency considered in this paper) is a mentionworthy consequence of the monotone-light factorization theorem. The second proof is based on the fact that Peano subcontinua can replace subcontinua in the definition of confluency for maps with locally connected codomain (Theorem 4.1). This pattern is also followed by semiconfluent maps; that is, Theorem 4.1 also holds for these maps (Theorem 6.2) and we derive from it that Freudenthal extensions preserve semiconfluency (Theorem 6.4). However, Theorem 4.1 does not hold for weakly confluent maps (Example 5.1) but still we can prove the analogue of Theorem 3.6 for this class of maps (Theorem 5.5).

2. The Freudenthal compactification

We restrict to the category of metrizable locally compact and σ-compact (*admissible spaces*, for short) and proper maps. More precisely we focus our interest on connected admissible spaces (*generalized continua*). Local compactness together with σ-compactness yield the existence of increasing sequences of compact subsets $K_n \subset X$ such that $X = \bigcup_{n=1}^{\infty} K_n$ with $K_n \subset \operatorname{int} K_{n+1}$. Such a sequence $\{K_n\}_{n\geq 1}$ is called an *exhausting sequence*.

Given an exhausting sequence $\{K_n\}_{n\geq 1}$ of a generalized continuum X, the ends of X are the elements $\varepsilon = (Q_n)_{n\geq 1}$ of the inverse limit

$$\mathcal{F}(X) = \varprojlim Q(X - \operatorname{int} K_n)$$

where $Q(X - \operatorname{int} K_n)$ is the space of quasicomponents of $X - \operatorname{int} K_n$ and the bonding maps are induced by inclusions. The space $\mathcal{F}(X)$ turns to be homeomorphic to a closed subset of the middle-third Cantor set. The *Freudenthal compactification*[1] of X is the set $\widehat{X} = X \cup \mathcal{F}(X)$ endowed with the compact topology

[1]In fact the Freudenthal compactification can be defined for the much larger class of rim-compact spaces (i.e., topological spaces whose points posses a neighborhood basis consisting of open sets with compact frontiers). Moreover, the Freudenthal compactification of a rim-compact space X is homeomorphic to the Stoilow-Kerékjártó compactification which is obtained as the

generated by the open sets of X together with the sets

$$\Omega^* = \Omega \bigcup \{(Q_n)_{n\geq 1};\ \text{such that } Q_n \subset \Omega \text{ for } n \text{ large }\}$$

where Ω ranges over the family of all open sets with compact frontier in X. See [8] and [2] for details. Moreover, the Freudenthal compactification of a generalized continuum is a continuum ([7]; Thm. VI). We have the following useful description of the basic open set Ω^*. Given any set $A \subset X$, let $A^{\mathcal{F}} = \overline{A}^{\widehat{X}} \cap \mathcal{F}(X)$.

Lemma 2.1. *For any basic open set $\Omega^* \subset \widehat{X}$, the equality $\Omega^* = \Omega \cup \Omega^{\mathcal{F}}$ holds.*

Proof. The inclusion $\Omega^* \subset \Omega \cup \Omega^{\mathcal{F}}$ readily follows from the definition of the Freudenthal topology. In order to check $\Omega^{\mathcal{F}} \subset \Omega^*$, assume that there is an end $\varepsilon = (Q_n)_{n\geq 1} \in \overline{\Omega}^{\widehat{X}} - \Omega^*$, that is, $Q_n \not\subset \Omega$ for all n. As $\operatorname{Fr}\Omega$ is compact we have $\operatorname{Fr}\Omega \subset \operatorname{int} K_{n_0}$ for some n_0 in an exhausting sequence of X, $\{K_n\}_{n\geq 1}$. Thus, $\Omega_n = \Omega - \operatorname{int} K_n$ is an open and closed set in $X - \operatorname{int} K_n$ for all $n \geq n_0$. As $Q_n \subset X - \operatorname{int} K_n$ is a quasicomponent, necessarily

$$Q_n \subset (X - \operatorname{int} K_n) - \Omega = W_n \quad (n \geq n_0). \tag{2.A}$$

Since W_n is open and closed in $X - \operatorname{int} K_n$ we derive that $U_n = W_n \cap (X - K_{n+1})$ is open in $X - K_{n+1} \subset X - \operatorname{int} K_n$ and so U_n is an open set in X. Finally, $Q_{n+2} \subset U_n$, yields $\varepsilon \in U_n^*$ and $U_n \cap \Omega \neq \varnothing$ by hypothesis, this leads to a contradiction since $\Omega \cap U_n = \varnothing$ by (2.A). □

Recall that a continuous map $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact subset $K \subset Y$. It is well known that proper maps between admissible spaces are closed ([4]; 3.7.18). Any proper map $f : X \longrightarrow Y$ between generalized continua extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ which restricts to a continuous map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Namely if $\varepsilon = (Q_n)_{n\geq 1}$, $\widehat{f}(\varepsilon) = f_*(\varepsilon) = (Q'_k)_{k\geq 1}$ where $Q'_k \subset Y$ is a quasicomponent such that $f(Q_{n_k}) \subset Q'_k$ for some increasing subsequence $(Q_{n_k})_{k\geq 1}$ of ε. We say that a proper map $f : X \to Y$ is *end-faithful* if $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a bijection. The map $\widehat{f}$ is termed the *Freudenthal extension* of f.

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quotient space of the Stone-Čech compactification $\beta(X)$ by factoring out the components of $\beta(X) - X$; see ([9], p. 115) and ([18], Corollary 1).

Locally connected (generalized) continua are called *(generalized) Peano continua*. For these spaces, quasicomponents coincide with components; so Freudenthal ends may be described as decreasing sequences of components. It is well-known that any generalized Peano continuum is arc-connected ([20]; 4.2.5). Furthermore, for generalized Peano continua an exhausting sequence can be chosen to be formed of subcontinua. More precisely, we have the following lemma.

Lemma 2.2 ([1]; 2.4(d))**.** *Let X be a generalized Peano continuum, then there exists an exhausting sequence $\{X_n\}_{n\geq 1}$ in X where each $X_n \subset X$ is a Peano subcontinuum.*

In the following lemmas we present some basic properties that will be used throughout this paper.

Lemma 2.3. *Let C be a continuum and $A \neq C$ be a non-trivial subcontinuum of C. If $K \subset C$ is a compact subset such that $A \cap K = \varnothing$, then there exists a continuum B in C with $K \cap B = \varnothing$ and $A \subsetneq B$.*

Proof. If $d(A,K) = \delta > 0$, we consider the open set $G = \{x \in C;\, d(x,A) < \frac{\delta}{2}\}$ for which $A \subset G$ and $K \cap \overline{G} = \varnothing$. Applying ([10]; Thm. 4, p. 173) we get a continuum B with $A \subsetneq B \subset \overline{G}$. In particular $K \cap B = \varnothing$. □

Lemma 2.4. *Let X be a generalized continuum and $\{K_n\}_{n\geq 1}$ be an exhausting sequence of X. For each point $p \in X$ there exists a sequence of continua $\{C_n\}_{n\geq 1} \subset X$ such that $p \in C_1$, $C_{n+1} - K_n \neq \emptyset$ and $C_n \subset C_{n+1}$ for all $n \geq 1$.*

Proof. We can assume without loss of generality that $p \in \operatorname{int} K_1$. Let C_1 be the component of p in K_1. Since the Freudenthal compactification $\widehat{X}$ is a continuum, the bumping boundary theorem ([10]; Thm. 1, p. 172) applied to $\widehat{X}$ yields $C_1 \cap \operatorname{Fr} K_1 \neq \varnothing$. Consider $p_1 \in C_1 \cap \operatorname{Fr} K_1$ and choose C_2 to be the component of p_1 in K_2. Notice that $C_1 \subset C_2$ and also $C_2 \cap \operatorname{Fr} K_2 \neq \varnothing$, and hence $C_2 - K_1 \neq \varnothing$. Proceeding inductively we obtain the desired sequence of continua. □

Lemma 2.5. *Let X be a generalized continuum. If H is a connected open subset of the Freudenthal compactification $\widehat{X}$, then $H - \mathcal{F}(X)$ is also connected.*

This lemma is an immediate consequence of the fact that the Stone-Čech compactification satisfies the corresponding analogue ([21]; 9.8) and that the Freudenthal compactification is perfect; that is, the canonical map $\beta X \to \widehat{X}$ is monotone; see Footnote 1. For the sake of completeness we give here a direct proof.

Proof of Lemma 2.5. Suppose for a moment that $H - \mathcal{F}(X) = V \cup W$ is the union of two disjoint open subsets in $H - \mathcal{F}(X)$ and hence in X. Clearly $H \subset$

$\overline{V}^{\widehat{X}} \cup \overline{W}^{\widehat{X}}$, and the connectedness of H yields the existence of some $p \in \overline{V}^{\widehat{X}} \cap \overline{W}^{\widehat{X}}$. Moreover, as V and W are disjoint we get that $p \in \mathcal{F}(X)$ is necessarily an end; that is, $p = (Q_n)_{n \geq 1}$ is a decreasing sequence of quasicomponents $Q_n \subset X - \operatorname{int} K_n$ where $\{K_n\}_{n \geq 1}$ is an exhausting sequence of X. Next we use that H is an open set in $\widehat{X}$ to find an open set $G \subset X$ with compact frontier such that $p \in G^* \subset H$.

Finally we observe that for any compact set $K \subset X$ with $\operatorname{Fr} G \subset \operatorname{int} K$ we have that $G - K = \overline{G}^X - K$ is an open and closed subset in $X - K$ and then $V_0 = (G - K) \cap V$ and $W_0 = (G - K) \cap W$ are open and closed subsets in $G - K \subset H - \mathcal{F}(X)$. Therefore, V_0 and W_0 are disjoint open an closed subsets in $X - K$. In particular V_0 and W_0 are open sets of X with compact frontiers $\operatorname{Fr} V_0 \cup \operatorname{Fr} W_0 \subset K$. We claim that $p \in V_0^* \cap W_0^*$ and there would be an integer n_0 greater enough such that $Q_n \subset V_0 \cap W_0$ for all $n \geq n_0$. This contradiction finishes the proof.

It remains to show the claim. For this, notice that by Lemma 2.1 it will suffice to check that $p \in \overline{V_0}^{\widehat{X}} \cap \overline{W_0}^{\widehat{X}}$. Let U^* be any basic open neighborhood of p in $\widehat{X}$ with $U^* \subset (G - K)^* \subset G^*$. Then $U \subset G - K$ and $p \in \overline{V}^{\widehat{X}}$ implies $V_0 \cap U = V \cap U = V \cap U^* \neq \varnothing$ and so $p \in \overline{V_0}^{\widehat{X}}$. Similarly $p \in \overline{W_0}^{\widehat{X}}$ and we are done. □

3. Confluent maps I

In this section we study the behavior of the Freudenthal extensions of confluent maps. We start by observing that any *monotone* proper surjection $f : X \to Y$ (i.e., $f^{-1}(y)$ connected for all $y \in Y$) between admissible spaces is end-faithful; see ([6]; 4.2) for a proof. As an immediate consequence we get the following proposition. Here X^+ represents the Alexandroff one-point compactification of X.

Proposition 3.1. *Let $f : X \to Y$ be a proper surjection between generalized continua. The following statements are equivalent:*

(a) *f is monotone;*
(b) *The Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is monotone;*
(c) *The Alexandroff extension $f^+ : X^+ \to Y^+$ is monotone.*

For the class of open maps we also have analogous properties.

Proposition 3.2. *Let $f : X \to Y$ be a proper surjection between generalized continua. The following statements are equivalent:*

(a) *f is open;*

(b) *The Freudenthal extension* $\widehat{f} : \widehat{X} \to \widehat{Y}$ *is open;*
(c) *The Alexandroff extension* $f^+ : X^+ \to Y^+$ *is open.*

Proof. It is obvious that (b) or (c) yields (a). Moreover, if f is an open map and $U \subset X^+$ is a basic open set then either U is open in X and then $f^+(U) = f(U)$ is open in Y and so in Y^+; or $U = X^+ - K$ for some compact set $K \subset X$, and then $f^+(U)$ is neighborhood of all points in the open set $f(X - K) = f^+(U) - \{\infty\}$ as well as neighborhood of $\infty \in Y^+ - f(K) \subset f^+(U)$. For the case of the Freudenthal extension, it will be enough to check the equality $\widehat{f}(\Omega^*) = (f(\Omega))^*$ for any basic open set $\Omega^* \subset \widehat{X}$, and this easily follows from Lemma 2.1 since any sequence $x_n \in \Omega$ such that $y_n = f(x_n)$ converges to some end $\eta \in (f(\Omega))^{\mathcal{F}}$ contains a subsequence converging to some end $\varepsilon \in \Omega^{\mathcal{F}}$ and hence $\eta \in \widehat{f}(\Omega^{\mathcal{F}})$. □

Monotone maps as well as open maps are examples of the so-called confluent maps. Recall that, given two spaces X and Y, by a *confluent map* we mean a continuous surjection $f : X \to Y$ such that for any subcontinuum $B \subset Y$ we have $f(A) = B$ for each connected component $A \subset f^{-1}(B)$.

The origin of this definition due to J. J. Charatonik is an old result of G. T. Whyburn establishing that open proper maps are confluent maps ([22]; 11.1). Since preimages of connected sets by monotone closed maps are connected ([4]; 6.1.29), it is obvious that monotone proper maps are also confluent. Incidentally, the following proposition shows that monotone and confluent maps coincide for proper real functions.

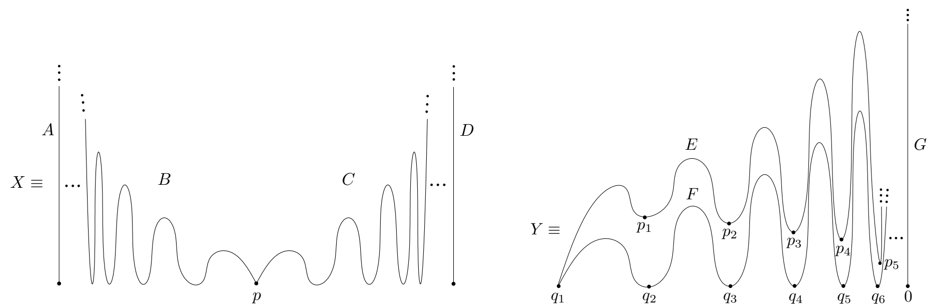
Proposition 3.3. *Any proper confluent surjection* $f : \mathbb{R} \to \mathbb{R}$ *is monotone.*

Proof. Assume that $f^{-1}(y_0)$ is not connected for some $y_0 \in \mathbb{R}$ so that there is $z \in \mathbb{R} - f^{-1}(y_0)$ such that $f^{-1}(y_0)$ meets both open intervals (z, ∞) and $(-\infty, z)$. Moreover, as f is onto then so is the induced map $f_* : \mathcal{F}(\mathbb{R}) = \{\pm\infty\} \to \mathcal{F}(\mathbb{R}) = \{\pm\infty\}$. Hence f_* is necessarily one-to-one. Assume $f_*(\infty) = \infty$ and $f_*(-\infty) = -\infty$.

On the other hand, $z \notin f^{-1}(y_0)$ yields $y_0 < f(z)$ or $y_0 > f(z)$. Assume the former and apply the confluency of f to the closed interval $[f(z), \infty)$ to get a connected non-compact closed set, that is, an interval $[a, \infty)$ with $z \in [a, \infty)$ and $f([a, \infty)) = [f(z), \infty)$. Here we use the assumption $f_*(\infty) = \infty$. Then $[z, \infty) \subset [a, \infty)$ and so there is $x_0 \in f^{-1}(y_0) \cap [a, \infty)$, for which we reach to the contradiction $y_0 = f(x_0) \in [f(z), \infty)$. Similar arguments work for the other choices $f_*(\infty) = -\infty$ and $y_0 > f(z)$. □

In contrast with open maps and monotone maps, the Freudenthal extension of a confluent map needs not be confluent as the following easy example shows.

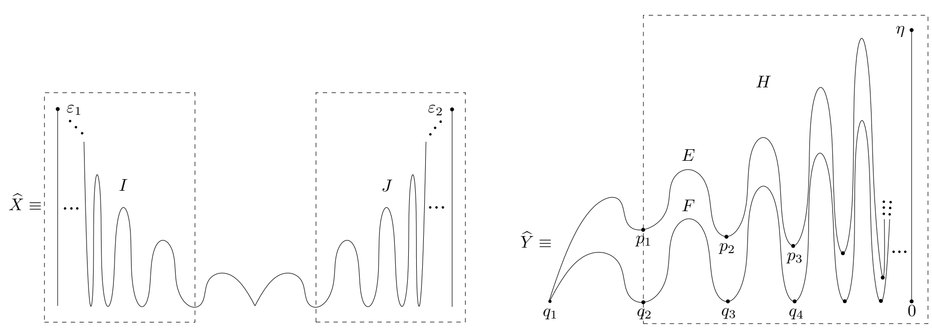
Example 3.4. Consider the generalized continua X and Y in $\mathbb{R}^2$ depicted in the following picture



(a) (b)

Figure 1

where the sequences $\{p_n\}_{n\geq 1} \subset E$ and $\{q_n\}_{n\geq 1} \subset F$ converge to 0. We choose a map $f : X \to Y$ which carries both straight half-lines A and D homeomorphically onto the straight half-line G, and the sinusoidal half-lines on the left and right-hand side of the point p, B and C, respectively, homeomorphically onto the sinusoidal half-lines E and F starting at q_1, respectively. It is readily checked that such a map is proper and confluent. However, its Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is not confluent.



(a) (b)

Figure 2

Indeed, we observe that X has two ends, say ε_1 and ε_2 while Y has only one end, say η. Then, given the continuum $H \subset \widehat{Y}$ in Figure 2(b), which is the complement of the open arc $(p_1, q_1) \subset \widehat{Y}$, we have that $f^{-1}(H) = I \sqcup J$ is the disjoint union of the components I and J sketched in Figure 2(a) whose images do not coincide with H because $\widehat{f}(I) \cap F = \varnothing$ and $\widehat{f}(J) \cap E = \varnothing$.

However, for light confluent proper maps whose codomains are generalized Peano continua we have the following theorem (compare ([16]; 13.29)).

Theorem 3.5. *Let $f : X \to Y$ be a light proper surjection between generalized continua and assume in addition that Y is locally connected. Then, f is confluent if and only if f is open.*

Recall that a map $f : X \to Y$ is termed *light* if each fiber $f^{-1}(y)$ is totally disconnected.

Proof. As mentioned above, open maps are always confluent. In order to prove the converse, let $U \subset X$ be an arbitrary open set and take any $y \in f(U)$. As Y is locally connected, y admits a countable neighborhood basis consisting of nested Peano subcontinua $\{V_n\}_{n\geq 1}$ ([16]; 13.19). Moreover, given $x \in U$ with $f(x) = y$, the confluency of f yields $f(H_n)V_n$ for the component of x $H_n \subset f^{-1}(V_n)$ $(n \geq 1)$. Notice that $H_{n+1} \subset H_n$. Therefore the intersection $H \bigcap_{n\geq 1} H_n$ is a continuum ([10]; Thm. 4, p. 170) containing x and so $y \in f(H) \subset \bigcap_{n\geq 1} V_n\{y\}$. Hence H is contained in the totally disconnected fiber $f^{-1}(y)$, whence H reduces to the singleton $H = \{x\} \subset U$. From this, one finds n_0 such that $H_n \subset U$ if $n \geq n_0$ ([16]; 1.7). Thus, $y = f(x) \in f(H_n) = V_n \subset f(U)$ shows that $f(U)$ is a neighborhood of y for all $y \in f(U)$; that is, $f(U)$ is an open set and f is an open map. □

The previous theorem leads to the following partial analogue of Propositions 3.1 and 3.2 for confluent maps.

Theorem 3.6. *Let X and Y be generalized continua and assume in addition that Y is locally connected. Then, any proper surjection $f : X \to Y$ is confluent if and only if its Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is confluent.*

Proof. The "if" part is obvious. Assume that f is confluent, then the monotone-light factorization theorem ([4]; 6.2.22) yields that $f = f_2 \circ f_1$ is the composite of a monotone proper surjection $f_1 : X \to Z$ and a light proper surjection $f_2 : Z \to Y$. Notice that Z is also a generalized continuum since any proper map preserves metrizability and local compactness ([4]; 4.4.15 and 3.7.21). Moreover, since f is confluent so is f_2 ([16]; 13.27(2)) and then Theorem 3.5 yields that f_2 is an open map. Therefore, the Freudenthal extensions $\widehat{f_1}$ and $\widehat{f_2}$ are monotone and open by Propositions 3.1 and 3.2, respectively. Thus, $\widehat{f_1}$ and $\widehat{f_2}$ are confluent maps and so is $\widehat{f} = \widehat{f_2} \circ \widehat{f_1}$ by ([16]; 13.27(1)). □

In [13], A. Lelek and E. D. Tymchatyn introduced a different notion of a confluent map $f : X \to Y$ by requiring that for any connected closed set $C \subset Y$ and for any quasicomponent $Q \subset f^{-1}(C)$ the equality $f(Q) = C$ holds; see ([13]; 1.4.(c')). Since quasicomponents coincide with components for compact spaces ([10]; Thm. 2, p. 169), both definitions are equivalent for such spaces. The following corollary extends this fact to the non-compact case.

Corollary 3.7. *Let $f : X \to Y$ be a proper surjection between generalized continua where Y is locally connected. Then the following statements are equivalent:*

(a) *f is Lelek-Tymchatyn confluent;*
(b) *f is confluent;*
(c) *For any connected subset $\Gamma \subset Y$ and any quasicomponent $Q \subset f^{-1}(\Gamma)$, $f(Q) = \Gamma$.*

Proof. (a)⇒(b) is immediate since the components and the quasicomponents of $f^{-1}(C)$ are the same whenever $f^{-1}(C)$ is compact. Here we use that f is proper. Also (c)⇒(a) is trivial. In order to show (b)⇒(c) we notice that the Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is confluent by Theorem 3.6 and so Lelek-Tymchatyn confluent. Now we apply ([13]; 3.1.(iii)) to get that any quasicomponent $Q \subset \widehat{f}^{-1}(\Gamma)f^{-1}(\Gamma)$ verifies that $f(Q) = \Gamma$. □

4. Confluent maps II

This section contains an alternative proof of Theorem 3.6 based on Theorem 4.1 below which extends ([16]; 13.22) and ([16]; 13.20) to the non-compact setting. Besides the possible interest of Theorem 4.1 on its own, this section collects most of the technical lemmas used in the proofs of the main results for semi and weakly confluent maps for which we do not have the corresponding analogues of Theorem 3.5, crucial in the short proof of Theorem 3.6 in Section 3.

If $\mathcal{C}$ denotes a family of connected subsets of the space Y, we will say that a continuous surjection $f : X \to Y$ is *$\mathcal{C}$-confluent* if for any $B \in \mathcal{C}$ and any component $C \subset f^{-1}(B)$, the equality $f(C) = B$ holds. The following theorem shows that other families of connected subsets can replace the family of all subcontinua in order to test the confluency of a proper map with local connected codomain.

Theorem 4.1. *Let $f : X \to Y$ be a proper surjection with Y a generalized Peano continuum. Then f is confluent if and only if f is $\mathcal{C}_i$-confluent for any of the following classes of connected subsets of Y $(1 \leq i \leq 3)$:*

(a) *$\mathcal{C}_1$ the class of Peano subcontinua;*
(b) *$\mathcal{C}_2$ the class of generalized Peano subcontinua;*

(c) $\mathcal{C}_3$ *the class of connected open subsets (or, equivalently, open generalized Peano subcontinua).*

In the proof of Theorem 4.1 we will use the following lemmas.

Lemma 4.2 (c.f. ([16]; 13.19))**.** *Let X be a generalized Peano continuum and B be a subcontinuum of X. Then, there exists a decreasing sequence of connected open neighborhoods of B, $\{W_n\}_{n\geq 1}$, such that the closures $B_n = \overline{W_n}$ are Peano subcontinua and $B = \bigcap_{n=1}^{\infty} B_n$.*

Proof. We choose an exhausting sequence of X consisting of Peano subcontinua (Lemma 2.2) $\{K_s\}_{s\geq 1}$ for which we can assume without loss of generality that $B \subset K_1$, and follow essentially the proof of ([16]; 13.19) with $Y = K_1$. Namely, by ([16]; 8.4) and ([16]; 8.9), given $n \geq 1$ one finds a cover of B by connected open subsets $U_1, \dots, U_m$ with the following properties for each $1 \leq k \leq m = m(n)$: (i) $\operatorname{diam}(U_k) < \frac{1}{n}$; (ii) $U_k \cap B \neq \varnothing$; and (iii) each U_k is locally connected (in fact, it verifies the so called property $\mathcal{S}$; see ([16]; 8.3)). Consider the open subset $W_n = \bigcup_{k=1}^{m} U_k$. Notice that W_n is connected by (ii). In addition, $B_n = \overline{W_n} = \bigcup_{k=1}^{m} \overline{U_k}$ is locally connected and hence a Peano continuum because each $\overline{U_k}$ is so by ([16]; 8.5). Here we also use any finite union of closed locally connected sets is locally connected ([10]; Thm. 1, p. 230). By (i), it follows that $B = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} W_n$. $\square$

Lemma 4.3. *Let $f : X \to Y$ be a $\mathcal{C}_3$-confluent proper surjection with Y a generalized Peano continuum. Given a connected open subset $U \subset Y$ and a component $H \subset f^{-1}(U)$, the restriction $g = f|_H : H \to f(H) = U$ is also $\mathcal{C}_3$-confluent.*

Proof. Let $V \subset U$ be a connected open set and $L \subset g^{-1}(V) = f^{-1}(V) \cap H$ be a component. It is readily checked that L is a component of $f^{-1}(V)$. Indeed, if L_0 is the component of $f^{-1}(V)$ containing L, then $L_0 \cap H \neq \varnothing$ and so $L_0 \subset H$. Hence $L = L_0$ is a component of $g^{-1}(V)$. Therefore, by assumption, $f(L) = V \subset U = f(H)$, and so $g(L) = V$. $\square$

Proof of Theorem 4.1. If f is confluent, then it is obvious that f is $\mathcal{C}_1$-confluent. Also (b)$\Rightarrow$(c) is immediate. Next we prove (a)$\Rightarrow$(b). For this, given a generalized Peano continuum $C \subset Y$, let $C = \bigcup_{n=1}^{\infty} C_n$ be an exhausting sequence of Peano subcontinua of C (Lemma 2.2). Then, for any component $A \subset f^{-1}(C)$ and $x \in A$ there exists n_0 with $f(x) \in C_n$ for all $n \geq n_0$. Hence for the components $A_n \subset f^{-1}(C_n)$ containing x we have $A_n \subset A_{n+1} \subset \dots$ and $A_n \subset A$ for all $n \geq n_0$. Moreover, by (a) $f\left(\bigcup_{n\geq n_0} A_n\right) = \bigcup_{n\geq n_0} C_n = C \subset f(A)$ and so $f(A) = C$.

Finally we check that if f is $\mathcal{C}_3$-confluent then it is also confluent. Indeed, Lemma 4.2 guarantees that each subcontinuum $C \subset Y$ can be written as an intersection $C = \bigcap_{n=1}^{\infty} B_n$ where $B_n = \overline{U}_n$ is a Peano subcontinuum, U_n is a connected open set and $C \subset U_{n+1} \subset U_n$ for all $n \geq 1$. If $A \subset f^{-1}(C) = \bigcap_{n=1}^{\infty} f^{-1}(U_n)$ is a component, we have that $A \subset H_1 \subset f^{-1}(U_1)$ for certain component H_1; moreover, the $\mathcal{C}_3$-confluency yields, $f(H_1) = U_1$. By applying Lemma 4.3 to the restriction $f_1 = f|_{H_1} : H_1 \to U_1$ we get $f(H_2) = U_2$ for any component $H_2 \subset f_1^{-1}(U_2) \subset H_1$. Proceeding inductively, we construct a sequence $\{H_n\}_{n\geq 1}$ such that for each $n \geq 1$, H_{n+1} is a component of $f_n^{-1}(U_{n+1}) \subset H_n$ and $f(H_n) = U_n$. Here $f_n = f|_{H_n} : H_n \to U_n$ is the corresponding restriction.

Observe that if $D_n \subset f^{-1}(B_n)$ is the component of $f^{-1}(B_n)$ containing H_n, we have $D_{n+1} \subset D_n$ $(n \geq 1)$ because $H_{n+1} \subset H_n$ and $f^{-1}(B_{n+1}) \subset f^{-1}(B_n)$. Furthermore, the intersection $\bigcap_{n=1}^{\infty} D_n$ is a continuum ([10]; Thm. 4, p. 170) and $A \subset \bigcap_{n=1}^{\infty} D_n \subset f^{-1}\left(\bigcap_{n=1}^{\infty} B_n\right) = f^{-1}(C)$. Hence $A = \bigcap_{n=1}^{\infty} D_n$ since $A \subset f^{-1}(C)$ is assumed to be a component. In addition, given $y \in C = \bigcap_{n=1}^{\infty} U_n$ we have that $f^{-1}(y) \cap H_n \neq \varnothing$ for all y because $f(H_n) = U_n$. Since $U_{n+1} \subset U_n$ for all $n \geq 1$, it follows by compactness of $f^{-1}(y)$ that the intersection of nested closed sets

$$\bigcap_{n=1}^{\infty} (f^{-1}(y) \cap D_n) = f^{-1}(y) \cap \left(\bigcap_{n=1}^{\infty} D_n \right) \neq \varnothing$$

is not empty. That is; there exits $x \in A = \bigcap_{n=1}^{\infty} D_n$ with $f(x) = y$. Hence, $f(A) = C$ and f is confluent. □

Remark. If we drop the local connectedness of Y from Theorem 4.1, then the result does not hold in general. For instance, the confluent proper map $f : X \to Y$ between generalized continua in Example 3.4 is not $\mathcal{C}_2$-confluent. Indeed, keeping the notation of Example 3.4, the generalized subcontinuum $E = F \cup G \subset Y$ satisfies that $f^{-1}(E)$ is the disjoint union of two components, A and $C \cup D$, and $f(A)$ is strictly contained in E.

In the second proof of Theorem 3.6 we will also need the following lemma.

Lemma 4.4. *Let $f : X \to Y$ be a proper surjection between generalized continua and H be an open set in $\widehat{Y}$. Given any component $D \subset \widehat{f}^{-1}(H)$ with $D \cap \mathcal{F}(X) \neq \varnothing$, the equality*

$$D = \overline{D - \mathcal{F}(X)}^{\widehat{f}^{-1}(H)}$$

holds. In particular, $D \cap X = D - \mathcal{F}(X) \neq \varnothing$ is always a non-empty set.

Proof. We start by fixing an exhausting sequence $\{K_n\}_{n\geq 1}$ in X and setting $A = \overline{D - \mathcal{F}(X)}^{\widehat{f}^{-1}(H)}$. It is clear that $D - \mathcal{F}(X) \subset A \subset D$. Here we use that D is closed in $\widehat{f}^{-1}(H)$.

Moreover, given any end $\varepsilon = (Q_n)_{n\geq 1} \in D$ where $Q_n \subset X - \operatorname{int} K_n$ is a quasicomponent and $Q_{n+1} \subset Q_n$, we consider any open neighborhood of ε, $W \subset \widehat{f}^{-1}(H)$. As $\widehat{f}^{-1}(H)$ is an open set of $\widehat{X}$, so is W. Therefore, the topology of $\widehat{X}$ provides us with some $n_0 \geq 1$ and a closed and open set N in $X - \operatorname{int} K_{n_0}$ such that $N^* = \overline{N}^{\widehat{X}}$ is a neighborhood of ε with $N^* \subset W$. Notice that $Q_n \subset N$ for all n large enough. On the other hand, we can apply ([6]; 2.2) to find a continuum $M \subset \widehat{H}$ such that

$$M \subset \overline{Q_{n_0}}^{\widehat{X}} \subset N^* \subset W \subset \widehat{f}^{-1}(H)$$

with $\varepsilon \in M$ and $M \cap \operatorname{Fr} K_{n_0} \neq \varnothing$. Hence $M \subset D$ by definition of component and

$$\varnothing \neq M \cap X \subset (W \cap D) \cap X = W \cap (D - \mathcal{F}(X))$$

shows that $\varepsilon \in A$, whence $D \subset A$ and the proof is finished. □

Lemma 4.5. *Let $\widehat{f} : \widehat{X} \to \widehat{Y}$ be Freudenthal extension of a proper surjection between generalized continua. If $H \subset \widehat{Y}$ is an open set, then any component $D \subset \widehat{f}^{-1}(H)$ is the closure in $\widehat{f}^{-1}(H)$ of the union of the family*

$$\mathcal{C}_D = \{C \text{ component of } f^{-1}(H - \mathcal{F}(X)) \text{ with } C \subset D\}.$$

Proof. As D is closed in $Z = \widehat{f}^{-1}(H)$, then $\overline{\bigcup_{C\in\mathcal{C}_D} C}^Z \subset D$. Conversely, any component $U \subset D - \mathcal{F}(X)$ is contained in some component $C \subset f^{-1}(H - \mathcal{F}(X)) \subset Z$, and so $U \subset C \subset D$. Then Lemma 4.4 yields $D = \overline{D - \mathcal{F}(X)}^Z \subset \overline{\bigcup_{C\in\mathcal{C}_D} C}^Z$. □

We will also use the following lemma whose proof is an easy exercise.

Lemma 4.6 ([4]; 1.4.C)**.** *Let $f : X \to Y$ be a continuous closed map. Then for any $A \subset X$ we have that $f(\overline{A}) = \overline{f(A)}$. Moreover, if $A, B \subset X$, then $f(A) \subset f(B)$ implies $f(\overline{A}) \subset f(\overline{B})$.*

Proof of Theorem 3.6 (Alternative). Sufficiency is obvious. Assume that f is confluent. By Theorem 4.1, in order to show that $\widehat{f}$ is confluent it will be enough to check that $\widehat{f}$ is $\mathcal{C}_3$ -confluent. For this, let $H \subset \widehat{Y}$ be any connected open subset. Then, by Lemma 2.5, $H - \mathcal{F}(Y)$ is also a connected open set in Y and Theorem 4.1 yields

$$f(C) = H - \mathcal{F}(Y) \tag{4.A}$$

for any component $C \subset f^{-1}(H - \mathcal{F}(Y))$. Next we apply Lemma 4.5 to obtain for any component $D \subset \widehat{f}^{-1}(H) = Z$

$$\widehat{f}(D) = \widehat{f}\left(\overline{\bigcup_{C \in \mathcal{C}_D} C}^{Z}\right) \overset{(I)}{=} \overline{f\left(\bigcup_{C \in \mathcal{C}_D} C\right)}^{H} = \overline{H - \mathcal{F}(Y)}^{H} \overset{(II)}{=} H$$

where $\mathcal{C}_D$ is the family of all components $C \subset f^{-1}(H - \mathcal{F}(X))$ contained in D. Here (I) follows from Lemma 4.6 applied to the restriction $\widehat{f} : Z = \widehat{f}^{-1}(H) \to H$ which is a proper map. Moreover, (II) follows from being $H - \mathcal{F}(Y)$ dense in the open set H. □

5. Weakly confluent maps

As a broad generalization of confluent maps, Lelek introduced in [11] the class of weakly confluent maps; see also [12]. Recall a continuous surjection $f : X \to Y$ is called *weakly confluent* if for any subcontinuum $B \subset Y$ there exists a component $C \subset f^{-1}(B)$ such that $f(C) = B$. It is obvious that confluent maps are weakly confluent. More generally, given a family of connected subsets of Y we say that f is *weakly $\mathcal{C}$-confluent* if for any $B \in \mathcal{C}$ there exists a component $C \subset f^{-1}(B)$ with $f(C) = B$.

The following easy example shows that Theorem 4.1 does not hold for the class of weakly confluent maps.

Example 5.1. Consider the generalized continuum $X \subset \mathbb{R}^2$ sketched in Figure 3(a):

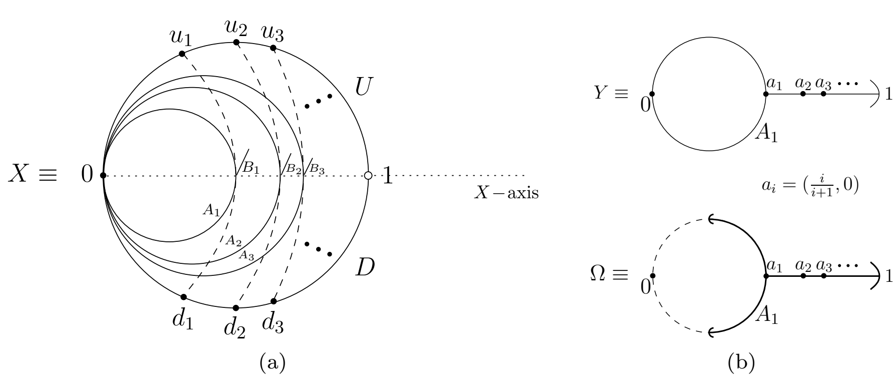


Figure 3

More precisely, we have $X = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} B_i) \cup U \cup D$ where A_i denotes the circle centered at $(\frac{i}{2i+2}, 0)$ and radius $\frac{i}{2i+2}$; moreover the B_i's are segments of positive slope and decreasing length converging to the point $(1, 0)$ such that $A_i \cap B_j = \varnothing$ if $i \neq j$ and $A_i \cap B_i = \{a_i\}$ with $a_i = (\frac{i}{i+1}, 0)$. Finally, U and D are the upper and lower open arcs with $S^1 = U \cup D \cup \{(1, 0)\}$ the unit circle.

On the other hand, we will consider the infinite graph depicted in Figure 3(b) which is the union of A_1 and the interval $[\frac{1}{2}, 1)$.

Next we proceed to define a weakly confluent proper map $f : X \to Y$ which is not weakly $\mathcal{C}_3$-confluent for the class $\mathcal{C}_3$ of connected open sets. For this we denote by Γ_i $(i \geq 1)$ the arcs depicted in dotted lines in Figure 3(a) whose extremes are u_i and d_i, and set Γ_0 to be the origin. Notice that $\Gamma_i \cap A_i = \{a_i\}$.

With the notation above, the map f is the identity on A_1 and fixes each a_i for all $i \geq 1$; moreover f carries the segment B_i $(i \geq 1)$ homeomorphically onto the interval $[a_i, a_{i+1}]$. In addition, for each $i \geq 1$, the arc in A_i running upwards from the origin to Γ_1 is mapped by f homeomorphically onto the upper semicircle $A_{1+} \subset A_1$ in the obvious way. Similarly, the lower arc running downwards is mapped onto the lower semicircle $A_{1-} \subset A_1$. Finally for each $i \geq 2$, f carries homeomorphically the arcs in the intersection of $\bigcup_{j \geq i} A_j$ with the region delimited by S^1 and the arcs Γ_{i-1} and Γ_i onto the interval $[a_{i-1}, a_i]$. If $A \subset f^{-1}(\Omega)$ is a compact component it is obvious that $f(A) \neq \Omega$; otherwise if A is not compact, then necessarily either $A \subset U$ or $A \subset D$. Here we use that f is proper. Thus either $f(A) = \Omega_+ \cup [1, +\infty) \neq \Omega$ or $f(A) = \Omega_- \cup [1, +\infty) \neq \Omega$ where $\Omega_\pm = \Omega \cap A_{1\pm}$.

Nevertheless, we can still prove the following partial result.

Theorem 5.2. *Any weakly $\mathcal{C}_3$-confluent proper surjection $f : X \to Y$ between generalized continua where Y is locally connected is weakly confluent.*

The proof of Theorem 5.2 as well as other proofs in this and subsequent sections use the following fact: Given a sequence of subcontinua $\{C_n\}_{n \geq 1}$ in a compact space X, the convergence of a sequence $x_n \in C_n$ guarantees that the lower limit $\mathrm{Li}\, C_n$ is non-empty, and then ([10]; Thm. 6, p.171) yields

Fact 5.3. *The upper limit $\mathrm{Ls}\, C_n$ is a subcontinuum of X.*

Proof of Theorem 5.2. Let $C \subset Y$ be a subcontinuum. By Lemma 4.2 there exists a decreasing sequence of connected open neighborhoods of C, U_n such that $B_n = \overline{U_n}$ is a Peano subcontinuum for each n and $C = \bigcap_{n \geq 1} B_n$. By assumption, for each n there exists $W_n \subset f^{-1}(U_n)$ with $f(W_n) = U_n$ and so $f\left(\overline{W_n}\right) = \overline{U_n} = B_n$. Here we use Lemma 4.6. Consider $y_0 \in C$ and $x_n \in \overline{W_n}$ with

$f(x_n) = y_0$. Since $f^{-1}(y_0)$ is compact, we can assume that the sequence $\{x_n\}_{n\geq 1}$ converges to some $x_0 \in f^{-1}(y_0)$ (always a subsequence does). Let $L_0 = \operatorname{Ls} \overline{W}_n$ be the continuum in $f^{-1}\left(\overline{U}_1\right)$ given by Fact 5.3. We claim that

$$f(L_0) = C. \tag{5.A}$$

Indeed; consider $z \in L_0$. Then there exists a sequence $z_s \in \overline{W_{n_s}} \subset f^{-1}\left(\overline{U_{n_s}}\right)$ converging to z. Since $\bigcap_{s\geq 1} f^{-1}\left(\overline{U_{n_s}}\right) = f^{-1}(C)$ it follows that $f(z) \in C$. Conversely, for all $y \in C$ we have that $f^{-1}(y) \cap \overline{W_n} \neq \varnothing$ for all n and we can choose a sequence $\{p_n\}_{n\geq 1}$ with $p_n \in f^{-1}(y) \cap \overline{W_n}$. This sequence admits a subsequence converging to some $p_y \in f^{-1}(y) \cap L_0$ by compactness. So, $y \in f(L_0)$ and the equality (5.A) holds. Now, it is readily checked that $f(H_0) = C$ for the component $H_0 \subset f^{-1}(C)$ that contains L_0. □

If the domain (and hence the codomain by ([1]; 1.1)) of a weakly confluent proper surjection $f : X \to Y$ is a generalized Peano continuum we obtain the analogue of Theorem 4.1. Namely, the following theorem holds.

Theorem 5.4. *Let $f : X \to Y$ a proper surjection with X a generalized Peano continuum. Then f is weakly confluent if and only if f is weakly $\mathcal{C}_i$-confluent $(1 \leq i \leq 3)$ for any of the classes $\mathcal{C}_i$ of connected subsets of Y in Theorem 4.1.*

Proof. We will prove

$$\begin{array}{ccc} \text{w. confluent} & \xrightarrow{(1)} & \text{w. } \mathcal{C}_1\text{-confluent} \\ \Uparrow (3) & & \Downarrow \\ \text{w. } \mathcal{C}_3\text{-confluent} & \xleftarrow[(2)]{} & \text{w. } \mathcal{C}_2\text{-confluent} \end{array}$$

Notice that (1) and (2) are obvious. Moreover (3) is Theorem 5.2. Finally, assume that f is weakly $\mathcal{C}_1$-confluent and let $C \subset Y$ be a generalized Peano subcontinuum. By Lemma 2.2 we can find an exhausting sequence of C, $\{C_n\}_{n\geq 1}$, consisting of Peano subcontinua. By assumption there exists a component H_n of $K_n = f^{-1}(C_n)$ with $f(H_n) = C_n$.

Observe that $\{K_n\}_{n\geq 1}$ form an exhausting sequence of $f^{-1}(C)$, and by connectedness of H_n we can find $x_n \in H_n \cap \operatorname{Fr} K_1 \neq \varnothing$ for all $n \geq 2$. Then, by compactness, there exists a sequence $\{x_{n_j}\}_{j\geq 1}$ converging to some $x_0 \in \operatorname{Fr} K_1$. In addition, $x_0 \in \operatorname{int} K_2$ and we can choose a connected open neighborhood of x_0, $\Theta \subset \operatorname{int} K_2$ and j_0 large enough such that $x_{n_j} \in \Theta \cap H_{n_j}$ for all $j \geq j_0$. Here we use that X is locally connected. As $\Theta \subset K_n$ $(n \geq 2)$, we have that $\Theta \subset H_{n_j}$ for $j \geq j_0$ and therefore

$$\Theta \subset H_{n_{j_0}} \subset H_{n_{j_0+1}} \subset H_{n_{j_0+2}} \subset \dots$$

Let $H \subset f^{-1}(C)$ be the component such that $\Theta \subset H$; then $\bigcup_{j=j_0}^{\infty} H_{n_j} \subset H$ and

$$C = \bigcup_{j=j_0}^{\infty} C_{n_j} = \bigcup_{j=j_0}^{\infty} f(H_{n_j}) \subset f(H) \subset C$$

so $f(H) = C$ and hence f is weakly $\mathcal{C}_2$-confluent. □

In spite of the failure of the analogue of Theorem 4.1, Theorem 3.6 extends to weakly confluent proper maps.

Theorem 5.5. *Let X and Y be generalized continua with Y locally connected. Then, any proper surjection $f : X \to Y$ is weakly confluent if and only if its Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is weakly confluent.*

Proof. The "if" part is obvious. Assume that f is weakly confluent and let $C \subset \widehat{Y}$ be a continuum. By Lemma 4.2 there exists a decreasing sequence $\{U_n\}_{n\geq 1} \subset \widehat{Y}$ of connected open neighborhoods of C, U_n such that (i) $P_n = \overline{U_n}^{\widehat{Y}}$ is a Peano subcontinuum; and (ii) $C = \bigcap_{n\geq 1} P_n$. Furthermore, Lemma 2.5 yields that $W_n = U_n - \mathcal{F}(Y)$ is a connected open set in Y for all $n \geq 1$. Therefore W_n is a generalized Peano continuum and then, for each n, there exists an exhausting sequence consisting of Peano subcontinua $K_i^n \subset W_n$ (Lemma 2.2). Then, since f is weakly confluent, for each pair (n, i) there exists a component $H_i^n \subset f^{-1}(K_i^n)$ such that $f(H_i^n) = K_i^n$. Given $p_1 \in K_1^n$, since f is proper, the non-empty intersection $f^{-1}(p_1) \cap H_i^n \neq \varnothing$ is compact for all $i \geq 1$. This way, any sequence $x_i \in f^{-1}(p_1) \cap H_i^n$ contains a subsequence converging to some point $z_0 \in f^{-1}(p_1)$. To ease the writing we can assume that z_0 is the limit of the whole sequence. In that case, the continuum $L_n = \operatorname{Ls} H_i^n \subset \widehat{X}$ given by Fact 5.3 satisfies

$$L_n \subset \overline{\widehat{f}^{-1}(W_n)}^{\widehat{X}} \subset \widehat{f}^{-1}\left(\overline{W_n}^{\widehat{Y}}\right) \widehat{f}^{-1}(P_n).$$

We claim that $W_n \subset \widehat{f}(L_n)$ and so $\widehat{f}(L_n) = P_n$ for all $n \geq 1$. Indeed; consider $x \in W_n$ and let i_0 be the smallest i such that $x \in K_i^n$. Pick $a_i \in f^{-1}(x) \cap H_i^n \neq \varnothing$ for each $i \geq i_0$. The compactness of $f^{-1}(x)$ implies that the sequence $\{a_i\}_{i\geq i_0}$ admits a subsequence converging to some $a \in L_n \cap f^{-1}(x)$. Whence, $x = f(a) \in \widehat{f}(L_n)$. On the other hand, (ii) above yields that given $z \in C$ there exists $q_n \in L_n \subset \widehat{f}^{-1}(P_n) \subset \widehat{f}^{-1}(P_1)$ with $\widehat{f}(q_n) = z$. Thus, the sequence

$\{q_n\}_{n\geq 1} \subset \widehat{f}^{-1}(z)$ contains a subsequence (it is harmless to assume the entire sequence) converging to some $q_0 \in \widehat{f}^{-1}(z)$. Hence can we apply again Fact 5.3 to construct the continuum $L_0 = \operatorname{Ls} L_n \subset \widehat{f}^{-1}(P_1)$. Next we show

$$\widehat{f}(L_0) = C. \tag{5.B}$$

For this, we observe that any $x \in L_0$ is the limit of a sequence $x_s \in L_{n_s}$. Hence, by continuity, $\widehat{f}(x)$ is the limit of the sequence $\widehat{f}(x_s) \in P_{n_s}$ lying in the nested sequence of Peano subcontinua $\{P_n\}_{n\geq 1}$, and so $\widehat{f}(x) \in \bigcap_{n\geq 1} P_n = C$. Conversely, for any $y \in C$, we use (ii) above to choose a sequence $y_n = f(x_n) \in P_n = \widehat{f}(L_n)$ converging to y with $x_n \in L_n$. Since f is proper, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ converging to some $x_0 \in L_0$. From this it readily follows $y = \widehat{f}(x_0) \in \widehat{f}(L_0)$.

From (5.B), $\widehat{f}(H) = C$ for the component $H \subset \widehat{f}^{-1}(C)$ containing L_0 and we are done. □

Remark. In a similar way as done for confluent maps, a more general definition of a weakly confluent map is given in [13]. Explicitly, a continuous surjection $f : X \to Y$, with $f^{-1}(y)$ compact for all $y \in Y$, is called weakly confluent if for any connected closed subset $C \subset Y$ there exists a quasicomponent $Q \subset f^{-1}(C)$ such that $f(Q) = C$ ([13]; 1.4.(w')). In case the codomain of the proper surjection $f : X \to Y$ is locally connected then the following statements are equivalent: (a) f is Lelek-Tymchatyn weakly confluent; (b) f is weakly confluent; and (c) for any connected subset $\Gamma \subset Y$ there exists a quasicomponent $Q \subset f^{-1}(\Gamma)$ such that $f(Q) = \Gamma$. The proof is similar to the one give in Corollary 3.7 with the obvious changes.

6. Semi-confluent maps

Semi-confluent maps, introduced by T. Maćkowiak in [14], form an intermediate class between confluent and weakly confluent maps. Recall that a surjection $f : X \to Y$ is called *semi-confluent* if for any subcontinuum $B \subset Y$ and any pair of components $C, D \subset f^{-1}(B)$ we have either $f(C) \subset f(D)$ or $f(D) \subset f(C)$. It readily follows from the definitions that any confluent proper map is semi-confluent. The relation between semi-confluent and weakly confluent maps is not obvious. We need the following lemma, analogous to a result of T. Maćkowiak ([14]; 3.1) for continua, to show that semi-confluent maps are weakly confluent. The proof is the same as in the compact case since the compactness of preimages of continua is guaranteed for proper maps.

Lemma 6.1 (c.f. ([14]; 3.1))**.** *Let $f : X \to Y$ a semi-confluent proper surjection where X and Y are admissible spaces. For any subcontinuum $B \subset Y$ and each family $\mathscr{C}$ of components of $f^{-1}(B)$ such that the union $\bigcup\{C : C \in \mathscr{C}\}$ is closed in X, there exists a component $C' \in \mathscr{C}$ whose image under f is maximal in the sense that $f(C') = f\left(\bigcup\{C; C \in \mathscr{C}\}\right)$.*

From this lemmas it readily follows that semi-confluent proper maps are weakly confluent.

In this section we deal with the Freudenthal extensions of semi-confluent proper maps. Notice that the Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ of the confluent map in Example 3.4 is not semi-confluent. In contrast with weakly confluent maps, the results on confluent maps in Section 4 extend to semi-confluent maps. In particular, the analogue of Theorem 4.1 still holds for semi-confluent maps. Namely, if we define for any class $\mathcal{C}$ of connected subspaces the notion of $\mathcal{C}$-semi-confluent map as done in Section 5 for weakly confluent maps, we can state and prove the following theorem.

Theorem 6.2. *Let $f : X \to Y$ be a proper surjection and Y be a generalized Peano continuum. Then $f : X \to Y$ is semi-confluent if and only if is $\mathcal{C}_i$-semi-confluent ($1 \leq i \leq 3$) for any of the classes $\mathcal{C}_i$ in Theorem 4.1.*

In the proof of Theorem 6.2 we will use the following easy lemma.

Lemma 6.3. *Let $f : X \to Y$ be a continuous map and consider the intersection $Z = \bigcap_{n\geq 1} Z_n$ of a decreasing sequence of compact subsets $Z_n \subset X$. Then $f(Z) = \bigcap_{n\geq 1} f(Z_n)$.*

Proof. It is clear that $f(Z) \subset \bigcap_{n\geq 1} f(Z_n)$. Conversely, consider $x \in \bigcap_{n\geq 1} f(Z_n)$, then $x = f(z_n)$ where $z_n \in Z_n \subset Z_1$. Since Z_1 is compact, there exists a subsequence $\{z_{n_j}\}_{j\geq 1}$ converging to some $w \in Z_1$. As the intersection of the Z_n's is Z, it is easily checked that $w \in Z$, and so $x = f(w) \in f(Z)$. □

Proof of Theorem 6.2. Recall that $\mathcal{C}_1 \subset \mathcal{C}_2 \supset \mathcal{C}_3$ are the classes of Peano subcontinua, generalized Peano subcontinua, and connected open subsets of Y, respectively. Thus, it is clear that semi-confluent maps are $\mathcal{C}_1$-semi-confluent and that $\mathcal{C}_2$-semi-confluent maps are $\mathcal{C}_3$-semi-confluent.

Assume now that f is $\mathcal{C}_1$-semi-confluent and take any $C \in \mathcal{C}_2$. If for two components $A, B \subset f^{-1}(C)$ one has $f(A) - f(B) \neq \varnothing \neq f(B) - f(A)$, there exist $y = f(a) \in f(A) - f(B)$ and $y' = f(b) \in f(B) - f(A)$ with $a \in A$ and $b \in B$. Let $\Gamma \subset C$ be any arc running from y to y'. Here we use that C is arc-connected. If D and D' are the components of a and b in $f^{-1}(\Gamma)$, respectively, then either

$f(D) \subset f(D') \subset f(B)$ and hence $y = f(a) \in f(B)$ or $f(D') \subset f(D) \subset f(A)$ and hence $y' = f(b) \in f(A)$. This leads to a contradiction and so f is $\mathcal{C}_2$-semi-confluent.

Finally assume that f is $\mathcal{C}_3$-semi-confluent and let $P, Q \subset f^{-1}(C)$ be any two components where $C \subset Y$ is an arbitrary subcontinuum. By applying Lemma 4.2 we can write $C = \bigcap_{n \geq 1} C_n$ as an intersection of Peano subcontinua $C_n = \overline{U_n}$ which are the closures of a nested sequence of connected open neighborhoods $U_n \supset C$. Therefore, for each $n \geq 1$ we find components $P_n, Q_n \subset f^{-1}(U_n)$ with $P \subset P_n$ and $Q \subset Q_n$. Moreover, as

$$P \subset \bigcap_{n=1}^{\infty} \overline{P_n} \subset \bigcap_{n=1}^{\infty} f^{-1}(C_n) = f(C),$$

it follows that the connected intersection $\bigcap_{n=1}^{\infty} \overline{P_n}$ coincides with the component P. Similarly $Q = \bigcap_{n=1}^{\infty} \overline{Q_n}$.

Furthermore, by assumption, for each $n \geq 1$, either $f(P_n) \subset f(Q_n)$ or $f(Q_n) \subset f(P_n)$ and so, $f\left(\overline{P_n}\right) \subset f\left(\overline{Q_n}\right)$ or $f\left(\overline{Q_n}\right) \subset f\left(\overline{P_n}\right)$ by Lemma 4.6.

Assume that there exists a subsequence $\{n_j\}_{j \geq 1}$ with $f\left(\overline{P_{n_j}}\right) \subset f\left(\overline{Q_{n_j}}\right)$ for each $j \geq 1$. Then Lemma 6.3 yields

$$f(P) \subset \bigcap_{j=1}^{\infty} f\left(\overline{P_{n_j}}\right) \subset \bigcap_{j=1}^{\infty} f\left(\overline{Q_{n_j}}\right) = f(Q).$$

Otherwise, there is n_0 that $f\left(\overline{Q_n}\right) \subset f\left(\overline{P_n}\right)$ for $n \geq n_0$ and, similarly, Lemma 6.3 yields $f(Q) \subset f(P)$. This shows that f is semi-confluent and the proof is finished. □

As done for confluent maps we use Theorem 6.2 to show the following theorem.

Theorem 6.4. *Let X be a generalized continuum and Y be a generalized Peano continuum. Any proper surjection $f : X \to Y$ is semi-confluent if and only if its Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is semi-confluent.*

Proof. The sufficiency is obvious. Assume that f is semi-confluent. By Theorem 6.2 it will be enough to check that $\widehat{f}$ is $\mathcal{C}_3$-semi-confluent. For this, let $H \subset \widehat{Y}$ be any connected open set. Then $H - \mathcal{F}(X)$ is connected by Lemma 2.5 and as f is $\mathcal{C}_3$-semi-confluent (Theorem 6.2) we have for any two components $C, C' \subset f^{-1}(H - \mathcal{F}(Y))$

$$f(C) \subset f(C') \text{ or } f(C') \subset f(C). \tag{6.A}$$

On the other hand, given two components $D, D' \subset Z = \widehat{f}^{-1}(H)$ we apply Lemma 4.5 to obtain

$$D = \overline{\bigcup_{C \in \mathcal{C}_D} C}^{Z} \text{ and } D' = \overline{\bigcup_{C' \in \mathcal{C}_{D'}} C'}^{Z} \tag{6.B}$$

where $\mathcal{C}_D$ is the family of components $C \subset f^{-1}(H - \mathcal{F}(X))$ with $C \subset D$. Similarly $\mathcal{C}_{D'}$. Assume that

$$\widehat{f}(D) - \widehat{f}(D') \neq \varnothing \text{ and } \widehat{f}(D') - \widehat{f}(D) \neq \varnothing$$

and let $d \in D, d \in D'$ with $\widehat{f}(d) \notin \widehat{f}(D')$ and $\widehat{f}(d') \notin \widehat{f}(D)$. By (6.B) we find sequences $x_n \in C_n \in \mathcal{C}_D$ and $x'_n \in C'_n \in \mathcal{C}_{D'}$ converging to d and d', respectively. Then (6.A) yields a subsequence $\{n_j\}_{j \geq 1}$ such that for all $j \geq 1$

$$\text{either } f(C_{n_j}) \subset f(C'_{n_j}) \subset H \text{ or } f(C'_{n_j}) \subset f(C_{n_j}) \subset H. \tag{6.C}$$

Assume the first case holds. Then

$$\widehat{f}(d) \in \overline{f\left(\bigcup_{j=1}^{\infty} C_{n_j}\right)}^{H} \subset \overline{f\left(\bigcup_{j=1}^{\infty} C'_{n_j}\right)}^{H} \overset{(I)}{=} \widehat{f}\left(\overline{\bigcup_{j=1}^{\infty} C'_{n_j}}^{Z}\right) \overset{(II)}{\subset} \widehat{f}(D') \tag{6.D}$$

which is a contradiction. Here we apply Lemma 4.6 to the restriction $\widehat{f} : Z = \widehat{f}^{-1}(H) \to H$ to get (I), while (II) follows from (6.B) above. Similarly for the second case in (6.C) and we are done. □

7. The special case of end-faithful maps

As noted in the first paragraph of Section 3, monotone proper maps are end-faithful and hence, the monoticity of Freudenthal extensions of monotone proper maps is immediate. In this section we will prove that Freudenthal extensions preserve the confluency and the weak confluency of end-faithful maps between generalized continua. In contrast, this is not true for semi-confluent maps.

The crucial property of Freudenthal extensions of end-faithful maps is established in the following proposition.

Proposition 7.1. *Let $f : X \to Y$ be any weakly confluent end-faithful surjection between generalized continua. Then, for each subcontinuum $C \subset \widehat{Y}$ with $C \cap \mathcal{F}(Y) \neq \varnothing$ the family $\mathcal{J} = \mathcal{J}_C$ of components of $\widehat{f}^{-1}(C)$ which meet $\mathcal{F}(X)$ reduces to a unique element D for which $\widehat{f}(D) = C$.*

In the proof of Proposition 7.1 we will use the following lemmas.

Lemma 7.2. *Any two distinct components $D_j \in \mathcal{J}$ $(j = 1, 2)$ have disjoint images $\widehat{f}(D_1) \cap \widehat{f}(D_2) = \varnothing$.*

Proof. Assume on the contrary that $E = \widehat{f}(D_1) \cap \widehat{f}(D_2)$ contains at least one point $p \in E$. Notice that, necessarily, $p \in Y$ since otherwise $D_1 = D_2$ by the end-faithfulness of f. Let $\{K_i\}_{i\geq 1}$ be an exhausting sequence of Y for which $p \in \operatorname{int} K_1$. Then the bumping boundary theorem ([10]; Thm. 1, p. 172) applied to each continuum $M_j = \widehat{f}(D_j)$ $(j = 1, 2)$ shows that the component of p in $M_i^j = M_j \cap K_i$, say Γ_i^j, necessarily meets the frontier

$$\operatorname{Fr}_{M_j} M_i^j \subset F_i^j = M_i^j \cap \operatorname{Fr} K_i.$$

For each $i \geq 1$, the union $\Gamma_i = \Gamma_i^1 \cup \Gamma_i^2$ is a continuum such that

$$\Gamma_i \cap F_n^j \neq \varnothing \text{ for } i \geq n \text{ and } j = 1, 2. \tag{7.A}$$

Furthermore, as f is weakly confluent, for every $i \geq 1$ there is a component $A_i \subset f^{-1}(\Gamma_i)$ with $f(A_i) = \Gamma_i$. In particular, $f^{-1}(p) \cap A_i \neq \varnothing$ for all $i \geq 1$ and there is a sequence $\{a_i\}_{i\geq 1}$ with $a_i \in A_i$ and $f(a_i) = p$. As f is proper, we can assume without loss of generality that this sequence converges to some $a_0 \in f^{-1}(p)$. Let $L = \operatorname{Ls} A_i \subset \widehat{f}^{-1}(C)$ denote the subcontinuum of $\widehat{X}$ provided by Fact 5.3, and let $D_0 \subset \widehat{f}^{-1}(C)$ be the component containing L. If $M_0 = \widehat{f}(D_0)$ we claim that each intersection

$$M_0 \cap M_j \neq \emptyset \quad (j = 1, 2) \tag{7.B}$$

contains at least one end, and so the assumption that f is end-faithful will yield $D_1 = D_0 = D_2$ and the intersection E must be empty whenever $D_1 \neq D_2$.

Let n be any fixed $n \geq 1$. By (7.A) above we can find a sequence $\{y_i^n\}_{i\geq n} \subset \Gamma_i \cap F_n^1$, and hence a sequence $\{x_i^n\}_{i\geq n}$ with $x_i^n \in A_i$ and $f(x_i^n) = y_i^n$.

Since f is proper and $\operatorname{Fr} K_n$ compact, the sequence $\{x_i^n\}_{i\geq n} \subset f^{-1}(F_n^1)$ contains a subsequence converging to some $x_0^n \in L \cap f^{-1}(F_n^1) \subset D_0 \cap f^{-1}(F_n^1)$. Here we use that $L = \operatorname{Ls} A_i$ is an upper limit. Notice that

$$y_0^n = f(x_0^n) \in M_0 \cap F_n^1 \subset M_0 \cap M_1 \cap \operatorname{Fr} K_n \quad (n \geq 1)$$

form an unbounded sequence, and so it contains a subsequence converging to some end in $M_0 \cap M_2$. Similarly, $M_0 \cap M_2$ contains at least one end. This shows the claim in (7.B) and the proof is finished. □

Lemma 7.3. *The disjoint union*

$$Z = \bigsqcup \{D;\, D \in \mathcal{J}\} \tag{7.C}$$

is closed and hence compact in $\widehat{f}^{-1}(C)$. In particular, the disjoint union $\widehat{f}(Z) = \bigsqcup_{D\in\mathcal{J}} \widehat{f}(D)$ given by Lema 7.2 is a compact set in C and so the restriction $\widehat{f_Z} : Z \to \widehat{f}(Z)$ is a closed map.

Proof. For this we consider any sequence $x_n \in D_n \in \mathcal{J}$ converging to some $x_0 \in \widehat{f}^{-1}(C)$ and choose an end $\varepsilon_n \in \mathcal{F}(X) \cap D_n$ for each n. The upper limit $L = \operatorname{Ls} D_n$ is a subcontinuum of $\widehat{f}^{-1}(C)$ containing x_0 (5.3) and by compactness of $\mathcal{F}(X)$, the sequence $\{\varepsilon_n\}_{n\geq 1}$ contains a subsequence converging to some end ε_0. Then ε_0 lies in the upper limit L and if D_0 is the component of x_0 in $\widehat{f}^{-1}(C)$ we get $\varepsilon_0 \in L \subset D_0$ and hence $x_0 \in Z$. This shows that Z is a closed set. □

Lemma 7.4. *With the notation of Lemma 7.3 the equality* $\widehat{f}(Z) = C$ *holds.*

Proof. Indeed, any end $\varepsilon \in C \cap \mathcal{F}(Y)$ lies in the image of Z by definition of the family $\mathcal{J}$. Otherwise, given $y \in C - \mathcal{F}(Y)$ we choose an exhausting sequence $\{Y_n\}_{n\geq 1}$ of compact subsets in Y with $y \in Y_1$. Applying the bumping boundary theorem ([10]; Thm. 1, p. 172), for each n, the component of y, say $C_n \subset Y_n$, meets the frontier $\operatorname{Fr} Y_n$. As f is weakly confluent, there exists a component $B_n \subset f^{-1}(C_n)$ such that $f(B_n) = C_n$. Notice that $B_n \cap f^{-1}(y) \neq \varnothing$ for all $n \geq 1$ and hence there exists a sequence $b_n \in B_n$ with $f(b_n) = y$. As $f^{-1}(y)$ is compact, we can assume without loss of generality that $\{b_n\}_{n\geq 1}$ converges to some $b_0 \in f^{-1}(y)$. Let $B = \operatorname{Ls} B_n$ be the continuum in $\widehat{f}^{-1}(C)$ obtained by using Fact 5.3. Obviously, $b_0 \in B$ and $y = f(b_0) \in f(B)$. Furthermore, as $C_n \cap \operatorname{Fr} Y_n \neq \varnothing$ for all $n \geq 1$ there exists a subsequence $\{n_s\}_{s\geq 1}$ and points $q_s \in C_{n_s} \cap \operatorname{Fr} Y_{n_s}$ converging to some end $\eta \in \mathcal{F}(Y)$. Since $f(B_n) = C_n$ $(n \geq 1)$ and f is proper one can find a sequence

$$\{x_t\}_{t\geq 1} \subset \widehat{f}^{-1}(\{q_s\}_{s\geq 1} \cup \{\eta\}) \subset \widehat{f}^{-1}(C)$$

with $f(x_t) = q_{s_t}$ converging to some end $\mu \in \mathcal{F}(X) \cap B$. Therefore, $\widehat{f}(\mu) = \eta$ and so $B \cap \mathcal{F}(X) \neq \varnothing$. Thus the component of $f^{-1}(C)$ that contains B belongs to the family $\mathcal{J}$, and $y \in f(B) \subset f(Z)$. □

Proof of Proposition 7.1. We first observe that since each $D \in \mathcal{J}$ is a component of $\widehat{f}^{-1}(C)$, $\mathcal{J}$ is the family of all components of the compact set Z in Lemma 7.3. Hence ([10]; Thm. 3, p.148) yields a continuous map $\varphi : Z \to \mathfrak{C}$ into the Cantor set $\mathfrak{C}$, such that the fibers of φ are the quasicomponent (= components by compactness, see ([10]; Thm. 3, p.169)) of Z; that is, the sets $D \in \mathcal{J}$.

Let $C/\mathcal{R}$ denote the quotient space defined by the relation $y\mathcal{R}y'$ if $y, y' \in \widehat{f}(D)$ for some $D \in \mathcal{J}$. Here we use Lemma 7.2. As the restriction $\widehat{f_Z} : Z \to C$ is a closed surjection by Lemmas 7.3 and 7.4, it is a quotient map and so is the composite $\pi \circ \widehat{f_Z} : Z \to C/\mathcal{R}$. Moreover, the map $\psi : C/\mathcal{R} \to \mathfrak{C}$ given by $\psi([x]) = \varphi(z)$ whenever $\pi\widehat{f}(z) = [x]$ is well-defined since $\widehat{f}(z), \widehat{f}(z') \in D$ if and only if $z, z' \in D$ by Lemma 7.2. This way the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathfrak{C} \\ {\scriptstyle \widehat{f_Z}}\downarrow & & \uparrow{\scriptstyle \psi} \\ C & \xrightarrow{\pi} & C/\mathcal{R} \end{array}$$

yields that ψ is continuous ([4]; 2.4.2) and then, the connectedness of C shows that $\varphi(Z) = \psi\pi(C)$ is constant in the totally disconnected space $\mathfrak{C}$. Thus, $\mathcal{J} = \{D\}$ reduces to a single component D and $\widehat{f}(D) = C$. $\square$

Next we use Proposition 7.1 to prove our next theorem.

Theorem 7.5. *Let $f : X \to Y$ be an end-faithful proper surjection between generalized continua. Then f is (weakly) confluent if and only if so is its Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$.*

Proof. Clearly if $\widehat{f}$ is (weakly) confluent so is f. Moreover, the converse in case of weak confluency is an immediate consequence of Proposition 7.1.

Now, let us assume that f is confluent and let $C \subset \widehat{Y}$ be any subcontinuum containing at least one end. Then, $\widehat{f}^{-1}(C)$ reduces to the unique component given by Proposition 7.1. Indeed, if $D \subset \widehat{f}^{-1}(C)$ is a component missing $\mathcal{F}(X)$, the image $\widehat{f}(D) = f(D) \subset C$ is a continuum in $\widehat{Y}$ with $f(D) \cap \mathcal{F}(Y) = \varnothing$. By applying Lemma 2.3 we find a continuum $A \subsetneq C$ such that $f(D) \subsetneq A$ and $A \cap \mathcal{F}(Y) = \varnothing$. Moreover, as D is a component of $\widehat{f}^{-1}(C)$, it is also a component of $f^{-1}(A)$ and from the confluency of f, necessarily $f(D) = A$ which is a contradiction. $\square$

The following proposition is the counterpart of Proposition 7.1 for the one-point compactification $f^+ : X^+ = X \cup \{\infty\} \to Y^+ = Y \cup \{\infty\}$ of any proper map $f : X \to Y$ regarded as an ∞-faithful map.

Proposition 7.6. *Let $f : X \to Y$ be a weakly confluent map between admissible spaces. Given any subcontinuum $C \subset Y^+$ with $\infty \in C$, we have $f^+(D_\infty) = C$ for the component $\infty \in D_\infty \subset (f^+)^{-1}(C)$.*

Proof. Let $\{K_n\}_{n\geq 1}$ be an exhausting sequence of Y. Given $p \in C - \{\infty\}$, we apply Lemma 2.4 to the component of p, $C_p \subset C - \{\infty\}$, to get an increasing sequence of continua $\{C^p_n\}_{n\geq 1}$ such that $p \in C^p_1$, $C^p_{n+1} - K_n \neq \varnothing$ and $C^p_n \subset C^p_{n+1} \subset C_p$.

By assumption, for each continuum C^p_n there exists a component $D^p_n \subset f^{-1}(C^p_n)$ such that $f(D^p_n) = C^p_n$. Notice each D^p_n is a continuum since f is proper. Hence, for each n there exists $d^p_n \in D^p_n$ such that $f(d^p_n) = p$. Since

$\{d_n^p\}_{n\geq 1} \subset f^{-1}(p)$ is a sequence in the compact set $f^{-1}(p)$, there is a subsequence, that we assume to be the whole sequence, converging to some $d_0^p \in f^{-1}(p)$. Let $D_0^p = \operatorname{Ls} D_n^p$ be the continuum in ${f^+}^{-1}(C)$ provided by Fact 5.3. Observe that $\infty \in D_0^p$ for all p since $f(D_{n+1}^p) - K_n = C_{n+1}^p - K_n \neq \emptyset$ for all $n \geq 1$ and D_0^p is closed in X^+. In particular, the set

$$D_0 = \bigcup_{p\in C-\{\infty\}} D_0^p$$

is connected an hence $D_0 \subset D_\infty$. Moreover, for each $p \in C - \{\infty\}$, $f(d_0^p) = p$, and then $C \subset f(D_0) \subset f(D_\infty) \subset C$, whence $f(D_\infty) = C$. □

As an immediate consequence of Proposition 7.6 one gets the following theorem.

Theorem 7.7. *Let $f : X \to Y$ be a proper surjection between admissible spaces. Then f is (weakly) confluent if and only if so is $f^+ : X^+ \to Y^+$.*

Proof. The "if" part is obvious. Moreover, the converse for weakly confluent maps is an immediate consequence of Proposition 7.6. For confluency, the same arguments as in the proof of Theorem 7.5 show that necessarily $(f^+)^{-1}(C)$ is connected. □

The following example shows that both Theorems 7.5 and 7.7 fail for semi-confluent maps.

Example 7.8. Consider the generalized continuum $Y \subset \mathbb{R}^2$ consisting of the spiral S_Y and the open arc E_Y and its homeomorphic copy $X \subset \mathbb{R}^2$ depicted in Figure 4(b) and (a), respectively.

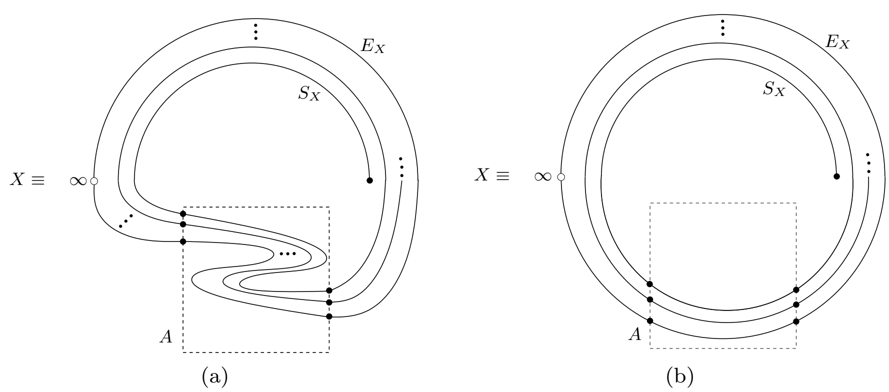


Figure 4

Let $h : X \to Y$ be a homeomorphism such that its restriction to the intersections with the boundary of the square A is order-preserving. Then we define the proper map $f : X \to Y$ as h outside the square A and the map $A \cap X \to A \cap Y$ defined componentwise by the projection sketched in Figure 5(a).

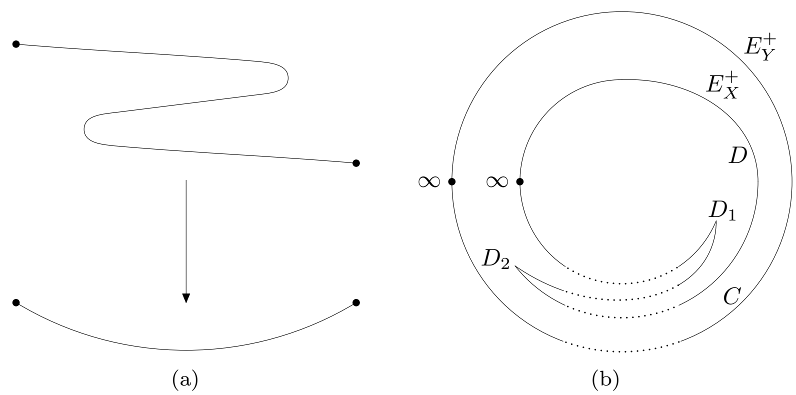


Figure 5

It is readily checked that f is a semi-confluent map but $f^+ : X^+ \to Y^+$ is not. For this we observe that the counterimage of the continuum $C \subset E_Y^+ \cong S^1$ in Figure 5(b) consists of the component D given by Proposition 7.6 and two further components $D_1, D_2 \subset (f^+)^{-1}(C)$ whose images are not related by inclusion.

We finish the paper by presenting two simple consequences of Theorem 7.7.

Corollary 7.9 (c.f. ([19]; Thm. 4)). *Let $f : X \to Y$ be a proper surjection between generalized continua where Y is ray-type or line-type (in particular $Y = \mathbb{R}_{\geq 0}$ or $Y = \mathbb{R}$), then f is weakly confluent.*

Recall that Y is said to be *ray-type* or *line-type*, respectively, if $Y = \varprojlim\{Y_n, f_n\}$ is an inverse limit where all bonding maps f_n are proper and $Y_n = \mathbb{R}_{\geq 0}$ ($Y = \mathbb{R}$; respectively) is the euclidean half-line for all $n \geq 1$.

Proof of Corollary 7.9. Assume that Y is ray-type. Then Y^+ is arc-like ([6]; 3.3) and by ([19]; Thm. 4) the extension $f^+ : X^+ \to Y^+$ is weakly confluent and from Theorem 7.7 it follows that f is weakly confluent.

If $Y = \varprojlim_p\{\mathbb{R}, g_n\}$ is line-type, then Y is a generalized continuum with two ends ([6]; 6.1) and it readily follows that the surjection $\widehat{Y} \to Z = \varprojlim\{\widehat{\mathbb{R}}, \widehat{g_n}\}$

in ([6]; 4.1) is an homeomorphism and hence $\widehat{Y}$ is arc-like. Now, applying ([19]; Thm. 4) again we have that the Freudenthal extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ is weakly confluent and hence f is weakly confluent ([5]; 2). □

In ([17]; 3.1) and ([17]; 3.2) S. B. Nadler characterizes the locally connected metric spaces which are the continuous image of the euclidean (half-)line by a confluent map. The following corollary of Theorem 7.7 gives an alternative proof for the case of proper images.

Corollary 7.10. *The confluent proper image of a half-line is a half-line. Moreover, the confluent proper image of an euclidean line is either a half-line or a line.*

Proof. Let $f : \mathbb{R} \to Y$ be a confluent proper surjection. By Theorem 7.7 $f^+ : S^1 \cong \mathbb{R}^+ \to Y^+$ is confluent, and then Y^+ is either a circle or an arc by ([16]; 13.31). In the first case, $Y = Y^+ - \{\infty\} \cong \mathbb{R}$. Otherwise, as $Y = Y^+ - \{\infty\}$ is connected, the point ∞ is one of the extremes of $Y^+ \cong [0,1]$, and so $Y \cong \mathbb{R}_{\geq 0}$.

Similarly, if we replace $\mathbb{R}$ by the half-line $\mathbb{R}_{\geq 0}$, then $Y^+ \cong [0,1]$ is now homeomorphic to an arc ([16]; 13.31), and the connectedness of Y yields $Y \cong \mathbb{R}_{\geq 0}$. □

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References

[1] R. Ayala, M. J. Chávez, A. Quintero, *A Hahn-Mazurkiewicz Theorem for generalized Peano continua*, Arch. Math. **71** (1998), 325–330.

[2] B. J. Ball, *Quasicompactifications and shape theory*, Pacific J. Math. **2** (1979), 251–259.

[3] J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fund. Math. **56** (1964), 213–220.

[4] R. Engelking, *General Topology*, Sigma Ser. Pure Math., Vol. VI Heldermann Verlag, 1989.

[5] T. Fernández, A. Quintero, *Homotopic properties of confluent maps in the proper category*, Topology Appl. **156** (2009), 2960–2970.

[6] ———, *Inverse sequences with proper bonding maps*, to appear in Colloq. Math.

[7] H. Freudenthal, *Neuaufbau der Endentheorie. Annals of Math*, **43** (1942), 261–279.

[8] ———, *Über die topologischen Raüme un Gruppen*, Math. Z. **33** (1931), 692–713.

[9] J. Isbell, Uniform Spaces, Amer. Math. Soc., Providence, 1964.

[10] K. Kuratowski, *Topology*, Volume II. Academic Press, 1968.

[11] A. Lelek, *A classification of mappings pertinent to curve theory*, Proceedings, Univ. of Oklahoma Topology Conference, Norman, (1972), 97–103.

[12] ———, D. R. Read, *Compositions of confluent mappings and some other classes of functions*, Colloq. Math. **29** (1974), 101–112.

[13] ———, E. D. Tymchatyn, *Pseudo-confluent mappings and classification of continua*, Can. J. Math. **6** (1975), 1336–1348.

[14] T. Maćkowiak, *Semi-confluent mappings and their invariants*, Fund. Math. **79** (1973), 251–264.
[15] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J., **39** (1972), 465–473.
[16] S. B. Nadler, *Continuum Theory*, Marcel Dekker, Inc., New York 1992.
[17] ———, *The metric confluent images of half-lines and lines*, Fund. Math. **102** (1979), 183–194.
[18] M. C. Rayburn, *On the Stoilow-Kerékjártó compactification*, J. London. Math. Soc. **6** (1973), 193–196.
[19] D. R. Read, *Confluent and related mappings*, Colloq. Math. **29** (1974), 233–239.
[20] A. W. Schurle, *Topics in Topology*, Noth-Hollan, 1979.
[21] R. C. Walker, The Stone-Čech Compactification, Springer, New York, 1974.
[22] G. T. Whyburn, *Open mappings on locally compact spaces*, Memoirs Amer. Math. Soc., 1, 19 (1950).

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