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DIRECTED PSEUDO-GRAPHS AND LIE ALGEBRAS OVER FINITE FIELDS

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Abstract. The main goal of this paper is to show an application of Graph Theory to classifying Lie algebras over finite fields. It is rooted in the representation of each Lie algebra by a certain pseudo-graph. As partial results, it is deduced that there exist, up to isomorphism, four, six, fourteen and thirty-four 2-, 3-, 4-, and 5-dimensional algebras of the studied family, respectively, over the field $\mathbb{Z}/2\mathbb{Z}$. Over $\mathbb{Z}/3\mathbb{Z}$, eight and twenty-two 2- and 3-dimensional Lie algebras, respectively, are also found. Finally, some ideas for future research are presented.

Keywords: directed pseudo-graph; adjacency matrix; Lie algebra

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1. Introduction

Nowadays, a novel research subject is being developed. It links Discrete Mathematics with diverse topics of Pure Mathematics and Physics. Its core idea lies in the representation of some algebraic objects by using certain graphs or pseudo-graphs. So Graph Theory provides new tools to obtain properties of the initial objects, through the analysis of some characteristics present in the graphs.

In the particular case of Lie Theory, this method was first introduced in 1968 by Hamelink (see [9]). This author analyzed two cases where Graph Theory was useful in classifying simple Lie algebras. Very few authors linked both theories for more than 40 years; however, in the last years several attempts on this topic have emerged. Indeed, [1], [2] show two different ways of using graphs as a tool to study some types

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of Lie algebras. More recent studies following these techniques are, for instance, [3], [4], [5], [7], [10], [11].

This paper has two main goals. The first is to show an application of simple directed pseudo-graphs to classifying a particular type of Lie algebras over $\mathbb{Z}/2\mathbb{Z}$. In this way, we perform this task up to dimension 5. The second goal is to extend the previous idea to classify Lie algebras over $\mathbb{Z}/3\mathbb{Z}$. In that case we show an application of directed pseudo-graphs to classifying Lie algebras up to dimension 4. In both cases, the main interest is not the computations or the results but the strategies provided to find out properties of Lie algebras.

In the classifications given, that can also be obtained without using graphs or computers, we are generalizing our own procedure presented in [1]. The family of Lie algebras dealt with in this paper is not common in the literature, but it is relevant by itself, since these algebras, in some sense, are precursors of the filiform Lie algebras. Specifically, these algebras belong to the n-dimensional family $\mathscr{F}_p$ defined over the field $\mathbb{Z}/p\mathbb{Z}$, with p prime, and having a basis $\{u_1, \ldots, u_n\}$ such that if $r, s < n$, then $[u_r, u_s] = 0$; and $[u_r, u_n]$ is a linear combination of basis elements with coefficients over the field $\mathbb{Z}/p\mathbb{Z}$ (note that u_n does not appear in this last linear combination). We must add that the theoretical interest in dealing with these algebras comes from the fact that they are those having a codimension-1 abelian ideal.

It is also convenient to note that although the classification of these algebras in low dimension, over $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, could be easily computed in a direct way, our purpose is not that one, but to show a suitable technique to be used for larger dimensions, which is relevant due to the interest in these classifications themselves. Remember that de Graff, for instance, classified solvable Lie algebras over all fields, only in dimensions $\leqslant 4$ (see [6]), and Patera and Zassenhaus [12] classified solvable Lie algebras of dimensions $\leqslant 4$ over perfect fields.

Finally, we propose our method to ease the study of other characteristics of Lie algebras that may be difficult to set and/or prove without the use of graphs.

2. Basic concepts

In this section we recall some concepts from both the Lie and Graph Theories. We believe they are useful for a total understanding of the paper. In the first place, we specifically consider the family of Lie algebras described before.

Definition 2.1. Let $\mathscr{F}_p$ be the n-dimensional family of Lie algebras defined over the field $\mathbb{Z}/p\mathbb{Z}$, with p prime, which has a basis $\{u_1, \ldots, u_n\}$, such that

▷ if $r, s < n$, then $[u_r, u_s] = 0$;

▷ $[u_r, u_n]$ is a linear combination of some basis elements, $u_1, \dots, u_{n-1}$, with coefficients in the field $\mathbb{Z}/p\mathbb{Z}$.

Note that a Lie algebra of $\mathscr{F}_2$ can be represented by a binary $(n-1) \times (n-1)$ square matrix, where the element in the i-th row and j-th column is the coefficient of u_j in the bracket $[u_i, u_n]$. Analogously, a Lie algebra of $\mathscr{F}_3$ can be represented by an $(n-1) \times (n-1)$ square matrix, where the element in the i-th row and j-th column is the coefficient of u_j in the bracket $[u_i, u_n]$, which can only be 0, 1, or 2.

For a more in-depth review on Lie algebras, the reader can consult [13], for instance. Let us now recall some concepts from Graph Theory (check [8] for more information).

Definition 2.2. A *directed pseudo-graph* is a pair $G = (V, D)$, where V is a finite nonempty set of elements called *vertices* and D is a set of ordered pairs of elements of V, called *directed edges*, admitting *loops* which are edges with the two vertices equal. When multiple edges are not allowed, we refer to these graphs as *simple directed pseudo-graphs*.

Definition 2.3. The *adjacency matrix of a directed pseudo-graph* with n vertices is an $n \times n$ matrix, where the element in the i-th row and j-th column is α when there exist α edges from vertex i to vertex j (for $1 \leqslant i, j \leqslant n$). The *adjacency matrix of a simple directed pseudo-graph* corresponds to the particular case $\alpha = 1$.

Definition 2.4. A *morphism* between two directed pseudo-graphs, $G_1 = (V_1, D_1)$ and $G_2 = (V_2, D_2)$, is a map $\varphi\colon V_1 \to V_2$ such that if $\langle u, v\rangle \in D_1$, then $\langle \varphi(u), \varphi(v)\rangle \in D_2$. An *isomorphism* between directed pseudo-graphs is a bijective morphism whose converse is a morphism.

3. Linking the two theories

Next we present some results obtained when pseudo-graphs are used as a tool to classify specific families of Lie algebras.

The classification over $\mathbb{Z}/2\mathbb{Z}$ is explained in Subsection 3.1, where an equivalence relation is introduced in the set of simple directed pseudo-graphs. In this Subsection, the most significant results from [1] are summarized. However, this procedure can be generalized and improved to classify Lie algebras over $\mathbb{Z}/3\mathbb{Z}$; for this latter finite field, another equivalence relation appears in the set of directed pseudo-graphs. Subsection 3.2 is therefore devoted to the definition and usage of this last relation.

3.1. Results over $\mathbb{Z}/2\mathbb{Z}$. Let $\mathscr{S}_n$ be the set of non-isomorphic simple directed pseudo-graphs with n vertices. Obviously, $\mathscr{S}_1$ is constituted by two simple directed pseudo-graphs (an isolated vertex and a vertex with a loop), and $\mathscr{S}_2$ is formed by the 10 simple directed pseudo-graphs in Figure 1. Let us see that these simple directed pseudo-graphs are useful to classify the family of Lie algebras $\mathscr{F}_2$ described in Definition 2.1 for each dimension.

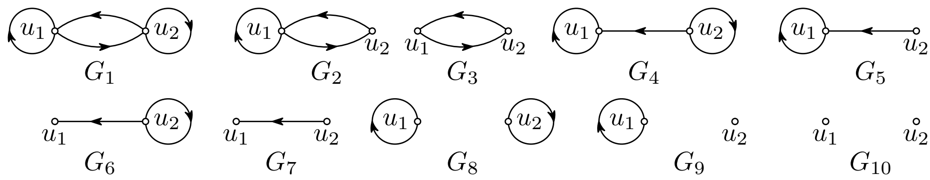

Figure 1. Non-isomorphic simple directed pseudo-graphs with 2 vertices.

Next, we establish an equivalence relation $\mathscr{R}_2$ in $\mathscr{S}_n$. First, we define the map (p, q) (with $1 \leqslant p \neq q \leqslant n$), which transforms the simple directed pseudo-graph G into the simple directed pseudo-graph $G(p, q)$ according to the following rules. If G has the adjacency matrix coefficients (α_{rs}), then $G(p, q)$ has the same vertex set as G and its adjacency matrix coefficients are

$$\beta_{rs} = \begin{cases} \alpha_{rs}, & \text{if } r \neq p \text{ and } s \neq q, \\ \alpha_{rq} + \alpha_{rp}, & \text{if } r \neq p \text{ and } s = q, \\ \alpha_{ps} + \alpha_{qs}, & \text{if } r = p \text{ and } s \neq q, \\ \alpha_{pq} + \alpha_{qq} + \alpha_{pp} + \alpha_{qp}, & \text{if } r = p \text{ and } s = q. \end{cases}$$

More concretely, the edge $\langle r, s\rangle$ belongs to the graph $G(p, q)$ if one of the following conditions is satisfied:

- ▷ The edge $\langle r, s\rangle$ belongs to the graph G, with $r \neq p$ and $s \neq q$.
- ▷ The edge $\langle r, q\rangle$ or the edge $\langle r, p\rangle$ belongs to the graph G, with $r \neq p$ and $s = q$.
- ▷ The edge $\langle p, s\rangle$ or the edge $\langle q, s\rangle$ belongs to the graph G, with $r = p$ and $s \neq q$.
- ▷ One or three of the edges $\langle p, q\rangle$, $\langle q, q\rangle$, $\langle p, p\rangle$, $\langle q, p\rangle$ belong to the graph G, with $r = p$ and $s = q$.

For instance, in $\mathscr{S}_2$, $G_1 = G_7(1, 2)$, $G_4 = G_3(1, 2)$, $G_5 = G_9(2, 1)$, and $G_6 = G_9(1, 2)$ (see Figure 1).

Now, we say that two elements of $\mathscr{S}_n$ are related by $\mathscr{R}_2$ if there exists a sequence of maps $(p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)$ which transforms one of them into the other. In this way, it is easy to prove that *if two simple directed pseudo-graphs are isomorphic, then they are related* (since, by construction, if two simple directed pseudo-graphs are isomorphic then their associated algebras are isomorphic and, according to Theorem 2.4 in [1], if two algebras are isomorphic then their associated simple directed pseudo-graphs are related). However, the converse is not necessarily true, as can be observed from the graphs G_1 and G_7. In order to find a representative element of each class of $\mathscr{S}_n$ for $n \leqslant 4$, we use the representation of the simple directed pseudo-graph by its adjacency matrix.

Let us consider an n-dimensional Lie algebra of $\mathscr{F}_2$ defined by its square binary matrix $(n-1) \times (n-1)$ (where the element in the i-th row and j-th column is the coefficient of u_j in the bracket $[u_i, u_n]$). This matrix can also be seen as the adjacency matrix of a certain simple directed pseudo-graph. Hence, in a natural way, we can define a map F, between $\mathscr{F}_2$ and $\mathscr{S}_{n-1}$, which associates an algebra with the simple directed pseudo-graph whose adjacency matrix is the matrix of the algebra. By considering this map, the next results were obtained.

Theorem 3.1. *Two simple directed pseudo-graphs in $\mathscr{S}_n$ are related (by $\mathscr{R}_2$) if and only if their corresponding Lie algebras are isomorphic.*

Proof. See [1] if any additional explanation is needed. □

By using Theorem 3.1 and Mathematica symbolic package, we get the following classifications of Lie algebras of $\mathscr{F}_2$ of dimensions 2, 3, 4, and 5, respectively.

Theorem 3.2. *There are two Lie algebras of dimension 2 in $\mathscr{F}_2$, defined by $[u_1, u_2] = 0$ and $[u_1, u_2] = u_1$, respectively, and associated with the simple directed pseudo-graphs of $\mathscr{S}_1$.*

Theorem 3.3. *There are six Lie algebras of dimension 3 in $\mathscr{F}_2$, associated with the following simple directed pseudo-graphs of $\mathscr{S}_2$:*

$$[u_1, u_3] = 0,\ [u_2, u_3] = 0 \leftrightarrow G_{10}; \qquad [u_1, u_3] = u_1,\ [u_2, u_3] = 0 \leftrightarrow G_9;$$
$$[u_1, u_3] = 0,\ [u_2, u_3] = u_1 \leftrightarrow G_7; \qquad [u_1, u_3] = u_2,\ [u_2, u_3] = u_1 \leftrightarrow G_3;$$
$$[u_1, u_3] = u_1 + u_2,\ [u_2, u_3] = u_1 \leftrightarrow G_2; \qquad [u_1, u_3] = u_1,\ [u_2, u_3] = u_2 \leftrightarrow G_8.$$

Theorem 3.4. *There are fourteen Lie algebras of dimension 4 in $\mathscr{F}_2$, associated with the simple directed pseudo-graphs from the classification of $\mathscr{S}_3$ given in Figure 2. We describe their laws below by the triple (a, b, c), where a, b, and c represent the values of the brackets $[u_1, u_4]$, $[u_2, u_4]$, and $[u_3, u_4]$, respectively. The laws of these algebras are:*

$\mathbf{h_4^1}$: $(0, 0, 0)$; $\mathbf{h_4^2}$: $(u_1, 0, 0)$; $\mathbf{h_4^3}$: $(0, u_1, 0)$;
$\mathbf{h_4^4}$: $(u_2, u_1, 0)$; $\mathbf{h_4^5}$: $(u_1 + u_2, u_1, 0)$; $\mathbf{h_4^6}$: $(0, u_1, u_2)$;
$\mathbf{h_4^7}$: $(u_1, 0, u_2)$; $\mathbf{h_4^8}$: $(u_1, u_2, 0)$; $\mathbf{h_4^9}$: (u_2, u_1, u_3);
$\mathbf{h_4^{10}}$: (u_3, u_1, u_2); $\mathbf{h_4^{11}}$: $(u_1, u_1 + u_3, u_2)$; $\mathbf{h_4^{12}}$: $(u_1 + u_3, u_1, u_2)$;
$\mathbf{h_4^{13}}$: $(u_3, u_1, u_1 + u_2)$; $\mathbf{h_4^{14}}$: (u_1, u_2, u_3).

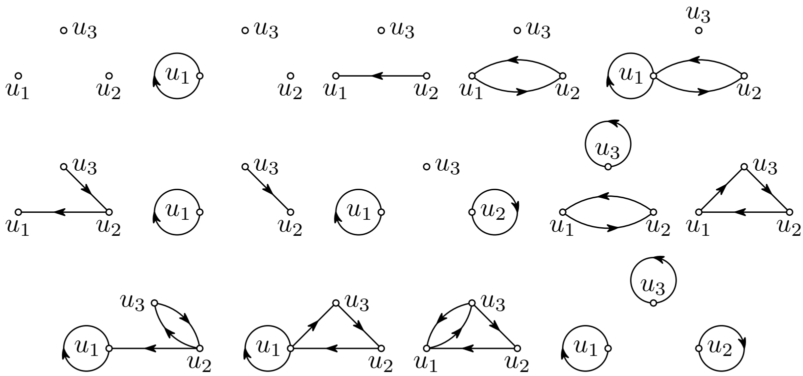

Figure 2. Classification of $\mathscr{S}_3$.

Theorem 3.5. *There are thirty-four Lie algebras of dimension* 5 *in* $\mathscr{F}_2$. *These algebras, which correspond to the* 34 *simple directed pseudo-graphs from the classification of* $\mathscr{S}_4$, *are the following:*

$[u_1, u_5] = 0,\ [u_2, u_5] = 0,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = 0,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = 0,\ [u_2, u_5] = u_1,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1 + u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = 0,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = 0,\ [u_3, u_5] = u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_2,\ [u_3, u_5] = 0,\ [u_4, u_5] = 0.$

$[u_1, u_5] = 0,\ [u_2, u_5] = 0,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_1.$

$[u_1, u_5] = u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_3,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_1 + u_3,\ [u_3, u_5] = u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = 0,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_2,\ [u_3, u_5] = 0,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = 0,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_1 + u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_1 + u_2,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_1 + u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = 0,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = 0,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_2,\ [u_3, u_5] = u_3,\ [u_4, u_5] = 0.$

$[u_1, u_5] = u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_4,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_1 + u_4,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_4,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_3.$

$[u_1, u_5] = u_1,\ [u_2, u_5] = u_1 + u_3,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_4.$

$[u_1, u_5] = u_4,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_1 + u_3.$
$[u_1, u_5] = u_4,\ [u_2, u_5] = u_1 + u_2,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_1 + u_3.$
$[u_1, u_5] = u_4,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_1 + u_2,\ [u_4, u_5] = u_3.$
$[u_1, u_5] = u_1 + u_4,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_3.$
$[u_1, u_5] = u_1 + u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_4.$
$[u_1, u_5] = u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_1 + u_2,\ [u_4, u_5] = u_4.$
$[u_1, u_5] = u_1 + u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_3 + u_4,\ [u_4, u_5] = u_3.$
$[u_1, u_5] = u_3,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_2,\ [u_4, u_5] = u_4.$
$[u_1, u_5] = u_2,\ [u_2, u_5] = u_1,\ [u_3, u_5] = u_3,\ [u_4, u_5] = u_4.$
$[u_1, u_5] = u_1,\ [u_2, u_5] = u_2,\ [u_3, u_5] = u_3,\ [u_4, u_5] = u_4.$

3.2. Results over $\mathbb{Z}/3\mathbb{Z}$. Once the method has been showen, we adapt and apply it to the case of $\mathbb{Z}/3\mathbb{Z}$. As the computations are straightforward and lengthy, we only present the differences from the previous cases (over $\mathbb{Z}/2\mathbb{Z}$).

Let $\mathscr{P}_{n-1}$ be the set of non-isomorphic directed pseudo-graphs with $n-1$ vertices which have only allowed simple and double edges and loops. For convenience in this subsection, we introduce the following definition which permits us to simplify the terminology.

Definition 3.1. Let G be a directed pseudo-graph of $\mathscr{P}_{n-1}$ with adjacency matrix (α_{rs}). We define the *opposite graph* of G (denoted by $O(G)$) to be the directed pseudo-graph of $\mathscr{P}_{n-1}$ whose adjacency matrix is $(2\alpha_{rs})$.

From the directed pseudo-graph G, we first select two of its vertices, called u_p and u_q (possibly, it could be $u_p = u_q$). Let us consider the map $((p, q))$ which transforms G into $G((p, q))$, whose vertex set is the one of G and the coefficients of its adjacency matrix are

$$\beta_{rs} = \begin{cases} \alpha_{rs}, & \text{if } r \neq p, \text{ and } s \neq q, \\ \alpha_{ps} + \alpha_{qs}, & \text{if } r = p \text{ and } s \neq q, \\ \alpha_{rq} + 2\alpha_{rp}, & \text{if } p \neq q, r \neq p \text{ and } s = q, \\ \alpha_{pq} + \alpha_{qq} + 2\alpha_{pp} + 2\alpha_{qp}, & \text{if } p \neq q, r = p \text{ and } s = q, \\ 2\alpha_{rp}, & \text{if } p = q, r \neq p \text{ and } s = q, \\ \alpha_{pp}, & \text{if } p = q, r = p \text{ and } s = q. \end{cases}$$

Next, we define the following equivalence relation $\mathscr{R}_3$ on $\mathscr{P}_{n-1}$: two directed pseudo-graphs G and H are equivalent when there exists a sequence of maps $((p, q))$ transforming G or $O(G)$ into H.

The previous relation, together with the following result, will allow us to set the main theorem, which will be used later to obtain the pursued classifications.

Proposition 3.1. *Let $u_1, \dots, u_n$ be the vectors of the canonical basis in the vector space $\mathscr{K}^n$, where $\mathscr{K}$ is any field. Let $\mathscr{L}$ and $\mathscr{L}^*$ be two n-dimensional isomorphic Lie algebras over $\mathscr{K}$ whose brackets are, respectively:*

$$\mathscr{L} \equiv [u_p, u_q] = \begin{cases} 0 & \text{if } 1 \leqslant p < q \leqslant n-1, \\ \sum\limits_{k=1}^{n-1} \alpha_{pk} u_k & \text{if } 1 \leqslant p \leqslant n-1 \text{ and } q = n; \end{cases}$$

$$\mathscr{L}^* \equiv [u_p, u_q]_* = \begin{cases} 0 & \text{if } 1 \leqslant p < q \leqslant n-1, \\ \sum\limits_{k=1}^{n-1} \beta_{pk} u_k & \text{if } 1 \leqslant p \leqslant n-1 \text{ and } q = n. \end{cases}$$

Then there exists an isomorphism $\varphi\colon \mathscr{L} \to \mathscr{L}^*$ defined as

$$\varphi(u_p) = \begin{cases} \sum\limits_{k=1}^{n-1} \varepsilon_{pk} u_k & \text{if } 1 \leqslant p \leqslant n-1, \\ \varepsilon_{nn} u_n & \text{if } p = n, \quad \text{with } \varepsilon_{nn} \neq 0. \end{cases}$$

The proof of Proposition 3.1 is omitted because it is purely algebraic, not complicated but lengthy and uninteresting. It consists in distinguishing two cases according to $\dim([\mathscr{L}, \mathscr{L}])$: (i) less than or equal to 1 in the first case, and (ii) greater than or equal to 2 in the second one. Moreover, certain technical lemmas are also used.

Among them, we only need the following result.

Corollary 3.1. *If $\mathscr{K} = \mathbb{Z}/3\mathbb{Z}$, then the isomorphism φ from Proposition 3.1 can be decomposed, in a straightforward way, into the addition of elementary isomorphisms of two types:*

$$\varphi_{pq}(u_k) = \begin{cases} u_p + u_q, & \text{if } k = p \text{ and } 1 \leqslant p,\ q \leqslant n-1, \\ u_k, & \text{if } k \neq p; \end{cases}$$

$$\varphi(u_p) = \begin{cases} u_p, & \text{if } 1 \leqslant p \leqslant n-1, \\ 2u_n, & \text{if } p = n. \end{cases}$$

Now, by taking into account the previous proposition, the main result is obtained.

Theorem 3.6 (Main Theorem)**.** *Two directed pseudo-graphs in $\mathscr{P}_{n-1}$, G and H, are equivalent (by $\mathscr{R}_3$) if and only if their respective associated Lie algebras are isomorphic.*

P r o o f. First of all, if G and H are equivalent, then there is a sequence of maps $((p_1, q_1)), \dots, ((p_h, q_h))$ which transforms G into H. Let us denote by $\mathscr{L}$ and $\mathscr{L}^*$

the Lie algebras associated with G and H, respectively. Each map $((p_i, q_i))$ matches to an isomorphism φ_i between the corresponding Lie algebras associated with two simple directed pseudo-graphs in each step. By composing these isomorphisms, we obtain an isomorphism $(\varphi_h \circ \ldots \circ \varphi_1)$ between $\mathscr{L}$ and $\mathscr{L}^*$.

Conversely, if $\mathscr{L}$ and $\mathscr{L}^*$ are isomorphic, then by Corollary 3.1, the isomorphism between them can be decomposed into the addition of elementary isomorphisms of two types:

$$\varphi_{pq}(u_k) = \begin{cases} u_p + u_q, & \text{if } k = p \text{ and } 1 \leqslant p,\ q \leqslant n-1, \\ u_k, & \text{if } k \neq p. \end{cases}$$

$$\varphi(u_p) = \begin{cases} u_p, & \text{if } 1 \leqslant p \leqslant n-1, \\ 2u_n, & \text{if } p = n. \end{cases}$$

Now, every φ_{pq} produces a map $((p, q))$ between the corresponding simple directed pseudo-graphs. Thus, the sequence of maps between G (or $O(G)$) and H appears, and this completes the proof. $\square$

Then, by using Theorem 3.6 and Mathematica symbolic package, the following classifications of Lie algebras of $\mathscr{F}_3$, for dimensions 3 and 4, respectively, are obtained.

Theorem 3.7. *Up to isomorphism, there exist eight Lie algebras of dimension 3 over $\mathbb{Z}/3\mathbb{Z}$ belonging to the family $\mathscr{F}_3$, whose laws are given by the following brackets:*

$\mathbf{g_3^1}$: $[u_1, u_3] = 0, [u_2, u_3] = 0$; $\quad$ $\mathbf{g_3^2}$: $[u_1, u_3] = 0, [u_2, u_3] = u_2$;
$\mathbf{g_3^3}$: $[u_1, u_3] = 0, [u_2, u_3] = u_1$; $\quad$ $\mathbf{g_3^4}$: $[u_1, u_3] = u_2, [u_2, u_3] = u_1$;
$\mathbf{g_3^5}$: $[u_1, u_3] = u_2, [u_2, u_3] = u_1 + u_2$; $\quad$ $\mathbf{g_3^6}$: $[u_1, u_3] = u_2, [u_2, u_3] = 2u_1$;
$\mathbf{g_3^7}$: $[u_1, u_3] = u_2, [u_2, u_3] = 2u_1 + u_2$; $\quad$ $\mathbf{g_3^8}$: $[u_1, u_3] = u_1, [u_2, u_3] = u_2$.

Let us note that these algebras are associated with the pseudo-graphs of $\mathscr{P}_2$ shown in Figure 3.

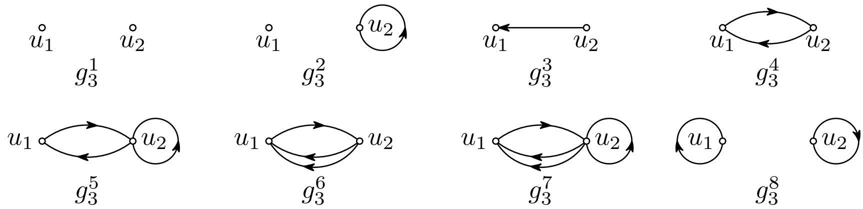

Figure 3. Each Lie algebra of dimension 3 in the family $\mathscr{F}_3$, represented by its associated graph.

Theorem 3.8. *Up to isomorphism, there exist twenty-two Lie algebras of dimension* 4 *over* $\mathbb{Z}/3\mathbb{Z}$ *belonging to the family* $\mathscr{F}_3$. *Their laws, represented by the triple* (a, b, c), *where* a, b, *and* c *represent the values of the brackets* $[u_1, u_4]$, $[u_2, u_4]$, *and* $[u_3, u_4]$, *respectively, are:*

$\mathbf{g_4^1}$: $(0, 0, 0)$;	$\mathbf{g_4^2}$: $(0, 0, u_3)$;
$\mathbf{g_4^3}$: $(0, 0, u_2)$;	$\mathbf{g_4^4}$: $(0, u_3, u_2)$;
$\mathbf{g_4^5}$: $(0, u_3, u_2 + u_3)$;	$\mathbf{g_4^6}$: $(0, u_3, 2u_2)$;
$\mathbf{g_4^7}$: $(0, u_3, 2u_2 + u_3)$;	$\mathbf{g_4^8}$: $(0, u_3, u_1)$;
$\mathbf{g_4^9}$: $(0, u_3, u_1 + u_3)$;	$\mathbf{g_4^{10}}$: $(0, u_2, u_3)$;
$\mathbf{g_4^{11}}$: (u_3, u_2, u_1);	$\mathbf{g_4^{12}}$: $(u_3, u_2, u_1 + u_3)$;
$\mathbf{g_4^{13}}$: $(u_3, u_2, u_1 + 2u_3)$;	$\mathbf{g_4^{14}}$: $(u_3, u_2, u_1 + u_2)$;
$\mathbf{g_4^{15}}$: $(u_3, u_2, 2u_1)$;	$\mathbf{g_4^{16}}$: $(u_3, u_2, 2u_1 + 2u_3)$;
$\mathbf{g_4^{17}}$: $(u_3, u_2, 2u_1 + u_2 + 2u_3)$;	$\mathbf{g_4^{18}}$: $(u_3, u_2 + u_3, u_1 + u_2)$;
$\mathbf{g_4^{19}}$: $(u_3, u_2 + u_3, u_1 + u_2 + u_3)$;	$\mathbf{g_4^{20}}$: $(u_3, u_2 + u_3, u_1 + 2u_2)$;
$\mathbf{g_4^{21}}$: $(u_3, u_2 + u_3, u_1 + 2u_2 + 2u_3)$;	$\mathbf{g_4^{22}}$: $(u_1, u_2, 2u_3)$.

4. Certain conclusions

By generalizing the procedure given in [1] which classified a selected family of Lie algebras over $\mathbb{Z}/2\mathbb{Z}$, we have also obtained as a new result a classification of Lie algebras over $\mathbb{Z}/3\mathbb{Z}$ up to dimension 4. However, when the dimension is increased, the computations are getting more and more complex, so we do not see a great interest in extending the problem. Moreover, we think it would be positive to obtain a general formula, or recursive relation, to know at least the number of algebras in dimension n, over $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. This would be, in our opinion, the main open problem.

In fact, authors think that classifying a family of Lie algebras over some finite fields using techniques from the graph theory is an appealing subject, although the approach could seem too computational, which unfortunately may hide the general picture. Perhaps, a more conceptual approach would open the way to using more efficient tools in the future.

Other related open problems can be listed here. First, as pointed out in [1], it is necessary to refine the techniques used in order to obtain $\mathscr{S}_n$ with $n \geqslant 5$. A previous step to do this implies the design of a suitable efficient algorithm.

Secondly, we have also observed that the classification of Lie algebras can help in the understanding of directed pseudo-graphs. For instance, by a converse process to the one shown here, we could study if two directed pseudo-graphs are not isomorphic by comparing their associated Lie algebras.

Finally, the study of Lie algebras defined over the field $\mathbb{Z}/p\mathbb{Z}$ with $p > 3$ prime is still a worthy unsolved problem.

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