Marcus Giaquinto. *The Search for Certainty. A philosophical account of foundations of mathematics*, Oxford, Clarendon Press, 2002.

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From the Preface: "This book is a philosophical examination of one of the most brilliant intellectual explorations ever, aimed at justifying mathematics in the wake of the class paradoxes." The sentence encapsulates the aims of the book under review, and justly emphasises the importance of its topic. The second part of the sentence may be judged, from a historical point of view, to underscore one of the book's weaknesses (more on this below), but generally speaking, *The Search for Certainty* is a very clear and up-to-date exposition, remarkable for its conciseness, the independence of thought, and the level of reflection deployed by its author. Clearly it is the result of many years' work.
My comments will be directed first at historical aspects of relevance, and then at the philosophical views advanced by the author.

The story told by Giaquinto has as its milestones the set-theoretical clarification of mathematical analysis, the paradoxes that appeared around 1900, the foundational work of Russell & Whitehead and its comparative weaknesses with respect to Zermelo–Fraenkel set theory, the Hilbert programme, and Gödel's underivability theorems. Throughout, the emphasis is not on a true depiction of the historical development ("this book is not a history") but on the main philosophical ideas and arguments, together with their indispensable mathematical background. In a nutshell, the key ingredients in the tale are indicated by three names: Russell, Hilbert, Gödel. Connoisseurs will recognise that, to this extent, Giaquinto's approach is in line with a long tradition in philosophy, a tradition that has been contested by historians of mathematics and logic. That is not merely an academic debate, since it proves relevant for the philosophical lessons to be extracted from efforts to find certainty in the foundations of mathematics.

The philosophical tradition to which Giaquinto is heir can be traced back to Russell himself, and to Russellians such as Carnap or Quine. The Russellians have long underscored the importance of the line Frege–Russell–Hilbert–Gödel in the development of foundational studies. Indeed, Bertrand Russell was not just a prominent logician and proponent of an influential version of logicism. Through his prolific writing and his influence as a philosopher, he also established a standard tradition of interpretation of the recent history of logic, foundational studies, and logicism itself. The class paradoxes played a pivotal role in this interpretation, as they do in the book under review and so many previous treatises. Thus, we read that Zermelo's axiomatic work was motivated by the paradoxes (p. 119), while Moore and others have underscored the key importance of debate on the axiom of choice and Zermelo's proof of the well-ordering theorem. If we take this latter interpretive turn, the philosophical lesson changes. Zermelo's axiomatisation was not just a response to difficulties in logical theory, but it was crucially motivated by problems and sceptical doubts regarding the new, abstract methodology favoured by mathematicians such as Dedekind, Cantor and Hilbert.

A lesser point of historical detail has to do with reconstructing the development of logicism. According to Russell, the logicist project began only with Frege's work, and was reshaped by Russell himself in such a way as to escape from the class and linguistic paradoxes. (All this is given excellent detailed explanation by Giaquinto.) Meanwhile, it is well known that Dedekind, a very influential mathematician by 1890, proposed a logicist conception of pure mathematics. But as Giaquinto writes, following Frege and Russell, Dedekind "did not have in mind any positive conception of logic" (p. 30). It is true that Dedekind spent almost no ink explaining his understanding of logical theory, and that he offered no system of *formal* logic, but it is nonetheless a fact that he proposed very positive views concerning those layers of logical theory that were crucial for the logicist reduction of pure mathematics – the theory of sets and mappings. Dedekind relied on the contemporary understanding of logic, and the very good reception accorded to his proposal by authors such as Schröder, Peano, Peirce, and with some provisos even Frege himself, shows that he was well acquainted with the state of the art.

But I would not like to give the impression that Giaquinto is too much prey to traditional lore. On the contrary, as I have mentioned, his work is remarkable for its independence of thought. Examples abound, for instance in the many critical reflections accompanying his explanation of the various systems of type theory in parts II and III. Another example is the remarks on the role and forms of the principle of comprehension (pp. 56, 119, 249 note 2), which in my opinion are much better than what can usually be found in similar works. Also remarkable is the fact that Giaquinto does not follow the confusing (and once again Russellian) tradition of saying that Zermelo's axiom system rests on a principle of "limitation of size."

Of particular merit for its carefulness and abundance of detail is the discussion and further elaboration of Hilbert's epistemological finitism (chap. 3, part IV), the compact presentation of Gödel's incompleteness theorems, and the discussion of their philosophical implications. With respect to the first incompleteness theorem, Giaquinto discusses its implications for (a) the relations between mathematical truth and formal derivability, (b) the relations between formal consistency and existence, with respect to first- and second-order axiomatic systems, and (c) the problematic character of finitary truth in its relations with derivability. With respect to the second theorem, frequently known as the underivability of consistency, he deals at length with the implications for Hilbert's programme, examining critically (a) the possibility, raised by Detlefsen, of using "non-standard" formal systems whose consistency is guaranteed by definition, and (b) the possibility that finitary reasoning may go beyond what is expressible in a formal system of arithmetic. In both cases, Giaquinto shows that the hopes to revitalize Hilbert's programme by such means have so far been illusory, and that there exist good reasons of principle supporting the view that Gödel's second theorem effectively blocks the road to a purely formal proof, of the kind envisioned by Hilbert, of the reliability of formal systems of mathematics (see p. 190). Admittedly, these arguments fall short of a strict mathematical proof of impossibility, but they are the best one can obtain in the absence of a fully precise delimitation of the scope of finitary reasoning.

In part VI, Giaquinto moves toward the conclusion that, if certainty about classical mathematics is beyond reach, nonetheless a high degree of confidence in its reliability is warranted. He begins by exposing and endorsing the iterative conception of sets, first proposed by Gödel, as providing a coherent conception that is expressed (at least partly) in the axioms of set theory. However, it can be argued that the iterative conception alone does not suffice to justify all of the ZFC axioms, not even the axiom of extensionality and the axiom of infinity. It is difficult to defend the view that ZFC is the expression of a single coherent conception – in fact it seems more like the result of a combination of different conceptions and strands (e.g., the classical concept of function which led to the combinatorial conception of sets, classical sources for the assumption of infinite sets, and the iterative conception).

That much having been said, still Giaquinto's intended conclusion is not completely blocked. I believe it possible to argue that those different strands that merged into set theory combine into a coherent overall picture. Coherent enough, I would say, to make it highly implausible that ZFC may lead to inconsistencies. As Giaquinto writes, "if ... the conception is coherent, an expression of it (or any part of it) will be consistent, whether or not it is instantiated" (p. 213, see also 221). This means that, whether or not we are realists about the universe of sets, the coherence of the conception on which the axiom system is based suffices to eliminate fear of inconsistency. This may be the reason why, in time, the pressing need of justification in the style of the Hilbert programme fell away.

Thus, although Giaquinto's argument needs refinement and would become more involved, its basic gist can be preserved. To this reviewer, at least, it seems reasonable and plausible to argue that our convictions of the consistency not only of set theory, but also of analysis, stem in fact from the coherence of the underlying conceptions. In the case of analysis, what is meant is above all the conception of the linear continuum, which certainly is much more basic to human thought and science than the iterative conception (and obviously much older).

Our author next presents a "solution" (which in his terminology is much more than a mere blockage) to the class paradoxes, based on ideas that Zermelo presented in his 1930 paper dealing with models of set theory. Zermelo proposed, on the basis of the second-order ZFC axiom system, the view of an unlimited sequence of ever more inclusive models of set theory, each one fitting with the previous ones in its lower reaches. An essential part of Zermelo's perspective was the idea that there is no all-inclusive universe of sets, and this suffices to explain why the class paradoxes are faulty arguments: they assumed that certain predicates had an absolute reference, but this is incompatible with the Zermelian picture. Giaquinto discusses also the solution to the Liar paradox that this approach offers, and concludes: "we can have rational confidence in the coherence of set theory and, therefore, of … classical mathematics" (p. 221).

A last part, Aftermath, brings several aspects of the story to the present. The reader will find here an interesting, compact discussion of results in modern developments of proof theory such as the programme of so-called "reverse mathematics" and Feferman's work on predicativism. While less ambitious than Hilbert's programme, these lines of study have led to interesting results on the possibility of proof-theoretical reduction of much of classical mathematics to very limited formal systems (which in some cases have been proved to be finitarily reliable). Such results give further support to Giaquinto's philosophical conclusion.

The book will be much welcome by readers who wish to obtain a deep grasp of key issues in the foundations of mathematics and their philosophical consequences, without having to study in full detail issues in set theory or proof theory. It can also be used as a very commendable philosophical companion to detailed studies of mathematical logic. It must be said, however, that the writing presupposes a good grasp of the symbolism of mathematical logic, and readers will need some command of abstract mathematical thought. Thus, in many cases *The Search for Certainty* will prove difficult to follow for readers who lack those competencies. But detailed knowledge and philosophical reflection are never bought at low prices: it is certain that the effort will prove rewarding for those with a real interest in these matters.