

# On the uniqueness of weak solutions of the two-dimensional primitive equations

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## ABSTRACT

The uniqueness of weak solutions of the primitive equations with Dirichlet boundary conditions at the bottom is an open problem even in the two dimensional case. The aim of this paper is to prove the uniqueness of weak solutions when we replace the Dirichlet boundary condition at the bottom by a friction condition. With this bottom boundary condition at the bottom, we establish an additional regularity for the vertical derivative of the horizontal velocity which allows us to conclude.

**Keywords.** Navier-type boundary conditions, primitive equations, uniqueness.

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## 1 Introduction

This paper is devoted to the proof of the uniqueness of weak solutions of the primitive equations in two space dimensions, with a traction by wind at the surface and the friction condition at the bottom derived in [2]. The

three dimensional version of these anisotropic equations is widely used in geophysics and the primitive equations have been extensively studied from the mathematical point of view starting the works by J.-L. Lions, R. Temam and S. Wang, see for instance [8] and [9]. The interested reader is referred for example to [1], [3], [6] and [13].

The equations are obtained from the anisotropic Navier-Stokes equations by an asymptotic analysis as the aspect ratio  $\delta = \text{depth}/\text{width}$  of the domain goes to 0. This asymptotic analysis has been recently done in [2] with an anisotropic Navier-type condition at the bottom and a traction condition by wind at the surface. A Navier-type boundary condition for the primitive equations has been obtained. We will use this boundary condition to establish the uniqueness in space dimension two.

With an homogeneous Dirichlet boundary condition at the bottom, the existence of a weak solution has been obtained in [8], by means of a Galerkin approximation, for three dimensional domains without beaches. In [5], a “weak/strong” uniqueness result with this kind of boundary conditions is proved. This result is in the same spirit as the result for the three dimensional Navier-Stokes equations, see [11], where a “weak/strong” uniqueness result is given, see also [12]. For results concerning the incompressible Navier-Stokes equations, we refer the interested reader to [10].

The uniqueness of weak solutions of the Navier-Stokes equations in space dimension two is well known, see [7]. On the contrary, the question of uniqueness of weak solutions of the primitive equations is still an open problem, even in two dimensional domains, see [6] for the question in a coupled system. The reader interested in a review on energy equality and uniqueness questions related to Navier-Stokes is referred to [4] page 22 and the references cited therein.

Here we consider the non standard boundary condition at the bottom for the two dimensional primitive equations established in [2]. This boundary condition corresponds to a vorticity-velocity condition. By this way we are able to give, to the best of our knowledge, the first uniqueness result for “weak” solutions of primitive equations in two space dimensions assuming that the initial data are regular with respect the vertical variations.

The paper is organized as follows: In Section 2 we recall the primitive equations in two space dimensions and the Navier-type boundary condition obtained in [2]. In Section 3, we give some definitions, hypothesis and anisotropic inequalities that we will use frequently in the sequel. The weak formulation related to the system, a weak existence theorem established in [2] and the main results are given in Section 4. In Section 5, we give an outline of the proof of the main Theorem. Section 6 is devoted to the regu-

larity of the pressure and finally, in Section 7, we establish the existence of a weak-vorticity solution which implies the uniqueness result.

## 2 Primitive equations and boundary conditions

Let us consider the domain  $\Omega$  of  $\mathbb{R}^2$ :

$$\Omega = \{(x, z) : x \in s, -h(x) < z < 0\},$$

where the horizontal section  $s$  is an open interval and  $h$  is a non negative continuous function on  $\bar{s}$  vanishing on  $\partial s$ . The boundary of  $\Omega$  is  $\partial\Omega = \overline{\Gamma_b} \cup \Gamma_s \cup \partial\Gamma_s$  where the bottom  $\Gamma_b$ , the surface  $\Gamma_s$  and the shore  $\partial\Gamma_s$  are defined by:

$$\Gamma_b = \{(x, -h(x)) : x \in s\}, \quad \Gamma_s = \{(x, 0) : x \in s\}, \quad \partial\Gamma_s = \{(x, 0) : x \in \partial s\}.$$

Note that the domain under consideration has a vanishing depth on  $\partial s$ ; the case of non vanishing depth can be treated as well.

We consider a fluid governed by the following primitive equations with traction by the wind at the surface and Navier-type condition at the bottom. More precisely we assume that the velocity of the fluid  $u = (v, w)$  and the pressure  $p$  satisfy the following primitive equations

$$\begin{cases} \partial_t v + v \partial_x v + w \partial_z v - \nu_h \partial_x^2 v - \nu_v \partial_z^2 v + \partial_x p = 0, \\ \partial_z p = 0, \quad w(t, x, z) = \int_z^0 \partial_x v(t, x, \xi) d\xi, \\ \langle v \rangle = 0 \text{ on } s, \end{cases} \quad (1)$$

where we denote

$$\langle \varphi \rangle(x) = \int_{-h(x)}^0 \varphi(x, z) dz.$$

The boundary and initial conditions are:

$$\begin{cases} \nu_v \partial_z v|_{\Gamma_s} = \alpha |v_{air}| (v_{air} - v|_{\Gamma_s}), \\ \nu_v \partial_z v|_{\Gamma_b} = \beta(x) v|_{\Gamma_b}, \\ v|_{t=0} = v_0 \end{cases} \quad (2)$$

where  $v_{air}$  is the horizontal velocity of the wind at the surface of the ocean and  $v_0$  is the initial horizontal velocity. We consider an anisotropic viscosity

$(\nu_h, \nu_v)$ ,  $\alpha \in \mathbb{R}$  is a positive constant and  $\beta = \beta(x)$  is a positive function defined on  $s$ .

We give here some comments on the condition  $\langle v \rangle = 0$  on  $s$ . It comes from the divergence free condition, the normal velocity condition at the boundary and the no flux condition on the shore usually used and given respectively by

$$\partial_x v + \partial_z w = 0 \text{ in } \Omega, \quad (v, w) \cdot n = 0 \text{ on } \partial\Omega, \quad \langle v \rangle = 0 \text{ on } \partial s.$$

Indeed, by integration with respect to  $z$  and using that  $w = 0$  on  $\Gamma_s$ , the divergence free condition gives

$$w(t, x, z) = \int_z^0 \partial_x v(t, x, \xi) d\xi.$$

With the condition  $(v, w) \cdot n = 0$  on  $\Gamma_b$ , this implies

$$\partial_x \langle v \rangle = 0 \text{ in } s.$$

Using now the no flux condition  $\langle v \rangle = 0$  on  $\partial s$ , we obtain  $\langle v \rangle = 0$  in  $s$ . Let us remark that we recover the condition  $(v, w) \cdot n = 0$  on  $\partial\Omega$  from the second and third equations on (1) since the surface of the domain corresponds to  $z = 0$ .

### 3 Functional spaces and anisotropic inequalities

This section is devoted to the functional setting of the primitive equations (1), (2). We introduce the following space:

$$\mathcal{V} = \left\{ \varphi \in C_s^\infty(\overline{\Omega}) : \langle \varphi \rangle = 0 \text{ in } s \right\},$$

where  $C_s^\infty(\overline{\Omega})$  is the space of all smooth ( $C^\infty$ ) functions on  $\overline{\Omega}$  that vanish in a neighbourhood of  $\partial\Gamma_s$ . Then the space  $H$  (resp.  $V$ ) is the closure of  $\mathcal{V}$  in  $L^2(\Omega)$  (resp.  $H^1(\Omega)$ ). We can easily check that

$$H = \left\{ \varphi \in L^2(\Omega) : \langle \varphi \rangle = 0 \text{ in } s \right\}, \quad V = \left\{ \varphi \in H^1(\Omega) : \langle \varphi \rangle = 0 \text{ in } s \right\}.$$

Throughout the paper, we will assume:

$$h \in H^2(S) \cap H_0^1(S) \text{ such that } |h'| > 0 \text{ on } \partial s, \quad \frac{h}{|h'|} \geq c \text{ dist}(x, \partial s), \quad (3)$$

$$\beta \in H_0^1(s), \quad v_{air} \in W^{1,\infty}(0, T; H_0^1(s)), \quad (4)$$

and

$$v_0 \in H, \quad \partial_z v_0 \in L^2(\Omega). \quad (5)$$

*Remark.* The assumptions on  $\beta$  and  $v_{air}$  are not optimal. It is possible to assume, for instance,  $\beta/h \in L^2(s)$  instead of  $\beta \in H_0^1(s)$ . This gives better assumptions, for example, in the presence of sidewalls. An exterior force may be added without major modifications in the proofs.

*Remark.* In the case of the presence of side-walls, we may assume that  $h \in H^2(s)$  and define the lateral boundary  $\Gamma_l = \{(x, z) : x \in \partial s \text{ and } -h(x) < z < 0\}$ . We choose an homogeneous Dirichlet boundary condition for  $v$  on this boundary. The space  $\mathcal{C}_s^\infty(\bar{\Omega})$  should be replaced by the space of smooth functions on  $\bar{\Omega}$  vanishing in a neighbourhood of  $\partial\Gamma_s \cup \Gamma_l$  and hence  $V$  is given by  $V = \{\varphi \in H^1(\Omega) : \langle \varphi \rangle = 0 \text{ in } s, \varphi = 0 \text{ on } \Gamma_l\}$ . In the sequel, we consider the case without sidewalls since the case with sidewalls is easier.

Let us now recall some anisotropic inequalities, proved in [5], that we will use in this work.

**Definition 1.** For  $p, q \in [1, +\infty]$ , we say that a function  $u$  belongs to  $L_x^p L_z^q(\Omega)$  if:

$$u(x, \cdot) \in L^q(-h(x), 0) \quad \text{and} \quad \|u(x, \cdot)\|_{L^q(-h(x), 0)} \in L^p(s),$$

and its norm is given by

$$\|u\|_{L_x^p L_z^q} = \left\| \|u(x, \cdot)\|_{L^q(-h(x), 0)} \right\|_{L^p(s)}. \quad \square$$

For simplicity, we will denote  $L_x^p L_z^q$  instead of  $L_x^p L_z^q(\Omega)$ .

**Lemma 2.** For  $v \in H^1(\Omega)$ , we have  $v \in L_x^\infty L_z^2 \cap L_x^2 L_z^\infty$ , and the inequalities:

$$\|v\|_{L_x^\infty L_z^2} \leq c \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2}$$

and

$$\|v\|_{L_x^2 L_z^\infty} \leq c \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2}$$

where  $c$  a positive constant depending only on  $\Omega$ . In the first inequality,  $\|v\|_{H^1(\Omega)}$  can be replaced by  $\|\partial_x v\|_{L^2(\Omega)}$  if  $v|_{\Gamma_b} = 0$ , and in the second one  $\|v\|_{H^1(\Omega)}$  can be replaced by  $\|\partial_z v\|_{L^2(\Omega)}$  if  $v|_{\Gamma_b} = 0$  (or  $v|_{\Gamma_s} = 0$ ).

Moreover, for all  $(v, w)$  such that  $\partial_x v \in L^2(\Omega)$  and  $w = \int_z^0 \partial_x v$ , we have

$$\|w\|_{L_x^2 L_z^\infty} \leq h_{\max}^{1/2} \|\partial_x v\|_{L^2(\Omega)},$$

where  $h_{\max} = \max_{\bar{s}} h$ .  $\square$

## 4 Definitions and the main results

The weak formulation of the primitive equations is given by

**Definition 3. (Weak solution)** *We say that  $v$  is a weak solution of (1)–(2) in  $(0, T)$  if:*

$$v \in L^\infty(0, T; H) \cap L^2(0, T; V),$$

*satisfies the variational formulation:  $\forall \varphi \in C^1([0, T]; \mathcal{V})$  with  $\varphi(T) = 0$ ,*

$$\left\{ \begin{array}{l} - \int_0^T \int_\Omega (\partial_t \varphi + v \partial_x \varphi + w \partial_z \varphi) v + \int_0^T \int_\Omega (\nu_h \partial_x v \partial_x \varphi + \nu_v \partial_z v \partial_z \varphi) \\ + \int_0^T \int_s \beta(x) \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) v|_{\Gamma_b} \varphi|_{\Gamma_b} + \int_0^T \int_s \alpha |v_{air}| v|_{\Gamma_s} \varphi|_{\Gamma_s} \\ = \int_\Omega v_0 \varphi(0) + \int_0^T \int_s \alpha |v_{air}| v_{air} \varphi|_{\Gamma_s} + \nu_h \int_0^T \int_s v|_{\Gamma_b} \partial_x [\varphi|_{\Gamma_b} h'(x)], \end{array} \right. \quad (6)$$

*with  $w = \int_z^0 \partial_x v$  and  $v$  satisfies the following energy inequality*

$$\left\{ \begin{array}{l} \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 + \nu_h \int_0^t \|\partial_x v(s)\|_{L^2(\Omega)}^2 + \nu_v \int_0^t \|\partial_z v(s)\|_{L^2(\Omega)}^2 \\ + \int_0^t \int_s \gamma(x) |v|_{\Gamma_b}|^2 + \frac{1}{2} \int_0^t \int_s \alpha |v_{air}| |v|_{\Gamma_s}|^2 \\ \leq \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^t \int_s \alpha |v_{air}|^3 \end{array} \right. \quad (7)$$

*with  $\gamma(x) = \beta(x) \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) - \frac{\nu_h}{2} h''(x)$ .  $\square$*

From (7), in order to ensure that the system is dissipative, we assume the additional relation between  $h$ ,  $\beta$ ,  $\nu_h$  and  $\nu_v$

$$\gamma(x) = \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) \beta(x) - \frac{\nu_h}{2} h''(x) \geq 0. \quad (8).$$

Let us now recall an existence result of a global weak solution established in [2].

**Theorem 4.** *Let (3)–(4) and (8) be satisfied and  $v_0 \in H$ . There exists a weak solution  $v \in L^2(0, T; V) \cap L^\infty(0, T; H)$  of (1)–(2).  $\square$*

This Theorem is proved by an asymptotic analysis on the anisotropic Navier-Stokes equations with anisotropic Navier boundary condition on the bottom, cf. [2]. Let us remark that in fact the hypothesis  $\beta \in L^\infty$ ,  $v_0 \in H$

and  $v_{\text{air}} \in L^3(0, T; L^3(\Omega))$  is sufficient to ensure an existence result. Given a weak solution  $v$ , De Rham's theorem gives the existence of a pressure  $p$  (as a Lagrange multiplier) such that  $(v, w, p)$  solves the primitive equations (1) in the distribution sense (cf. [2]).

Moreover, one has:

$$\left\{ \begin{array}{l} - \int_0^T \int_{\Omega} (\partial_t \varphi + v \partial_x \varphi + w \partial_z \varphi) v + \int_0^T \int_{\Omega} (\nu_h \partial_x v \partial_x \varphi + \nu_v \partial_z v \partial_z \varphi) \\ + \int_0^T \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) v|_{\Gamma_b} \varphi|_{\Gamma_b} \\ + \int_0^T \int_s \alpha |v_{\text{air}}| (v|_{\Gamma_s} - v_{\text{air}}) \varphi|_{\Gamma_s} \\ = \int_{\Omega} v_0 \varphi(0) + \nu_h \int_0^T \int_s v|_{\Gamma_s} \partial_x (\varphi|_{\Gamma_b} h') + \int_0^T \int_{\Omega} p \nabla \cdot (\varphi, \psi), \end{array} \right. \quad (9)$$

for all  $\varphi \in C^1([0, T]; C_s^\infty(\overline{\Omega}))$ , such that  $\varphi(T) = 0$ , and for all  $\psi$  regular enough satisfying  $(\varphi, \psi) \cdot n_{\partial\Omega} = 0$ .

Let us now give the definition of a weak-vorticity solution.

**Definition 5. (Weak-vorticity solution)** *We say that  $v$  is a weak-vorticity solution of (1)–(2) in  $(0, T)$  if it is a weak solution and it satisfies the additional regularity:*

$$\partial_z v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

*This additional regularity implies that  $v$  satisfies the boundary conditions at the bottom and at the surface in the trace sense.  $\square$*

*Remark.* Note that  $\partial_z v$  may be seen as the vorticity associated to the velocity in the primitive equations. This is the reason why use the name weak-vorticity solution. Indeed, if we consider  $u_{NS} = (v_{NS}, w_{NS})$  the weak solution of the two dimensional Navier-Stokes equations, the vorticity is given by  $\omega_{NS} = \partial_z v_{NS} - \partial_x w_{NS}$ . But, since the primitive equations are obtained from Navier-Stokes equations by an asymptotic analysis as  $\delta \rightarrow 0$  assuming  $x \approx 1$ ,  $z \approx \delta$ ,  $v_{NS} \approx 1$  and  $w_{NS} \approx \delta$ . Then, we have that  $\partial_z v_{NS} - \partial_x w_{NS} \approx \partial_z v$ .  $\square$

The goal of the paper is to prove the following result

**Theorem 6.** *Let (3)–(5) and (8) be satisfied. Any weak solution of primitive equations (1)–(2) is a weak-vorticity solution.  $\square$*

A consequence of this Theorem is the following uniqueness result of weak solutions with more regular initial data.

**Theorem 7.** *Let (3)–(5) and (8) be satisfied. There exists a unique weak solution of the primitive equations (1)–(2). Moreover this solution is a weak-vorticity solution.  $\square$*

The existence result of weak solutions has been established in [2]. In order to prove the uniqueness of such weak solution, we use Theorem 6 and derive the energy estimates for the difference between any two solutions with the same initial data. We control the nonlinear terms with the regularity  $\partial_z v \in L^4(0, T, L^4(\Omega))$  coming from Theorem 6.

## 5 Outline of the proof of Theorem 6

In order to prove Theorem 6, we establish the equation satisfied by  $\partial_z v$  where  $v$  is a weak solution of (1)–(2). Let  $\eta \in \mathcal{C}^1([0, T]; C^\infty(\Omega))$  such that  $\eta = 0$  on a neighbourhood of  $\partial Q_T$  where  $Q_T = (0, T) \times \Omega$ . We choose  $\varphi = \partial_z \eta$  as a test function in the variational formulation of Definition 3. Since  $\varphi \in \mathcal{C}^1([0, T]; \mathcal{V})$ , this gives :

$$\int_0^T \int_\Omega -\partial_t \eta \partial_z v - v \partial_x \eta \partial_z v - w \partial_z \eta \partial_z v - \nu_h \partial_z v \partial_x^2 \eta + \nu_v \partial_z v \partial_z^2 \eta = 0$$

This implies that  $\partial_z v$  satisfies in  $\mathcal{D}'((0, T) \times \Omega)$ :

$$\partial_t(\partial_z v) + \partial_x(v \partial_z v) + \partial_z(w \partial_z v) - \nu_h \partial_x^2(\partial_z v) - \nu_v \partial_z^2(\partial_z v) = 0.$$

Therefore  $\partial_z v$  is solution of a parabolic equation and thus we hope to prove the weak regularity

$$\partial_z v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

Since  $\partial_z v$  satisfies a non homogeneous Dirichlet boundary condition on the boundary, *see* (2), we have to choose an appropriate lifting in order to be able to obtain some energy estimates. Let us consider the function

$$\psi = \nu_v \partial_z v - \phi(x, z)v - e$$

where  $v$  is a weak solution of (1)–(2),

$$\phi(t; x, z) = - \left( 1 + \frac{z}{h(x)} \right) \alpha |v_{air}(t; x)| - \frac{z}{h(x)} \beta(x)$$

and

$$e(t; x, z) = \alpha |v_{air}(t; x)| v_{air}(t; x) \left( 1 + \frac{z}{h(x)} \right).$$



Since  $(\phi v)|_{\Gamma_s} = -\alpha|v_{air}|v$ ,  $(\phi v)|_{\Gamma_b} \equiv \beta v$ ,  $e|_{\Gamma_s} = \alpha|v_{air}|v_{air}$  and  $e|_{\Gamma_b} = 0$ , we get that  $\psi$  is solution of the following linear problem:

$$\begin{cases} \partial_t \psi + v \partial_x \psi + w \partial_z \psi - \nu_h \partial_x^2 \psi - \nu_v \partial_z^2 \psi = G & \text{in } (0, T) \times \Omega, \\ \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi|_{t=0} = \nu_v \partial_z v_0 - (\phi v)|_{t=0} + e|_{t=0} & \text{in } \Omega \end{cases} \quad (10)$$

where

$$\begin{aligned} G = & -v \partial_t \phi + 2\nu_h \partial_x \phi \partial_x v + \nu_h \partial_x^2 \phi v + 2\nu_v \partial_z \phi \partial_z v \\ & - v \partial_x \phi v - w \partial_z \phi v - \partial_t e + \nu_h \partial_x^2 e - v \partial_x e - w \partial_z e + \phi \partial_x p. \end{aligned}$$

Then the proof of Theorem 6 is divided in two parts. In the first part, we prove that there exists a weak solution  $\psi$  of (10) that means

$$\psi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

In the second part, we prove that if we define  $\tilde{v}$  such that

$$\nu_v \partial_z \tilde{v} = \psi + \phi v + e, \quad \langle \tilde{v} \rangle = 0$$

then

$$\tilde{v} = v.$$

This implies

$$\partial_z v = \psi + \phi v + e.$$

Using now the weak regularity of  $v$ ,  $\psi$  and Hypothesis (3)–(5), any weak solution corresponding to the initial data  $v_0$  satisfying (5) is a weak-vorticity solution that means

$$\partial_z v \in L^2(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

For the second part, we need to prove some regularity of the pressure  $p$  since the term  $\phi \partial_x p$  appears in the right-hand side of (10). This is the goal of the following section.

## 6 Regularity of the pressure

At first, let us remark that we can identify the pressure  $p$  as a distribution on  $s$  since  $p$  does not depend on the vertical coordinate. Indeed, let us choose  $\phi \in \mathcal{D}(s)$  and let us define  $h_{\min}$  the minimum of  $h$  on the support of  $\phi$ . It

suffices to define  $\psi \in \mathcal{D}(-h_{\min}, 0)$  with  $\int_{-h_{\min}}^0 \psi = 1$  and to choose  $\phi\psi$  as test function in  $\mathcal{D}(\Omega)$ . It suffices to write

$$\langle p, \phi \rangle_{\mathcal{D}'(s), \mathcal{D}(s)} = \langle p, \phi\psi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

Let us now prove the following result

**Theorem 8.** *Let (3)–(5) be satisfied. If  $(v, p)$  is a weak solution of (1)–(2), we get*

$$\sqrt{h}\partial_x p \in L^2(0, T; H^{-1}(s)). \quad \square$$

*Remark.* The regularity  $\sqrt{h}\partial_x p \in L^2(0, T; H^{-1}(s))$  means that the map from  $L^2(0, T; H_0^1(s))$  into  $\mathbb{R}$  defined by:

$$\varphi \mapsto \int_0^T \int_s p \partial_x (\sqrt{h}\varphi)$$

is linear and continuous.  $\square$

Let us give here a lemma, proved using Hardy's inequality, which allows us to choose  $\varphi/\sqrt{h}$  as test function in the mixte formulation (8) to prove Theorem 8.

**Lemma 9.** *Let (3) be satisfied. Let  $\varphi \in H_0^1(s)$ , then  $\tilde{\varphi}$  defined by  $\tilde{\varphi} = \varphi/\sqrt{h}$  and considered as a function of  $x$  and  $z$  belongs to  $H^1(\Omega)$ .  $\square$*

**Proof.** We have

$$\partial_x \tilde{\varphi} = \partial_x \left( \frac{\varphi}{\sqrt{h}} \right) = \left( \frac{\partial_x \varphi}{\sqrt{h}} \right) - \frac{1}{2} \left( \frac{\varphi h'}{h^{3/2}} \right).$$

Therefore, using Hypothesis (3), we deduce

$$\begin{aligned} \left\| \partial_x \left( \frac{\varphi}{\sqrt{h}} \right) \right\|_{L^2(\Omega)} &\leq \left\| \partial_x \varphi \right\|_{L^2(s)} + C \left\| \frac{\varphi}{h^{1/2} \operatorname{dist}(x, \partial s)} \right\|_{L^2(\Omega)} \\ &\leq \left\| \partial_x \varphi \right\|_{L^2(s)} + C \left\| \frac{\varphi}{\operatorname{dist}(x, \partial s)} \right\|_{L^2(s)} \end{aligned}$$

and we conclude using Hardy's inequality.  $\square$

**Proof of Theorem 8.** Let us consider the mixte formulation (9). Let us choose  $\tilde{\varphi} = \varphi/\sqrt{h}$  as test function with  $\varphi \in C_0^1([0, T]; C_0^\infty(S))$ . Using that  $\langle v \rangle = 0$  and that  $\partial_z \tilde{\varphi} = 0$ , we get

$$\begin{aligned} \int_0^T \int_s p \partial_x (\sqrt{h} \varphi) &= - \int_0^T \int_\Omega \partial_x \left( \frac{\varphi}{\sqrt{h}} \right) v^2 + \nu_h \int_0^T \int_\Omega \partial_x v \partial_x \left( \frac{\varphi}{\sqrt{h}} \right) \\ &+ \int_0^T \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) v|_{\Gamma_b} \frac{\varphi}{\sqrt{h}} + \int_0^T \int_s \alpha |v_{\text{air}}| (v|_{\Gamma_s} - v_{\text{air}}) \frac{\varphi}{\sqrt{h}} \\ &- \nu_h \int_0^T \int_s v|_{\Gamma_b} \partial_x \left[ \varphi(t; x) \frac{h'(x)}{\sqrt{h}} \right]. \end{aligned}$$

We conclude checking that all the terms on the right-hand side define linear continuous forms with respect to  $\varphi \in L^2(0, T; H_0^1(s))$  using (3)–(5) and the fact that  $v$  is a weak solution.

The first two terms are treated by using that  $v^2, \partial_x v \in L^2(0, T; L^2(\Omega))$ . The third and the fourth terms are treated by using that  $\beta/h$  and  $v_{\text{air}}/h$  belong to  $L^2(s)$  (using Hardy's inequality) and that  $v|_{\Gamma_s}$  and  $v|_{\Gamma_b} \in L^2(0, T; L^2(s))$ .

For the last term, using the weak regularity of  $v$  and the inequality, (cf. [2]),

$$\|v|_{\Gamma_b}\|_{H^{1/2}(s)} \leq c \|v\|_{H^1(\Omega)},$$

we get  $v|_{\Gamma_b} \in L^2(0, T; H^{1/2}(s))$ , then it suffices to prove that we have  $\partial_x(\varphi h'/\sqrt{h}) \in L^2(0, T; H^{-1/2}(s))$ . We conclude since we have

$$\left\| \frac{\varphi h'}{\sqrt{h}} \right\|_{H^{1/2}(s)} \leq C \left\| \frac{\varphi h'}{\sqrt{h}} \right\|_{H^1(\Omega)} \leq C \|\varphi\|_{H_0^1(s)}$$

using Hardy's inequality.  $\square$

## 7 Existence of weak-vorticity solution

In this section, we prove the existence of a weak solution  $\psi$  of (10) and we prove the identification between  $\tilde{v}$  and  $v$  where  $\tilde{v}$  is defined Section 5.

1) *Existence of a weak solution  $\psi$  of (10).* This existence result follows from a classical Galerkin method. We just prove the relevant *a priori* estimates.

Let us take (at least formally)  $\psi$  as a test function in (10), we obtain:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\psi(t)\|_{L^2(\Omega)}^2 + \nu_h \|\partial_x \psi(t)\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z \psi(t)\|_{L^2(\Omega)}^2 \\
&= - \int_{\Omega} \partial_t \phi v \psi + 2\nu_h \int_{\Omega} (\partial_x \phi)(\partial_x v) \psi + \nu_h \int_{\Omega} (\partial_x^2 \phi) v \psi \\
&+ 2\nu_v \int_{\Omega} (\partial_z \phi)(\partial_z v) \psi d\Omega - \int_{\Omega} (\partial_x \phi) v^2 \psi - \int_{\Omega} w(\partial_z \phi) v \psi \\
&- \langle \partial_t e, \psi \rangle_{\Omega} - \nu_h \int_{\Omega} (\partial_x e)(\partial_x \psi) - \int_{\Omega} v(\partial_x e) \psi - \int_{\Omega} w(\partial_z e) \psi \\
&- \int_s p \partial_x \langle \phi \psi \rangle = \sum_{i=1}^{11} I_i
\end{aligned} \tag{11}$$

We bound each terms in the right hand-side by  $a(t) + b(t) \|\psi\|_{L^2(\Omega)}^2 + \nu_h \|\partial_x \psi\|_{L^2(\Omega)}^2 / 20$  with  $a, b \in L^1(0, T)$  using the anisotropic estimates in Lemma 2 and Hardy's inequality on  $s$ . Let us prove this kind of estimates on three terms for the reader's convenience. The remaining terms are bounded in the same way.

The first term is bounded as follows

$$\begin{aligned}
| \int_{\Omega} \partial_t \phi v \psi | &\leq c \|\partial_t |v_{air}|\|_{L_x^2 L_z^{\infty}} \|v\|_{L^2} \|\psi\|_{L_x^{\infty} L_z^2} \\
&\leq c \|\partial_t v_{air}\|_{L^2(S)}^2 \|v\|_{L^2(\Omega)}^2 + \frac{\nu_h}{20} \|\partial_x \psi\|_{L^2(\Omega)}^2.
\end{aligned}$$

Concerning the fourth term, we write

$$\begin{aligned}
| \int_{\Omega} \partial_z \phi \partial_z v \psi | &\leq c \left( \left\| \frac{v_{air}}{h} \right\|_{L_x^2 L_z^{\infty}} + \left\| \frac{\beta}{h} \right\|_{L_x^2 L_z^{\infty}} \right) \|\partial_z v\|_{L^2} \|\psi\|_{L_x^{\infty} L_z^2} \\
&\leq c \left( \|v_{air}\|_{H_0^1(S)} + \|\beta\|_{H_0^1(S)} \right) \|\partial_z v\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}^{1/2} \|\partial_x \psi\|_{L^2(\Omega)}^{1/2} \\
&\leq c \|\partial_z v\|_{L^2(\Omega)}^2 + c \left( \|v_{air}\|_{H_0^1(S)}^4 + \|\beta\|_{H_0^1(S)}^4 \right) \|\psi\|_{L^2(\Omega)}^2 + \frac{\nu_h}{20} \|\partial_x \psi\|_{L^2(\Omega)}^2.
\end{aligned}$$

The ninth term is controlled by

$$\begin{aligned}
| \int_{\Omega} v \partial_x e \psi | &\leq c \left( \|\partial_x (|v_{air}| v_{air})\|_{L_x^2 L_z^{\infty}} + \left\| \frac{h' |v_{air}| v_{air}}{h} \right\|_{L_x^2 L_z^{\infty}} \right) \|\psi\|_{L_x^{\infty} L_z^2} \|v\|_{L^2} \\
&\leq c \|\psi\|_{L^2(\Omega)}^{1/2} \|\partial_x \psi\|_{L^2(\Omega)}^{1/2} \|v\|_{L^2(\Omega)} \|v_{air}\|_{H_0^1(\Omega)}^2 \\
&\leq c \|v\|_{L^2(\Omega)}^4 + \|v_{air}\|_{H_0^1(\Omega)}^4 \|\psi\|_{L^2(\Omega)}^2 + \frac{\nu_h}{20} \|\partial_x \psi\|_{L^2(\Omega)}^2
\end{aligned}$$

Let us now control the term coming from the pressure. One has

$$\begin{aligned}
\int_{\Omega} p \partial_x \langle \phi \psi \rangle &= \langle \sqrt{h} \partial_x p, \frac{1}{\sqrt{h}} \int_{-h}^0 \phi \psi \rangle_{H^{-1}(S) \times H_0^1(S)} \\
&\leq \|\sqrt{h} \partial_x p_s\|_{H^{-1}(S)} \|\partial_x (\frac{1}{\sqrt{h}} \int_{-h}^0 \phi \psi)\|_{L^2(S)}.
\end{aligned}$$

Taking into account that  $\psi|_{\Gamma_b} = 0$ , we get

$$\|\partial_x(\frac{1}{\sqrt{h}} \int_{-h}^0 \phi\psi)\|_{L^2(s)} \leq \|\frac{1}{\sqrt{h}} \int_{-h}^0 \partial_x(\phi\psi)\|_{L^2(s)} + \|\frac{h'}{h^{3/2}} \int_{-h}^0 \phi\psi\|_{L^2(s)}. \quad (12)$$

Thus using the expression of  $\phi$ , the fact that  $\psi \in L^2(0, T; H_0^1(\Omega))$  and the hypothesis on  $\beta$ ,  $v_{\text{air}}$  and  $h$ , we get, by the use of Hardy's inequality:

$$\int_{\Omega} p \partial_x \langle \phi \psi \rangle \leq c(\|v_{\text{air}}\|_{H_0^1(s)}^2 + \|\beta\|_{H_0^1(s)}^2) \|\sqrt{h} \partial_x p_s\|_{H^{-1}(s)}^2 + \frac{\nu_h}{20} \|\partial_x \psi\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2.$$

Remark that, in order to prove the previous estimate, we have used the anisotropic estimate  $L_x^\infty L_z^2$  on  $\psi$ . Indeed, for example, the second term in (12) is controlled as follows

$$\int_s(\int_{-h}^0 \frac{h'}{h^{3/2}} \phi\psi)^2 \leq 2 \int_s(\int_{-h}^0 \alpha \frac{h+z}{h} \frac{h'}{h^{3/2}} |v_{\text{air}}| |\psi|)^2 + 2 \int_s(\int_{-h}^0 \frac{z}{h} \frac{h'}{h^{3/2}} |\beta| |\psi|)^2.$$

Thus

$$\int_s(\int_{-h}^0 \frac{h'}{h^{3/2}} \phi\psi)^2 \leq C \int_s |\frac{v_{\text{air}} h'}{h}|^2 \|\psi\|_{L_z^2}^2 \int_{-h}^0 \frac{(h+z)^2}{h^3} + C \int_s |\frac{\beta h'}{h}|^2 \|\psi\|_{L_z^2}^2 \int_{-h}^0 \frac{z^2}{h^3}$$

which implies, using  $L_x^\infty L_z^2$  estimate on  $\psi$ ,

$$\int_s(\int_{-h}^0 \frac{h'}{h^{3/2}} \phi\psi)^2 \leq C(\|v_{\text{air}}\|_{H_0^1(s)}^2 + \|\beta\|_{H_0^1(s)}^2) \|\psi\|_{L^2(\Omega)} \|\partial_x \psi\|_{L^2(\Omega)}.$$

In conclusion, we obtain from (11):

$$\frac{d}{dt} \|\psi(t)\|_{L^2(\Omega)}^2 + \nu_h \|\partial_x \psi(t)\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z \psi(t)\|_{L^2(\Omega)}^2 \leq a(t) + b(t) \|\psi\|_{L^2(\Omega)}^2,$$

with  $a, b \in L^1(0, T)$ . The Gronwall Inequality gives the *a priori* estimates.

II) *Identification of  $v$  and  $\tilde{v}$ .* Let us introduce  $a = \psi + \phi v + e$  and define

$$\tilde{v} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$$

such that

$$\nu_v \partial_z \tilde{v} = a, \quad \langle \tilde{v} \rangle = 0.$$

We choose for example

$$\tilde{v}(x, z) = -\frac{1}{\nu_v} \int_z^0 a(x, s) ds + \frac{1}{\nu_v} \frac{1}{h(x)} \left( \int_{-h(x)}^0 \left( \int_z^0 a(x, s) ds \right) dz \right).$$

Remark that  $\tilde{v}$  satisfies the boundary conditions  $\nu_v \partial_z \tilde{v} = \alpha |v_{air}| (v_{air} - v)$  on  $\Gamma_s$  and  $\nu_v \partial_z \tilde{v} = \beta v$  on  $\Gamma_b$  and that  $\tilde{v}(0) = v_0$ . If we prove that  $\tilde{v} = v$  then we will have

$$\nu_v \partial_z v = \psi + \phi v + e$$

and therefore using the weak regularity of  $\psi$ ,  $v$  and Hypothesis (3)–(5), we get

$$\partial_z v \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

This gives Theorem 6.

At first we establish the equation satisfied by  $\tilde{v}$ . Let us choose  $\int_z^0 \eta(x, s) ds$  as test functions in the variational formulation satisfied by  $\psi$  with  $\langle \eta \rangle = 0$  and  $\eta \in \mathcal{D}(\Omega)$ . We have

$$\int_\Omega \partial_t a \left( \int_z^0 \eta(x, s) ds \right) = \nu_v \int_\Omega \partial_z (\partial_t \tilde{v}) \left( \int_z^0 \eta(x, s) ds \right) = \nu_v \int_\Omega \partial_t \tilde{v} \eta$$

and

$$\begin{aligned} \int_\Omega [\partial_x(v a) + \partial_z(w a)] \left( \int_z^0 \eta(x, s) ds \right) &= - \int_\Omega \left( \int_z^0 [\partial_x(v a) + \partial_z(w a)](x, s) ds \right) \eta \\ &= -\nu_v \int_\Omega \left( \int_z^0 \partial_x(v \partial_z \tilde{v})(x, s) ds \right) \eta + \nu_v \int_\Omega w \partial_z \tilde{v} \eta. \end{aligned}$$

Moreover

$$\begin{aligned} \nu_h \int_\Omega \partial_x a \partial_x \left( \int_z^0 \eta(x, s) ds \right) &= \nu_v \nu_h \int_\Omega \partial_z (\partial_x \tilde{v}) \left( \int_z^0 \partial_x \eta(x, s) ds \right) \\ &= \nu_v \nu_h \int_\Omega \partial_x \tilde{v} \partial_x \eta. \end{aligned}$$

Using that  $0 = \langle \partial_x \eta \rangle$ , we get

$$\begin{aligned} \nu_v \int_\Omega \partial_z a \partial_z \left( \int_z^0 \eta(x, s) ds \right) &= -\nu_v^2 \int_\Omega \partial_z (\partial_z \tilde{v}) \eta \\ &= \nu_v^2 \int_\Omega \partial_z \tilde{v} \partial_z \eta \end{aligned}$$

Let us now use the expressions above. Dividing them by  $\nu_v$ , we obtain, using De Rham's Theorem, that there exists  $\tilde{p}_s$  such that  $(\tilde{v}, \tilde{p}_s)$  satisfies:

$$\begin{cases} \partial_t \tilde{v} + v \partial_x \tilde{v} + w \partial_z \tilde{v} - \nu_h \partial_x^2 \tilde{v} - \nu_v \partial_z^2 \tilde{v} + \partial_x \tilde{p}_s \\ \quad = v \partial_x \tilde{v} + \int_z^0 \partial_x(v \partial_z \tilde{v})(x, s) ds \\ \langle \tilde{v} \rangle = 0. \end{cases} \quad (13)$$

Using now the boundary conditions satisfied by  $\tilde{v}$ , we obtain that  $\tilde{v}$  satisfies the following variational formulation:  $\forall \eta \in C^1([0, T]; \mathcal{V})$

$$\left\{ \begin{aligned} & \langle \partial_t \tilde{v}, \eta \rangle_\Omega + \int_0^t \int_\Omega \left\{ - \left( \int_z^0 \partial_x (v \partial_z \tilde{v})(x, s) ds \right) \eta + w \partial_z \tilde{v} \eta \right\} \\ & + \int_0^t \int_\Omega \nu_h \partial_x \tilde{v} \partial_x \eta + \nu_v \partial_z \tilde{v} \partial_z \eta + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) \eta|_{\Gamma_s} \\ & + \int_0^t \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) v|_{\Gamma_b} \eta|_{\Gamma_b} = \nu_h \int_0^t \int_s v|_{\Gamma_b} \partial_x [\eta|_{\Gamma_b} h'(x)]. \end{aligned} \right. \quad (14)$$

Let us now prove that  $\tilde{v} = v$ . We know that  $v$  satisfies the following variational formulation:  $\forall \varphi \in C^1([0, T]; \mathcal{V})$

$$\left\{ \begin{aligned} & \langle v(t), \varphi(t) \rangle_\Omega - \int_0^t \int_\Omega (\partial_t \varphi + v \partial_x \varphi + w \partial_z \varphi) v \\ & + \int_0^t \int_\Omega (\nu_h \partial_x v \partial_x \varphi + \nu_v \partial_z v \partial_z \varphi) + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) \varphi|_{\Gamma_s} \\ & + \int_0^t \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) v|_{\Gamma_b} \varphi|_{\Gamma_b} \\ & = \int_\Omega v_0 \varphi(0) + \nu_h \int_0^t \int_s v|_{\Gamma_b} \partial_x [\varphi|_{\Gamma_b} h'(x)], \end{aligned} \right. \quad (15)$$

All the calculations made below could be justified by using mollifiers in time and passing to the limit.

If we take  $\tilde{v}$  as test function in (15), taking into account that we have  $\nu_v \partial_z \tilde{v} = a \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ , we obtain the equality:

$$\begin{aligned} & \langle \tilde{v}(t), v(t) \rangle_\Omega - \int_0^t \int_\Omega (\partial_t \tilde{v} + v \partial_x \tilde{v} + w \partial_z \tilde{v}) v \\ & + \int_0^t \int_\Omega (\nu_h \partial_x v \partial_x \tilde{v} + \nu_v \partial_z v \partial_z \tilde{v}) \\ & + \int_0^t \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) v|_{\Gamma_b} \tilde{v}|_{\Gamma_b} + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) \tilde{v}|_{\Gamma_s} \\ & = \int_\Omega |v_0|^2 dx + \nu_h \int_0^t \int_s v|_{\Gamma_b} \partial_x [\tilde{v}|_{\Gamma_b} h'(x)] \end{aligned} \quad (16)$$

Secondly, multiplying (13) by  $v$ , we get:

$$\begin{aligned}
& \int_0^t \langle \partial_t \tilde{v}, v \rangle_\Omega + \int_0^t \int_\Omega \{v \partial_x \tilde{v} + w \partial_z \tilde{v}\} v \\
& + \int_0^t \int_\Omega (\nu_h \partial_x \tilde{v} \partial_x v + \nu_v \partial_z \tilde{v} \partial_z v) + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) v|_{\Gamma_s} \\
& + \int_0^t \int_s \beta(x) \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) |v|_{\Gamma_b}|^2 \\
& = \nu_h \int_0^t \langle \partial_x [v|_{\Gamma_b} h'(x)], \tilde{v}|_{\Gamma_b} \rangle_{H^{-1/2}(S) \times H^{1/2}(S)} \\
& \quad + \int_0^t \int_\Omega \left\{ v \partial_x \tilde{v} + \left( \int_z^0 \partial_x (v \partial_z \tilde{v})(x, s) ds \right) \right\} v.
\end{aligned} \tag{17}$$

Adding (16) to (17), the terms  $\int_0^t \langle \partial_t \tilde{v}, v \rangle_\Omega ds$  and  $\int_\Omega (v \partial_x \tilde{v} + w \partial_z \tilde{v}) v d\Omega ds$  vanish, we get: a.e.  $t \in (0, T)$

$$\begin{aligned}
& \langle \tilde{v}(t), v(t) \rangle_\Omega + 2 \int_0^t \int_\Omega (\nu_h \partial_x \tilde{v} \partial_x v + \nu_v \partial_z \tilde{v} \partial_z v) \\
& + \int_0^t \int_s \beta(x) \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) v|_{\Gamma_b} (v|_{\Gamma_b} + \tilde{v}|_{\Gamma_b}) \\
& + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) (v + \tilde{v})|_{\Gamma_s} = \|v_0\|_{L^2(\Omega)}^2 \\
& + \nu_h \int_0^t \int_s v|_{\Gamma_b} \tilde{v}|_{\Gamma_b} h''(x) + \int_0^t \int_\Omega \left\{ v \partial_x \tilde{v} + \int_z^0 \partial_x (v \partial_z \tilde{v})(x, s) ds \right\} v.
\end{aligned} \tag{18}$$

Then, if we multiply (13) by  $\tilde{v}$  and integrate in  $(0, T) \times \Omega$ , we obtain the energy equality:

$$\begin{aligned}
& \frac{1}{2} \|\tilde{v}(t)\|_{L^2(\Omega)}^2 + \int_0^t (\nu_h \|\partial_x \tilde{v}\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z \tilde{v}\|_{L^2(\Omega)}^2) \\
& + \int_0^t \int_s \beta(x) \left(1 + \frac{\nu_h}{\nu_v} |h'(x)|^2\right) v|_{\Gamma_b} \tilde{v}|_{\Gamma_b} \\
& + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) \tilde{v}|_{\Gamma_s} = \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 \\
& + \frac{\nu_h}{2} \int_0^t \int_s |\tilde{v}|_{\Gamma_b}|^2 h''(x) + \int_0^t \int_\Omega \left\{ v \partial_x \tilde{v} + \int_z^0 \partial_x (v \partial_z \tilde{v})(x, s) ds \right\} \tilde{v}.
\end{aligned} \tag{19}$$



Finally, if we take  $v$  as a test function in (15), we get the energy inequality:

$$\begin{aligned}
& \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 + \int_0^t \left( \nu_h \|\partial_x v\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z v\|_{L^2(\Omega)}^2 \right) \\
& + \int_0^t \int_s \beta(x) \left( 1 + \frac{\nu_h}{\nu_v} |h'(x)|^2 \right) |v|_{\Gamma_b}|^2 \\
& + \int_0^t \int_s \alpha |v_{air}| (v|_{\Gamma_s} - v_{air}) v|_{\Gamma_s} \leq \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \frac{\nu_h}{2} \int_0^t \int_s |v|_{\Gamma_b}|^2 h''(x).
\end{aligned} \tag{20}$$

Then, doing (19)+(20)-(18), we get: a.e.  $t \in (0, T)$ ,

$$\begin{aligned}
& \frac{1}{2} \|v(t) - \tilde{v}(t)\|_{L^2(\Omega)}^2 \\
& + \int_0^t \left( \nu_h \|\partial_x (v - \tilde{v})(s)\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z (v - \tilde{v})(s)\|_{L^2(\Omega)}^2 \right) \leq I + J
\end{aligned} \tag{21}$$

with

$$I = \int_0^t \int_{\Omega} \left\{ v \partial_x \tilde{v} + \int_z^0 \partial_x (v \partial_z \tilde{v})(x, s) ds \right\} (\tilde{v} - v)$$

and

$$J = \frac{\nu_h}{2} \int_0^t \int_s |v|_{\Gamma_b} - \tilde{v}|_{\Gamma_b}|^2 h''(x).$$

This inequality will give  $\tilde{v} = v$  using Gronwall's Lemma. We just have to estimate the right-hand side of (21). The last term is bounded as follows

$$J \leq c \int_0^t \|v - \tilde{v}\|_{L^2(\Omega)} \|\nabla(v - \tilde{v})\|_{(L^2(\Omega))^2}. \tag{22}$$

Let us now look at the first term  $I$ . We will strongly use the fact that  $\langle v \rangle = 0$  and  $\langle \tilde{v} \rangle = 0$ . For example, using this properties, we have

$$\int_{\Omega} \left( \int_z^0 \partial_x (\tilde{v} \partial_z \tilde{v})(x, s) ds \right) (\tilde{v} - v) = - \int_{\Omega} \tilde{v} \partial_x \tilde{v} (\tilde{v} - v).$$

Thus:

$$I = - \int_0^t \int_{\Omega} \left\{ (v - \tilde{v}) \partial_x \tilde{v} + \int_z^0 \partial_x ((v - \tilde{v}) \partial_z \tilde{v})(x, s) ds \right\} (v - \tilde{v}).$$

Using that

$$\partial_z((v - \tilde{v}) \partial_x \tilde{v}) - \partial_x((v - \tilde{v}) \partial_z \tilde{v}) = \partial_z(v - \tilde{v}) \partial_x \tilde{v} - \partial_x(v - \tilde{v}) \partial_z \tilde{v},$$

we obtain

$$\begin{aligned}
\int_z^0 \partial_x((v - \tilde{v}) \partial_z \tilde{v}) & = ((v - \tilde{v}) \partial_x \tilde{v})(x, 0) - ((v - \tilde{v}) \partial_x \tilde{v})(x, z) \\
& + \int_z^0 \partial_x(v - \tilde{v}) \partial_z \tilde{v} - \int_z^0 \partial_z(v - \tilde{v}) \partial_x \tilde{v}.
\end{aligned}$$

Therefore, we get

$$I = - \int_0^t \int_{\Omega} \left( \int_z^0 [\partial_z \tilde{v} \partial_x (v - \tilde{v}) - \partial_x \tilde{v} \partial_z (v - \tilde{v})] (x, s) ds \right) (v - \tilde{v}).$$

Integrating by parts with respect to  $z$ , this gives:

$$I = \int_0^t \int_{\Omega} \{ \partial_z \tilde{v} \partial_x (v - \tilde{v}) - \partial_x \tilde{v} \partial_z (v - \tilde{v}) \} \left( \int_z^0 (v - \tilde{v})(x, s) ds \right) = I_1 + I_2. \quad (23)$$

Let us remark that

$$\begin{aligned} I_1 &\leq \int_0^t \|\partial_z \tilde{v}\|_{L_x^\infty L_z^2} \|\partial_x (v - \tilde{v})\|_{L^2(\Omega)} \left\| \int_z^0 (v - \tilde{v})(x, s) ds \right\|_{L_x^2 L_z^\infty} \\ &\leq C \int_0^t \|\partial_z \tilde{v}\|_{L^2(\Omega)}^{1/2} \|\partial_z \tilde{v}\|_{H^1(\Omega)}^{1/2} \|\partial_x (v - \tilde{v})\|_{L^2(\Omega)} \|v - \tilde{v}\|_{L^2(\Omega)} \\ &\leq \frac{\nu_h}{8} \int_0^t \|\partial_x (v - \tilde{v})\|_{L^2(\Omega)}^2 + C(\nu_h) \int_0^t \|\partial_z \tilde{v}\|_{L^2(\Omega)} \|\partial_z \tilde{v}\|_{H^1(\Omega)} \|v - \tilde{v}\|_{L^2(\Omega)}^2 \end{aligned} \quad (24)$$

Splitting  $I_2$  in two terms, we get:

$$I_2 = - \int_0^t \int_{\Omega} \partial_x \tilde{v} |v - \tilde{v}|^2 d\Omega + \int_0^t \int_{\Omega} \partial_x (\partial_z \tilde{v}) (v - \tilde{v}) \left( \int_z^0 (v - \tilde{v})(x, s) ds \right) d\Omega = A + B \quad (25)$$

Now we can bound  $A$  and  $B$  as follows:

$$\begin{aligned} A &\leq \int_0^t \|\partial_x \tilde{v}\|_{L^2(\Omega)} \|v - \tilde{v}\|_{L^4(\Omega)}^2 \\ &\leq \int_0^t \|\partial_x \tilde{v}\|_{L^2(\Omega)} \|v - \tilde{v}\|_{H^1(\Omega)} \|v - \tilde{v}\|_{L^2(\Omega)} \\ &\leq \frac{\min\{\nu_h, \nu_v\}}{8} \int_0^t \|v - \tilde{v}\|_{H^1(\Omega)}^2 + C(\nu_h, \nu_v) \int_0^t \|\partial_x \tilde{v}\|_{L^2(\Omega)}^2 \|v - \tilde{v}\|_{L^2(\Omega)}^2 \end{aligned} \quad (26)$$

and

$$\begin{aligned} B &\leq \int_0^t \|\partial_x (\partial_z \tilde{v})\|_{L^2(\Omega)} \|v - \tilde{v}\|_{L_x^\infty L_z^2} \left\| \int_z^0 (v - \tilde{v})(x, s) ds \right\|_{L_x^2 L_z^\infty} \\ &\leq C \int_0^t \|\partial_x (\partial_z \tilde{v})\|_{L^2(\Omega)} \|v - \tilde{v}\|_{L^2(\Omega)}^{3/2} \|v - \tilde{v}\|_{H^1(\Omega)}^{1/2} \\ &\leq \frac{\min\{\nu_h, \nu_v\}}{8} \int_0^t \|v - \tilde{v}\|_{H^1(\Omega)}^2 + C(\nu_h, \nu_v) \int_0^t \|\partial_x (\partial_z \tilde{v})\|_{L^2(\Omega)}^{4/3} \|v - \tilde{v}\|_{L^2(\Omega)}^2. \end{aligned} \quad (27)$$

Therefore, using (21)–(27), we get:

$$\begin{aligned} & \|v(t) - \tilde{v}(t)\|_{L^2(\Omega)}^2 + \int_0^t \left( \nu_h \|\partial_x(v - \tilde{v})(s)\|_{L^2(\Omega)}^2 + \nu_v \|\partial_z(v - \tilde{v})(s)\|_{L^2(\Omega)}^2 \right) \\ & \leq C \int_0^t \left( 1 + \|\partial_z \tilde{v}\|_{L^2(\Omega)} \|\partial_z \tilde{v}\|_{H^1(\Omega)} + \|\partial_x \tilde{v}\|_{L^2(\Omega)}^2 + \|\partial_x(\partial_z \tilde{v})\|_{L^2(\Omega)}^{4/3} \right) \|v - \tilde{v}\|_{L^2(\Omega)}^2. \end{aligned} \tag{28}$$

Then, using the regularity of  $\partial_z \tilde{v}$ , the initial data satisfied by  $v$  and  $\tilde{v}$  and using the Gronwall inequality, we conclude that  $\tilde{v} = v$ . The proof of Theorem 6 is then complete using Parts I) and II).  $\square$

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