On a double penalized smectic-A model.

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Abstract

In smectic-A liquid crystals a unity director vector $\boldsymbol{n}$ appear, modeling an average preferential direction of the molecules and also the normal vector of the layer configuration. In the E's model [5], the Ginzburg-Landau penalization related to the constraint $|\boldsymbol{n}| = 1$ is considered and, assuming the constraint $\nabla \times \boldsymbol{n} = 0$, $\boldsymbol{n}$ is replaced by the so-called layer variable φ such that $\boldsymbol{n} = \nabla\varphi$.

In this paper, a double penalized problem is introduced related to a smectic-A liquid crystal flows, considering a Cahn-Hilliard system to model the behavior of $\boldsymbol{n}$. Then, the issue of the global in time behavior of solutions is attacked, including the proof of the convergence of the whole trajectory towards a unique equilibrium state.

Keywords: Smectic-A liquid crystals, Navier-Stokes equations, Cahn-Hilliard system, coupled non-linear parabolic system, convergence to equilibrium.

1 Introduction

The original equations in the continuum theory of liquid crystals models was developed during the period of 1958 through 1968 by Ericksen and Leslie. Smectic crystals are a liquid-crystalline phase, where the molecules of the liquid crystal have a certain orientational order (as in the nematic case) and also have a certain positional order (layer structure). In the uni-axial case, the molecules of a liquid crystal have a preferred orientation modeled by an unit vectorial function, $\boldsymbol{d}$. In smectic case, the molecules are arranged in almost incompressible layers of almost constant width. In smectic-A case, the single optical axis perpendicular to

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the layer, $\boldsymbol{n}$, is proportional to $\boldsymbol{d}$ because the preferred direction of molecules is perpendicular to the layers. The incompressibility of the layers is modeled by the conditions $\nabla \times \boldsymbol{n} = 0$ and $|\boldsymbol{n}| = 1$. Then, in particular $\boldsymbol{d} = \boldsymbol{n}$.

Notice that, since $\nabla \times \boldsymbol{n} = 0$ then $\boldsymbol{n}$ can by written like $\boldsymbol{n} = \nabla\varphi$, where the level sets of the potential function φ represent the layer structure. In this way, a model for smectic-A liquid crystals is presented by E in [5] written in the velocity-pressure variables $(\boldsymbol{u}, p)$ and the layer variable φ. This model has a decreasing in time energy. In particular, a fourth order φ-equation is considered. Existence of global weak solutions of the E's model is deduced by Liu in [8], proving also the global in time regularity in the case of viscosity coefficient large enough. Existence of weak solutions bounded up to infinity time and time-periodic weak solutions for time-dependent boundary Dirichlet data for φ are proved in [4], just as existence and uniqueness of regular solutions (up to infinity time) for both problems (the initial value problem and the time-periodic one), assuming a viscosity coefficient large enough. Finally, in a recent paper [9], Segatti and Wu prove the convergence of each trajectory towards equilibrium states of this E's model.

By the contrary, in this paper we will introduced a double penalized model written in the "primitive" variables $(\boldsymbol{u}, p)$ and $\boldsymbol{n}$, jointly to an auxiliary variable $\boldsymbol{w}$ related to the Euler-Lagrange system for a (double penalized) elastic energy (see (8) below). This new model also has a decreasing in time energy. Afterwards, the issue of the global in time behavior of weak and strong solutions is attacked, including the convergence of the whole trajectory towards a unique equilibrium state, without imposing viscosity coefficient large enough.

More concretely, we study the following PDE system in $\Omega \times (0, +\infty)$:

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \Delta \boldsymbol{u} - \lambda(\nabla \boldsymbol{n})^t \boldsymbol{w} + \nabla q = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$\partial_t \boldsymbol{n} + \boldsymbol{u} \cdot \nabla \boldsymbol{n} - \gamma \Delta \boldsymbol{w} = 0, \tag{3}$$

$$\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) + \boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) - \boldsymbol{w} = 0, \tag{4}$$

where $\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) = \dfrac{1}{\varepsilon_1^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{d}$ and $\mathcal{A}_{\varepsilon_2}(\boldsymbol{n})$ is an elliptic operator such that

$$(\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}), \overline{\boldsymbol{n}}) := a_{\varepsilon_2}(\boldsymbol{n}, \overline{\boldsymbol{n}}) := (\nabla \boldsymbol{n}, \nabla \overline{\boldsymbol{n}}) + \frac{1}{\varepsilon_2^2}(\nabla \times \boldsymbol{n}, \nabla \times \overline{\boldsymbol{n}}) \qquad \forall \boldsymbol{n}, \overline{\boldsymbol{n}} \in \boldsymbol{H}_0^1(\Omega).$$

This system models a Smectic-A liquid crystal confined in an open bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with regular boundary $\partial\Omega$ during the time interval $[0, +\infty)$. Here, $\boldsymbol{u} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^N$ is the flow velocity, $q : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ describes the fluid pressure plus other potential terms, $\boldsymbol{n}$ models the orientation of the crystal molecules and $\boldsymbol{w}$ is a variable related to the equilibrium equation. The constants ρ, ν, λ, and γ are positive, representing respectively, the density and viscosity of the fluid, the ratio between the kinetic energy and the elastic one, and the elastic relaxation time.

The rest of the paper is organized as follows. After some notations, we derive the model in Section 2 and introduce some preliminary results in Section 3. Two time differential inequalities in weak and strong norms are proved in Section 4 and Section 5, respectively, concluding the existence of global in time weak solutions of the system and the existence and uniqueness of strong solutions in large times. In Section 6, the limiting process when the relaxation parameter ε_2 of the curl-free condition goes to zero is studied. Finally, in Section 7 the behavior at infinite time of the double penalized problem is analyzed, proving the convergence of the whole trajectory towards a unique equilibrium state.

1.1 Notations

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(X)$ the Banach space $L^p(0, T; X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$.

- The L^p-norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, the H^m-norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$). The inner product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The boundary $H^s(\partial\Omega)$-norm is denoted by $\|\cdot\|_{s;\partial\Omega}$.

- We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \cdot \boldsymbol{u} = 0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \qquad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

- The notation rotational is used in the following sense:

  For 2D domains (i.e. $N = 2$), $\nabla \times \boldsymbol{n} = \nabla^\perp \cdot \boldsymbol{n} = -\partial_2 n_1 + \partial_1 n_2$ (with $\nabla^\perp = (-\partial_2, \partial_1)^t$) and
  $$\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) = \mathcal{A}^1(\boldsymbol{n}) + \frac{1}{\varepsilon_2^2}\mathcal{A}^2(\boldsymbol{n}) = -\Delta \boldsymbol{n} - \frac{1}{\varepsilon_2^2}\nabla^\perp(\nabla^\perp \cdot \boldsymbol{n})$$
  where
  $$\mathcal{A}^2(\boldsymbol{n}) = -\nabla^\perp(\nabla^\perp \cdot \boldsymbol{n}) = (\partial_2(\nabla^\perp \cdot \boldsymbol{n}), -\partial_1(\nabla^\perp \cdot \boldsymbol{n}))^t.$$

  For 3D domains (i.e. $N = 3$), $\nabla \times \boldsymbol{n} = (\partial_2 n_3 - \partial_3 n_2,\ \partial_3 n_1 - \partial_1 n_3,\ \partial_1 n_2 - \partial_2 n_1)$ and

$\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) = \mathcal{A}^1(\boldsymbol{n}) + \frac{1}{\varepsilon_2^2}\mathcal{A}^2(\boldsymbol{n})$ where $\mathcal{A}^1(\boldsymbol{n}) = \Delta \boldsymbol{n}$ and

$$\begin{aligned}\mathcal{A}^2(\boldsymbol{n}) = (&\partial_2(\partial_1 n_2 - \partial_2 n_1) - \partial_3(\partial_3 n_1 - \partial_1 n_3),\\ &\partial_3(\partial_2 n_3 - \partial_3 n_2) - \partial_1(\partial_1 n_2 - \partial_2 n_1),\\ &\partial_1(\partial_3 n_1 - \partial_1 n_3) - \partial_2(\partial_2 n_3 - \partial_3 n_2))^t.\end{aligned}$$

- We will consider Ω regular enough to have the following equivalent norms:

$$\|\boldsymbol{n}\|_1^2 \approx a_{\varepsilon_2}(\boldsymbol{n},\boldsymbol{n}) + \|\boldsymbol{n}_{\partial\Omega}\|^2_{1/2;\partial\Omega} \tag{5}$$

$$\|\boldsymbol{n}\|_2^2 \approx |\mathcal{A}_{\varepsilon_2}(\boldsymbol{n})|_2^2 + \|\boldsymbol{n}_{\partial\Omega}\|^2_{3/2;\partial\Omega} \tag{6}$$

$$\|\boldsymbol{n}\|_2^2 \approx \|\mathcal{A}_{\varepsilon_2}(\boldsymbol{n})\|_1^2 + \|\boldsymbol{n}_{\partial\Omega}\|^2_{5/2;\partial\Omega} \tag{7}$$

- In the sequel, $C > 0$ will denote different constants, depending only on the fixed data of the problem.

2 Derivation of the Model

It is usual to consider an approximation by Ginzburg-Landau penalization,

$$F_{\varepsilon_1}(\boldsymbol{n}) = \frac{1}{4\varepsilon_1^2}(|\boldsymbol{n}|^2 - 1)^2$$

for the non-convex constraint $|\boldsymbol{n}| = 1$ ($|\boldsymbol{n}| = |\boldsymbol{n}(t,x)|$ denotes the point-wise euclidean norm) [1]. In the equations, the function

$$\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F_{\varepsilon_1}(\boldsymbol{n}) = \frac{1}{\varepsilon_1^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{d} \quad (\varepsilon_1 > 0)$$

appear.

On the other hand, with respect to the constraint $\nabla \times \boldsymbol{n} = 0$, we consider the penalization function

$$G_{\varepsilon_2}(\boldsymbol{n}) = \frac{1}{2\varepsilon_2^2}|\nabla \times \boldsymbol{n}|^2 \quad (\varepsilon_2 > 0).$$

The molecule configuration is determined by minimizing the convex functional (called *Dirichlet energy*), $\frac{1}{2}\int_\Omega |\nabla \boldsymbol{n}|^2$ with the non-convex constraint $|\boldsymbol{n}| = 1$ and the linear constraint $\nabla \times \boldsymbol{n} = 0$. This problem can be replaced by a problem without constraints by minimizing the (double) penalized energy (called *elastic energy*):

$$E_e = \frac{1}{2}\int_\Omega |\nabla \boldsymbol{n}|^2 + \int_\Omega F_{\varepsilon_1}(\boldsymbol{n}) + \int_\Omega G_{\varepsilon_2}(\boldsymbol{n}). \tag{8}$$

The Euler-Lagrange system obtained is $\boldsymbol{w} = 0$ where

$$\boldsymbol{w} \equiv \mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) + \boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}). \tag{9}$$

We consider a system for $\boldsymbol{n}$ of Cahn-Hilliard type:

$$\partial_t \boldsymbol{n} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{n} - \gamma \nabla \boldsymbol{w}) = 0, \tag{10}$$

where the positive constant γ is an elastic relaxation time. From conservation of linear momentum, we have the system

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \nabla \cdot (\sigma^d + \lambda\, \sigma^e) + \frac{\lambda}{\varepsilon_2}(\nabla \boldsymbol{n})^t \mathcal{A}^2_{\varepsilon_2}(\boldsymbol{n}) + \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \tag{11}$$

where the term $\frac{\lambda}{\varepsilon_2}(\nabla \boldsymbol{n})^t \mathcal{A}^2_{\varepsilon_2}(\boldsymbol{n})$ corresponds to the zero rotational constraint and the Cauchy stress tensor has been split, besides the pressure term ∇p, in a dissipative (or viscous) tensor σ^d plus the elastic tensor of Ericsen-Leslie's theory σ^e:

$$\sigma^d = \mu_4 D(\boldsymbol{u}), \qquad\qquad \sigma^e = -\nabla \cdot ((\nabla \boldsymbol{n})^t \nabla \boldsymbol{n}). \tag{12}$$

Here $D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u})$ denotes the symmetric tensor of the velocity gradient. Taking into account that

$$\nabla \cdot ((\nabla \boldsymbol{n})^t \nabla \boldsymbol{n}) = \nabla \left(\frac{|\nabla \boldsymbol{n}|^2}{2} + F_{\varepsilon_1}(\boldsymbol{n}) \right) + (\nabla \boldsymbol{n})^t \left(\Delta \boldsymbol{n} - \boldsymbol{f}(\boldsymbol{n}) \right)$$

and, since $\nabla \cdot \boldsymbol{u} = 0$, $\nabla \cdot D(\boldsymbol{u}) = \frac{\mu_4}{2} \Delta \boldsymbol{u} = \nu \Delta \boldsymbol{u}$ for $\nu = \mu_4/2$, then (9), (10), (11) can be rewritten as (1)-(4) defining the potential function

$$q = p + \lambda \left(\frac{|\nabla \boldsymbol{n}|^2}{2} + F_{\varepsilon_1}(\boldsymbol{n}) \right).$$

The system (1)-(4) is completed with the (Dirichlet) boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}, \quad \boldsymbol{w}|_{\partial\Omega} = 0 \tag{13}$$

the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{n}(0) = \boldsymbol{n}_0 \quad \text{in } \Omega \tag{14}$$

Without loss of generality, we fix the constants $\nu = \rho = \lambda = \gamma = 1$.

3 Some preliminary results and definitions

The two following results are proved in [3]

Lemma 1 *Let $E, \Phi \in L^1_{loc}(0,+\infty)$ be two functions in $\mathbb{R}$ satisfying a.e. $t \in (0,+\infty)$:*

$$E(t),\ \Phi(t) \geq 0, \quad E'(t) + \Phi(t) \leq 0.$$

Then, $E \in C_b[0,+\infty)$, is a decreasing function and

$$\exists \lim_{t\to+\infty} E(t) = E_\infty \geq 0.$$

Moreover, $\Phi \in L^1(0,+\infty)$

Lemma 2 *Let $\Phi \in L^1(0,+\infty)$ be a function satisfying $\Phi'(t) \leq C(\Phi(t)^3 + 1)$. Then, $\Phi(t)$ is a function asymptotically stable to 0, that is, $\lim_{t\to+\infty} \Phi(t) = 0$. In particular, there exists $t^* \geq 0$ such that $\Phi \in C_b[t^*,+\infty)$, that is it is a continuous and bounded function.*

One can prove the following Lojasiewicz-Simon inequality modifying slightly the proof of Lemma 6.3.4 in [11]. In fact, in [11], the homogeneous condition $\boldsymbol{n}|_{\partial\Omega} = 0$ is assumed. See [10] for a non homogeneus Dirichlet boundary condition.

Lemma 3 *Let $\mathcal{E}$ be the following set of equilibrium points:*

$$\mathcal{E} = \{\boldsymbol{n} : \mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) + \boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) = 0, \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}\}$$

and $\overline{\boldsymbol{n}} \in \mathcal{E}$. Then there are two positive constants β and $\theta \in (0,1/2)$ depending on $\overline{\boldsymbol{n}}$ such that for all $\boldsymbol{n} \in \boldsymbol{H}^2$ with $\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}$ and $\|\boldsymbol{n} - \overline{\boldsymbol{n}}\|_2 \leq \beta$, it holds

$$|E_e(\boldsymbol{n}) - E_e(\overline{\boldsymbol{n}})|^{1-\theta} \leq C|\boldsymbol{w}|_2$$

Definition 4 *We say that $(\boldsymbol{u}, \boldsymbol{n}, \boldsymbol{w})$ is a weak solution of (1)-(4), (13), (14) in $[0,T]$ if*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}), \qquad \boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1), \qquad \boldsymbol{n} \in L^\infty(0,T;\boldsymbol{H}^1),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \boldsymbol{n}(0) = \boldsymbol{n}_0 \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}, \quad \boldsymbol{w}|_{\partial\Omega} = 0$$

and

$$(\partial_t \boldsymbol{u}, \bar{\boldsymbol{u}}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \bar{\boldsymbol{u}}) + (\nabla\boldsymbol{u}, \nabla\bar{\boldsymbol{u}}) - ((\nabla\boldsymbol{n})^t\boldsymbol{w}, \bar{\boldsymbol{u}}) = 0 \quad \forall \bar{\boldsymbol{u}} \in \boldsymbol{V}, \tag{15}$$

$$(\partial_t \boldsymbol{n}, \bar{\boldsymbol{w}}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{n}, \bar{\boldsymbol{w}}) + (\nabla\boldsymbol{w}, \nabla\bar{\boldsymbol{w}}) = 0 \quad \forall \bar{\boldsymbol{w}} \in \boldsymbol{H}^1_0, \tag{16}$$

$$a_{\varepsilon_2}(\boldsymbol{n}, \bar{\boldsymbol{n}}) + (f_{\varepsilon_1}(\boldsymbol{n}), \bar{\boldsymbol{n}}) - (\boldsymbol{w}, \bar{\boldsymbol{n}}) = 0 \quad \forall \bar{\boldsymbol{n}} \in \boldsymbol{H}^1_0. \tag{17}$$

Moreover, from the regularity of $\boldsymbol{w}$ and (7), we can obtain $\boldsymbol{n} \in \boldsymbol{L}^2(\boldsymbol{H}^3)$ whenever $\boldsymbol{n}_{\partial\Omega} \in \boldsymbol{H}^{5/2}(\partial\Omega)$.

Definition 5 *We say that a weak solution* $(\boldsymbol{u}, \boldsymbol{n}, \boldsymbol{w})$ *of (1)-(4), (13), (14) in* $[0,T]$ *is a strong solution if*

$$\boldsymbol{u} \in L^\infty(\boldsymbol{H}^1) \cap L^2(\boldsymbol{H}^2), \qquad \boldsymbol{w} \in L^\infty(\boldsymbol{H}^1), \qquad \partial_t \boldsymbol{n} \in L^2(\boldsymbol{H}^1)$$

and the fully differential system (1)-(4) is verified a.e. in $[0,T] \times \Omega$.

Moreover, from the regularity of $\boldsymbol{w}$ and (7), we can obtain $\boldsymbol{n} \in L^\infty(\boldsymbol{H}^3)$ whenever $\boldsymbol{n}_{\partial\Omega} \in \boldsymbol{H}^{5/2}(\partial\Omega)$.

4 Energy Equality and Weak Estimates

If $(\boldsymbol{u}, \boldsymbol{n}, \boldsymbol{w})$ is a regular enough solution of (1)-(4), (13), (14), we can carry out the following argument. By taking $\bar{\boldsymbol{u}} = \boldsymbol{u}$, $\bar{\boldsymbol{w}} = \boldsymbol{w}$ and $\bar{\boldsymbol{n}} = \partial_t \boldsymbol{n}$ as test function in (15), (16) and (17) respectively (observe that $\partial_t \boldsymbol{n} \in \boldsymbol{H}_0^1$ because $\boldsymbol{u}_{\partial\Omega}$ does not depend on time), one has

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}|\boldsymbol{u}|_2^2 + |\nabla \boldsymbol{u}|_2^2 - ((\nabla \boldsymbol{n})^t \boldsymbol{w}, \boldsymbol{u}) = 0, \\ (\partial_t \boldsymbol{n}, \boldsymbol{w}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{n}, \boldsymbol{w}) + |\nabla \boldsymbol{w}|_2^2 = 0, \\ \dfrac{d}{dt}\left(\dfrac{1}{2} a_{\varepsilon_2}(\boldsymbol{n}, \boldsymbol{n}) + \displaystyle\int_\Omega F_{\varepsilon_1}(\boldsymbol{n})\right) - (\boldsymbol{w}, \partial_t \boldsymbol{n}) = 0. \end{cases}$$

Adding and canceling the nonlinear convective term $(\boldsymbol{u}\cdot\nabla \boldsymbol{n}, \boldsymbol{w})$ with the elastic term $-((\nabla \boldsymbol{n})^t \boldsymbol{w}, \boldsymbol{u})$, one arrives at the following *energy equality*:

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}a_{\varepsilon_2}(\boldsymbol{n}, \boldsymbol{n}) + F_{\varepsilon_1}(\boldsymbol{n})\right) + \int_\Omega \left(|\nabla \boldsymbol{u}|^2 + |\nabla \boldsymbol{w}|^2\right) = 0, \tag{18}$$

which shows the dissipative character of the model. Moreover, assuming the initial estimates $|\boldsymbol{u}_0|_2^2 \le C$ and $\|\boldsymbol{n}_0\|_1^2 \le C$ and taking into account (5) and that $\boldsymbol{w}|_{\partial\Omega} = 0$, one has the following weak estimates (which are uniform bounds in the infinite time interval $[0, +\infty)$):

$$\boldsymbol{u} \text{ in } L^\infty(0,+\infty; \boldsymbol{H}) \cap L^2(0,+\infty; \boldsymbol{V}), \quad \boldsymbol{w} \text{ in } L^2(0,+\infty; \boldsymbol{H}^1), \quad \boldsymbol{n} \text{ in } L^\infty(0,+\infty; \boldsymbol{H}^1). \tag{19}$$

Moreover, from the bound of $\boldsymbol{w}$ in $L^2(\boldsymbol{H}^1)$ and (7), one has

$$\boldsymbol{n} \text{ is uniformly bounded in } L^2(0,T; \boldsymbol{H}^3),\ \forall T > 0. \tag{20}$$

5 Strong Estimates

By taking $A\boldsymbol{u}$ as test function in the $\boldsymbol{u}$-system (1) (A being the Stokes operator), applying Hölder and Young's inequalities and the weak estimate $\|\boldsymbol{n}(t)\|_1 \le C$, one obtains:

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}|\nabla\boldsymbol{u}|_2^2 + |A\boldsymbol{u}|_2^2 \le C\left(|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|_2 + |(\nabla\boldsymbol{n})^t\boldsymbol{w}|_2\right)\|\boldsymbol{u}\|_2 \\ \le C\left(|\boldsymbol{u}|_6|\nabla\boldsymbol{u}|_3 + |\nabla\boldsymbol{n}|_3|\boldsymbol{w}|_6\right)\|\boldsymbol{u}\|_2 \le C\left(\|\boldsymbol{u}\|_1^{3/2}\|\boldsymbol{u}\|_2^{3/2} + \|\boldsymbol{n}\|_2^{1/2}|\nabla\boldsymbol{w}|_2\|\boldsymbol{u}\|_2\right) \\ \le \frac{1}{2}|A\boldsymbol{u}|_2^2 + C\left(|\nabla\boldsymbol{u}|_2^6 + \|\boldsymbol{n}\|_2|\nabla\boldsymbol{w}|_2^2\right)\end{gathered}$$

Then,

$$\frac{d}{dt}|\nabla\boldsymbol{u}|_2^2 + |A\boldsymbol{u}|_2^2 \le C\left(|\nabla\boldsymbol{u}|_2^6 + \|\boldsymbol{n}\|_2|\nabla\boldsymbol{w}|_2^2\right). \tag{21}$$

By taking $\partial_t\boldsymbol{w}$ as test function in the $\boldsymbol{w}$-system of (3), deriving the $\boldsymbol{n}$-system (4) respect to t, taking $\partial_t\boldsymbol{n}$ as test function, adding both equalities and canceling the term $(\partial_t\boldsymbol{n}, \partial_t\boldsymbol{w})$ one has:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla\boldsymbol{w}|_2^2 + a_{\varepsilon_2}(\partial_t\boldsymbol{n}, \partial_t\boldsymbol{n}) = -(\boldsymbol{u}\cdot\nabla\boldsymbol{n}, \partial_t\boldsymbol{w}) - (\partial_t\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}), \partial_t\boldsymbol{n}) \\ &\le \|\boldsymbol{u}\cdot\nabla\boldsymbol{n}\|_1\|\partial_t\boldsymbol{w}\|_{-1} + |\nabla_n\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n})|_3|\partial_t\boldsymbol{n}|_2\|\partial_t\boldsymbol{n}\|_1.\end{aligned} \tag{22}$$

By making the t-derivative of the $\boldsymbol{n}$-system (4):

$$\|\partial_t\boldsymbol{w}\|_{-1} \le C(\|\partial_t\boldsymbol{n}\|_1 + |\nabla_{\boldsymbol{n}}\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n})|_3|\partial_t\boldsymbol{n}|_2),$$

and, using the weak estimate $\|\boldsymbol{n}(t)\|_1 \le C$:

$$|\nabla_n\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n})|_3 \le C(1 + |\boldsymbol{n}|_6^2) \le C.$$

Therefore, from (22) we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla\boldsymbol{w}|_2^2 + a_{\varepsilon_2}(\partial_t\boldsymbol{n}, \partial_t\boldsymbol{n}) \le \varepsilon\|\partial_t\boldsymbol{n}\|_1^2 + C(\|\boldsymbol{u}\cdot\nabla\boldsymbol{n}\|_1^2 + |\partial_t\boldsymbol{n}|_2^2). \tag{23}$$

The second term on the right hand side of (23) can be bounded as

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{n}\|_1^2 \le C\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2\|\boldsymbol{n}\|_2^2 \le \delta\|\boldsymbol{u}\|_2^2 + C\|\boldsymbol{u}\|_1^2\|\boldsymbol{n}\|_2^4$$

(for $\delta > 0$) and the third one as

$$|\partial_t\boldsymbol{n}|_2^2 \le \|\partial_t\boldsymbol{n}\|_{-1}\|\partial_t\boldsymbol{n}\|_1 \le \varepsilon\|\partial_t\boldsymbol{n}\|_1^2 + C(\|\boldsymbol{u}\cdot\nabla\boldsymbol{n}\|_{-1}^2 + |\nabla\boldsymbol{w}|_2^2).$$

Therefore, from the inequality $K\|\partial_t\boldsymbol{n}\|_1^2 \le a_{\varepsilon_2}(\partial_t\boldsymbol{n}, \partial_t\boldsymbol{n})$ and taking $\varepsilon \le K/4$,

$$\frac{d}{dt}|\nabla\boldsymbol{w}|_2^2 + K\|\partial_t\boldsymbol{n}\|_1^2 \le \delta\|\boldsymbol{u}\|_2^2 + C\left(\|\boldsymbol{u}\|_1^2\|\boldsymbol{n}\|_2^4 + |\nabla\boldsymbol{w}|_2^2\right). \tag{24}$$

By using the $\boldsymbol{n}$-system (4),

$$\|\boldsymbol{n}\|_2 \le |\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n})|_2 + |\boldsymbol{w}|_2 + \|\boldsymbol{n}_{\partial\Omega}\|_{3/2} \le C(1 + |\boldsymbol{w}|_2).$$

Then, from (21) and (24) we obtain (taking δ small enough):

$$\begin{aligned} &\frac{d}{dt}(|\nabla \boldsymbol{u}|_2^2 + |\nabla \boldsymbol{w}|_2^2) + \frac{1}{2}|A\boldsymbol{u}|_2^2 + K\|\partial_t \boldsymbol{n}\|_1^2 \\ &\le C\Big(|\nabla \boldsymbol{u}|_2^6 + |\nabla \boldsymbol{w}|_2^2(1 + |\boldsymbol{w}|_2) + \|\boldsymbol{u}\|_1^2(1 + |\boldsymbol{w}|_2^4)\Big) \\ &\le C\Big(1 + (|\nabla \boldsymbol{u}|_2^2 + |\nabla \boldsymbol{w}|_2^2)^3\Big). \end{aligned} \tag{25}$$

Fixed the initial datum $(\boldsymbol{u}_0, \boldsymbol{n}_0) \in \boldsymbol{H} \times \boldsymbol{H}^1$ and assuming boundary data $\boldsymbol{n}_{\partial\Omega} \in H^{3/2}(\partial\Omega)$, by using a Galerkin Method and proceeding in analogous way to [7], [2], one can prove existence of weak solutions of (1)-(4),(13),(14) in $(0, +\infty)$, and existence (and uniqueness) of strong solution of (1)-(4),(13),(14) in $(t^*, +\infty)$ for a big enough time $t^* \ge 0$. This last result is based in a small initial data argument associated to (25). Since $\nabla \boldsymbol{u}, \nabla \boldsymbol{w} \in L^2(0, +\infty; \boldsymbol{L}^2(\Omega))$, in particular, there exists a big enough time $t*$ such that $|\nabla \boldsymbol{u}(t^*)|_2$ and $|\nabla \boldsymbol{w}(t^*)|_2$ are small enough.

6 The limit model as ε_2 goes to zero

In this part, $C > 0$ denote a generic constant independent of ε_2. For each $\varepsilon_2 > 0$, let us consider a weak solution $(u_{\varepsilon_2}, n_{\varepsilon_2}, \boldsymbol{w}_{\varepsilon_2})$ of the ε_2-approximate problem(1)-(4), (13), (14). The goal of this section is to take limits as ε_2 goes to zero.

Let $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{n}_0 \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{n}_0|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}$. We can repeat Section 4 to obtain the energy inequality,

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\boldsymbol{u}_{\varepsilon_2}|^2 + \frac{1}{2}a_{\varepsilon_2}(\boldsymbol{n}_{\varepsilon_2}, \boldsymbol{n}_{\varepsilon_2}) + F_{\varepsilon_1}(\boldsymbol{n}_{\varepsilon_2})\right) + \int_\Omega \left(|\nabla \boldsymbol{u}_{\varepsilon_2}|^2 + |\nabla \boldsymbol{w}_{\varepsilon_2}|^2\right) \le 0.$$

If we suppose that

$$|\nabla \times \boldsymbol{n}_0|_2^2 \le C\,\varepsilon_2^2,$$

in particular,

$$\frac{1}{2}|\boldsymbol{u}_0|^2 + \frac{1}{2}a_{\varepsilon_2}(\boldsymbol{n}_0, \boldsymbol{n}_0) + F_{\varepsilon_1}(\boldsymbol{n}_0) \le C$$

and then the following bounds (independent of ε_2) hold:

$$\begin{aligned} &a_{\varepsilon_2}(\boldsymbol{n}_{\varepsilon_2}, \boldsymbol{n}_{\varepsilon_2}) && \text{is bounded in } L^\infty(0, +\infty), \\ &\boldsymbol{u}_{\varepsilon_2} && \text{is bounded in } L^\infty(0, +\infty; \boldsymbol{H}) \cap L^2(0, +\infty; \boldsymbol{V}), \\ &\boldsymbol{w}_{\varepsilon_2} && \text{is bounded in } L^2(0, +\infty; \boldsymbol{H}^1). \end{aligned}$$

From the bound of $a_{\varepsilon_2}(\cdot,\cdot)$ and (5),

$$\boldsymbol{n}_{\varepsilon_2} \text{ is bounded in } L^\infty(0,T;\boldsymbol{H}^1), \quad \forall T>0$$

whenever $\boldsymbol{n}_{\partial\Omega} \in \boldsymbol{H}^{1/2}(\partial\Omega)$. In particular,

$$\frac{1}{\varepsilon_2^2}|\nabla\times\boldsymbol{n}_{\varepsilon_2}|_2^2 \text{ is bounded in } L^\infty(0,+\infty). \tag{26}$$

Moreover, since $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is bounded in $L^{4/3}(0,+\infty;\boldsymbol{H}^{-1})$ and $(\nabla\boldsymbol{n})^t\boldsymbol{w}$ is bounded in $L^\infty(0,T;\boldsymbol{L}^{3/2})$, from the $\boldsymbol{u}$-system (1),

$$\partial_t\boldsymbol{u}_{\varepsilon_2} \text{ is bounded in } L^{4/3}(0,T;\boldsymbol{V}'),$$

and from the $\boldsymbol{w}$-system (3),

$$\partial_t\boldsymbol{n}_{\varepsilon_2} \text{ is bounded in } L^2(0,T;\boldsymbol{H}^{-1}).$$

Consequently, there exists subsequences (for simplicity, equally denoted) and limit functions $\boldsymbol{u}$, $\boldsymbol{n}$, $\boldsymbol{w}$ such that

$$\begin{aligned}
&\boldsymbol{u}_{\varepsilon_2}\to\boldsymbol{u} \quad \text{weakly in } L^2(0,+\infty;\boldsymbol{V}), \text{ strongly in } L^2(0,T;\boldsymbol{H}) \text{ and a.e. in } (0,T)\times\Omega,\\
&\boldsymbol{n}_{\varepsilon_2}\to\boldsymbol{n} \quad \text{weakly-}\star \text{ in } L^\infty(0,T;\boldsymbol{H}^1), \quad \text{strongly in } C(0,T;\boldsymbol{L}^2) \text{ and a.e. in } (0,T)\times\Omega,\\
&\boldsymbol{w}_{\varepsilon_2}\to\boldsymbol{w} \quad \text{weakly in } L^2(0,+\infty;\boldsymbol{H}^1),\\
&\partial_t\boldsymbol{u}_{\varepsilon_2}\to\partial_t\boldsymbol{u} \quad \text{weakly in } L^{4/3}(0,T;\boldsymbol{H}^{-1}),\\
&\partial_t\boldsymbol{n}_{\varepsilon_2}\to\partial_t\boldsymbol{n} \quad \text{weakly in } L^2(0,T;\boldsymbol{H}^{-1}),
\end{aligned}$$

This allows to pass to the limit when ε_2 goes to zero, in each term of ε_2-approximate problem. First of all, from (26) the limit vector $\boldsymbol{n}$ verifies the constraint $\nabla\times\boldsymbol{n}=0$.

Moreover, observe that for test functions $\bar{\boldsymbol{u}}\in\boldsymbol{V}$, $\bar{\boldsymbol{n}}\in\boldsymbol{H}_0^1$ such that $\nabla\times\bar{\boldsymbol{n}}=0$, and for any $T>0$, we have that

$$\int_0^T\langle(\nabla\boldsymbol{n}_{\varepsilon_2})^t\boldsymbol{w}_{\varepsilon_2},\bar{\boldsymbol{u}}\rangle = -\int_0^T\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{w}_{\varepsilon_2}\cdot\boldsymbol{n}_{\varepsilon_2} \longrightarrow -\int_0^T\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{n}$$

and

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \nabla\boldsymbol{n}_{\varepsilon_2}:\nabla\bar{\boldsymbol{n}} + \frac{1}{\varepsilon_2^2}(\nabla\times\boldsymbol{n}_{\varepsilon_2})\cdot(\nabla\times\bar{\boldsymbol{n}}) + f_{\varepsilon_1}(\boldsymbol{n}_{\varepsilon_2})\cdot\bar{\boldsymbol{n}} - \boldsymbol{w}_{\varepsilon_2}\cdot\bar{\boldsymbol{n}}\\
&\longrightarrow \int_0^T\int_\Omega \nabla\boldsymbol{n}:\nabla\bar{\boldsymbol{n}} + f_{\varepsilon_1}(\boldsymbol{n})\cdot\bar{\boldsymbol{n}} - \boldsymbol{w}\cdot\bar{\boldsymbol{n}} = 0.
\end{aligned}$$

Therefore, taking advantage of the De Rham results, we arrive at the following limit problem:

$$\begin{aligned}
\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} - (\nabla\boldsymbol{n})^t\boldsymbol{w} + \nabla q &= 0,\\
\nabla\cdot\boldsymbol{u} &= 0,\\
\partial_t\boldsymbol{n} + \boldsymbol{u}\cdot\nabla\boldsymbol{n} - \Delta\boldsymbol{w} &= 0,\\
-\Delta\boldsymbol{n} + \boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) - \boldsymbol{w} + \nabla^\perp\psi &= 0,\\
\nabla\times\boldsymbol{n} &= 0,
\end{aligned}$$

where q and ψ are the Lagrange multipliers corresponding to the constraints $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \times \boldsymbol{n} = 0$.

7 Behavior at infinite time

Now, we study the large time behavior of the ε_2-problem, (1)-(4),(13),(14), for each ε_2 fixed.

We define respectively the kinetic, elastic and total energy as:

$$E_k(\boldsymbol{u}(t)) = \frac{1}{2}\int_\Omega |\boldsymbol{u}(t)|^2, \qquad E_e(\boldsymbol{n}(t)) = \int_\Omega \left(\frac{1}{2}a_{\varepsilon_2}(\boldsymbol{n}(t),\boldsymbol{n}(t)) + F_{\varepsilon_1}(\boldsymbol{n}(t))\right),$$

$$E(\boldsymbol{u}(t),\boldsymbol{n}(t)) = E_k(\boldsymbol{u}(t)) + E_e(\boldsymbol{n}(t))$$

Theorem 6 *Assume that $(\boldsymbol{u}_0,\boldsymbol{n}_0) \in \boldsymbol{H} \times \boldsymbol{H}^1$. Fixed $(\boldsymbol{u}(t),\boldsymbol{n}(t),\boldsymbol{w}(t))$ a weak solution of (1)-(4),(13),(14) in $(0,+\infty)$ which is a strong solution in $(t^*,+\infty)$ for some $t^* > 0$, then there exists a number $E_\infty \geq 0$ such that the total energy satisfies*

$$E(\boldsymbol{u}(t),\boldsymbol{n}(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{as } t \uparrow +\infty. \tag{27}$$

Moreover

$$\boldsymbol{u}(t) \to 0 \text{ in } \boldsymbol{H}_0^1 \quad \text{and} \quad \boldsymbol{w}(t) \to 0 \text{ in } \boldsymbol{H}^1 \quad \text{as } t \uparrow +\infty. \tag{28}$$

Proof. From energy equality (18) and Lemma 1 we obtain (27). If we denote

$$\Phi(t) = \|\boldsymbol{u}\|_1^2 + |\nabla \boldsymbol{w}|_2^2, \qquad \Psi(t) = \|\boldsymbol{u}\|_2^2 + \|\partial_t \boldsymbol{n}\|_1^2,$$

from (25), we obtain

$$\Phi' + C_1\Psi \leq C_2(\Phi^3 + 1).$$

By applying Lemma 2 we have that $\lim_{t\to+\infty} \Phi(t) = 0$, that is, (28). ∎

Let S be the set of equilibrium points of (1)-(4):

$$S = \{(0,\overline{\boldsymbol{n}}) : \mathcal{A}_{\varepsilon_2}(\overline{\boldsymbol{n}}) + \boldsymbol{f}_{\varepsilon_1}(\overline{\boldsymbol{n}}) = 0, \overline{\boldsymbol{n}}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}\}.$$

On the other hand, the ω-limit set of $(\boldsymbol{u}_0,\boldsymbol{n}_0) \in \boldsymbol{V} \times \boldsymbol{H}^2$ is defined as follows:

$$\omega(\boldsymbol{u}_0,\boldsymbol{n}_0) = \{(\boldsymbol{u}_\infty,\boldsymbol{n}_\infty) \in \boldsymbol{V} \times \boldsymbol{H}^3 : \exists\{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n),\boldsymbol{n}(t_n)) \to (\boldsymbol{u}_\infty,\boldsymbol{n}_\infty) \text{ in } \boldsymbol{H}^1 \times \boldsymbol{H}^3\}.$$

Theorem 7 *Under hypothesis of Theorem 6, $\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)$ is a nonempty bounded subset of $\boldsymbol{V} \times \boldsymbol{H}^3$ and $\omega(\boldsymbol{u}_0,\boldsymbol{n}_0) \subset S$. Moreover, for any $(0,\overline{\boldsymbol{n}}) \in S$ such that $(0,\overline{\boldsymbol{n}}) \in \omega(\boldsymbol{u}_0,\boldsymbol{n}_0)$, it holds $E_e(\overline{\boldsymbol{n}}) = E_\infty$.*

Proof.

Step 1: We will see that $\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)\neq\emptyset$ and $\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)\subset S$.

From weak estimates, $(\boldsymbol{u},\boldsymbol{n})\in L^\infty(0,+\infty;\boldsymbol{H}\times\boldsymbol{H}^1)$, hence there exists $\{t_n\}\uparrow+\infty$ and $(\boldsymbol{u}_\infty,\boldsymbol{n}_\infty)$ such that $(\boldsymbol{u}(t_n),\boldsymbol{n}(t_n))\to(\boldsymbol{u}_\infty,\boldsymbol{n}_\infty)$ weakly in $\boldsymbol{H}\times\boldsymbol{H}^1$. From (28), $\boldsymbol{u}_\infty=0$ and $\boldsymbol{u}(t_n)\to 0$ in $\boldsymbol{H}^1_0$. On the other hand, $\boldsymbol{n}_\infty$ is a weak solution of the equilibrium equation. Indeed, $\boldsymbol{n}(t_n)\to\boldsymbol{n}_\infty$ a.e. in Ω and strongly in L^p for all $p<6$, therefore $\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))\to\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)$ a.e. in Ω and $|\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))|_2\le C\|\boldsymbol{n}(t_n)\|_1\le C$, hence $\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))\to\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)$ strongly in $\boldsymbol{L}^2$. By taking into account that $\boldsymbol{n}(t_n)\to\boldsymbol{n}_\infty$ weakly in $\boldsymbol{H}^1$ and $\boldsymbol{w}(t)\to 0$ in $\boldsymbol{H}^1$, it suffices take limits in (17) as $\{t_n\}\uparrow+\infty$ to obtain that $\boldsymbol{n}_\infty$ verifies the equilibrium equation.

Now, we are going to prove the convergence $\boldsymbol{n}(t_n)\to\boldsymbol{n}_\infty$ in $\boldsymbol{H}^2$. Indeed, by using $\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}_\infty)-\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)=0$, one has

$$\begin{aligned}|\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}(t_n))-\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}_\infty)|_2 &\le |\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}(t_n))-\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))+|\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))-\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)|_2\\ &=|\boldsymbol{w}(t_n)|_2+|\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))-\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)|_2\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Therefore, from the H^2-continuous dependence of the elliptic problem associated to $\mathcal{A}_{\varepsilon_2}$,

$$\|\boldsymbol{n}(t_n)-\boldsymbol{n}_\infty\|_2\le C|\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}(t_n))-\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}_\infty)|_2\to 0.$$

Now, by using the convergences of $\boldsymbol{n}(t_n)\to\boldsymbol{n}_\infty$ in $\boldsymbol{H}^2$ and $\boldsymbol{w}(t)\to 0$ in $\boldsymbol{H}^1$ in the inequality

$$\|\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}(t_n))-\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}_\infty)\|_1\le\|\boldsymbol{w}(t_n)\|_1+\|\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}(t_n))-\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}_\infty)\|_1,$$

and the H^3-continuous dependence of the elliptic problem associated to $\mathcal{A}_{\varepsilon_2}$, we obtain the convergence $\boldsymbol{n}(t_n)\to\boldsymbol{n}_\infty$ in $\boldsymbol{H}^3$.

Step 2: If $(0,\overline{\boldsymbol{n}})\in\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)$ then $E_e(\overline{\boldsymbol{n}})=E_\infty$

From Step 1, there exists $\{t_n\}\uparrow+\infty$ such that $(\boldsymbol{u}(t_n),\boldsymbol{n}(t_n))\to(0,\overline{\boldsymbol{n}})$ in $\boldsymbol{H}^1\times\boldsymbol{H}^3$ as $n\uparrow+\infty$. Then, from (27) we obtain that

$$E_\infty=\lim_{n\to+\infty}E(\boldsymbol{u}(t_n),\boldsymbol{n}(t_n))=E_e(\overline{\boldsymbol{n}}).$$

■

Although the set of critical points $\overline{\boldsymbol{n}}$ (with the same elastic energy) might be a continuum, we are going to prove the uniqueness of limit of the whole trajectory of $\boldsymbol{n}(t)$.

Theorem 8 *Under conditions of Theorem 7, $\boldsymbol{n}(t)\to\overline{\boldsymbol{n}}$ in $\boldsymbol{H}^3$ as $t\uparrow+\infty$. In particular, $\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)=\{(0,\overline{\boldsymbol{n}})\}$.*

Proof. Let $(0,\overline{\boldsymbol{n}})\in\omega(\boldsymbol{u}_0,\boldsymbol{n}_0)\subset S$. In particular, there exists $t_n\uparrow+\infty$ such that $\boldsymbol{u}(t_n)\to 0$ in $\boldsymbol{H}^1$ and $\boldsymbol{n}(t_n)\to\overline{\boldsymbol{n}}$ in $\boldsymbol{H}^3$.

Step 1: Let us suppose there exists $t_\star > 0$ such that

$$\|\boldsymbol{n}(t) - \overline{\boldsymbol{n}}\|_2 \le \beta \quad \text{and} \quad |\boldsymbol{u}(t)|_2 \le 1 \quad \forall\, t \ge t_\star$$

($\beta > 0$ being the constant appearing in Lemma 3), then the following inequalities hold:

$$\frac{d}{dt}\Big((E(\boldsymbol{u}(t), \boldsymbol{n}(t)) - E_e(\overline{\boldsymbol{n}}))^\theta\Big) + C\,\theta\,(|\nabla \boldsymbol{u}(t)|_2 + |\nabla \boldsymbol{w}(t)|_2) \le 0, \quad \forall\, t \ge t_\star \tag{29}$$

$$\int_{t_0}^{t_1} \|\partial_t \boldsymbol{n}\|_{-1} \le \frac{C}{\theta}(E(\boldsymbol{u}(t_0), \boldsymbol{n}(t_0)) - E_e(\overline{\boldsymbol{n}})))^\theta, \qquad \forall\, t_0, t_1 \ge t_\star, \tag{30}$$

where $\theta \in (0, 1/2]$ is the constant appearing in Lemma 3 (of Lojasiewicz-Simon). Indeed, the energy equality (18) is written as

$$\frac{dE}{dt} = -C\left(|\nabla \boldsymbol{u}|_2^2 + |\nabla \boldsymbol{w}|_2^2\right).$$

Then, by taking the time derivative of the function

$$H(t) := (E(\boldsymbol{u}(t), \boldsymbol{n}(t)) - E_\infty)^\theta,$$

we have

$$-\frac{dH(t)}{dt} = \theta(E(\boldsymbol{u}(t), \boldsymbol{n}(t)) - E_\infty)^{\theta-1} C\big(|\nabla \boldsymbol{u}(t)|_2^2 + |\nabla \boldsymbol{w}(t)|_2^2\big). \tag{31}$$

On the other hand, recalling that the unique critical point of the kinetic energy is $\boldsymbol{u} = 0$, taking into account that $|E_k(\boldsymbol{u}) - E_k(0)| = \dfrac{1}{2}|\boldsymbol{u}|_2^2$ and since $2(1-\theta) > 1$ and $|\boldsymbol{u}(t)|_2 \le 1$, then

$$|E_k(\boldsymbol{u}(t)) - E_k(0)|^{1-\theta} = C|\boldsymbol{u}(t)|_2^{2(1-\theta)} \le C|\boldsymbol{u}(t)|_2 \quad \forall\, t \ge t_\star$$

Therefore, by using the Lojasiewicz-Simon inequality (given in Lemma 3), we have,

$$|E(\boldsymbol{u}(t), \boldsymbol{n}(t)) - E_\infty)|^{1-\theta} \le |E_k(\boldsymbol{u}(t)) - E_k(0)|^{1-\theta} + |E_e(\boldsymbol{n}(t)) - E_e(\overline{\boldsymbol{n}})|^{1-\theta} \le C(|\boldsymbol{u}(t)|_2 + |\boldsymbol{w}(t)|_2).$$

Hence, by using the Poincare inequality in (31) we obtain

$$-\frac{dH(t)}{dt} \ge C\theta(|\boldsymbol{u}(t)|_2 + |\boldsymbol{w}(t)|_2)^{-1}(|\nabla \boldsymbol{u}(t)|_2^2 + |\nabla \boldsymbol{w}(t)|_2^2) \ge C\theta(|\nabla \boldsymbol{u}(t)|_2 + |\nabla \boldsymbol{w}(t)|_2) \quad \forall\, t \ge t_\star$$

and (29) is proved. Integrating (29) in $[t_0, t_1]$ (for any $t_0, t_1 \ge t_\star$) one gets

$$C(E(\boldsymbol{u}(t_1), \boldsymbol{n}(t_1)) - E_\infty)^\theta + \theta \int_{t_0}^{t_1} (|\nabla \boldsymbol{u}|_2 + |\nabla \boldsymbol{w}|_2) \le C(E(\boldsymbol{u}(t_0), \boldsymbol{n}(t_0)) - E_\infty)^\theta. \tag{32}$$

On the other hand, since $\partial_t \boldsymbol{n} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{n} - \nabla \boldsymbol{w}) = 0$, by using the weak estimate $\|\boldsymbol{n}(t)\|_1 \le C$, in particular

$$\|\partial_t \boldsymbol{n}\|_{-1} \le C(|\boldsymbol{u} \otimes \boldsymbol{n}|_2 + |\nabla \boldsymbol{w}|_2) \le C(|\nabla \boldsymbol{u}|_2 + |\nabla \boldsymbol{w}|_2)$$

Integrating in $[t_0, t_1]$ and applying (32), we obtain (30).

Step 2: There exists n_0 big enough such that $\|\boldsymbol{n}(t)-\overline{\boldsymbol{n}}\|_2 \leq \beta$ and $|\boldsymbol{u}(t)|_2 \leq 1$ for all $t \geq t_{n_0}$

The second bound becomes from $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}_0^1$ given in (28). Then, we will see the first bound for $\boldsymbol{n}(t)$. Since $\boldsymbol{n}(t_n) \to \overline{\boldsymbol{n}}$ in $\boldsymbol{H}^3$ and $E(\boldsymbol{u}(t_n), \boldsymbol{n}(t_n)) \to E_\infty = E_e(\overline{\boldsymbol{n}})$, then for any $\varepsilon \in (0, \beta)$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$:

$$\|\boldsymbol{n}(t_n)-\overline{\boldsymbol{n}}\|_2 \leq \varepsilon \quad \text{and} \quad \frac{C}{\theta}(E_e(\boldsymbol{u}(t_n), \boldsymbol{n}(t_n)) - E_\infty)^\theta \leq \varepsilon \tag{33}$$

For each $n \geq N(\varepsilon)$, we define

$$\bar{t}_n = \sup\{t : t > t_n,\ \|\boldsymbol{n}(s)-\overline{\boldsymbol{n}}\|_2 < \beta \quad \forall s \in [t_n, t)\}.$$

It suffices to prove that $\bar{t}_n = +\infty$. Assume by contradiction that $t_n < \bar{t}_n < +\infty$. Observe that $\|\boldsymbol{n}(\bar{t}_n)-\overline{\boldsymbol{n}}\|_2 = \beta$ and $\|\boldsymbol{n}(t)-\overline{\boldsymbol{n}}\|_2 < \beta$ for all $t \in [t_n, \bar{t}_n)$. By step 1, for all $t \in [t_n, \bar{t}_n]$, from (30) and (33) we obtain that

$$\int_{t_n}^{\bar{t}_n} \|\partial_t \boldsymbol{n}\|_{-1} \leq C\varepsilon.$$

Therefore,

$$\|\boldsymbol{n}(\bar{t}_n)-\overline{\boldsymbol{n}}\|_{-1} \leq \|\boldsymbol{n}(t_n)-\overline{\boldsymbol{n}}\|_{-1} + \int_{t_n}^{\bar{t}_n} \|\partial_t \boldsymbol{n}\|_{-1} \leq C\varepsilon,$$

which implies that $\lim_{n\to+\infty} \|\boldsymbol{n}(\bar{t}_n)-\overline{\boldsymbol{n}}\|_{-1} = 0$. Since $\boldsymbol{n}$ is bounded in $L^\infty(t^*, +\infty; \boldsymbol{H}^3)$ then, $\boldsymbol{n}(t)$ is relatively compact in $\boldsymbol{H}^2$. Therefore, there exists a subsequence of $\boldsymbol{n}(\bar{t}_n)$, still denoted $\boldsymbol{n}(\bar{t}_n)$ converging to $\overline{\boldsymbol{n}}$ in $\boldsymbol{H}^2$. Hence, for n sufficiently large $\|\boldsymbol{n}(\bar{t}_n)-\overline{\boldsymbol{n}}\|_2 < \beta$, which contradicts the definition of $\bar{t}_n$.

Step 3: $\boldsymbol{n}(t)$ converges to $\overline{\boldsymbol{n}}$ in $\boldsymbol{H}^3$ as $t \uparrow +\infty$.

By using Steps 1 and 2, we have from (30) that $\boldsymbol{n}(t)_{t\geq t_{n_0}}$ is a Cauchy sequence in $\boldsymbol{H}^{-1}$ as $t \uparrow +\infty$. This and the strong $\boldsymbol{H}^3$-convergence by sequences of $\boldsymbol{n}(t)$, gives the convergence of the whole trajectory of $\boldsymbol{n}(t)$ towards $\overline{\boldsymbol{n}}$ in $\boldsymbol{H}^3$. ■

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