STABILITY FOR NEMATIC LIQUID CRYSTALS WITH STRETCHING TERMS

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Dedicated to the memory of Valery S. Melnik

We study a nematic crystal model appearing in [Liu *et al.*,2007] modeling stretching effects depending on the different shape of microscopic molecules of the material, under periodic boundary conditions. The aim of the present article is twofold: to extend the results given in [Sun & Liu, 2009], to a model with more complete stretching terms and to obtain some stability and asymptotic stability properties for this model.

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1. Introduction

The Nematic Liquid Crystal system is a Navier-Stokes type model for incompressible fluids respect to the macroscopic variables, that takes into account the crystallinity of the microscopic molecules of the material. It can be obtained coupling Navier-Stokes equations with the Ginzburg-Landau equations, being its unknowns the solenoidal velocity $\mathbf{u}(t,\mathbf{x})$, the pressure of the fluid $p(t,\mathbf{x})$, and the director field $\mathbf{d}(t,\mathbf{x})$, that represents the orientation of the liquid crystal molecules. Moreover, we suppose that the fluid is confined in a domain $\Omega \subset \mathbb{R}^3$.

We deal with an Ericksen-Leslie type formulation. A simplified model was analyzed by F. H. Lin & C. Liu in [Lin & Liu, 1995]. In fact, this model is a penalized one depending on the Ginzburg-Landau function:

$$\mathbf{f}_\epsilon(\mathbf{d}) = \frac{1}{\epsilon^2}\left(|\mathbf{d}|^2 - 1\right)\mathbf{d},$$

where $|\mathbf{d}|$ denotes the euclidean norm in $\mathbb{R}^3$ and $\epsilon > 0$ is a penalization parameter. This penalization function has a potential structure, i. e. there exists the function $F_\epsilon(\mathbf{d}) = \frac{1}{4\,\epsilon^2}\left(|\mathbf{d}|^2 - 1\right)^2$ such that $\mathbf{f}_\epsilon(\mathbf{d}) = \nabla_{\mathbf{d}}(F_\epsilon(\mathbf{d}))$ for all $\mathbf{d} \in \mathbb{R}^3$.

We denote $Q = (0,+\infty)\times\Omega$ and $\Sigma = (0,+\infty)\times\partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a smooth enough domain and $\partial\Omega$ its boundary. We consider the EDP system appearing in [Liu *et al.*,2007; system (1.9), p. 1187], that reads as:

$$(LC)\begin{cases} D_t\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p - \lambda\nabla\cdot\sigma^e &= \mathbf{0} \quad \text{in } Q,\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } Q,\\ \mathcal{D}_t\mathbf{d} + \gamma\mathbf{w} &= \mathbf{0} \quad \text{in } Q, \end{cases}$$

being

$$D_t\mathbf{u} = \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

the material derivative of $\mathbf{u}$,

$$\sigma^e = -(\nabla\mathbf{d})^t\nabla\mathbf{d} - \beta\mathbf{w}\,\mathbf{d}^t - (1+\beta)\mathbf{d}\,\mathbf{w}^t \qquad (1)$$

the elastic stress tensor ($\beta\in\mathbb{R}$) and

$$\mathbf{w} = -\Delta\mathbf{d} + \mathbf{f}_\epsilon(\mathbf{d})$$

the Euler-Lagrange system derived from the minimization problem respect to the elastic energy

$$E_e(\mathbf{d}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2 + \int_\Omega \mathbf{F}_\epsilon(\mathbf{d}). \qquad (2)$$

The term

$$\mathcal{D}_t\mathbf{d} = D_t\mathbf{d} + C(\mathbf{d},\nabla\mathbf{u})$$

describes a general derivative containing the material derivative $D_t\mathbf{d} = \partial_t\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d}$ and the quadratic term

$$C(\mathbf{d},\nabla\mathbf{u}) = \beta(\nabla\mathbf{u})\mathbf{d} + (1+\beta)(\nabla\mathbf{u})^t\mathbf{d}$$

modeling the so-called stretching effects, depending on the form of the molecules [Liu et al., 2007]. In fact, the constant $\beta = -\alpha$ is associated with the aspect ratio r of the ellipsoid particles. The case of α near to 1 corresponds to rod like particles (then the transport is purely covariant stretching), the case of α near to 0 corresponds to disc like particles (then the transport is anti-stretching) and the case of α near to 1/2 corresponds to the spherical shape (the transport is the rigid rotation of the center of the mass).

Finally, $\nu > 0$ is the fluid viscosity, $\lambda > 0$ is the elasticity constant and $\gamma > 0$ is a relaxation in time constant.

The theoretical analysis of a simplified model without stretching effects, i.e for $C(\mathbf{d},\nabla\mathbf{u}) = 0$ and the corresponding elastic tensor $\sigma^e = -(\nabla\mathbf{d})^t\nabla\mathbf{d}$, was made in [Lin & Liu, 1995] obtaining existence of global weak solution, i. e.

$$\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega))\cap L^2(0,T;\mathbf{H}^1(\Omega)),$$

$$\mathbf{d}\in L^\infty(0,T;\mathbf{H}^2(\Omega))\cap L^2(0,T;\mathbf{H}^3(\Omega)),$$

for all $T>0$, and the existence (and uniqueness) of local strong solution, i. e.

$$\mathbf{u}\in L^\infty(0,T_*;\mathbf{H}^1(\Omega))\cap L^2(0,T_*;\mathbf{H}^2(\Omega)),$$

$$\mathbf{d}\in L^\infty(0,T_*;\mathbf{H}^2(\Omega))\cap L^2(0,T_*;\mathbf{H}^3(\Omega)),$$

with $T_*\le T$ (small enough) or $T_* = T$ (for each $T>0$) for big enough viscosity coefficient ν or for two-dimensional domains. All these previous results are given for the time-independent Dirichlet boundary data:

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{d} = \mathbf{h} \quad \text{on } \Sigma, \quad (\mathbf{h}\neq\mathbf{h}(t))$$

and for the initial-value boundary problem with initial condition:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0 \quad \text{in } \Omega. \qquad (3)$$

When time-dependent Dirichlet data for $\mathbf{d}$ is considered ($\mathbf{h} = \mathbf{h}(t)$), the existence of weak time-periodic solution, that is solutions obtained by changing (3) by $\mathbf{u}(0) = \mathbf{u}(T)$ and $\mathbf{d}(\mathbf{0}) = \mathbf{d}(T)$, is obtained in [Climent-Ezquerra *et al.*,]. The strong regularity up to infinite time for big enough viscosity ν jointly with the strong regularity of time-periodic solutions are obtained in [Climent-Ezquerra *et al.*,].

The results corresponding to the initial-value boundary problem are extended in [Lin & Liu, 2000] to a much more complete model respect to the dissipative tensor and considering the particular stretching effects for the case of spherical molecules, i.e. taking $\beta = -1/2$ in (1).

Recently, a liquid crystal model with a stretching term for the case of rod like particles (taking $\beta = -1$ in (1)) and periodic boundary conditions for both $\mathbf{u}$ and $\mathbf{d}$ has been studied in [Sun & Liu, 2009], obtaining global weak solution and local strong solution (which is global for large enough viscosity)

The aim of the present article is twofold: to extend the last results of [Sun & Liu, 2009] to a model with more complete stretching terms and to obtain some stability and asymptotic stability properties for this model.

2. General Framework.

Assume that we have the following situation, a.e. $t\in(t_0,+\infty)$:

$$E(t),\, F(t)\ge 0, \quad E'(t) + F(t) \le 0. \qquad (4)$$

Then, $E\in C_b[t_0,+\infty)$, is a decreasing function and there exists

$$\lim_{t\to+\infty} E(t) = E_\infty \ge 0.$$

On the other hand, $F \in L^1(t_0,+\infty)$, that is,

$$\int_{t_0}^{+\infty} F(t)\,dt < +\infty.$$

In this case, for any $\delta > 0$, there exists a large enough time $t_1^* = t_1^*(\delta) \geq t_0$ such that:

$$\int_{t_1^*}^{+\infty} F(t)\,dt \leq \delta. \tag{5}$$

In particular, we can say that for each $\delta > 0$ there exists a large enough time $t_1^*(\delta) \geq t_0$ such that

$$\frac{1}{\tau}\int_t^{t+\tau} F(t)\,dt \leq \frac{\delta}{\tau}, \quad \forall \tau > 0,\ \forall t \geq t_1^*(\delta). \tag{6}$$

Lemma 2.1. *Let $F \in L^1(t_0,+\infty)$, $F \geq 0$ in $(t_0,+\infty)$, satisfying (6). Then, $\forall \delta > 0$, $\forall t \geq t_1^*(\delta)$ and $\forall \tau > 0$ there exists a time $\bar t \in [t,t+\tau]$ such that:*

$$F(\bar t) \leq \frac{2\delta}{\tau}. \tag{7}$$

Indeed, the set of points $\bar t \in [t,t+\tau]$ satisfying (7) has measure $\geq \tau/2$.

Proof. We focus on the proof in the interval $[t_1^*, t_1^*+\tau]$. The proof for another interval of length τ contained in $[t_1^*,+\infty)$ is similar.

Indeed, we define:

$$A = \{s \in [t_1^*, t_1^*+\tau] / \, F(s) \geq \frac{2\delta}{\tau}\}.$$

Therefore,

$$\int_A F(t)\,dt + \int_{A^c} F(t)\,dt \leq \delta,$$

and thus

$$\frac{2\delta}{\tau}\,|A| \leq \delta \quad \Rightarrow \quad |A| \leq \frac{\tau}{2}.$$

That is, $|A^c| \geq \tau/2$. ■

Now, we assume that the following differential inequality for $F(t)$ holds:

$$F'(t) \leq C_2(F(t)^3 + 1). \tag{8}$$

Lemma 2.2. *Let $F \in L^1(t_0,+\infty)$ be a function satisfying the differential inequality (8). For any $\varepsilon < 1$, if $F(t_0) \leq \varepsilon/3$, then $F(t) \leq \varepsilon$ $\forall t \in [t_0, t_0 + T_*(\varepsilon)]$, where $T_*(\varepsilon) = \dfrac{\varepsilon}{3C_2}$.*

Proof. We argue by contradiction: Suppose that there exists a time $t_1 \in [t_0, t_0+T_*(\varepsilon)]$ such that $F(t) < \varepsilon$ in $[t_0,t_1)$ and $F(t_1) = \varepsilon$. Then, from eq. (8) we obtain that $F' < 2C_2$ en $[t_0,t_1]$. Integrating in $[t_0,t_1]$, we get:

$$\begin{aligned} F(t_1) &< F(t_0) + 2C_2\,(t_1 - t_0) \\ &\leq F(t_0) + 2C_2 T_*(\varepsilon) \\ &\leq \frac{\varepsilon}{3} + 2C_2\frac{\varepsilon}{3C_2} = \varepsilon. \end{aligned}$$

This fact contradicts the starting hypothesis. ■

2.1. *Asymptotic stability.*

Theorem 2.3. *Let $\varepsilon < 1$, and $F \in L^1(0,+\infty)$, $F \geq 0$, such that both (8) and inequality (6) hold for $\delta = \dfrac{\varepsilon^2}{36C_2}$, $t_1^* = t_1^*(\delta)$ and $\tau = \dfrac{T_*(\varepsilon)}{2}$. Then,*

$$F(t) \leq \varepsilon,\ \forall\, t \geq t_2^* = t_1^* + \frac{T_*(\varepsilon)}{2} = t_1^*(\delta) + \frac{\varepsilon}{6C_2}. \tag{9}$$

Remark 2.4. In particular, $F \in W^{1,1}(t_2^*,+\infty) \hookrightarrow C[t_2^*,+\infty)$.

Proof. We argue by contradiction: Assume that there exists a time $\widetilde t > t_2^*$ such that $F(t) > \varepsilon$. We consider the interval $[\widetilde t - T_*(\varepsilon)/2, \widetilde t] \subset [t_1^*,+\infty)$.

From Lemma 2.1 we conclude that for each interval of length $T_*(\varepsilon)/2$ contained in $[t_1^*,+\infty)$ and $\forall t \geq t_1^*$ there exists a time $\bar t_1 \in [\widetilde t - T_*(\varepsilon)/2, \widetilde t]$ such that:

$$F(\bar t_1) \leq \frac{2\delta}{\tau} = \frac{\varepsilon^2/(18C_2)}{\varepsilon/(6C_2)} = \frac{\varepsilon}{3}.$$

Thus, applying Lemma 2.2, one verifies:

$$F(t) \leq \varepsilon, \quad \forall t \in [\bar t_1, \bar t_1 + T_*(\varepsilon)].$$

Observe that $\widetilde t \in [\bar t_1, \bar t_1 + T_*(\varepsilon)]$, which gives us to contradiction. ■

Corollary 2.5. *Let $F \in L^1(t_0,+\infty)$ be a function satisfying eq. (8). Then, $F(t)$ is a function* **asymptotically stable to** 0*, that is,*

$$\lim_{t\to+\infty} F(t) = 0.$$

2.2. Stability until infinite time.

If we assume that:

$$(H1) \qquad E(t_0) \le \delta(\varepsilon) = \frac{\varepsilon^2}{36C_2},$$

then, from (4) we get: $\forall t_1 > t_0$

$$\int_{t_0}^{t_1} F(t)\,dt \quad \le \quad E(t_0) - E(t_1) \le \delta(\varepsilon)$$

In fact, one has (5) for $t_1^*(\varepsilon) = t_0$. Then, applying Theorem 2.3, we obtain:

$$F(t) \le \varepsilon \quad \forall t \ge t_0 + \frac{\varepsilon}{6C_2}\left(= t_0 + \frac{T_*(\varepsilon)}{2}\right) \qquad (10)$$

If, moreover,

$$(H2) \qquad F(t_0) \le \frac{\varepsilon}{3},$$

then applying Lemma 2.2, we get:

$$F(t) \le \varepsilon \quad \forall t \in [t_0, t_0 + T_*(\varepsilon)] \qquad (11)$$

In summary, assuming $(H1)$ and $(H2)$, one has:

$$E(t) \le \delta(\varepsilon), \quad F(t) \le \varepsilon, \quad \forall t \ge t_0.$$

3. The Nematic Liquid Crystal Model

3.1. Weak estimates

If we consider both $\mathbf{u}(t)$ and $\mathbf{w}(t)$ as test functions in the $\mathbf{u}$-system and $\mathbf{d}$-system of (LC) respectively, taking into account the equality:

$$\nabla \cdot ((\nabla \mathbf{d})^t \nabla \mathbf{d}) = -(\nabla \mathbf{d})^t \mathbf{w} + \nabla E_e(\mathbf{d}),$$

we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\nabla\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &+ - \lambda\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{d}]\cdot\mathbf{w}\,d\mathbf{x} \\ &-\lambda\int_\Omega \left[\beta\mathbf{w}^t\mathbf{d} + (1+\beta)\mathbf{d}^t\mathbf{w}\right] : \nabla\mathbf{u}\,d\mathbf{x} = 0 \end{aligned} \qquad (12)$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\nabla\mathbf{d}(t)\|^2_{\mathbf{L}^2(\Omega)} + F_\epsilon(\mathbf{d})(t)\right) + \gamma\|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \\ &+ \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{d}]\cdot\mathbf{w}\,d\mathbf{x} + \int_\Omega C(\mathbf{d},\nabla\mathbf{u})\cdot\mathbf{w}\,d\mathbf{x} = 0 \end{aligned} \qquad (13)$$

for any boundary conditions for $(\mathbf{u},\mathbf{d})$ given in the Introduction (that is, Dirichlet, Neumann or periodic for $\mathbf{d}$). Then, adding (12) to (13) multiplied by λ, the last two terms of (12) and (13) cancel and the so-called *energy equality* holds:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \lambda E_e(\mathbf{d}(t))\right] \\ &+\nu\|\nabla\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \lambda\gamma\|\mathbf{w}(t)\|^2_{\mathbf{L}^2(\Omega)} = 0 \end{aligned} \qquad (14)$$

(recall that $E_e(\mathbf{d})$ is given in (2)).

Note that, defining the time functions $E(t)$ and $F(t)$ as:

$$E(t) \;=\; \frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \lambda E_e(\mathbf{d}(t)), \qquad (15)$$

$$F(t) \;=\; \nu\|\nabla\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \gamma\lambda\|\mathbf{w}(t)\|^2_{\mathbf{L}^2(\Omega)}, \qquad (16)$$

the inequality (4) is deduced from (14). As a consequence, we can deduce the existence of weak solutions of the problem. Moreover, by using this weak regularity and the $\mathbf{H}^2(\Omega)$ and $\mathbf{H}^3(\Omega)$ regularity of the elliptic problem $-\Delta\mathbf{d} + f_\epsilon(\mathbf{d}) = \mathbf{w}$ with appropriate boundary conditions, we can deduce [Climent-Ezquerra *et al.*,]:

$$\begin{aligned} \|\mathbf{d}\|_{\mathbf{H}^2(\Omega)} &\le C\left(\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + 1\right), \\ \|\mathbf{d}\|_{\mathbf{H}^3(\Omega)} &\le C\left(\|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} + 1\right). \end{aligned} \qquad (17)$$

In the next section, we will use repeatedly these estimates.

3.2. Strong estimates

In this section, we only consider the periodic boundary conditions case for all variables $(\mathbf{u}, p, \mathbf{d})$. Taking both $-\Delta\mathbf{u}$ and $-\lambda\Delta\mathbf{w}$ as test functions in the $\mathbf{u}$-system and in the $\mathbf{d}$-system of (LC) respectively, one can obtain ([Sun & Liu, 2009]):

$$\begin{aligned} &\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \\ &\le \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C\frac{\lambda^2}{\nu}\|(\nabla\mathbf{d})^t\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \\ &+\lambda\int_\Omega \left\{\beta\mathbf{w}\mathbf{d}^t + (1+\beta)\mathbf{d}\mathbf{w}^t\right\}\,\nabla(\Delta\mathbf{u})\,d\mathbf{x} \end{aligned} \qquad (18)$$

and

$$\lambda \int_\Omega \nabla \partial_t \mathbf{d} : \nabla \mathbf{w}\, d\mathbf{x} + \lambda\gamma \|\nabla \mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} + \lambda \int_\Omega \nabla((\mathbf{u}\cdot\nabla)\mathbf{d}) : \nabla \mathbf{w}\, d\mathbf{x} + \lambda \int_\Omega \nabla C(\mathbf{d},\nabla\mathbf{u}) : \nabla \mathbf{w}\, d\mathbf{x} = 0 \quad (19)$$

Observe that the first term of (19) can be rewritten as:

$$\int_\Omega \nabla \partial_t \mathbf{d} : \nabla \mathbf{w}\, d\mathbf{x} = -\int_\Omega \Delta(\partial_t \mathbf{d})\mathbf{w}\, d\mathbf{x} = \int_\Omega \partial_t \mathbf{w}\,\mathbf{w}\, d\mathbf{x} - \int_\Omega f'_\epsilon(\mathbf{d})(\partial_t\mathbf{d})\mathbf{w}\, d\mathbf{x} = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} - \int_\Omega f'_\epsilon(\mathbf{d})(\partial_t\mathbf{d})\mathbf{w}\, d\mathbf{x} \quad (20)$$

Now, using the $\mathbf{d}$-system of (LC), that is, $\partial_t \mathbf{d} = -(\mathbf{u}\cdot\nabla)\mathbf{d} - C(\mathbf{d},\nabla\mathbf{u}) - \gamma\mathbf{w}$, one has:

$$-\int_\Omega f'_\epsilon(\mathbf{d})(\partial_t\mathbf{d})\mathbf{w}\, d\mathbf{x} = \int_\Omega f'_\epsilon(\mathbf{d})\left((\mathbf{u}\cdot\nabla)\mathbf{d} + C(\mathbf{d},\nabla\mathbf{u}) + \gamma\mathbf{w}\right)\mathbf{w}\, d\mathbf{x} \le \varepsilon\left(\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}\right) + C_\varepsilon\left(F^3(t)+1\right) \quad (21)$$

for ε small enough.

The more nonlinear terms of the last term of (19) are manipulated as follows (here, the periodic boundary conditions are again applied):

$$\lambda \int_\Omega \left(\beta\nabla(\nabla\mathbf{u})\mathbf{d} + (1+\beta)\nabla(\nabla\mathbf{u})^t\mathbf{d}\right) : \nabla\mathbf{w}\, d\mathbf{x} \le -\lambda\int_\Omega \nabla(\Delta\mathbf{u}) : \left\{\beta\mathbf{w}\mathbf{d}^t + (1+\beta)\mathbf{d}\mathbf{w}^t\right\}\, d\mathbf{x} + C\|D^2\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{d}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^3(\Omega)} \quad (22)$$

Note that the first term on the right-hand side of (22) cancels with the last term in (18). The remaining part of the last term of (19) can be written as:

$$\lambda\int_\Omega (\beta(\nabla\mathbf{d}\cdot\nabla)\mathbf{u} : \nabla\mathbf{w} + (1+\beta)(\nabla\mathbf{w}\cdot\nabla)\mathbf{u} : \nabla\mathbf{d})\, d\mathbf{x} \le C(\lambda,\beta)\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{d}\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{u}\|_{\mathbf{L}^3(\Omega)} \quad (23)$$

being $C(\lambda,\beta)$ a constant depending on λ and β.

Therefore, adding (18) to (19) and taking into account estimates (20)-(23), we obtain:

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \lambda\|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}\right) + \nu\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \lambda\gamma\|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \\ &\le \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C\frac{\lambda^2}{\nu}\|(\nabla\mathbf{d})^t\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \\ &+\lambda\int_\Omega \nabla((\mathbf{u}\cdot\nabla)\mathbf{d}) : \nabla\mathbf{w}\, d\mathbf{x} \\ &+C\left(F^3(t)+1\right) \\ &+C\|D^2\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{d}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^3(\Omega)} \\ &+C(\lambda,\beta)\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{d}\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{u}\|_{\mathbf{L}^3(\Omega)} \\ &= \textstyle\sum_{i=1}^6 I_i \end{aligned} \quad (24)$$

Using Sobolev's inequalities, the estimates for the I_i-terms can be summarized as follows:

$$\begin{aligned} I_1 &\le \|\mathbf{u}\|^2_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{u}\|^2_{\mathbf{L}^3(\Omega)} \le \varepsilon\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C_\varepsilon\|\nabla\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)} \\ I_2 &\le \|\nabla\mathbf{d}\|^2_{\mathbf{L}^6(\Omega)}\|\mathbf{w}\|^2_{\mathbf{L}^3(\Omega)} \le \varepsilon\|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} + C_\varepsilon\|\mathbf{w}\|^6_{\mathbf{L}^2(\Omega)} \end{aligned}$$

$$I_3 \le \|\nabla\mathbf{u}\|_{\mathbf{L}^3(\Omega)}\|\nabla\mathbf{d}\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|\Delta\mathbf{d}\|_{\mathbf{L}^3(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)} = I_{31} + I_{32}$$

where term I_{31} as the same type of estimates as term I_6.

$$I_{32} \le \|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|\Delta\mathbf{d}\|_{\mathbf{L}^3(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \le \varepsilon\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + C_\varepsilon\|\nabla\mathbf{u}\|^4_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}$$

$$I_5 \le \varepsilon\left(\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}\right) + C_\varepsilon\|\mathbf{w}\|^6_{\mathbf{L}^2(\Omega)}$$

$$I_6 \le \varepsilon\left(\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}\right) + C_\varepsilon\|\mathbf{w}\|^4_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$$

Now, we introduce function $G(t)$ defined as:

$$G(t) = \nu\|\Delta\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \lambda\gamma\|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \quad (25)$$

Observe that, finally, (24) can be written in the following form:

$$F'(t) + G(t) \le C\,\left(1 + F^3(t)\right),$$

being $F(t)$ and $G(t)$ the functions defined in (16) and (25) respectively. Therefore, function $F(t)$ satisfies (8).

4. Applications of the General Framework to Nematic Liquid Crystal.

4.1. Asymptotic stability.

Let $(\mathbf{u}_0, \mathbf{d}_0) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega)$ be two given functions and $(\mathbf{u}(t), \mathbf{d}(t))$ a weak solution of system (LC) with periodic boundary conditions and initial data $(\mathbf{u}_0, \mathbf{d}_0)$.

Since (4) and (8) hold, applying the results from Section 2, one has:

$$\begin{cases} E(t) \downarrow E_\infty\,(\geq 0) \quad \text{in } \mathbb{R}, & t \uparrow +\infty \\ F(t) \to 0 \quad \text{in } \mathbb{R}, & t \uparrow +\infty \end{cases}$$

hence:

$$\begin{cases} E'(t) \to 0 & \text{in } \mathbb{R} & t \uparrow +\infty \\ \mathbf{u}(t) \to 0 & \text{in } \mathbf{H}^1_0(\Omega) & t \uparrow +\infty \\ \mathbf{w}(t) \to 0 & \text{in } \mathbf{L}^2(\Omega) & t \uparrow +\infty \end{cases}$$

Moreover, for each subsequence $t_j \uparrow +\infty$, there exists a subsequence $(t_{j_k}) \subset (t_j)$ such that:

$$\mathbf{d}(t_{jk}) \rightharpoonup \bar{\mathbf{d}} \quad \text{in } \mathbf{H}^2(\Omega)\text{-weak} \quad \text{for } k \uparrow +\infty.$$

being $\bar{\mathbf{d}}$ a critical point of the elastic energy $E_e(\mathbf{d})$, that is, a solution for the stationary problem:

$$-\Delta\bar{\mathbf{d}} + f_\varepsilon(\bar{\mathbf{d}}) \;=\; \mathbf{0} \quad \text{in } \Omega,$$

with periodic boundary conditions on $\partial\Omega$. Note that,

$$E_\infty = \frac{\lambda}{2}\left(|\nabla\bar{\mathbf{d}}(t)|_2^2 + 2\int_\Omega F(\bar{\mathbf{d}}(t))\right) = \lambda E_e(\bar{\mathbf{d}})$$

that is, every possible limit of the director field $\bar{\mathbf{d}}$ when $t \uparrow +\infty$ is a critical point of the elastic energy and all these possible limits have the same elastic energy E_∞.

4.2. Stability for constant director fields.

If $(\mathbf{u}_0, \mathbf{d}_0)$ are such that:
$(H1)$

$$\frac{1}{2}\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{\lambda}{2}\|\nabla\mathbf{d}_0\|^2_{\mathbf{L}^2(\Omega)} + \lambda\int_\Omega F_\epsilon(\mathbf{d}_0) \leq \delta(\varepsilon)$$

(in particular, $\|\mathbf{d}_{cte} - \mathbf{d}_0\|^2_{\mathbf{H}^1(\Omega)} \leq C\,\varepsilon^2$ for a constant vector $\bar{\mathbf{d}}_{cte}$ with $|\bar{\mathbf{d}}_{cte}| = 1$) and

$$(H2) \qquad \nu\|\nabla\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \gamma\lambda\|\mathbf{w}_0\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{\varepsilon}{3},$$

where $\mathbf{w}_0 = -\Delta\mathbf{d}_0 + \mathbf{f}_\epsilon(\mathbf{d}_0)$ then for each $t \geq t_0$, applying the results from Section 2 one has:

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\lambda}{2}\|\nabla\mathbf{d}(t)\|^2_{\mathbf{L}^2(\Omega)} + \lambda\int_\Omega F_\epsilon(\mathbf{d}(t)) \leq \delta(\varepsilon)$$

and

$$\nu\|\nabla\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \gamma\lambda\|\mathbf{w}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{\varepsilon}{3}.$$

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