On the structure of the positive solutions of the logistic equation with nonlinear diffusion

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1 Introduction

In this work we study the structure of the positive solutions of the degenerate logistic equation, i.e. of the elliptic boundary value problem

$$\begin{cases} d\mathcal{L}w^m = \sigma w - b(x)w^r & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, $\mathcal{L}$ is a general second order uniformly elliptic operator, b is a positive function, $m \geq 1$, $r > 1$, d is a positive constant and σ is a real parameter. Eq. (1) was introduced in biological models by Gurtin-McCamy [7], see also [13] and [14], in describing the dynamics of biological populations whose mobility is density dependent. In (1), Ω is the inhabiting region, $w(x)$ represents the density of a species and we are assuming that Ω is fully surrounded by inhospitable areas, since the population density is subject to homogeneous Dirichlet boundary conditions. The operator $\mathcal{L}$ measures the diffusivity and the external transport effects of the species. In the case $m > 1$ the diffusion, i.e. the rate the moving of the species from high density regions to low density ones, is slower than in the linear case ($m = 1$), which gives to rise a "more realistic" model. Moreover, here $d > 0$ is the diffusion rate of the species, $b(x)$ and σ are associated with the limiting effect crowding in the population and the growth rate of the species, respectively.

An appropriate change of variable, see (5), transforms (1) into

$$\begin{cases} \mathcal{L}u = \lambda u^q - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $\lambda \in \mathbb{R}$, $0 < q < p$ and $q \leq 1$. The case $q = 1$ and $p \geq 1$ has been widely studied in the recent years. When $q = 1$ and $p > 1$, it is well known that there exists a unique positive solution θ_λ of (2) if, and only if, $\lambda > \sigma_1[\mathcal{L}]$, where $\sigma_1[\mathcal{L}]$ is the principal eigenvalue of $\mathcal{L}$ in Ω subject to homogeneous Dirichlet boundary conditions. Moreover, there exists a continuum of positive solutions of (2) bifurcating from $(\lambda, u) = (\sigma_1[\mathcal{L}], 0)$ which is unbounded. In the particular case $q = p = 1$ a vertical bifurcation diagram appears at $\lambda = \sigma_1[\mathcal{L} + b]$. Figure 1 shows these cases.

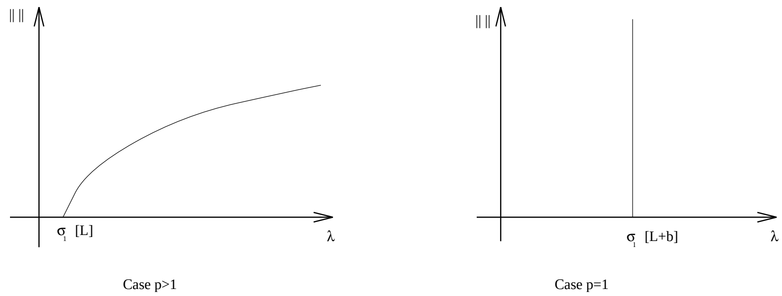

Figure 1: Bifurcation diagrams with q=1

When $q < 1$, in our knowledge only partial results are known about existence and uniqueness of positive solutions of (2). Indeed, when $\mathcal{L} = -\Delta$ and $b(x) = b \in \mathbb{R}$, it was proved in [12], Corollary 1, that there exists a unique positive solution of (2) if, and only if, $\lambda > 0$. When b is a function in x and $\mathcal{L} = -\Delta$, Pozio and Tesei [16] showed that if $\lambda > 0$ there exists a positive solution of (2). Moreover, if $p \geq 1$ or $p < 1$ and λ large enough, then the positive solution is unique, see Theorem 5 of [16]. Similar results were obtained by Leung and Fan in [10], see Theorem 2.1. We improve these results in two ways: when $\mathcal{L}$ is a second order uniformly elliptic operator not necessarily selfadjoint and b is a function in x, we prove that there exists a unique positive solution of (2) if, and only if, $\lambda > 0$. This solution will be denoted by $\theta_{[\lambda,q,p]}$. Moreover, there exists a continuum of positive solutions of (2) bifurcating from the trivial solution $u = 0$ at $\lambda = 0$ which is unbounded, see Figure 2.

We can define the map

$$\mathcal{F}_q : \mathbb{R} \mapsto C_0^{2,\alpha}(\overline{\Omega}), \quad \mathcal{F}_q(\lambda) := \theta_{[\lambda,q,p]}$$

with $\mathcal{F}_q(\lambda) = 0$ if $\lambda \leq 0$. We focus on the study of the map $\mathcal{F}_q$, specifically we analyze the behaviour of $\mathcal{F}_q$ as $\lambda \downarrow 0^+$ and $\lambda \uparrow +\infty$, through the singular perturbation theory. We generalize

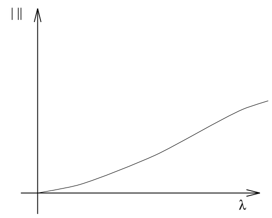

Figure 2: Bifurcation diagram with $q < 1$.

the results obtained when $q = 1$. Indeed, when $q < 1$, $q < p$, we prove that if $1 < p$,

$$\frac{\mathcal{F}_q(\lambda)}{\lambda^{1/(p-q)}} \to \left(\frac{1}{b(x)}\right)^{1/(p-q)} \text{ uniformly on compact subsets of } \Omega \text{ as } \lambda \uparrow +\infty \text{ and}$$

$$\mathcal{F}_q(\lambda) = O(\lambda^{1/(1-q)}) \text{ as } \lambda \downarrow 0^+;$$

if $p < 1$,

$$\mathcal{F}_q(\lambda) = O(\lambda^{1/(1-q)}) \text{ as } \lambda \uparrow +\infty \text{ and}$$

$$\frac{\mathcal{F}_q(\lambda)}{\lambda^{1/(p-q)}} \to \left(\frac{1}{b(x)}\right)^{1/(p-q)} \text{ uniformly on compact subsets of } \Omega \text{ as } \lambda \downarrow 0^+;$$

and if $p = 1$,

$$\mathcal{F}_q(\lambda) = \lambda^{1/(1-q)}\mathcal{F}_q(1).$$

These results are a first step to obtain non-existence and existence results of systems with nonlinear diffusion as already it was shown when the diffusion is linear in [4].

Finally, we study how the bifurcation diagram of Figure 2 varies when $q \uparrow 1$. We will show that if $p > 1$, $\theta_{[\lambda,q,p]} \to \theta_\lambda$ as $q \uparrow 1$. In the special case $p = 1$, we prove that if $\lambda < \sigma_1[\mathcal{L} + b]$ (resp. $\lambda > \sigma_1[\mathcal{L} + b]$) then $\theta_{[\lambda,q,p]}$ tends to 0 (resp. infinity) as $q \uparrow 1$.

An outline of this work is as follows. In Section 2 we study the existence and uniqueness of positive solution of (2), as well as some monotony properties of $\mathcal{F}_q$. In Section 3 we analyze the behaviour of the mapping $\mathcal{F}_q$ as $\lambda \downarrow 0^+$, $\lambda \uparrow +\infty$ (Theorem 3) and as $q \uparrow 1$.

2 Existence and comparison results

In this section we study the positive solutions of

$$\begin{cases} d\mathcal{L}w^m = \sigma w - b(x)w^r & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $m > 1$, $r > 1$, $d > 0$, $b \in C^\alpha(\overline{\Omega}), \alpha \in (0,1)$, with $b(x) > 0$ for all $x \in \overline{\Omega}$, σ is a real parameter and $\mathcal{L}$ is a second order operator of the form

$$\mathcal{L} := -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i}$$

with

$$a_{ij} \in C^{1,\alpha}(\overline{\Omega}),\ b_i \in C^\alpha(\overline{\Omega}) \quad a_{ij} = a_{ji}, \quad \text{with } 0 < \alpha < 1,$$

and uniformly elliptic in the sense that

$$\exists \rho > 0 \quad \text{such that} \quad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \rho|\xi|^2, \quad \forall \xi \in \mathbb{R}^N,\ \forall x \in \Omega. \tag{4}$$

In the sequel, given any function $f \in C^\alpha(\overline{\Omega})$ we shall denote

$$f_M := \sup_{\overline{\Omega}} f, \qquad f_L := \inf_{\overline{\Omega}} f.$$

If $r \neq m$, performing the change

$$w^m = d^{m/(r-m)}u, \tag{5}$$

(3) can be rewritten as

$$\begin{cases} \mathcal{L}u = \lambda u^q - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where p and q satisfy

$$(H) \qquad 0 < q < p, \qquad q < 1.$$

In the special case $r = m$, the change $w^m = u$ transforms (3) into

$$\begin{cases} (d\mathcal{L} + b(x))u &= \lambda u^q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

On the other hand, it is well-known that the linear eigenvalue problem

$$\begin{cases} (\mathcal{L} + f)u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{8}$$

with $f \in L^\infty(\Omega)$ has a principal eigenvalue $\sigma_1^\Omega[\mathcal{L} + f]$, with a corresponding eigenfunction $\varphi_1^\Omega[\mathcal{L} + f](x) > 0$ for all $x \in \Omega$, $\partial_n \varphi_1^\Omega[\mathcal{L} + f](x) < 0$ for all $x \in \partial\Omega$ where n is the outward unit normal on $\partial\Omega$ and normalized such that $\|\varphi_1^\Omega[\mathcal{L} + f]\|_\infty = 1$ (the superscript Ω will be omitted if no confusion arises).

The following results characterize the existence and uniqueness of positive solutions for (6) and (7).

Theorem 1 *Assume (H). Then* (6) *possesses a unique positive solution in* $C^{2,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$ *if, and only if,* $\lambda > 0$.

Proof. We use the sub-supersolution method with Hölder continuous functions, cf. [1] and Theorem 4.5.1 in [15]. It is not hard to show that $\overline{u} := (\lambda/b_L)^{1/(p-q)}$ is a supersolution of (6). Moreover, using the maximum principle we can prove that

$$\|u\|_\infty \leq \left(\frac{\lambda}{b_L}\right)^{\frac{1}{p-q}} \tag{9}$$

for any u solution of (6).

Take $\underline{u} =: \varepsilon\varphi_1[\mathcal{L}]$, with $\varepsilon > 0$ to choose. It is easy to check that we can take $\varepsilon > 0$ sufficiently small such that $\underline{u}$ is a subsolution of (6) and $\underline{u} \leq \overline{u}$. This proves the existence of positive solution of (6) in $C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. The maximum principle implies that $\lambda > 0$ is a necessary condition for the existence of positive solution of (6). For the uniqueness we are going to use a change of variable already used in [17], see also [3], in a slightly different context. We define

$$z := \frac{1}{1-q} u^{1-q}.$$

Then (6) is equivalent to

$$\left\{ \begin{array}{rcll} \mathcal{L}z - \dfrac{q}{(1-q)z} \displaystyle\sum_{i,j=1}^{N} a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} & = & \lambda - b(x)(1-q)^{(p-q)/(1-q)} z^{(p-q)/(1-q)} & \text{in } \Omega, \\ z & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{10}$$

Let z_2 be the maximal solution of (10), which exists by (9). Suppose there exists another solution z_1 of (10) with $z_1 \leq z_2$. We are going to prove that $z_1 \geq z_2$. We argue by contradiction. We suppose that there exists $P \in \Omega$ where

$$\Phi := z_1 - z_2$$

attains its negative minimum. Let $r > 0$ be such that $0 < z_1(x) < z_2(x)$ for all $x \in B(P,r)$, where $B(P,r)$ is the ball of radius r centered at P. It is not hard to show that Φ satisfies

$$\mathcal{L}\Phi - \frac{q}{1-q} \Big(\sum_{i,j=1}^{N} a_{ij} [\frac{1}{z_1} \frac{\partial z_1}{\partial x_i} \frac{\partial z_1}{\partial x_j} - \frac{1}{z_2} \frac{\partial z_2}{\partial x_i} \frac{\partial z_2}{\partial x_j}]\Big) = -b(x)(1-q)^{(p-q)/(1-q)} (z_1^{(p-q)/(1-q)} - z_2^{(p-q)/(1-q)}).$$

On the other hand, it can be proved that

$$\sum_{i,j=1}^{N} a_{ij} [\frac{1}{z_1} \frac{\partial z_1}{\partial x_i} \frac{\partial z_1}{\partial x_j} - \frac{1}{z_2} \frac{\partial z_2}{\partial x_i} \frac{\partial z_2}{\partial x_j}] = \sum_{i=1}^{N} c_i \frac{\partial \Phi}{\partial x_i} - c(x)\Phi$$

where

$$c_i = \sum_{j=1}^{N} a_{ij} \frac{1}{z_1} (\frac{\partial z_1}{\partial x_j} + \frac{\partial z_2}{\partial x_j}), \quad c(x) = \frac{1}{z_1 z_2} \sum_{i,j=1}^{N} a_{ij} \frac{\partial z_2}{\partial x_i} \frac{\partial z_2}{\partial x_j}.$$

So, Φ verifies

$$\mathcal{L}_1 \Phi + \frac{q}{1-q} c(x)\Phi = -b(x)(1-q)^{(p-q)/(1-q)} (z_1^{(p-q)/(1-q)} - z_2^{(p-q)/(1-q)}), \quad \text{in } B(P,r), \tag{11}$$

being

$$\mathcal{L}_1 = -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} (b_i - \frac{q}{1-q} c_i) \frac{\partial}{\partial x_i}.$$

By (4), $c(x) \geq 0$ in $B(P, r)$, and from (H) we have that $z_2^{(p-q)/(1-q)} > z_1^{(p-q)/(1-q)}$ in $B(P, r)$, and so by the strong maximun principle of Hopf, see for example Theorem 3.5 in [6], $\Phi = C < 0$ in $B(P, r)$ with C constant. Thus, the left hand side of (11) is non-positive and right one positive. This gives a contradiction and completes the proof. $\diamond$

The following result is well known when the operator is selfadjoint, see [2], [9], [10] and [17] for example, and its proof can be deduced by Theorem 1. So that, we only present an alternative uniqueness proof in which we use a singular eigenvalue problem.

Theorem 2 *If* $0 < q < 1$*, then* (7) *possesses a unique positive solution in* $C^{2,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$ *if, and only if,* $\lambda > 0$*.*

Proof. Let u_1, u_2, $u_1 \geq u_2$, u_1 the maximal positive solution of (7) and u_2 an arbitrary positive solution. Then

$$\sigma_1[d\mathcal{L} + b - \lambda u_i^{q-1}] = 0 \qquad i = 1, 2. \tag{12}$$

Observe that this principal eigenvalue is not in the setting of (8) because $u_i^{q-1} \notin L^\infty(\Omega)$. But, u_i is a positive function satisfying (7) and so, by the strong maximum principle, there exists a positive constant C such that

$$C d_\Omega(x) \leq u_i(x) \qquad \text{for all } x \in \overline{\Omega},$$

where $d_\Omega(x) := dist(x, \partial\Omega)$. Hence, $d_\Omega^{1-q}(x) u_i^{q-1}$ is bounded and so we can apply the results of [8] (see also [5] for selfadjoint operators) to define correctly $\sigma_1[d\mathcal{L} + b - \lambda u_i^{q-1}]$. Now, applying the mean value theorem

$$(d\mathcal{L} + b - \lambda q \xi^{q-1})(u_1 - u_2) = 0$$

for some $u_2 \leq \xi \leq u_1$. Hence,

$$0 = \sigma_1[d\mathcal{L} + b - \lambda q \xi^{q-1}] \geq \sigma_1[d\mathcal{L} + b - \lambda q u_2^{q-1}],$$

but from (12), we get that $\sigma_1[d\mathcal{L} + b - \lambda q u_2^{q-1}] > \sigma_1[d\mathcal{L} + b - \lambda u_2^{q-1}] = 0$, which gives a contradiction. $\diamond$

In the sequel we shall denote $\theta_{[\lambda,q,p]}$ the unique positive solution of (6) if (H) holds, with $\theta_{[\lambda,q,p]} = 0$ if $\lambda \le 0$.

The following result is well known and it will be very useful to compare positive solutions of different logistic boundary value problems.

Lemma 1 *Assume (H). Then:*

1. *If* $\lambda \le 0$, (6) *does not admit a positive subsolution.*
2. *If* $\lambda > 0$ *and* $\overline{u}$ *is a positive supersolution of* (6)*, then* $\theta_{[\lambda,q,p]} \le \overline{u}$.
3. *If* $\lambda > 0$ *and* $\underline{u}$ *is a positive subsolution of* (6)*, then* $\underline{u} \le \theta_{[\lambda,q,p]}$.

From Lemma 1 we obtain the following results. The first one shows the monotony of $\theta_{[\lambda,q,p]}$ with respect to the domain and the second one will be quite useful below.

Corollary 1 *Assume (H) and let* Ω_1 *be a subdomain of* Ω *with boundary* $\partial\Omega_1$ *sufficiently smooth. If we denote* $\theta^{\Omega}_{[\lambda,q,p]}$ *the unique positive solution of* (6) *in* Ω*, then*

$$\theta^{\Omega_1}_{[\lambda,q,p]} < \theta^{\Omega}_{[\lambda,q,p]} \qquad \text{in } \Omega_1.$$

Corollary 2 *Assume (H). Then there exists a constant* $K(\lambda) := K(\Omega, \lambda, q, p) > 0$ *such that*

$$K(\lambda)\varphi_1[\mathcal{L}] \le \theta_{[\lambda,q,p]} < \left(\frac{\lambda}{b_L}\right)^{\frac{1}{p-q}}. \tag{13}$$

Proof. We will prove that $K\varphi_1[\mathcal{L}]$ is a subsolution of (6). Then the first inequality of (13) follows from Lemma 1. Indeed, $K\varphi_1[\mathcal{L}]$ is a subsolution of (6) if, for example,

$$K^{1-q}\sigma_1[\mathcal{L}] + b_M K^{p-q} = \lambda. \tag{14}$$

Now, for fixed $\lambda > 0$, (14) has a unique positive solution which we denote $K(\lambda)$ and which satisfies

$$\lim_{\lambda\downarrow 0^+} K(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\uparrow +\infty} K(\lambda) = \infty.$$

The second inequality of (13) follows from (9) and the strong maximum principle. $\diamond$

Remark 1 *It is important to note:*

1. *If* $p = 1$,

$$K(\lambda) = \left(\frac{\lambda}{\sigma_1[\mathcal{L}] + b_M} \right)^{\frac{1}{1-q}}.$$

2. *If* $1 < p$,

$$K(\lambda) = O(\lambda^{1/(1-q)}) \quad \textit{if } \lambda \downarrow 0^+ \quad \textit{and} \quad K(\lambda) = O(\lambda^{1/(p-q)}) \quad \textit{if } \lambda \uparrow +\infty.$$

3. *If* $p < 1$,

$$K(\lambda) = O(\lambda^{1/(p-q)}) \quad \textit{if } \lambda \downarrow 0^+ \quad \textit{and} \quad K(\lambda) = O(\lambda^{1/(1-q)}) \quad \textit{if } \lambda \uparrow +\infty.$$

When $b(x) = b \in \mathbb{R}$, Lemma 1 can be used to prove some monotony properties of $\theta_{[\lambda,q,p]}$ with respect to λ.

Proposition 1 *Suppose (H) and that* $b(x) = b \in \mathbb{R}$, $\lambda, \mu > 0$. *The following assertions are true:*

1. *Assume* $1 \leq p$. *If* $\lambda \geq \mu$, *then*

$$\left(\frac{\lambda}{\mu}\right)^{1/(p-q)} \theta_{[\mu,q,p]} \leq \theta_{[\lambda,q,p]} \leq \left(\frac{\lambda}{\mu}\right)^{1/(1-q)} \theta_{[\mu,q,p]}.$$

2. *Assume* $p < 1$. *If* $\lambda \geq \mu$, *then*

$$\left(\frac{\lambda}{\mu}\right)^{1/(1-q)} \theta_{[\mu,q,p]} \leq \theta_{[\lambda,q,p]} \leq \left(\frac{\lambda}{\mu}\right)^{1/(p-q)} \theta_{[\mu,q,p]}.$$

Proof. We only prove the first part; the second one follows similarly. So, assume $1 \leq p$ and take $\eta := (\lambda/\mu)^{1/(p-q)}$. It can be showed that $\eta\theta_{[\mu,q,p]}$ is a subsolution of (6). Analogously, it can be proved that $(\lambda/\mu)^{1/(1-q)}\theta_{[\mu,q,p]}$ is a supersolution of (6). From Lemma 1, the result follows. $\diamond$

As an immediate consequence of Proposition 1, we obtain the following result:

Corollary 3 *Assume (H) and that* $b(x) = b \in \mathbb{R}$. *The following assertions are true:*

1. $\theta_{[\lambda,q,p]}$ *is increasing in* λ.

2. *If $1 < p$, then*

$$\frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(p-q)}} \quad \textit{is increasing in } \lambda \textit{ and } \quad \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(1-q)}} \quad \textit{is decreasing in } \lambda.$$

3. *If $p < 1$, then*

$$\frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(p-q)}} \quad \textit{is decreasing in } \lambda \textit{ and } \quad \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(1-q)}} \quad \textit{is increasing in } \lambda.$$

4. *If $p = 1$, then*

$$\frac{\theta_{[\lambda,q,1]}}{\lambda^{1/(1-q)}} \quad \textit{is constant in } \lambda.$$

Remark 2 *1. The case $p = 1$ is very special. In fact it holds*

$$\theta_{[\lambda,q,1]} = \lambda^{1/(1-q)}\theta_{[1,q,1]}. \tag{15}$$

2. In the very special case, $q = 1$ and $p = 2$, it was shown in [11] that $\theta_{[\lambda,1,2]}/\lambda$ is increasing in λ. Thus, our result is a generalization of that one.

3 Asymptotic behaviour of the branch $\theta_{[\lambda,q,p]}$

We will regard (6) as a bifurcation problem with λ as the bifurcation parameter. By the above results, from the trivial state $u = 0$ emanates a curve of positive solutions at $\lambda = 0$. This curve goes to the right and to infinity as $\lambda \uparrow +\infty$. Throughout this section $\omega_{[\lambda,q]}$ will denote the unique positive solution of (7) with $d = 1$ and $b \equiv 0$.

The main result of this section completes the information of Corollary 3.

Theorem 3 *Assume (H).*

1. *If $1 < p$, then*

$$\lim_{\lambda\downarrow 0^+} \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(1-q)}} = \omega_{[1,q]} \qquad \textit{in } C^2(\overline{\Omega}).$$

$$\lim_{\lambda\uparrow +\infty} \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(p-q)}} = \left(\frac{1}{b(x)}\right)^{1/(p-q)} \qquad \textit{uniformly on compacts of } \Omega.$$

2. If $p<1$, then

$$\lim_{\lambda\downarrow 0^+}\frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(p-q)}}=\left(\frac{1}{b(x)}\right)^{1/(p-q)} \qquad \textit{uniformly on compacts of } \Omega.$$

$$\lim_{\lambda\uparrow+\infty}\frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(1-q)}}=\omega_{[1,q]} \qquad \textit{in } C^2(\overline{\Omega}).$$

3. If $p=1$, then

$$\lim_{\lambda\downarrow 0^+}\frac{\theta_{[\lambda,q,1]}}{\lambda^{1/(1-q)}}=\lim_{\lambda\uparrow+\infty}\frac{\theta_{[\lambda,q,1]}}{\lambda^{1/(1-q)}}=\theta_{[1,q,1]}.$$

To prove this result we need some preliminaries. Consider the following problem

$$\begin{cases} d\mathcal{L}w &= w^q-b(x)w^p \quad \text{in } \Omega,\\ \quad w &= 0 \qquad\qquad\quad\ \text{on } \partial\Omega, \end{cases} \tag{16}$$

with $d>0$. Observe that this problem is in the setting of (3) and so, fixed $d>0$, there exists a unique positive solution of (16) which we will denote $\Phi_{[d,q,p]}$. The following result provides us with the behaviour of $\Phi_{[d,q,p]}$ as $d\uparrow+\infty$ and $d\downarrow 0^+$. This is a singular perturbation problem. In fact we give a proof that is a slight modification of the Theorem 3.4 in [4]; we include it for reader's convenience.

Theorem 4 *Assume (H) and let $\Phi_{[d,q,p]}$ be the unique positive solution of* (16)*. Then*

$$\lim_{d\downarrow 0^+}\Phi_{[d,q,p]}=\left(\frac{1}{b(x)}\right)^{\frac{1}{p-q}} \qquad \textit{uniformly on compact subsets of } \Omega,$$

$$\lim_{d\uparrow+\infty}\Phi_{[d,q,p]}=0 \qquad \textit{uniformly on } \Omega. \tag{17}$$

Proof. We consider $\overline{u}_d=d^{-1}\omega_{[1,q]}$. It is easy to show that $\overline{u}_d$ is a supersolution of (16) provided that

$$\omega_{[1,q]}^q(1-d^{-q}+b(x)d^{-p}\omega_{[1,q]}^{p-q})\geq 0.$$

Taking d sufficiently large and a further application of Lemma 1 gives (17).

Let $\mathcal{K}$ be a compact subset of Ω. We shall show that given $\varepsilon>0$ there exists $d_0=d_0(K,\varepsilon)>0$ such that for every $d<d_0$

$$\left(\frac{1}{b}\right)^{\frac{1}{p-q}}-\varepsilon\leq\Phi_{[d,q,p]}\leq\left(\frac{1}{b}\right)^{\frac{1}{p-q}}+\varepsilon \qquad \text{in } \mathcal{K}. \tag{18}$$

Let $\beta = \beta(\varepsilon)$ be such that

$$0 < \beta(\varepsilon) < \left(\left(\frac{1}{b}\right)^{1/(p-q)} + \varepsilon\right)^{p-q} - \frac{1}{b}.$$

Take $\Phi \in C^\infty(\overline{\Omega})$ such that

$$\left(\frac{1}{b} + \beta\right)^{\frac{1}{p-q}} \leq \Phi \leq \left(\frac{1}{b}\right)^{\frac{1}{p-q}} + \varepsilon \qquad \text{in } \Omega.$$

Then, we have

$$\Phi^q - b(x)\Phi^p = b(x)\Phi^q(1/b(x) - \Phi^{p-q}) \leq -\beta b(x)\Phi^q \leq d\mathcal{L}\Phi \qquad \text{in } \Omega,$$

for any $d < d_1$, for some $d_1(\varepsilon)$. Thus, for any $d < d_1$ the function Φ is a supersolution of (16) and from Lemma 1, we get

$$\Phi_{[d,q,p]} \leq \Phi \leq \left(\frac{1}{b}\right)^{\frac{1}{p-q}} + \varepsilon.$$

By a compactness argument, to complete the proof of (18) it suffices to show that given $x_0 \in \mathcal{K}$ there exist $r_0 > 0$ and $d_2 = d_2(x_0)$ such that for each $d < d_2$

$$\Phi_{[d,q,p]} \geq \left(\frac{1}{b}\right)^{\frac{1}{p-q}} - \varepsilon \qquad \text{in } B(x_0, r_0).$$

For any $B(x_0, r) \subset \Omega$, $r > 0$, from Corollary 1 we have

$$\Phi_{[d,q,p]}^{B(x_0,r)} \leq \Phi_{[d,q,p]} \qquad \text{in } B(x_0, r).$$

Thus, to complete the proof it remains to show that for any $d < d_2$,

$$\Phi_{[d,q,p]}^{B(x_0,2r_0)} \geq \left(\frac{1}{b}\right)^{\frac{1}{p-q}} - \varepsilon \qquad \text{in } B(x_0, r_0).$$

We consider two different cases:

Case 1: Suppose there exists $r_0 > 0$ such that $b(x) = b \in \mathbb{R}$ in $B_0 := B(x_0, 2r_0) \subset \Omega$. Let $\varphi_1^{B_0}[\mathcal{L}]$ normalized so that

$$\|\varphi_1^{B_0}[\mathcal{L}]\|_{\infty, B_0} = \frac{1}{2}. \tag{19}$$

Set $B_1 := B(x_0, r_0)$. Then, $\varphi_1^{B_0}[\mathcal{L}](x) > 0$ for each $x \in \overline{B}_1$ and there exists $\varphi_0 \in C^2(B_1)$ such that

$$\varphi_0(x_0) = 1\,, \qquad \|\varphi_0\|_{\infty, B_1} = 1\,, \qquad \varphi_0(x) > 0 \quad \forall x \in \overline{B}_1 \tag{20}$$

and the function $\Psi : B_0 \to \mathbb{R}$ defined by

$$\Psi(x) = \begin{cases} \varphi_1^{B_0}[\mathcal{L}](x) & \text{if } x \in B_0 \setminus B_1, \\ \varphi_0(x) & \text{if } x \in \overline{B}_1, \end{cases}$$

lies in $C^2(B_0)$. Given $\delta \in (0,1)$, we define

$$\Psi_\delta := \delta \left(\frac{1}{b}\right)^{\frac{1}{p-q}} \Psi,$$

Since $b \in \mathbb{R}$, then $\Psi_\delta \in C^2(B_0)$. It is not hard to show that Ψ_δ is a positive subsolution of (16) if, and only if,

$$\frac{\mathcal{L}\Psi}{\Psi^q} \le \frac{1}{d} b^{(1-q)/(p-q)} \delta^{q-1} (1 - \delta^{p-q} \Psi^{p-q}) \qquad \text{in } B_0, \tag{21}$$

and this inequality holds if d is sufficiently small. Indeed, observe that the left hand side of (21) is bounded above in B_0. From (19) and (20), we have that $\Psi \le \Psi^q$, and so

$$\frac{\mathcal{L}\Psi}{\Psi^q} \le \frac{\mathcal{L}\Psi}{\Psi} \le C,$$

for some $C > 0$. This last inequality follows by the strong maximum principle. Thus, since $\delta < 1$ and $0 \le \Psi \le 1$, it is sufficient to take d small to satisfy (21). From Lemma 1, we have that for d sufficiently small

$$\Psi_\delta \le \Phi_{[d,q,p]}^{B_0} \le \Phi_{[d,q,p]} \qquad \text{in } B_0.$$

Clearly, since $\Psi(x_0) = 1$ if δ is taken sufficiently close to 1, then Ψ_δ will be as close as we want to $(1/b)^{1/(p-q)}$ on some ball centered at x_0. This completes the proof in this case.

Case 2: Assume $b(x)$ is not constant in some ball centered at x_0. We have

$$d\mathcal{L}\Phi_{[d,q,p]}^{B_0} = (\Phi_{[d,q,p]}^{B_0})^q - b(x)(\Phi_{[d,q,p]}^{B_0})^p \ge (\Phi_{[d,q,p]}^{B_0})^q - b_{M,B_0}(\Phi_{[d,q,p]}^{B_0})^p$$

and so, $\Phi^{B_0}_{[d,q,p]}$ is a positive supersolution of (16) with $b(x) = b_{M,B_0} \in \mathbb{R}$, and so from Lemma 1 that

$$\Phi^{B_0}_{[d,q,p]} \geq \hat{\Phi}^{B_0}_{[d,q,p]},$$

where $\hat{\Phi}^{B_0}_{[d,q,p]}$ stands for the unique positive solution of (16) with $b(x) = b_{M,B_0} \in \mathbb{R}$. Thus, from the Case 1, there exists $r_1 > 0$ such that

$$\Phi^{B_0}_{[d,q,p]} \geq \hat{\Phi}^{B_0}_{[d,q,p]} \geq (1/b_{M,B_0})^{1/(p-q)} - \frac{\varepsilon}{2} \qquad \text{in } B(x_0, r_1).$$

Therefore, if B_0 is chosen so that for each $x \in B_0$

$$(1/b_{M,B_0})^{1/(p-q)} \geq (1/b(x))^{1/(p-q)} - \frac{\varepsilon}{2},$$

then

$$\Phi^{B_0}_{[d,q,p]} \geq \left(\frac{1}{b(x)}\right)^{1/(p-q)} - \varepsilon$$

for each $x \in B(x_0, r_1)$. This completes the proof. $\diamond$

We consider the equation

$$\begin{cases} \mathcal{L}w = w^q - db(x)w^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

From Theorem 1, given $d > 0$ there exists a unique positive solution $\Theta_{[d,q,p]}$ of (22). The following result provides us the behaviour of $\Theta_{[d,q,p]}$ as $d \downarrow 0^+$ and $d \uparrow +\infty$.

Theorem 5 *Assume (H) and let $\Theta_{[d,q,p]}$ be the unique positive solution of* (22). *Then,*

$$\lim_{d \downarrow 0^+} \Theta_{[d,q,p]} = \omega_{[1,q]} \qquad \textit{in } C^{2,\nu}(\overline{\Omega}), \textit{ for some } \nu \in (0,1)$$

$$\lim_{d \uparrow +\infty} \Theta_{[d,q,p]} = 0 \qquad \textit{uniformly on } \Omega.$$

Proof. By Corollary 2,

$$\Theta_{[d,q,p]} \leq (\frac{1}{db_L})^{1/(p-q)},$$

from which the second relation follows.

On the other hand, it is not hard to prove that $\overline{u} = \omega_{[1,q]}$ is a supersolution of (22) and hence,

$$\|\Theta_{[d,q,p]}\|_\infty \leq \|\omega_{[1,q]}\|_\infty = K \text{ (independent of } d\text{)}.$$

Thus, according to the L^s theory of elliptic equations, $\{\Theta_{[d,q,p]}\}_d$ is a bounded sequence in $W^{2,s}(\Omega)$, for $s > 1$, and so we can extract a convergent subsequence, again labeled by d, such that

$$\Theta_{[d,q,p]} \to \overline{w} \qquad \text{in } C^{1,\alpha}(\overline{\Omega}), \text{ where } 0 < \alpha = 1 - N/s < 1,$$

as $d \downarrow 0^+$. Using (22) we get

$$\Theta_{[d,q,p]} = (\mathcal{L})^{-1}(\Theta^q_{[d,q,p]} - db(x)\Theta^p_{[d,q,p]}),$$

and so

$$\begin{cases} \mathcal{L}\overline{w} = \overline{w}^q & \text{in } \Omega, \\ \overline{w} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, as in Corollary 2, we can get a constant $K = K(\Omega) > 0$, independent of d, such that

$$K(\Omega)\varphi_1[\mathcal{L}] \leq \Theta_{[d,q,p]}, \qquad \text{for all } d \in [0, d_0], \text{ for some } d_0 > 0.$$

In fact, in this case we can take K satisfying

$$db_M K^{p-q} + K^{1-q}\sigma_1[\mathcal{L}] = 1.$$

It can be proved that the map

$$d \in [0, d_0] \mapsto K(d)$$

is continuous, and so there exists the constant $K(\Omega)$. We can deduce that $\overline{w} = \omega_{[1,q]}$ and by Ascoli-Arzela's Theorem all sequence converges in $C^{2,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$ and the result follows. $\diamond$

Proof Theorem 3. Let us define

$$\Psi_{[\lambda,q,p]} := \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(p-q)}}.$$

It is easy to check that $\Psi_{[\lambda,q,p]}$ is the unique positive solution of the equation

$$\left\{\begin{array}{rcll} \frac{1}{\lambda^{(p-1)/(p-q)}}\mathcal{L}w &=& w^q - b(x)w^p & \text{in } \Omega, \\ w &=& 0 & \text{on } \partial\Omega, \end{array}\right.$$

included in the setting (16). Now, Theorem 4 proves two relations of Theorem 3.

If we write,

$$\chi_{[\lambda,q,p]} := \frac{\theta_{[\lambda,q,p]}}{\lambda^{1/(1-q)}},$$

then $\chi_{[\lambda,q,p]}$ is the unique positive solution of

$$\left\{\begin{array}{rcll} \mathcal{L}w &=& w^q - \lambda^{(p-1)/(1-q)}b(x)w^p & \text{in } \Omega, \\ w &=& 0 & \text{on } \partial\Omega. \end{array}\right.$$

From Theorem 5, the other relations follow.

Finally, for $p = 1$ the result follows by (15). The proof of Theorem 3 is completed. $\diamond$

Now, we denote θ_λ the unique positive solution of (6) for $q = 1$ and $p > 1$ if $\lambda > \sigma_1[\mathcal{L}]$, with $\theta_\lambda = 0$ if $\lambda \leq \sigma_1[\mathcal{L}]$. The next results provide us the behaviour of $\theta_{[\lambda,q,p]}$ as $q \uparrow 1$. We consider two different cases: $p > 1$ and $p = 1$.

Theorem 6 *Assume $p > 1 > q$ and $\lambda > 0$. Then*

$$\lim_{q\uparrow 1} \theta_{[\lambda,q,p]} = \theta_\lambda \qquad \text{in } C^{2,\nu}(\overline{\Omega}) \text{ for some } \nu \in (0,1).$$

Proof. Fix $\delta \in (0,1)$. We know from Corollary 2 that for $q \in [1-\delta, 1]$,

$$\|\theta_{[\lambda,q,p]}\|_\infty \leq \left(\frac{\lambda}{b_L}\right)^{\frac{1}{p-q}} \leq K \quad \text{(independent of } q.)$$

We can reason as in Theorem 5 and conclude that there exists a subsequence $\{\theta_{[\lambda,q,p]}\}_q$ such that

$$\theta_{[\lambda,q,p]} \to \overline{w} \geq 0 \qquad \text{in } C^{1,\alpha}(\overline{\Omega}), \text{ with } 0 < \alpha < 1,$$

as $q \uparrow 1$ with $\overline{w}$ satisfying

$$\left\{\begin{array}{rcll} \mathcal{L}\overline{w} &=& \lambda\overline{w} - b(x)\overline{w}^p & \text{in } \Omega, \\ \overline{w} &=& 0 & \text{on } \partial\Omega. \end{array}\right.$$

So, if $\lambda \leq \sigma_1[\mathcal{L}]$, $\overline{w} = 0$. On the other hand, if $\lambda > \sigma_1[\mathcal{L}]$, we can choose $K(\lambda)$, independent of q, such that

$$K(\lambda)\varphi_1[\mathcal{L}] \leq \theta_{[\lambda,q,p]}.$$

Again the Ascoli-Arzela's Theorem completes the proof. $\diamond$

The case $p = 1$ is more complicated. We are going to prove that $\theta_{[\lambda,q,p]}$ tends to 0 when $\lambda < \sigma_1[\mathcal{L}+b]$ and to infinity when $\lambda > \sigma_1[\mathcal{L}+b]$ as $q \uparrow 1$, showing that the bifurcation diagram with $q < 1$ (see Figure 2) "converges"to the one with $q = p = 1$ (see Figure 1).

Theorem 7 *Assume* $0 < q < p = 1$. *Then:*

1. *If* $\lambda < \sigma_1[\mathcal{L}+b]$, *then* $\|\theta_{[\lambda,q,1]}\|_\infty \to 0$ *as* $q \uparrow 1$.

2. *If* $\lambda > \sigma_1[\mathcal{L}+b]$, *then* $\|\theta_{[\lambda,q,1]}\|_\infty \to \infty$ *as* $q \uparrow 1$.

Proof. For the first part, we fix $\lambda < \sigma_1[\mathcal{L}+b]$. From the continuous dependence of $\sigma_1[\mathcal{L}+b]$ respect to the domain, there exists a regular domain $\Omega' \supset \Omega$ such that

$$\lambda < \sigma_1^{\Omega'}[\mathcal{L}+b] < \sigma_1^{\Omega}[\mathcal{L}+b]. \tag{23}$$

Let $\varphi_1' := \varphi_1^{\Omega'}[\mathcal{L}+b]$ be with $\|\varphi_1'\|_{\infty,\Omega'} = 1$. It is not difficult to see that $\overline{u} := M\varphi_1'$ is a supersolution of (6) being

$$M = \left(\frac{\lambda}{\sigma_1^{\Omega'}[\mathcal{L}+b]}\right)^{1/(1-q)} \frac{1}{(\varphi_1')_{L,\Omega}},$$

and so, by Lemma 1,

$$\|\theta_{[\lambda,q,1]}\|_{\infty,\Omega} \leq M\|\varphi_1'\|_{\infty,\Omega}.$$

Now, it suffices to use (23) and to tend $q \uparrow 1$.

For the second part, we are going to build a subsolution whose norm tends to infinity. We take $\varphi_1[\mathcal{L}+b]$ normalized such that $\|\varphi_1[\mathcal{L}+b]\|_\infty = 1$. It is easy to prove that $\underline{u} := C\varphi_1[\mathcal{L}+b]$ is a subsolution of (6) with

$$C = \left(\frac{\lambda}{\sigma_1[\mathcal{L}+b]}\right)^{1/(1-q)}.$$

Again, taking $q \uparrow 1$, the proof concludes since $\lambda > \sigma_1[\mathcal{L} + b]$. $\diamond$

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