A nonlinear age-dependent model with spatial diffusion ⋆

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Abstract

The main goal of this paper is to study the existence and uniqueness of positive solution for a nonlinear age-dependent equation with spatial diffusion. For that, we mainly use properties of an eigenvalue problem related to the equation and the sub-supersolution method. We justify that this method works for this kind of equation, in which appears a potential blowing-up and a non-local initial condition.

Key words: Sub-supersolution method, non-local initial condition, logistic age-dependent.

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1 Introduction

In this paper we study the existence of solution of the nonlinear problem

$$\begin{cases} u_a - \Delta u + q(x,a)u = g(x,a,u) & \text{in } Q := \Omega \times (0,A), \\ u = 0 & \text{on } \Sigma := \partial\Omega \times (0,A), \\ u(x,0) = \int_0^A \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $A > 0$, β a bounded, nonnegative and nontrivial function, q is a measurable function blowing up at $a = A$ and g is a measurable function with assumptions that will be detailed below. For that, we are going to employ the sub-supersolution method. Observe that, mathematically, apart from the nonlinearity, (1) has two main difficulties: the coefficient q is not bounded and the initial datum is non-local. So, we can not apply the classical sub-supersolution method for parabolic problem (see for instance [12] and [6]).

We assure that assuming the existence of an ordered pair of sub-supersolution of (1), there exists a solution between the sub and the supersolution provided of, basically, g is a lipschitz function in the variable u. Hence, our result generalizes the classical ones for parabolic problem in the two ways mentioned above. We would like to point out that although comparison results have been used in this framework (see for instance Lemma 2 in [7] and Lemma 4.5 in [9]), we have not found a sub-supersolution method developed for this kind of problem.

We apply the above result to study the existence and uniqueness of positive solution of the logistic equation

$$\begin{cases} u_a - \Delta u + q(x,a)u = \lambda u - u^2 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \int_0^A \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{2}$$

with $\lambda \in \mathbb{R}$. Equ. (2) provides us with the steady-state solutions of the corresponding time dependent problem, and so, $u(x,a)$ represents the density population of a species of age a localized in x, being thus Ω the habitat of the species. Observe that we are assuming that Ω is surrounded by inhospitable areas because the homogenous Dirichlet boundary condition. A is the maximal age for the species, $\lambda - q(x,a)$ is the birth (when positive) or death (when negative) rate function. In this context, the fact that $q(x,a)$ blows up at $a = A$ it will assure that the species dies out at the age of A. Finally, β is the rate fertility function.

In general, the study of the structure of positive solutions set of a problem similar to (2) is far from to be easy. In fact, to our knowledge only linear problem in u has been analyzed in [9], although in this case the equation also depends on the total population $P(x) = \int_0^A u(x,a)da$. Specifically in [9], the reaction term is the trivial function and q and β verify

$$q(x,a) = q_1(a) + q_2(P), \qquad \beta(x,a) = \beta_1(a),$$

and moreover P is the positive solution of the classical logistic elliptic equation, and so it is known (see Theorem 3.5 in [9]). Under these assumptions, the author proved that only separable solutions exist and he looked at them giving the explicit solution. Finally, we want to mention that nonlinear problems in u without diffusion have been studied previously, see for example [13] and [14].

For the nonexistence and uniqueness results of (2) we will study an eigenvalue problem related to (2) with the coefficient q depending of a and x. Specifically, the following eigenvalue problem is analyzed

$$\begin{cases} u_a - \Delta u + q(x,a)u = \lambda u & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \int_0^A \beta(x,a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{3}$$

For that, we follow the main idea of [8] but we have given another point of view to their results and some of them will be shaped. We prove that there exists a unique principal eigenvalue (in the sense that it is the unique with positive eigenfunction associated) denoted by $\lambda_0(q)$. Observe that great difference that there exists between problem (3) and the classical parabolic one (3) with $u(x,0) = u_0(x) > 0$ instead of the non-local initial condition, where a unique positive solution exists for all $\lambda \in \mathbb{R}$.

We apply all the above results to show that (2) possesses a positive solution if, and only if, $\lambda > \lambda_0(q)$. Moreover, if $\lambda \leq \lambda_0(q)$ the only nonnegative solution of (2) is the trivial one and if $\lambda > \lambda_0(q)$ there exists a unique positive solution. Again, a drastic change occurs with respect to the problem (2) with $u(x,0) = u_0(x)$, which possesses a unique positive solution for all $\lambda \in \mathbb{R}$.

An outline of the work is as follows: in Section 2 we analyze the eigenvalue problem related to (2), study the linear case and establish a strong maximum principle; in Section 3 we prove that the sub-supersolution method works and in the last section we apply these results to the logistic equation (2).

2 The eigenvalue problem

In this section we study the eigenvalue problem (3) assuming that

$(\mathcal{H}_q)$ q is a function such that $q \in L^\infty(\overline{\Omega} \times (0,r))$ for $r < A$ and

$$\int_0^r q_M(a)da < \infty, \qquad \int_0^A q_L(a)da = +\infty, \tag{4}$$

being $q_L(a) := \inf_{x\in\overline{\Omega}} q(x,a)$ and $q_M(a) := \sup_{x\in\overline{\Omega}} q(x,a)$.

$(\mathcal{H}_\beta)$ $\beta \in L^\infty(Q)$, $\beta \geq 0$, nontrivial and

$$mes\{a \in [0,A] : \beta_L(a) := \inf_{x\in\overline{\Omega}} \beta(x,a) > 0\} > 0.$$

Remark 1 *Condition* (4) *is necessary to have that* $\lim_{a\uparrow A} u(x,a) \equiv 0$, *for* u *solution of* (3), *see Remark 4 below.*

Definition 2 λ *is an eigenvalue of* (3) *if there exists* $u \in L^2(0,A;H_0^1(\Omega))$, $u_a + qu \in L^2(0,A;H^{-1}(\Omega))$ *with* $u \neq 0$ *solution of* (3) *in the sense that* $\forall v \in L^2(0,A;H_0^1(\Omega))$*:*

$$\int_0^A < u_a + qu, v > \, da + \int_Q \nabla u \cdot \nabla v \, da\, dx = \lambda \int_Q uv \, da\, dx,$$

$$u(x,a) = 0 \qquad on\ \Sigma,$$

$$u(x,0) = \int_0^A \beta(x,a)u(x,a)\, da, \quad in\ \Omega,$$

where $<,>$ *denotes the duality pairing between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$.

We say that λ *is a principal eigenvalue if* $u > 0$ *in* Q.

Before studying (3) we need to analyze the autonomous case, i.e.,

$$\begin{cases} u_a - \Delta u + m(a)u = \lambda u & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \int_0^A \gamma(a)u(x,a)da & \text{in } \Omega, \end{cases} \tag{5}$$

where

(H_m) $m \in L^\infty(0, r)$ for $r < A$ and

$$\int_0^A m(a)da = +\infty. \qquad (6)$$

(H_γ) $\gamma \in L^\infty(0, A)$, $\gamma \geq 0$ and nontrivial.

Theorem 3 *Assume (H_m) and (H_γ). Then, (5) possesses a positive solution if, and only if,*

$$\lambda = \lambda_1 + r_m,$$

where r_m is defined by

$$1 = \int_0^A \gamma(a) e^{r_m a - \int_0^a m(s)ds} da, \qquad (7)$$

and λ_1 is the principal eigenvalue of the $-\Delta$ under homogeneous Dirichlet boundary condition. Moreover, in this case the solution is

$$\varphi_0(x, a) = e^{r_m a - \int_0^a m(s)ds} \varphi_1(x),$$

being φ_1 a positive eigenfunction associated to λ_1.

PROOF. First, thanks to Theorem 3.5 of [9], any solution of (5) is separable. Observe that in the cited result, $A = \infty$, but we can adapt the proof to the case $A < \infty$. Take

$$u(x, a) = p(a)\varphi(x).$$

Then,

$$p_a + m(a)p = rp, \qquad r \in \mathbb{R};$$

and so,

$$p(a) = p_0 e^{ra - \int_0^a m(s)ds}.$$

It is not hard to show that p satisfies the initial condition

$$p(0) = \int_0^A \gamma(a)p(a)da,$$

if $r = r_m$. On the other hand,

$$-\Delta\varphi = (\lambda - r_m)\varphi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega,$$

and thus, $\lambda - r_m = \lambda_1$ and $\varphi = \varphi_1$. This completes the proof.

Remark 4 *(1) Observe that thanks to* (6) *we get that* $\lim_{a\uparrow A} \varphi_0(x,a) = 0$ *for all* $x \in \Omega$.

(2) A related result was proved in Theorem 1 of [4], using properties of the infinitesimal generator associated to (5).

To end the autonomous case, we establish a strong maximum principle.

Definition 5 *Denote by* $L^2_+(\Omega) := \{f \in L^2(\Omega) : f(x) \geq 0 \text{ a. e. } x \in \Omega\}$. *We say that* $u \in L^2_+(\Omega)$ *is quasi-interior point of* $L^2_+(\Omega)$, *and we write* $u \gg 0$, *if*

$$\int_\Omega u(x)f(x)dx > 0, \qquad \text{for all } f \in L^2_+(\Omega) \text{ and nontrivial.}$$

Lemma 6 *Assume* (H_m) *and that* u *is solution of*

$$\begin{cases} u_a - \Delta u + m(a)u = f(x,a) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \phi(x) & \text{in } \Omega, \end{cases} \tag{8}$$

and $f \geq 0$, $\phi \geq 0$ *and some of the inequalities strict. Then,* $u \gg 0$.

If $f \equiv \phi \equiv 0$, *then* $u \equiv 0$.

PROOF. Observe that if u is solution of (8), then

$$v := ue^{\int_0^a m(s)ds},$$

is the solution of the equation

$$\begin{cases} v_a - \Delta v = g(x,a) := f(x,a)e^{\int_0^a m(s)ds} & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(x,0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{9}$$

By [10] (see also [2]), $v \gg 0$ and so $u \gg 0$.

Similarly, if $f \equiv \phi \equiv 0$, then $v \equiv 0$, and so that u.

We state now a result for the non-autonomous linear case. Consider the problem

$$\begin{cases} u_a - \Delta u + q(x,a)u = f(x,a) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = \phi(x) & \text{in } \Omega, \end{cases} \tag{10}$$

where q satisfies $(\mathcal{H}_q)$ and $\phi \in L^2(\Omega)$.

Lemma 7 *Suppose that $f \in L^2(Q)$. Then, there exists a unique solution u of (10) such that $u \in L^2(0, A; H_0^1(\Omega))$ and $u_a + q(x,a)u \in L^2(0, A; H^{-1}(\Omega))$. Moreover, for each $0 < A_0 < A$ we have that $u \in C([0, A_0]; L^2(\Omega))$.*

Furthermore, we have the following comparison principles:

(1) If $f \geq 0$ and $\phi \geq 0$, then $u \geq 0$. If some of the inequalities is strict, we deduce that $u \gg 0$.

(2) If $f_1 \geq f_2 \geq 0$, $\phi_1 \geq \phi_2 \geq 0$ and $q_1 \leq q_2$ in their respective domains, then $u_1 \geq u_2$, where u_i, $i = 1, 2$, is the solution of (10) with $f = f_i$, $\phi = \phi_i$ and $q = q_i$.

PROOF. Under the change of variable

$$w = e^{-ka}u, \qquad k > 0,$$

w satisfies

$$\begin{cases} w_a - \Delta w + (q+k)w = g := fe^{-ka} & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = \phi(x) & \text{in } \Omega, \end{cases} \tag{11}$$

and so by $(\mathcal{H}_q)$, we can take k large such that $q + k/3 \geq 0$. We study now (11) instead of (10).

Define

$$q_n := \min\{q, n\}, \qquad n \in \mathbb{N},$$

and consider the problem

$$\begin{cases} w_a - \Delta w + (q_n(x,a) + k)w = g(x,a) & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{12}$$

Now, for each $n \in \mathbb{N}$, since $q_n + k$ is bounded, there exists a unique w_n solution of (12) with $w_n \in L^2(0,A;H_0^1(\Omega))$ and $(w_n)_a \in L^2(0,A;H^{-1}(\Omega))$. Multiplying (12) by w_n and integrating we obtain

$$\frac{1}{2}\frac{d}{da}\int_Q |w_n|^2 + \int_Q |\nabla w_n|^2 + \int_Q (q_n + k)w_n^2 = \int_Q g w_n,$$

and so, applying that $2ab \leq (\varepsilon^2 a^2 + (1/\varepsilon^2)b^2)$ we get

$$\frac{1}{2}\frac{d}{da}\int_Q |w_n|^2 + \int_Q |\nabla w_n|^2 + \int_Q (q_n + k/3)w_n^2 + (k/3)w_n^2 \leq C.$$

Now, we can extract a sequence (w_n) such that

$$\begin{aligned} & w_n \rightharpoonup w && \text{in } L^2(0,A;H_0^1(\Omega)), \\ & \sqrt{q_n + (k/3)}\, w_n \rightharpoonup h && \text{in } L^2(Q), \\ & (w_n)_a + (q_n + k/3)w_n \rightharpoonup z && \text{in } L^2(0,A;H^{-1}(\Omega)). \end{aligned}$$

On the other hand, for $\varphi \in C_c^\infty(0,A;H_0^1)$, and for n large enough, we get

$$\int_0^A < (w_n)_a + (q_n + k/3)w_n > \varphi = \int_0^A (-w_n\varphi_a + (q+k/3)w_n\varphi) \to$$

$$\to \int_0^A (-w\varphi_a + (q+k/3)w)\varphi),$$

and so

$$z = w_a + (q+k/3)w.$$

This shows that u is solution of (10).

The regularity $u \in C([0,A_0];L^2(\Omega))$, $A_0 < A$, follows considering the equation (10) in $Q_0 := \Omega \times (0,A_0)$, see for example Theorem X.1 of [3].

For the uniqueness, take two different solutions u_1 and u_2. Then, $w = u_1 - u_2$ satisfies that

$$w_a - \Delta w + q(x,a)w = 0, \qquad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \qquad w(x,0) = 0 \quad \text{in } \Omega.$$

It suffices to multiply this problem by w and obtain that $w \equiv 0$.

Now, assume that $f \geq 0$ and $\phi \geq 0$ and let u the solution of (10). Then, by the classical maximum principle (observe that the potential is bounded) applied to (12) it follows that $w_n \geq 0$, and so that $u \geq 0$. Moreover,

$$0 \leq f = u_a - \Delta u + q(x,a)u \leq u_a - \Delta u + q_M(a)u,$$

and so the fact of $u \gg 0$ follows by Lemma 6.

The main result of this section is:

Theorem 8 *Assume $(\mathcal{H}_q)$ and $(\mathcal{H}_\beta)$. Then, there exists a unique principal eigenvalue of* (3)*, denoted by $\lambda_0(q)$. Moreover, it is simple and the only eigenvalue having a positive eigenfunction. The positive eigenfunctions can be taken bounded. Furthermore, for any other eigenvalue λ of* (3)*, it holds that*

$$Re(\lambda) > \lambda_0(q). \tag{13}$$

Finally, the map

$$q \mapsto \lambda_0(q)$$

is increasing.

We need some preliminaries before proving this result. For each $\phi \in L^2(\Omega)$ we define z_ϕ the unique solution of (which exists by Lemma 7)

$$\begin{cases} z_a - \Delta z + q(x,a)z = 0 \text{ in } Q, \\ z = 0 \qquad\qquad\qquad\quad \text{on } \Sigma, \\ z(x,0) = \phi(x) \qquad\quad\;\, \text{in } \Omega, \end{cases} \tag{14}$$

and define the operator $\mathcal{B}_\lambda : L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{B}_\lambda(\phi) = \int_0^A \beta(x,a)e^{\lambda a} z_\phi(x,a)da.$$

The next result plays an important role in this work

Lemma 9 *(1) The operator $\mathcal{B}_\lambda$ is a well-defined, compact and positive operator.*

(2) It holds that

$$\mathcal{A}_\lambda(\phi) \leq \mathcal{B}_\lambda(\phi) \leq \mathcal{C}_\lambda(\phi) \qquad \forall \phi \geq 0, \tag{15}$$

where

$$\mathcal{A}_\lambda(\phi) := \int_0^A \beta_L(a)e^{\lambda a} w_\phi(x,a)da, \qquad \mathcal{C}_\lambda(\phi) := \int_0^A \beta_M(a)e^{\lambda a} y_\phi(x,a)da$$

being w_ϕ and y_ϕ the solutions of (14) *with $q(x,a) = q_M(a)$ and $q(x,a) = q_L(a)$, respectively (i. e., w_ϕ and y_ϕ are solutions of autonomous problems).*

(3) $\mathcal{B}_\lambda$ is an irreducible operator.

(4) If ϕ is a fixed point of $\mathcal{B}_\lambda$, then λ is an eigenvalue of (3).

(5) Conversely, if (λ, u) is a pair of eigenvalue-eigenfunction of (3)*, then $\phi(x) := u(x,0)$ is a fixed point of $\mathcal{B}_\lambda$.*

PROOF. That $\mathcal{B}_\lambda$ is well-defined follows by Lemma 7. The compactness is due to the properties of the mapping $\phi \mapsto z_\phi$, see also [8].

Paragraph b) follows by Lemma 7 b). Indeed, since $q_M(a) \geq q(x,a)$, $w_\phi \leq z_\phi$, and so $\mathcal{A}_\lambda \leq \mathcal{B}_\lambda$ because $\beta_L(a) \leq \beta(x,a)$.

Now, we are going to show that $\mathcal{B}_\lambda$ is an irreducible operator. Recall that a positive operator is irreducible if a power of the operator (eventually itself) is strongly positive. So, we will prove that it is strongly positive, i.e., if $\phi \geq 0$ and nontrivial then $\mathcal{B}_\lambda(\phi) \gg 0$. First, observe that

$$w_\phi \gg 0$$

by Lemma 6. As consequence, using $(\mathcal{H}_\beta)$ we have

$$\mathcal{B}_\lambda(\phi) \geq \mathcal{A}_\lambda(\phi) \gg 0.$$

This implies that $\mathcal{B}_\lambda$ is strongly positive.

Let ϕ be a fixed point of $\mathcal{B}_\lambda$. It is not difficult to show that $u = e^{\lambda a} z_\phi$ is an eigenfunction associated to λ.

Conversely, let (λ, u) be an eigenvalue and an associated eigenfunction of (3). By the regularity of u, see Lemma 7, we have that $\phi(x) := u(x,0) \in L^2(\Omega)$. Moreover,

$$z_\phi = z_{u(x,0)} = e^{-\lambda a} u(x,a), \tag{16}$$

and so $\mathcal{B}_\lambda \phi = \phi$. This completes the proof.

Now, define by

$$r(\mathcal{B}_\lambda),$$

the spectral radius of $\mathcal{B}_\lambda$. Since $\mathcal{B}_\lambda$ is a positive compact irreducible linear operator on a Banach lattice, $r(\mathcal{B}_\lambda)$ is positive, see Theorem 3 in [11]. By the Krein-Rutman Theorem (see Theorem 12.3 of [5] for a very general version),

$r(\mathcal{B}_\lambda)$ is an algebraically simple eigenvalue with a quasi-interior eigenfunction, and it is the only eigenvalue having a positive eigenfunction. So, we have the following result

Corollary 10 *λ_0 is a principal eigenvalue of* (3) *if, and only if,* $r(\mathcal{B}_{\lambda_0}) = 1$.

PROOF. Let $u_0 > 0$ be a principal eigenfunction associated to λ_0. By Lemma 6 it follows that $u_0 \gg 0$. Now, thanks to $(\mathcal{H}_\beta)$ we obtain that $\phi_0(x) := u_0(x,0) \gg 0$. Now, by Lemma 9 e), ϕ_0 is a strongly positive fixed point of $\mathcal{B}_{\lambda_0}$ and by Krein-Rutman Theorem we get that $r(\mathcal{B}_{\lambda_0}) = 1$.

Conversely, if $r(\mathcal{B}_{\lambda_0}) = 1$ there exists a strongly positive fixed point ϕ_0 of $\mathcal{B}_{\lambda_0}$. In this case, by Lemma 9 d) $u_0(x,a) = e^{\lambda_0 a} z_{\phi_0} \gg 0$ is a principal eigenfunction of (3).

As consequence of Theorem 3, we have for the autonomous problem that

Proposition 11 *Assume* (H_m) *and* (H_γ)*. Then,*

$$r(\mathcal{D}_{\lambda_1 + r_m}) = 1,$$

being

$$\mathcal{D}_\lambda(\phi) = \int_0^A \gamma(a) e^{\lambda a} p_\phi(x,a)\,da,$$

with p_ϕ the unique solution of (14) *with* $q(x,a) = m(a)$.

PROOF. Observe that for $\phi = \varphi_1$ we get that

$$p_{\varphi_1} = e^{-\lambda_1 a - \int_0^a m(s)ds} \varphi_1,$$

and so,

$$\mathcal{D}_\lambda(\varphi_1) = \varphi_1 \int_0^A \gamma(a) e^{(\lambda - \lambda_1)a - \int_0^a m(s)ds}\,da.$$

Thus,

$$\mathcal{D}_{\lambda_1 + r_m}(\varphi_1) = \varphi_1,$$

i.e., 1 is an eigenvalue with a positive eigenfunction. Then, by the Krein-Rutman Theorem $r(\mathcal{D}_{\lambda_1+r_m}) = 1$.

PROOF. [Theorem 8] First, recall that the map $\lambda \mapsto r(\mathcal{B}_\lambda)$ is increasing (see for instance Theorem 3.2 (v) in [1]). Second, we will prove that the map $\lambda \mapsto r(\mathcal{B}_\lambda)$ is continuous. Take $\lambda_n \to \lambda_0$ as $n \to +\infty$, then for all $\varepsilon > 0$ we have that $\lambda_0 - \varepsilon \leq \lambda_n \leq \lambda_0 + \varepsilon$ for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. We claim that

$$e^{-\varepsilon A} r(\mathcal{B}_{\lambda_0}) \leq r(\mathcal{B}_{\lambda_0 - \varepsilon}) \leq r(\mathcal{B}_{\lambda_n}) \leq r(\mathcal{B}_{\lambda_0+\varepsilon}) \leq e^{\varepsilon A} r(\mathcal{B}_{\lambda_0}), \qquad (17)$$

whence the continuity follows, and so if there exists a principal eigenvalue, this is unique.

Take ϕ_0 a principal eigenfunction associated to $r(\mathcal{B}_{\lambda_0})$. Then,

$$e^{\varepsilon A} r(\mathcal{B}_{\lambda_0})\phi_0 - \mathcal{B}_{\lambda_0+\varepsilon}(\phi_0) \geq 0,$$

and so by Theorem 3.2 (iv) in [1] we obtain that $e^{\varepsilon A} r(\mathcal{B}_{\lambda_0}) \geq r(\mathcal{B}_{\lambda_0+\varepsilon})$. This proves (17).

Now, applying Proposition 11 to $\mathcal{A}_\lambda$ and $\mathcal{C}_\lambda$ and Lemma 9 b), it follows that

$$1 = r(\mathcal{A}_{\lambda_1+r_{q_M}}) \leq r(\mathcal{B}_{\lambda_1+r_{q_M}}), \qquad 1 = r(\mathcal{C}_{\lambda_1+r_{q_L}}) \geq r(\mathcal{B}_{\lambda_1+r_{q_L}}),$$

and so, there exists $\lambda_0(q) \in (\lambda_1 + r_{q_L}, \lambda_1 + r_{q_M})$ such that $r(\mathcal{B}_{\lambda_0(q)}) = 1$. Again, the Krein-Rutman Theorem proves the character simple of $\lambda_0(q)$ and (13).

Now, take $q_1 \leq q_2$. Then the solutions of (14) with $q = q_1$ (resp. $q = q_2$), denoted by z_1 (resp. z_2), satisfy that

$$z_1 \geq z_2,$$

whence it follows that $\lambda_0(q_1) \leq \lambda_0(q_2)$.

We will prove that the positive eigenfunctions are bounded. Take φ an eigenfunction associated to $\lambda_0(q)$. Then, by (16) we get that

$$\varphi(x,a) = e^{\lambda_0(q)a} z_{\varphi(x,0)}(x,a).$$

On the other hand, it is not hard to prove that

$$z_{\varphi(x,0)}(x,a) \leq e^{-\int_0^a q_L(s)ds} c_{\varphi(x,0)}(x,a),$$

where $c_{\varphi(x,0)}$ denotes the solution of (9) with $g \equiv 0$ and $\phi(x) = \varphi(x,0) \in L^2(\Omega)$. Then,

$$\varphi(x,a) \leq e^{\lambda_0(q)a - \int_0^a q_L(s)ds} c_{\varphi(x,0)}(x,a),$$

and so, since $c_{\varphi(x,0)} \in C^\infty((0,A)\times\Omega)$ see Theorem X.1 of [3], it follows that φ is bounded.

3 The sub-supersolution method

Now, we want to study the nonlinear problem (1) where β and q satisfy $(\mathcal{H}_\beta)$ and $(\mathcal{H}_q)$ respectively, and $g : \Omega \times (0,A) \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function.

Definition 12 *(1) We say that a function* $u \in L^2(0,A;H_0^1(\Omega))$, $u_a + qu \in L^2(0,A;H^{-1}(\Omega))$, $g(x,a,u) \in L^2(Q)$ *is a solution of* (1) *if it satisfies that for all* $v \in L^2(0,A;H_0^1(\Omega))$

$$\int_0^A < u_a + qu, v > \, da + \int_Q \nabla u \cdot \nabla v \, da \, dx = \int_Q g(x,a,u) v \, da \, dx,$$

$$u(x,a) = 0 \qquad \text{on } \Sigma,$$

$$u(x,0) = \int_0^A \beta(x,a) u(x,a) \, da, \quad \text{in } \Omega.$$

(2) We say that a function $\overline{u} \in L^2(0,A;H^1(\Omega))$, $\overline{u}_a + q\overline{u} \in L^2(0,A;[H^1(\Omega)]')$, $g(x,a,\overline{u}) \in L^2(Q)$ *is a supersolution of* (1) *if it satisfies that:*

(a) For all $v \in L^2(0, A; H_0^1(\Omega)), v \geq 0$

$$\int_0^A < \overline{u}_a + q\overline{u}, v > \, da + \int_Q \nabla \overline{u} \cdot \nabla v \, da \, dx \geq \int_Q g(x, a, \overline{u}) v \, da \, dx,$$

(b) $\overline{u} \geq 0$ *on* Σ*,*

(c)

$$\overline{u}(x, 0) \geq \int_0^A \beta(x, a) \overline{u}(x, a) da \quad \text{in } \Omega.$$

Similar definition for a subsolution, interchanging the inequalities.

Theorem 13 *Assume* $(\mathcal{H}_\beta)$, $(\mathcal{H}_q)$ *and that*

$$|g(x, a, s_1) - g(x, a, s_2)| \leq L|s_1 - s_2|, \quad \text{for a.e. } x \in \Omega, \ a \in (0, A), \ s_1, s_2 \in \mathbb{R}. \tag{18}$$

Then, if there exists a pair of sub-supersolution of (1) *such that* $\underline{u} \leq \overline{u}$ *there exists a minimal* u_* *and maximal* u^* *solutions of* (1)*, in the sense that for any other solution* $u \in [\underline{u}, \overline{u}] := \{u \in L^2(Q) : \underline{u} \leq u \leq \overline{u}\}$*, it holds that*

$$\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u}.$$

PROOF. Take $M > 0$ a positive constant to be chosen later, and define the sequence u_n as $u_0 = \underline{u}$ and for $n \geq 1$

$$\begin{cases} (u_n)_a - \Delta(u_n) + q(x, a)u_n + Mu_n = g(x, a, u_{n-1}) + Mu_{n-1} & \text{in } Q, \\ u_n = 0 & \text{on } \Sigma, \\ u_n(x, 0) = \int_0^A \beta(x, a) u_{n-1}(x, a) da & \text{in } \Omega, \end{cases} \tag{19}$$

$u^0 = \overline{u}$ and u^n defined by

$$\begin{cases} (u^n)_a - \Delta(u^n) + q(x, a)u^n + Mu^n = g(x, a, u^{n-1}) + Mu^{n-1} & \text{in } Q, \\ u^n = 0 & \text{on } \Sigma, \\ u^n(x, 0) = \int_0^A \beta(x, a) u^{n-1}(x, a) da & \text{in } \Omega. \end{cases} \tag{20}$$

First, we show that u_n is well-defined. Since $g(x,a,u_0) = g(x,a,\underline{u}) \in L^2(Q)$, we can apply Lemma 7 and conclude the existence of u_1. Moreover, since

$$-L|u_1 - u_0| + g(x,a,u_0) \leq g(x,a,u_1) \leq L|u_1 - u_0| + g(x,a,u_0),$$

it follows that $g(x,a,u_1) \in L^2(Q)$, and so the existence of u_2, and analogously u_n. Similarly, it can proved the existence of u^n.

We will show that u_n (resp. u^n) is increasing (resp. decreasing) and that

$$\underline{u} \leq \ldots \leq u_n \leq u_{n+1} \leq u^{n+1} \leq u^n \leq \ldots \leq \overline{u}. \tag{21}$$

Indeed, taking $w := u_1 - u_0$, it satisfies

$$\begin{cases} w_a - \Delta w + q(x,a)w + Mw \geq 0 \text{ in } Q, \\ w \geq 0 & \text{on } \Sigma, \\ w(x,0) \geq 0 & \text{in } \Omega. \end{cases} \tag{22}$$

Using Lemma 7 we conclude that $w \geq 0$, i.e.,

$$\overline{u} = u_0 \leq u_1.$$

Now assume that $u_{n-1} \leq u_n$. Observe that

$$g(x,a,u_n) - g(x,a,u_{n-1}) + M(u_n - u_{n-1}) \geq (M - L)(u_n - u_{n-1}) \geq 0,$$

for $M > L$. Then $w := u_{n+1} - u_n$ satisfies

$$\begin{cases} w_a - \Delta w + q(x,a)w + Mw \geq 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = \int_0^A \beta(x,a)(u_n - u_{n-1})(x,a)da \geq 0 & \text{in } \Omega, \end{cases} \tag{23}$$

again Lemma 7 shows that $u_n \leq u_{n+1}$.

Similarly, it can be proved the rest of inequalities of (21).

Now, we multiply (19) by u_n and obtain

$$\frac{1}{2}\frac{d}{da}\int_Q |u_n|^2 + \int_Q |\nabla u_n|^2 + \int_Q (q+M)u_n^2 = \int_Q (g(x,a,u_{n-1}) + Mu_{n-1})u_n.$$

On the other hand, observe that

$$g(x,a,u_{n-1}) \leq L(u_{n-1} - \underline{u}) + g(x,a,\underline{u}) \leq L(\overline{u} - \underline{u}) + g(x,a,\underline{u}),$$

and

$$g(x,a,u_{n-1}) \geq g(x,a,\underline{u}) - L(u_{n-1} - \underline{u}) \geq g(x,a,\underline{u}) - L(\overline{u} - \underline{u}),$$

and so,

$$\frac{1}{2}\frac{d}{da}\int_Q |u_n|^2 + \int_Q |\nabla u_n|^2 + \int_Q (q_n+M)u_n^2 \leq C,$$

with C independent of n. With a similar reasoning to the used in Lemma 7, we can extract a subsequence (u_n) such that

$$\begin{aligned} &u_n \rightharpoonup u_* && \text{in } L^2(0,A;H_0^1(\Omega)),\\ &\sqrt{q_n}u_n \rightharpoonup w && \text{in } L^2(Q),\\ &(u_n)_a + q_n u_n \rightharpoonup z && \text{in } L^2(0,A;H^{-1}(\Omega)). \end{aligned}$$

By the monotony of u_n and the Monotone Convergence Theorem, we can conclude

$$u_n \to u_* \quad \text{in } L^2(Q). \tag{24}$$

Now, using that

$$-L(u_n - u_*) + g(x,a,u_*) \leq g(x,a,u_n) \leq L(u_n - u_*) + g(x,a,u_*),$$

and so, $g(x,a,u_n) \rightharpoonup g(x,a,u_*)$ weakly in $L^2(Q)$, it follows that u_* a solution of (1).

Finally, the continuity of the trace application on $a = 0$ and (24) imply that

$$u_*(x,a) = \int_0^A \beta(x,a)u_*(x,a)da.$$

That u_* is the minimal solution of (1) is not difficult to show. Indeed, if u is a solution of (1) such that $u \in [\underline{u}, \overline{u}]$, it can be shown that the sequence u_n built in (19) satisfies that $\overline{u} \leq u_n \leq u$. So,

$$u_n \uparrow u_* \leq u.$$

Similarly, we can reason with the sequence u^n and conclude the existence of a maximal solution u^* of (1). This ends the proof.

4 Application to the logistic equation

The main result of this section is:

Theorem 14 *Problem* (2) *possesses a positive solution if, and only if,* $\lambda > \lambda_0(q)$. *Moreover, in the case that the solution exists, then it is unique.*

PROOF. Suppose that $u > 0$ is solution of (2). Then, we can write the equation (2) as

$$u_a - \Delta u + (q(x,a) + u(x,a) - \lambda)u = 0, \quad u(x,0) > 0,$$

with $q+u-\lambda$ satisfying $(\mathcal{H}_q)$. Hence, by Lemma 7, $u \gg 0$, and so by Theorem 8, and taking into account that $u_a - \Delta u + (q(x,a) + u(x,a))u = \lambda u$, it follows that

$$\lambda = \lambda_0(q+u) \tag{25}$$

and by the monotony of the map $q \mapsto \lambda_0(q)$,

$$\lambda = \lambda_0(q+u) > \lambda_0(q).$$

Assume now that $\lambda > \lambda_0(q)$. Take

$$\underline{u} := \varepsilon\varphi(x,a)$$

with $\varepsilon > 0$ sufficiently small and φ a positive eigenfunction associated to $\lambda_0(q)$. It is not difficult to show that $\underline{u}$ is subsolution of (2). Indeed, $\underline{u} = 0$ on Σ and

$$\underline{u}(x,0) = \varepsilon\varphi(x,0) = \varepsilon \int_0^A \beta(x,a)\varphi(x,a)da = \int_0^A \beta(x,a)\underline{u}(x,a)da.$$

Finally,

$$\underline{u}_a - \Delta \underline{u} + q(x,a)\underline{u} \leq \lambda \underline{u} - \underline{u}^2,$$

provided that

$$\varepsilon\varphi(x,a) \leq \lambda - \lambda_0(q),$$

which is true taking ε sufficiently small (observe that φ is bounded, cf. Theorem 8).

We will build a supersolution. Define

$$F_\mu(a) := \mu a - \int_0^a q_L(s)ds, \qquad \mu \in \mathbb{R},$$

and take $\mu \in \mathbb{R}$ sufficiently large so that

$$\int_0^A e^{F_\mu(a)}da \geq \frac{1}{\overline{\beta}}. \tag{26}$$

where $\overline{\beta} := \sup_Q \beta(x,a)$. Consider the function

$$G(x) := \int_0^A \frac{e^{F_\mu(a)}}{1 + x\int_0^a e^{F_\mu(s)}ds}da.$$

Observe that G is a continuous function and by (26) we have that

$$\lim_{x\downarrow 0} G(x) \geq \frac{1}{\overline{\beta}}, \qquad \lim_{x\to+\infty} G(x) = 0,$$

and so, there exists $y_0 > 0$ such that $G(y_0) = 1/\overline{\beta}$, i.e.,

$$\int_0^A \frac{e^{F_\mu(a)}}{1 + y_0\int_0^a e^{F_\mu(s)}ds}da = \frac{1}{\overline{\beta}}. \tag{27}$$

Define $Y(a)$ the unique solution of the differential equation

$$y_a + q_L(a)y = \mu y - y^2, \qquad y(0) = y_0;$$

where y_0 is defined by (27). Solving the above equation, we get that

$$Y(a) = \frac{e^{F_\mu(a)}}{\frac{1}{y_0} + \int_0^a e^{F_\mu(s)} ds}. \tag{28}$$

Take

$$\overline{u}(a) := KY(a),$$

with K a positive constant large. It can be proved that $\overline{u}$ is a supersolution of (2) for μ large. Indeed, $\overline{u} > 0$ on Σ and

$$\overline{u}_a - \Delta \overline{u} + q(x,a)\overline{u} \geq \lambda \overline{u} - \overline{u}^2,$$

provided that

$$\mu - \lambda + q(x,a) - q_L(a) + (K-1)Y \geq 0,$$

which is satisfied if $\mu \geq \lambda$, and $K \geq 1$. On the other hand, using (27) we have that

$$\int_0^A \beta(x,a)\overline{u}(x,a)da \leq K y_0 \overline{\beta} \int_0^A \frac{e^{F_\mu(a)}}{1 + y_0 \int_0^a e^{F_\mu(s)} ds} da = K y_0 = \overline{u}(x,0).$$

Now, it is clear that we can choose $\varepsilon > 0$ and $K > 0$ such that $\underline{u} \leq \overline{u}$. This completes the proof of the existence of positive solution.

For the uniqueness we assume that there exist two different positive solutions u_1 and u_2. Define

$$w := u_2 - u_1 \neq 0.$$

It is clear that w satisfies

$$\begin{cases} w_a - \Delta w + (q(x,a) + u_1 + u_2)w = \lambda w & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = \int_0^A \beta(x,a) w(x,a) da & \text{in } \Omega, \end{cases} \tag{29}$$

and so, since $w \neq 0$, by (13), we have that

$$\lambda \geq \lambda_0(q + u_1 + u_2) > \lambda_0(q + u_1),$$

which is an absurdum. Indeed, since u_1 is positive solution of (1) we have that $\lambda = \lambda_0(q + u_1)$, see (25).

Remark 15 *(1) Observe that the unique solution u of* (2) *satisfies that*

$$\lim_{a \uparrow A} u(x, a) = 0, \qquad \text{for } x \in \Omega.$$

Indeed, by $(\mathcal{H}_q)$ it follows that $\lim_{a\uparrow A} F_\mu(a) = -\infty$, and thanks to (28) *we conclude the claim.*

(2) In the autonomous case, $q(x, a) = q(a)$ and $\beta(x, a) = \beta(a)$, we have shown that

$$\lambda_0(q) = \lambda_1 + r_q,$$

where r_q is defined in (7) *with $\gamma(a) = \beta(a)$.*

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