ON THE SYMBIOTIC LOTKA-VOLTERRA MODEL WITH DIFFUSION AND TRANSPORT EFFECTS

M. Delgado[1], J. López-Gómez[2] and A. Suárez[1]

[1]Dpto. de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
C. Tarfia s/n. 41012-Sevilla, Spain

[2]Departamento de Matemática Aplicada
Universidad Complutense
28040-MADRID, Spain

Abstract. In this work we analyze the existence, stability and multiplicity of coexistence states for a symbiotic Lotka-Volterra model with general diffusivities and transport effects. Global bifurcation theory, blowing up arguments for a priori bounds, singular perturbation results, singularity theory and fixed point index in cones are among the techniques used to get our results and to explain the drastic change of behavior exhibited by the dynamics of the model between the cases of weak and strong mutualism between the species. Our methodology works out to treat much more general classes of symbiotic models.

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1. Introduction. In this paper we analyze the existence, multiplicity and stability of coexistence states for the following problem

$$\begin{aligned} \mathcal{L}_1 u &= \lambda u - a(x)u^2 + b(x)uv \\ \mathcal{L}_2 v &= \mu v - d(x)v^2 + c(x)uv \end{aligned} \qquad \text{in } \Omega\,, \tag{1.1a}$$

$$u = v = 0 \qquad \text{on } \partial\Omega\,, \tag{1.1b}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 regularity, $\mathcal{L}_k$, $k=1\,,\,2$ are two second order uniformly elliptic operators of the form

$$\mathcal{L}_k = -\sum_{i,j=1}^{N} a_{ijk}(x)\partial_i\partial_j + \sum_{j=1}^{N} b_{jk}(x)\partial_j + c_k(x) \qquad k=1\,,\,2\,, \tag{1.2}$$

with

$$a_{ijk} \in C(\overline{\Omega})\,, \qquad b_{jk}\,,\, c_k \in L^\infty(\Omega)\,, \qquad i\,,j \in \{1,...,N\}\,, \quad k \in \{1,2\}\,, \tag{1.3}$$

and a, b, c, $d \in C(\overline{\Omega})$ satisfy $a(x)>0$, $d(x)>0$, for each $x\in\overline{\Omega}$, and $b\geq 0$, $c\geq 0$ in Ω, $b\neq 0$, $c\neq 0$; λ, $\mu\in\mathbb{R}$ will be regarded as bifurcation parameters. Under these assumptions, (1.1) provides us with a model for symbiotic species, where Ω is the inhabiting region, $u(x)$ and $v(x)$ are the densities of each of the species, $a(x)$ and $d(x)$ describe the limiting effects of crowding in each population, $b(x)$ and $c(x)$ are the interaction rates between the species, the operators $\mathcal{L}_k - c_k(x)$, $k=1,2$, measure the diffusivities and the external transport effects of the species, and $\lambda - c_1(x)$, $\mu - c_2(x)$ are the growth rates of the species, positive on favorable regions and negative on unfavorable ones. In this model we are assuming that Ω is fully surrounded by inhospitable areas, because both population densities are subject to homogeneous Dirichlet boundary conditions.

In this work our attention will be focused into the problem of analyzing the existence, stability and multiplicity of the non-negative solution couples (u,v) of (1.1). Due to the structure of (1.1) and thanks to the strong maximum principle, if (u,v) is a solution of (1.1) with $u\neq 0$ (resp. $v\neq 0$), then u (resp. v) is strongly positive in the sense of Section 2. Therefore, (1.1) admits three types of non-negative component-wise solution couples. Namely, the *trivial* one, $(0,0)$; those with one component positive and the other zero, $(u,0)$ or $(0,v)$, referred as the *semi-trivial* positive solutions, and those with both components positive, the *coexistence states*.

The symbiotic model has attracted much less attention in the literature than its competing and predator-prey counterparts, due basically to the absence of a priori bounds for the coexistence states in high spatial dimensions ($N\geq 6$) under strong mutualism ($bc-ad$ large). This lack of a priori bounds was observed originally in [16], where it was shown that the positive solutions of the parabolic problem associated with (1.1) may blow up in finite time when $\mathcal{L}_1=\mathcal{L}_2=-\Delta$ and $bc>ad$, and in [21], where it was shown that if in addition $\lambda=\mu$, then the coexistence states of (1.1) are given by the positive solutions of

$$-\Delta w = \lambda w + w^2 \qquad \text{in} \;\; \Omega\,, \qquad w|_{\partial\Omega}=0\,, \tag{1.4}$$

and that thanks to the results of [13], (1.4) possesses uniform a priori bounds in any compact subinterval of λ if, and only if, $2<\frac{N+2}{N-2}$, i.e. if $N\leq 5$.

The absence of a priori bounds for the coexistence states of (1.1) makes very involved the problem of finding out global sufficient conditions for the existence of a coexistence state, since most of the technical tools available to attack this kind of problems involve either degree theory, i.e. global bifurcation theory, or monotonicity techniques, where the existence of a priori bounds is needed. Nevertheless, although most of the attention has been focused into the very special case when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and a, b, c, d are constants, in recent years some substantial progress has been carried out into the analysis of these problems.

The study of symbiotic species actually started in [19], where it was constructed monotonic sequences which approximate the solutions of (1.1). In [17] the method of sub and supersolutions for systems, coming from [28], was used to show that if $\lambda > \sigma_1$ and $\mu > \sigma_1$, then (1.1) possesses a coexistence state if, and only if, $bc < ad$, where σ_1 is the principal eigenvalue of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions. This result was generalized in [20] to cover some more general classes of symbiotic kinetics. The first global result about the existence of coexistence states for the symbiotic model was found in [25] by using global bifurcation theory, where it was shown that if any of the semi-trivial positive solutions is linearly unstable, then the model possesses a coexistence state provided $bc < ad$; global in the sense that if some of the semi-trivial states is stable, then there are choices of the several parameters involved in the setting of (1.1) for which the model does not admit a coexistence state (cf. Section 11 here in for further details). Almost simultaneously, in [35] was found the same result included in [25], but this time using the method of sub and supersolutions. More recently, allowing the coefficients of the model to vary, the technique of decoupling was shown to work out to get the same result as in [25] and [35], [5]. In [21] and [23] fixed point index in cones and global bifurcation theory were shown to work out to get the corresponding results for wider classes of models.

Although the global results of [21] work out to show that a global continuum of coexistence states emanates from each of the surfaces of semi-trivial positive solutions along their curves of change of stability in the space of the parameters (λ, μ), the first global result in the case $bc > ad$ was found in [27], where it was shown that if $N \leq 5$ and some of the semi-trivial positive solutions is linearly stable, then the model possesses a coexistence state. We point out that this result was obtained for the special case when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and all coefficients are constant. In [27], the blowing up argument of [13] was adapted to show the existence of a priori bounds in case $N \leq 5$ and then the fixed point index in cones was used to complete the proof.

In this work we extend and complete all the previous features, obtaining in addition some optimal non-existence and multiplicity results for all ranges of the parameters in the general setting of (1.1), and in addition we analyze the bifurcation equations of (1.1) at $(\lambda, \mu) = (\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$. Hereafter, given an elliptic operator $\mathcal{L}$, $\sigma_1^\Omega[\mathcal{L}]$ will stand for the principal eigenvalue of $\mathcal{L}$ in Ω under homogeneous Dirichlet boundary conditions. Our analysis of the bifurcation equations at $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$ explains the drastic change of behavior of the global continuum of coexistence states as some of the interactions between the species, b or c, grows acrossing the critical value given by Theorem 10.1 in Section 10. Namely, the global manifold of coexistence states linking the two surfaces of semi-trivial positive solutions turns backwards in the parameter space (λ, μ) changing its relative position with respect to each the surfaces of semi-trivial solutions, as the amplitude of b, or c, grows.

To state our main results, we have to introduce some of notation. Given $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ (resp. $\mu > \sigma_1^\Omega[\mathcal{L}_2]$), $(\theta_\lambda, 0)$ (resp. $(0, \theta_\mu)$) will stand for the unique semi-trivial solution of (1.1) of the form $(u, 0)$, $u > 0$ (resp. $(0, v)$, $v > 0$). Moreover, for any $f \in L^\infty(\Omega)$ we denote

$$f_L := \text{ess} \inf_\Omega f \,, \qquad f_M := \text{ess} \sup_\Omega f \,.$$

Among our main results we list the following ones:

• If $b_M c_M < a_L d_L$ and any of the semitrivial positive solutions is linearly unstable, then (1.1) possesses a coexistence state. If in addition $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ and $\mu > \sigma_1^\Omega[\mathcal{L}_2]$, then there exists $I_0 > 0$ such that if

$$\min\{b_M \,,\, c_M\} < I_0 \,,$$

then the coexistence state is unique and exponentially asymptotically stable.

• If $b_M c_M < a_L d_L$ and for $(\lambda, \mu) = (\lambda_0, \mu_0)$ some of the semitrivial positive solutions is linearly stable and (1.1) possesses a coexistence state, then it possesses a coexistence state for each (λ, μ) satisfying $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and at least two coexistence states if $\lambda > \lambda_0$, $\mu > \mu_0$ and some of the semi-trivial positive solutions is linearly stable.

• If $b_M c_M < a_L d_L$, then for each $\lambda \in \mathbb{R}$, there exists $\mu_{ext}(\lambda) \in \mathbb{R}$ such that (1.1) does not admit a coexistence state if $\mu \leq \mu_{ext}(\lambda)$. Similarly, for each $\mu \in \mathbb{R}$, there exists $\lambda_{ext}(\mu) \in \mathbb{R}$ such that (1.1) does not admit a coexistence state if $\lambda \leq \lambda_{ext}(\mu)$.

• If $\mathcal{L}_1 = \mathcal{L}_2$, $N \leq 5$,

$$b_L c_L - a_M d_M > \max\{a_M b_M - a_L b_L \,,\, d_M c_M - d_L c_L\} \,, \tag{1.5}$$

and some of the semitrivial positive solutions is linearly stable, then (1.1) possesses a coexistence state.

• Assume that $\mathcal{L}_1 = \mathcal{L}_2$, $N \leq 5$, (1.5), and that there exists $(\lambda, \mu) = (\lambda_0, \mu_0)$ for which (1.1) possesses a coexistence state being any of the semi-trivial states linearly unstable. Then, (1.1) possesses a coexistence state for each (λ, μ) satisfying $\lambda \leq \lambda_0$ and $\mu \leq \mu_0$, and at least two coexistence states if $\lambda < \lambda_0$, $\mu < \mu_0$ and any of the semi-trivial states is linearly unstable.

• Assume $\mathcal{L}_1 = \mathcal{L}_2$, $N \leq 5$ and (1.5). Then, for each $\lambda \in \mathbb{R}$ there exists $\mu_{ext}(\lambda) \in \mathbb{R}$ such that (1.1) does not admit a coexistence state if $\mu \geq \mu_{ext}(\lambda)$. Similarly, for each $\mu \in \mathbb{R}$, there exists $\lambda_{ext}(\mu) \in \mathbb{R}$ such that (1.1) does not admit a coexistence state if $\lambda \geq \lambda_{ext}(\mu)$.

We now describe the distribution and contains of this paper. In Section 2 we give an extension of Theorem 2.5 in [22] to cover our general setting here in, and then use it to infer some basic monotonicity properties of principal eigenvalues. Most of these results come from Section 2 of [3].

In Section 3 we study the single boundary value problem

$$\mathcal{L}_1 u = \lambda\, u - a(x)\, u^2 \quad \text{in } \Omega\,, \qquad u|_{\partial\Omega} = 0\,. \tag{1.6}$$

A particular attention is paid to the behavior of its unique positive solution as $\lambda \uparrow \infty$, showing that

$$\lim_{\lambda\uparrow\infty} \frac{\theta_{[\mathcal{L}_1,\lambda,a]}}{\lambda} = a^{-1} \tag{1.7}$$

uniformly on any compact subset of Ω, where $\theta_{[\mathcal{L}_1,\lambda,a]}$ stands for the unique positive solution of (1.6). This result extends the corresponding singular perturbation result in Section 3 of [12] to our general setting here in, and it is the basic technical tool to get our non-existence results in Section 7.

In Section 4 we characterize the attractive character of each of the semitrivial positive solutions in terms of several parameters involved in the setting of (1.1) through by the principal eigenvalues of some related second order elliptic operators. Then, we analyze the shape of the curves of change of stability in the space of the parameters (λ, μ).

Section 5 is devoted to the abstract results concerning the existence of global continua of coexistence states emanating from the surfaces of semitrivial positive solutions along their respective curves of change of stability. The analysis throughout this work shows that these results are optimal, reducing the problem of finding out coexistence states for (1.1) to the problem of finding out a priori bounds for the component-wise positive solutions of (1.1). The methodology adopted in this section comes from the abstract theory developed in [21] for general systems with two species.

In Section 6 we analyze the existence of coexistence states for the case of small interaction coefficients. How small should they are is measured by condition

$$b_M c_M < c_L d_L\,. \tag{1.8}$$

Precisely, we will find out some non-existence results and then we will use the theory of Section 5 to show that (1.1) possesses a coexistence state if any of the semitrivial positive solutions is linearly unstable. The analysis of Section 11 for the case of constant coefficients will show the optimality of our results.

In Section 7 we analyze the existence of coexistence states for the case of large interaction coefficients. How large should they are is measured by condition (1.5). Notice that if any coefficient is assumed to be constant, then (1.5) becomes into

$$bc > ad\,. \tag{1.9}$$

By technical reasons for most of the results in this section we need assuming that $\mathcal{L}_1 = \mathcal{L}_2$, assumption needed in all previous references. We begin the section giving a necessary condition for the existence of coexistence states which is totally new even for the simplest symbiotic models where $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and any coefficient is constant. Namely, if $(0, \theta_{[\mathcal{L}_1,\mu,d]})$ (resp. $(\theta_{[\mathcal{L}_1,\lambda,a]})$) is linearly unstable, then (1.1) does not admit a coexistence state if μ (resp. λ) is sufficiently large (cf. Theorem 7.1 here in). This non-existence result is based upon (1.7), finding out the behavior of an eventual sequence of coexistence states for μ, or λ, large. Then, we adapt the blowing up argument of [13] to show that uniform a priori bounds for the coexistence states of (1.1) are available if $N \leq 5$. We should point out that our blowing up argument differs substantially from the corresponding argument of [27] and that we need a general Liouville type result much sharper than the corresponding result in [27]. These additional difficulties coming from the fact that in this work we are dealing with a general elliptic operator and with spatially varying coefficients. We refer to Section 7 for further details. Bringing together the non-existence results and the a priori bounds, it follows from the global results in Section 5 that if any of the semitrivial positive solutions is linearly stable, then (1.1) possesses a coexistence state.

In Section 8 we use the abstract theory of [2] to show that the method of sub and supersolutions is valid for (1.1). Then, we use it to analyze the structure of the set of λ's and μ's for which (1.1) possesses a coexistence state and to get our multiplicity results, those already stated in the list above.

In Section 9 we obtain simple readily computable conditions in terms of the several coefficients involved in the setting of (1.1) ensuring that (1.1) has a unique stable coexistence state, and then consider the parabolic problem associated with (1.1) to show that there is a dense subset of the set of initial data such that any solution starting there in converges to the coexistence state as time grows to infinity.

In Section 10, considering (λ, μ) as the main bifurcation parameters we describe the possible local bifurcation diagrams near the co-dimension two singularity

$$(\lambda, \mu) = (\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])\,.$$

For this, we apply the general results of [10] where one of the authors developed a singularity theory to deal with this type of two parameter bifurcation problems.

Finally, in Section 11 we restrict ourselves to the original Lotka-Volterra symbiotic model with diffusion, $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and a, b, c, d constants, for which we can give some sharper existence and non-existence results and can go further in the analysis of the bifurcation equation around the co-dimension two bifurcation point, obtaining in addition some global results about the nature of the local bifurcations to coexistence states from the surfaces of semitrivial positive solutions along their curves of change of stability. As a result from this analysis we can explain the drastic change of behavior of the global manifold of coexistence that links the two surfaces of semitrivial positive solutions along their curves of change of stability as bc acrosses the critical value ad passing from values where $bc < ad$ to values where $bc > ad$.

2. The maximum principle. Main properties of the principal eigenvalues. In this section we give an extension of Theorem 2.5 in [22] to cover our setting here and then we infer some basic properties of principal eigenvalues which will be used throughout this paper. We will consider a uniformly elliptic operator of the form

$$\mathcal{L} = -\sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j + \sum_{j=1}^{N} b_j(x)\partial_j + e(x)\,, \tag{2.1}$$

with

$$a_{ij} \in C(\overline{\Omega})\,, \qquad b_j\,,\, e \in L^\infty(\Omega)\,, \qquad i\,,j \in \{1,...,N\}\,, \tag{2.2}$$

and use the natural product order on $L^p(\Omega) \times L^p(\partial\Omega)$. Recall that $p > N$ implies $W^{2,p}(\Omega) \subset C^{2-\frac{N}{p}-\varepsilon}(\overline{\Omega})$ with compact imbedding for all $\varepsilon > 0$ and that each $u \in W^{2,p}(\Omega)$ is a.e. twice classically differentiable in Ω (e.g. Theorem VIII.1 of [33]).

Suppose that $p > N$. Then $u \in W^{2,p}(\Omega)$ is said to be strongly positive if $u(x) > 0$ for $x \in \Omega$ and $\partial_n u(x) < 0$ for all $x \in \partial\Omega$ with $u(x) = 0$, where n is the outward unit normal on $\partial\Omega$. The operator $\mathcal{L}$ is said to satisfy the strong maximum principle in Ω if $p > N$, $u \in W^{2,p}(\Omega)$, and $(\mathcal{L}u, u) > (0,0)$ imply that u is strongly positive. Consider the eigenvalue problem

$$\mathcal{L}u = \sigma u \quad \text{in} \ \ \Omega\,, \qquad u = 0 \quad \text{on} \ \ \partial\Omega\,, \tag{2.3}$$

in $W^{2,p}(\Omega)$ and let $\mathcal{L}_p$ denote the closure of the operator $\mathcal{L}|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)}$ in $L^p(\Omega)$. Then, (2.3) can be reformulated as the eigenvalue equation

$$\mathcal{L}_p u = \sigma u \qquad \text{in} \ \ L^p(\Omega)\,. \tag{2.4}$$

It is an easy consequence of standard regularity theory that the spectrum and the eigenspaces of $\mathcal{L}_p$ are independent of $p > N$. Moreover, from the strong maximum principle and the generalization of the Krein Rutman Theorem of [32] together with Theorem 3 in [29], the following result holds (cf. Section 2 of [3]).

Theorem 2.1. *There exists a least eigenvalue of* (2.4)*, denoted by* $\sigma_1^\Omega[\mathcal{L}]$ *and called principal eigenvalue of* $\mathcal{L}$ *in* Ω*. This eigenvalue is simple and possesses a unique eigenfunction, up to multiplicative constants, which can be taken positive, the so called principal eigenfunction of* $\mathcal{L}$ *in* Ω*. Moreover, the principal eigenfunction is strongly positive and* $\sigma_1^\Omega[\mathcal{L}]$ *is the only eigenvalue of* (2.4) *possessing a positive eigenfunction. Furthermore, any other eigenvalue* σ *of* (2.4) *satisfies*

$$Re\ \sigma > \sigma_1^\Omega[\mathcal{L}]$$

and $(\mathcal{L}_p + \nu)^{-1} \in \mathcal{L}(L^p(\Omega))$ *is positive, compact and irreducible for* $\nu > -\sigma_1^\Omega[\mathcal{L}]$.

If $p > N$ a function $\overline{u} \in W^{2,p}(\Omega)$ is said to be a positive supersolution of $\mathcal{L}$ in Ω if $\overline{u} \geq 0$ and $(\mathcal{L}\overline{u}, \overline{u}) \geq (0,0)$. If in addition $(\mathcal{L}\overline{u}, \overline{u}) > (0,0)$, then it is said that $\overline{u}$ is a positive strict supersolution. Similarly, a function $\underline{u} \in W^{2,p}(\Omega)$ is said to be a positive subsolution of $\mathcal{L}$ in Ω if $\underline{u} \geq 0$ and $(\mathcal{L}\underline{u}, \underline{u}) \leq (0,0)$. If in addition $(\mathcal{L}\underline{u}, \underline{u}) < (0,0)$, then it is said that $\underline{u}$ is a positive strict subsolution.

¿From the strong maximum principle it is easily seen that any positive strict supersolution is strongly positive. Moreover, the following characterization of the strong maximum principle holds (cf. Theorem 2.5 in [22] and Theorem 2.4 in [3]).

Theorem 2.2. *The following assertions are equivalent:*

(i) $\sigma_1^\Omega[\mathcal{L}] > 0$*;*

(ii) $\mathcal{L}$ *possesses a positive strict supersolution in* Ω*;*

(iii) $\mathcal{L}$ *satisfies the strong maximum principle in* Ω*.*

From this characterization we can readily get the following properties of $\sigma_1^\Omega[\mathcal{L}]$ which will be used throughout this work. For selfadjoint operators, these properties are easily obtained from the variational characterization of the principal eigenvalue.

Theorem 2.3. *(i) Monotonicity with respect to the potential: Let* V_1, $V_2 \in L^\infty(\Omega)$ *such that* $V_1 \le V_2$ *and* $V_1 < V_2$ *on a set of positive measure. Then,*

$$\sigma_1^\Omega[\mathcal{L} + V_1] < \sigma_1^\Omega[\mathcal{L} + V_2]\,. \tag{2.5}$$

(ii) Continuity with respect to the potential: If $V_n \in L^\infty(\Omega)$, $n \ge 1$ *is a sequence of potentials such that*

$$\lim_{n\to\infty} \|V_n - V\|_{\infty,\Omega} = 0\,,$$

then

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathcal{L} + V_n] = \sigma_1^\Omega[\mathcal{L} + V]\,.$$

(iii) If Ω_1 *is a proper subdomain of* Ω *with* $\partial\Omega_1$ *of class* C^2*, then*

$$\sigma_1^{\Omega_1}[\mathcal{L}] > \sigma_1^\Omega[\mathcal{L}]\,. \tag{2.6}$$

Proof. (i) Let φ_1 be the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L} + V_1]$. Then,

$$(\mathcal{L} + V_2)\varphi_1 = \sigma_1^\Omega[\mathcal{L} + V_1]\varphi_1 + (V_2 - V_1)\varphi_1 > \sigma_1^\Omega[\mathcal{L} + V_1]\varphi_1$$

on a set of positive measure, and hence φ_1 is a positive strict supersolution of $\mathcal{L} + V_2 - \sigma_1^\Omega[\mathcal{L} + V_1]$. Thus, thanks to Theorem 2.2, we find that

$$\sigma_1^\Omega[\mathcal{L} + V_2 - \sigma_1^\Omega[\mathcal{L} + V_1]] > 0\,.$$

This relation implies (2.5).

(ii) For any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$V - \varepsilon \le V_n \le V + \varepsilon \qquad \forall n \ge N_0\,.$$

Thus, by Part (i) we find that

$$\sigma_1^\Omega[\mathcal{L} + V] - \varepsilon \le \sigma_1^\Omega[\mathcal{L} + V_n] \le \sigma_1^\Omega[\mathcal{L} + V] + \varepsilon\,.$$

This completes the proof.

(iii) Let φ denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L}]$. Then,

$$(\mathcal{L} - \sigma_1^\Omega[\mathcal{L}])\varphi = 0$$

in Ω_1 and $\varphi > 0$ on $\partial\Omega_1$. Thus, φ is a positive strict supersolution of $\mathcal{L} - \sigma_1^\Omega[\mathcal{L}]$ in Ω_1 and hence, it follows from Theorem 2.2 that

$$\sigma_1^{\Omega_1}[\mathcal{L} - \sigma_1^\Omega[\mathcal{L}]] > 0\,.$$

This relation implies (2.6). $\square$

3. The logistic equation. The semi-trivial positive solutions of (1.1) are given by the positive solutions of a semilinear elliptic boundary value problem of the form

$$\begin{aligned} \mathcal{L}w = \gamma w - f(x)w^2 \qquad &\text{in } \ \Omega\,, \\ w = 0 \qquad &\text{on } \ \partial\Omega\,, \end{aligned} \tag{3.1}$$

where $\mathcal{L}$ is a second order uniformly elliptic operator of the form (2.1) with coefficients satisfying (2.2), $\gamma \in \mathbb{R}$, and $f \in C(\overline{\Omega})$ satisfies $f(x) > 0$ for each $x \in \overline{\Omega}$. If $p > N$ and $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a positive solution of (3.1), then

$$(\mathcal{L} + fw)w = \gamma w$$

and thanks to Theorem 2.1 we have that

$$\gamma = \sigma_1^\Omega[\mathcal{L} + fw] \tag{3.2}$$

and that w is strongly positive. Therefore, $w(x) > 0$ for each $x \in \Omega$ and $\partial_n w(x) < 0$ for each $x \in \partial\Omega$. The following result characterizes the existence of positive solutions for (3.1).

Theorem 3.1. *If $p > N$, then the problem* (3.1) *possesses a positive solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ if, and only if, $\gamma > \sigma_1^\Omega[\mathcal{L}]$. Moreover, it is unique if it exists. Let $\theta_{[\mathcal{L},\gamma,f]}$ denote it. Then,*

$$\lim_{\gamma \downarrow \sigma_1^\Omega[\mathcal{L}]} \theta_{[\mathcal{L},\gamma,f]} = 0 \tag{3.3}$$

uniformly in $\overline{\Omega}$.

Condition (3.3) says that the positive solutions bifurcate from the trivial state $w = 0$ at the critical value of the parameter $\gamma = \sigma_1^\Omega[\mathcal{L}]$. This result is well known under some additional regularity conditions on the several coefficients involved in the model setting, e.g. see [14]. By the sake of completeness we shall give a short self-contained proof of it.

Proof of Theorem 3.1. Let w be a positive solution of (3.1). Then, thanks to Theorem 2.1, we have (3.2) and hence Theorem 2.3(i) implies

$$\gamma = \sigma_1^\Omega[\mathcal{L} + fw] > \sigma_1^\Omega[\mathcal{L}]\,.$$

Therefore, $\gamma > \sigma_1^\Omega[\mathcal{L}]$ is necessary for the existence of a positive solution. Assume $\gamma > \sigma_1^\Omega[\mathcal{L}]$. It is easily seen that large positive constants provide us with supersolutions of (3.1) and that if $\varphi > 0$ stands for the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L}]$, then $\varepsilon\varphi$ provide us with arbitrarily small positive subsolutions if $\varepsilon > 0$ is sufficiently small. Therefore, (3.1) possesses at least a positive solution for each $\gamma > \sigma_1^\Omega[\mathcal{L}]$. We point out that the method of sub and supersolutions works out thanks to the validity of the strong maximum principle.

To show the uniqueness let w_1, w_2 be two arbitrary positive solutions of (3.1). Then,

$$(\mathcal{L}+f(w_1+w_2)-\gamma)(w_1-w_2)=0\,,$$

and therefore, 0 is an eigenvalue of $\mathcal{L}+f(w_1+w_2)-\gamma$ in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. On the other hand, it follows from Theorem 2.3(i) and (3.2) that

$$\sigma_1^\Omega[\mathcal{L}+f(w_1+w_2)-\gamma]>\sigma_1^\Omega[\mathcal{L}+fw_1-\gamma]=0$$

and hence, due to Theorem 2.1,

$$\text{Re }\sigma>\sigma_1^\Omega[\mathcal{L}+f(w_1+w_2)-\gamma]>0$$

for any other eigenvalue σ of $\mathcal{L}+f(w_1+w_2)-\gamma$. This contradiction completes the proof of the uniqueness. Condition (3.3) follows easily from the simplicity of $\sigma_1^\Omega[\mathcal{L}]$ and the uniqueness given by the local bifurcation theorem of [7]. This completes the proof. □

The following result will be very usefull to compare positive solutions of different logistic boundary value problems.

Lemma 3.2. *(i) If $\gamma\le\sigma_1^\Omega[\mathcal{L}]$, then* (3.1) *does not admit a positive subsolution.*

(ii) If $\gamma>\sigma_1^\Omega[\mathcal{L}]$ and $\overline{w}\in W^{2,p}(\Omega)$ is a positive strict supersolution of (3.1)*, then $\overline{w}-\theta_{[\mathcal{L},\gamma,f]}$ is strongly positive.*

(iii) Similarly, if $\gamma>\sigma_1^\Omega[\mathcal{L}]$ and $\underline{w}\in W^{2,p}(\Omega)$ is a positive strict subsolution of (3.1)*, then $\theta_{[\mathcal{L},\gamma,f]}-\underline{w}$ is strongly positive.*

Proof. (i) Assume that $\gamma\le\sigma_1^\Omega[\mathcal{L}]$ and that (3.1) possesses a positive subsolution. Then, since sufficiently large constants are supersolutions, (3.1) possesses a positive solution. By Theorem 3.1 this is impossible and therefore, (3.1) can not admit a positive subsolution.

(ii) Assume $\gamma>\sigma_1^\Omega[\mathcal{L}]$ and let $\overline{w}\in W^{2,p}(\Omega)$ be a positive strict supersolution of (3.1). Then,

$$(\,(\mathcal{L}+f(\overline{w}+\theta_{[\mathcal{L},\gamma,f]})-\gamma)(\overline{w}-\theta_{[\mathcal{L},\gamma,f]}),\overline{w}|_{\partial\Omega}\,)>(0,0)$$

and since

$$\sigma_1^\Omega[\mathcal{L}+f(\overline{w}+\theta_{[\mathcal{L},\gamma,f]})-\gamma]>\sigma_1^\Omega[\mathcal{L}+f\theta_{[\mathcal{L},\gamma,f]}-\gamma]=0\,,$$

Theorem 2.2 completes the proof. Similarly, (iii) follows. □

In the sequel, given any function $f\in L^\infty(\Omega)$ we shall denote

$$f_M:=\text{ess }\sup_\Omega f\,,\qquad f_L:=\text{ess }\inf_\Omega f\,.$$

Corollary 3.3. *Assume* $\sigma_1^\Omega[\mathcal{L}] < \gamma_1 < \gamma_2$. *Then,*

$$\theta_{[\mathcal{L},\gamma_2,f]} - \theta_{[\mathcal{L},\gamma_1,f]}$$

is strongly positive. It will simply said that

$$\theta_{[\mathcal{L},\gamma_1,f]} << \theta_{[\mathcal{L},\gamma_2,f]} \,.$$

Moreover,

$$\theta_{[\mathcal{L},\gamma,f]} \leq \frac{\gamma - e_L}{f_L} \,.$$

Proof. We have

$$\mathcal{L}\theta_{[\mathcal{L},\gamma_1,f]} = \gamma_1\theta_{[\mathcal{L},\gamma_1,f]} - f\theta^2_{[\mathcal{L},\gamma_1,f]} < \gamma_2\theta_{[\mathcal{L},\gamma_1,f]} - f\theta^2_{[\mathcal{L},\gamma_1,f]}$$

and hence, $\theta_{[\mathcal{L},\gamma_1,f]}$ is a positive strict subsolution of (3.1) with $\gamma = \gamma_2$. Lemma 3.2(iii) completes the proof of this part.

Now, observe that

$$(\mathcal{L} - e)\theta_{[\mathcal{L},\gamma,f]} = (\gamma - e)\theta_{[\mathcal{L},\gamma,f]} - f\theta^2_{[\mathcal{L},\gamma,f]} \leq (\gamma - e_L)\theta_{[\mathcal{L},\gamma,f]} - f_L\theta^2_{[\mathcal{L},\gamma,f]} \,,$$

and hence, thanks to Lemma 3.2, $\gamma - e_L > \sigma_1^\Omega[\mathcal{L} - e]$ and

$$\theta_{[\mathcal{L},\gamma,f]} \leq \theta_{[\mathcal{L}-e,\gamma-e_L,f_L]} \,.$$

Moreover, since $\frac{\gamma - e_L}{f_L}$ is a positive supersolution of

$$(\mathcal{L} - e)u = (\gamma - e_L)u - f_L u^2 \,,$$

a further application of Lemma 3.2 gives

$$\theta_{[\mathcal{L}-e,\gamma-e_L,f_L]} \leq \frac{\gamma - e_L}{f_L} \,.$$

Notice that any positive constant is a positive strict supersolution of $\mathcal{L} - e$ in Ω. Hence, $\sigma_1^\Omega[\mathcal{L} - e] > 0$ and $\gamma - e_L > 0$. This completes the proof. ☐

The following result provides us with the growth of $\theta_{[\mathcal{L},\gamma,f]}$ as $\gamma \uparrow \infty$.

Theorem 3.4. *The following holds*

$$\lim_{\gamma\uparrow\infty} \frac{\theta_{[\mathcal{L},\gamma,f]}}{\gamma} = f^{-1}$$

uniformly on compact subsets of Ω.

Proof. The new function Ψ_γ defined by

$$\theta_{[\mathcal{L},\gamma,f]} = \gamma\Psi_\gamma$$

is the unique positive solution of

$$\frac{1}{\gamma}\mathcal{L}\Psi = \Psi - f\Psi^2 \qquad \text{in } \Omega\,, \qquad \Psi|_{\partial\Omega} = 0\,. \tag{3.4}$$

It suffices to show that

$$\lim_{\gamma\uparrow\infty} \Psi_\gamma = f^{-1} \tag{3.5}$$

uniformly on compact subsets of Ω.

Let K be a compact subset of Ω. We shall show that given $\varepsilon > 0$ there exists $\gamma_=\gamma(K,\varepsilon) > 0$ such that for every $\gamma > \gamma(K,\varepsilon)$

$$f^{-1} - \varepsilon \le \Psi_\gamma \le f^{-1} + \varepsilon \qquad \text{in } K\,.$$

Fix $\varepsilon > 0$ and let $\overline{\Psi} \in C^\infty(\overline{\Omega})$ such that

$$f^{-1} + \frac{\varepsilon}{2} \le \overline{\Psi} \le f^{-1} + \varepsilon \qquad \text{in } \Omega\,.$$

Then, there exists $\gamma_0 = \gamma_0(\varepsilon)$ such that for any $\gamma > \gamma_0$ the following is satisfied

$$\overline{\Psi} - f\overline{\Psi}^2 = f\overline{\Psi}(f^{-1} - \overline{\Psi}) \le -\frac{\varepsilon}{2} f\overline{\Psi} \le \frac{1}{\gamma}\mathcal{L}\overline{\Psi} \qquad \text{in } \Omega\,.$$

Thus, for any $\gamma > \gamma_0$ the function $\overline{\Psi}$ is a supersolution of (3.4) and thanks to Lemma 3.2 we have

$$\Psi_\gamma \le \overline{\Psi} \le f^{-1} + \varepsilon\,.$$

Since K is compact, to complete the proof of (3.5) it suffices to show that given $x_0 \in K$ there exist a neighborhood $U(x_0)$ of x_0 and a $\gamma_1 = \gamma_1(x_0)$ such that

$$\Psi_\gamma \ge f^{-1} - \varepsilon \qquad \text{in } U(x_0)$$

for each $\gamma > \gamma_1$. For $R > 0$ such that $B_R(x_0) \subset \Omega$, where $B_R(x_0)$ is the ball of radius R centered at x_0, and γ sufficiently large $\Psi_\gamma^{B_R(x_0)}$ will stand for the unique positive solution of

$$\frac{1}{\gamma}\mathcal{L}\Psi = \Psi - f\Psi^2 \qquad \text{in } B_R(x_0)\,, \qquad \Psi|_{\partial B_R(x_0)} = 0\,. \tag{3.6}$$

Since Ψ_γ is a positive strict supersolution of (3.6), we find from Lemma 3.2 that

$$\Psi_\gamma^{B_R(x_0)} \le \Psi_\gamma \qquad \text{in } B_R(x_0)\,.$$

Thus, to complete the proof it remains to show that there exists γ_1 such that

$$\Psi_\gamma^{B_R(x_0)} \geq f^{-1} - \varepsilon \qquad \text{in } \; B_{\frac{R}{2}}(x_0)\,, \tag{3.7}$$

for each $\gamma > \gamma_1$. To prove this we consider two different cases.

Case 1: There exists $R > 0$ such that $f(x)$ is constant in $B_0 := B_R(x_0) \subset \Omega$. Let φ_0 denote the principal eigenfunction associated with $\sigma_1^{B_0}[\mathcal{L}]$ normalized so that

$$\|\varphi_0\|_{\infty,B_0} = \frac{1}{2}\,.$$

Set $B_1 := B_{\frac{R}{2}}(x_0)$. Then, $\varphi_0(x) > 0$ for each $x \in \overline{B}_1$ and there exists $\hat{\varphi}_0 \in W^{2,p}(B_1)$ such that

$$\hat{\varphi}_0(x_0) = 1\,, \qquad \|\hat{\varphi}_0\|_{\infty,B_1} = 1\,, \qquad \hat{\varphi}_0(x) > 0 \quad \forall x \in \overline{B}_1$$

and the function $\Phi : B_0 \to \mathbb{R}$ defined by

$$\Phi(x) = \begin{cases} \varphi_0(x) & \text{if } \; x \in B_0 \setminus B_1\,, \\[2ex] \hat{\varphi}_0(x) & \text{if } \; x \in \overline{B}_1\,, \end{cases}$$

lies in $W^{2,p}(B_0)$. Given $\delta \in (0,1)$ arbitrary set

$$\Phi_\delta := \delta f^{-1}\Phi \in W^{2,p}(B_0)\,.$$

We claim that Φ_δ is a positive subsolution of (3.6) if γ is sufficiently large. Indeed, the following relation holds

$$\gamma^{-1}\mathcal{L}\Phi_\delta \leq \Phi_\delta - f\Phi_\delta^2 \qquad \text{in } \; B_0$$

if, and only if,

$$\frac{\mathcal{L}\Phi}{\Phi} \leq \gamma(1-\delta\Phi) \qquad \text{in } \; B_0\,. \tag{3.8}$$

Since $\gamma > 0$, $\delta < 1$ and $0 \leq \Phi \leq 1$, the right hand side of (3.8) is bounded away from zero. Moreover, by the construction of Φ it is easily seen that $\frac{\mathcal{L}\Phi}{\Phi}$ is bounded above in B_0. Thus, (3.8) is satisfied for γ large enough. This shows the previous claim and hence, thanks to Lemma 3.2, we have that for γ sufficiently large

$$\Psi_\gamma \geq \Psi_\gamma^{B_R(x_0)} \geq \Phi_\delta \qquad \text{in } \; B_R(x_0)\,.$$

Clearly, if δ is taken sufficiently close to 1, then Φ_δ will be as close as we want to f^{-1} on some ball centered at x_0, since $\Phi(x_0) = 1$. This completes the proof in this case.

Case 2: If $f(x)$ is not constant in some ball centered at x_0, then we can compare $\Psi_\gamma^{B_0}$ with the positive solution of a problem with constant coefficients. Indeed, we have

$$\gamma^{-1}\mathcal{L}\Psi_\gamma^{B_0} = \Psi_\gamma^{B_0} - f(\Psi_\gamma^{B_0})^2 \geq \Psi_\gamma^{B_0} - \sup_{B_0} f(\Psi_\gamma^{B_0})^2$$

and so, $\Psi_\gamma^{B_0}$ is a positive supersolution of

$$\gamma^{-1}\mathcal{L}\Psi = \Psi - \sup_{B_0} f\Psi^2 \qquad \text{in } \ B_R(x_0)\,, \qquad \Psi|_{\partial B_R(x_0)} = 0\,. \tag{3.9}$$

Thus, it follows from Lemma 3.2 that

$$\Psi_\gamma^{B_0} \geq \hat{\Psi}_\gamma^{B_0}\,,$$

where $\hat{\Psi}_\gamma^{B_0}$ stands for the unique positive solution of (3.9). Thus, there exists a neighborhood $U(x_0)$ such that

$$\Psi_\gamma^{B_0} \geq \hat{\Psi}_\gamma^{B_0} \geq (\sup_{B_0} f)^{-1} - \frac{\varepsilon}{2}$$

in $U(x_0)$. Therefore, if B_0 is chosen so that for each $x \in B_0$

$$(\sup_{B_0} f)^{-1} \geq (f(x))^{-1} - \frac{\varepsilon}{2}\,,$$

then

$$\Psi_\gamma^{B_0} \geq (f(x))^{-1} - \varepsilon$$

for each $x \in U(x_0)$. This completes the proof. $\square$

4. Change of stability of semi-trivial positive solutions. By Theorem 3.1, (1.1) possesses a semi-trivial positive solution of the form $(u, 0)$ if, and only if, $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$. Moreover, in this case the semi-trivial state is $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$. Similarly, (1.1) possesses a semi-trivial positive solution of the form $(0, v)$ if, and only if, $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ and if this is the case, then it is given by $(0, \theta_{[\mathcal{L}_2,\mu,d]})$. The following result characterizes the linearized stability of each of these semi-trivial states.

Proposition 4.1. *Assume* $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$. *Then,* $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ *is linearly asymptotically stable if, and only if,*

$$\mu < \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]\,; \tag{4.1}$$

linearly unstable if, and only if,

$$\mu > \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]\,; \tag{4.2}$$

and linearly neutrally stable if

$$\mu = \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]\,. \tag{4.3}$$

Similarly, if we assume $\mu > \sigma_1^\Omega[\mathcal{L}_2]$, then $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ is linearly asymptotically stable if, and only if, $\lambda < \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]$; linearly unstable if, and only if, $\lambda > \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]$; and linearly neutrally stable if

$$\lambda = \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]\,. \tag{4.4}$$

Proof. The linearized stability of $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ is given by the sign of the real parts of the eigenvalues of the linearization of (1.1) at $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$, i.e. by the real parts of the τ's for which the following linear problem admits a solution $(u, v) \in (W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))^2 \setminus \{(0,0)\}$

$$\begin{aligned} \mathcal{L}_1 u &= (\lambda - 2a\theta_{[\mathcal{L}_1,\lambda,a]})u + b\theta_{[\mathcal{L}_1,\lambda,a]}v + \tau\, u\,, \\ \mathcal{L}_2 v &= (\mu + c\theta_{[\mathcal{L}_1,\lambda,a]})v + \tau\, v\,. \end{aligned} \tag{4.5}$$

If $v = 0$, then (4.5) becomes into

$$\mathcal{L}_1 u = (\lambda - 2a\theta_{[\mathcal{L}_1,\lambda,a]})u + \tau\, u\,. \tag{4.6}$$

On the other hand, from the definition of $\theta_{[\mathcal{L}_1,\lambda,a]}$ we find from Theorem 2.1 that

$$\sigma_1^\Omega[\mathcal{L}_1 + a\theta_{[\mathcal{L}_1,\lambda,a]} - \lambda] = 0\,.$$

Thus, Theorem 2.3 implies

$$\sigma_1^\Omega[\mathcal{L}_1 + 2a\theta_{[\mathcal{L}_1,\lambda,a]} - \lambda] > 0\,, \tag{4.7}$$

and hence, by Theorem 2.1 any eigenvalue τ of (4.6) satisfies

$$\mathrm{Re}\,\tau > \sigma_1^\Omega[\mathcal{L}_1 + 2a\theta_{[\mathcal{L}_1,\lambda,a]} - \lambda] > 0\,.$$

Thus, the eigenvalues with associated eigenfunctions of the form $(u, 0)$ have positive real part. If $v \neq 0$, then τ is an eigenvalue of $\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]} - \mu$. Assume (4.1). Then,

$$\sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]} - \mu] > 0$$

and due to Theorem 2.1 the real part of any eigenvalue of $\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]} - \mu$ must be positive. Hence, under condition (4.1) the real parts of any eigenvalue τ of (4.5) are positive and therefore, the state $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ is linearly asymptotically stable. Now, assume (4.2). Then,

$$\tau_1 := \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]} - \mu] < 0$$

is an eigenvalue to a positive eigenfunction, say ψ, of the second equation of (4.5). Since $\tau_1 < 0$, (4.7) implies

$$\sigma_1^\Omega[\mathcal{L}_1 + 2a\theta_{[\mathcal{L}_1,\lambda,a]} - \lambda - \tau_1] > 0\,,$$

and therefore, thanks to the strong maximum principle, the first equation of (4.5) with $\tau = \tau_1$ possesses a unique solution. Namely,

$$u = (\mathcal{L}_1 + 2a\theta_{[\mathcal{L}_1,\lambda,a]} - \lambda - \tau_1)^{-1}(b\theta_{[\mathcal{L}_1,\lambda,a]}\psi)\,.$$

Therefore, under condition (4.2) $\tau_1 < 0$ is an eigenvalue of (4.5) and hence the state $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ is linearly unstable. Finally if we assume (4.3), it is easily seen that $\tau_1 = 0$ is an eigenvalue of (4.5) and that any other eigenvalue has positive real part. Therefore, under condition (4.3) the state $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ is linearly neutrally stable.

The results concerning with the other semi-trivial state follow by symmetry interexchanging $\mathcal{L}_1$, λ, a and b by $\mathcal{L}_2$, μ, d and c, respectively. $\square$

By Proposition 4.1 we shall refer to the curve (4.3) in the (λ, μ)-plane as the curve of change of stability of the semi-trivial positive solution $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$. Similarly, the curve (4.4) will be refereed as the curve of change of stability of $(0, \theta_{[\mathcal{L}_2,\mu,d]})$. The following result provides us with the global behavior of these curves.

Proposition 4.2. *The mapping $F(\lambda)$ defined by*

$$F(\lambda) := \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]\,, \qquad \lambda > \sigma_1^\Omega[\mathcal{L}_1]\,, \tag{4.8}$$

is continuous strictly decreasing and satisfies

$$\lim_{\lambda \downarrow \sigma_1^\Omega[\mathcal{L}_1]} F(\lambda) = \sigma_1^\Omega[\mathcal{L}_2]\,, \qquad \lim_{\lambda \uparrow \infty} F(\lambda) = -\infty\,. \tag{4.9}$$

Similarly, the mapping $G(\mu)$ defined by

$$G(\mu) := \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]\,, \qquad \mu > \sigma_1^\Omega[\mathcal{L}_2]\,, \tag{4.10}$$

is continuous strictly decreasing and satisfies

$$\lim_{\mu \downarrow \sigma_1^\Omega[\mathcal{L}_2]} G(\mu) = \sigma_1^\Omega[\mathcal{L}_1]\,, \qquad \lim_{\mu \uparrow \infty} G(\mu) = -\infty\,. \tag{4.11}$$

Proof. The continuity and monotonicity of $F(\lambda)$ can be easily obtained from Theorem 3.1, Corollary 3.3 and Theorem 2.3(ii). The first relation of (4.9) follows from (3.3) and Theorem 2.3(ii). We now show the second relation of (4.9). Since $c \in C(\overline{\Omega})$, $c \geq 0$, $c \neq 0$, there exists a ball B with $\overline{B} \subset \Omega$ such that

$$c_L := \min_{\overline{B}} c > 0\,.$$

On the other hand, by Theorem 3.4

$$\lim_{\lambda\uparrow\infty}\frac{\theta_{[\mathcal{L}_1,\lambda,a]}}{\lambda}=a^{-1}\qquad \text{uniformly in } \overline{B}\,,$$

and hence, there exists λ_0 such that for $\lambda>\lambda_0$

$$\theta_{[\mathcal{L}_1,\lambda,a]}>\frac{\lambda}{2\max_{\overline{B}}a}\qquad \text{in } \overline{B}\,.$$

Therefore, Theorem 2.3 implies

$$F(\lambda)<\sigma_1^B[\mathcal{L}_2-c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]<\sigma_1^B[\mathcal{L}_2]-\frac{c_L}{2\max_{\overline{B}}a}\,\lambda$$

for each $\lambda>\lambda_0$. This completes the proof. The same argument shows the corresponding properties of $G(\mu)$. $\square$

By Proposition 4.2 the curves of change of stability of the semi-trivial positive solutions meet at $(\sigma_1^\Omega[\mathcal{L}_1],\sigma_1^\Omega[\mathcal{L}_2])$. The next result provides us with the tangents of these curves and their concavity or convexity character at this co-dimension two singularity.

Lemma 4.3. *Let φ_j , φ_j^* be the principal eigenfunctions associated with $\mathcal{L}_j$ and $\mathcal{L}_j^*$, respectively, $j=1,2$, where $*$ stands for the adjoint and*

$$\int_\Omega\varphi_j^2=1\,,\qquad \int_\Omega\varphi_j\varphi_j^*=1\,,\qquad j=1\,,2\,.$$

Then,

$$\begin{aligned}\theta_{[\mathcal{L}_1,\lambda,a]}&=(\lambda-\sigma_1^\Omega[\mathcal{L}_1])m_{a,1}^{-1}\varphi_1+(\lambda-\sigma_1^\Omega[\mathcal{L}_1])^2m_{a,1}^{-2}U_1+O((\lambda-\sigma_1^\Omega[\mathcal{L}_1])^3)\,,\\ \theta_{[\mathcal{L}_2,\mu,d]}&=(\mu-\sigma_1^\Omega[\mathcal{L}_2])m_{d,1}^{-1}\varphi_2+(\mu-\sigma_1^\Omega[\mathcal{L}_2])^2m_{d,1}^{-2}U_2+O((\mu-\sigma_1^\Omega[\mathcal{L}_2])^3)\,,\end{aligned}\tag{4.12}$$

$$\begin{aligned}\sigma_1^\Omega[\mathcal{L}_2-c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]&=\sigma_1^\Omega[\mathcal{L}_2]-m_{c,a}(\lambda-\sigma_1^\Omega[\mathcal{L}_1])-M_{c,a}(\lambda-\sigma_1^\Omega[\mathcal{L}_1])^2\\&\quad+O((\lambda-\sigma_1^\Omega[\mathcal{L}_1])^3)\,,\\ \sigma_1^\Omega[\mathcal{L}_1-b(x)\theta_{[\mathcal{L}_2,\mu,d]}]&=\sigma_1^\Omega[\mathcal{L}_1]-m_{b,d}(\mu-\sigma_1^\Omega[\mathcal{L}_2])-M_{b,d}(\mu-\sigma_1^\Omega[\mathcal{L}_2])^2\\&\quad+O((\mu-\sigma_1^\Omega[\mathcal{L}_2])^3)\,,\end{aligned}\tag{4.13}$$

as $\lambda\downarrow\sigma_1^\Omega[\mathcal{L}_1]$ and $\mu\downarrow\sigma_1^\Omega[\mathcal{L}_2]$, where

$$m_{a,1}:=\int_\Omega a\varphi_1^2\varphi_1^*>0\,,\qquad m_{d,1}:=\int_\Omega d\varphi_2^2\varphi_2^*>0\,,$$

$$m_{c,a}:=m_{a,1}^{-1}\int_\Omega c\varphi_1\varphi_2\varphi_2^*\,,\qquad m_{b,d}:=m_{d,1}^{-1}\int_\Omega b\varphi_2\varphi_1\varphi_1^*\,,$$

$$M_{c,a} := \int_\Omega c(x)(m_{a,1}^{-1}\psi_2\varphi_1 + m_{a,1}^{-2}U_1\varphi_2)\varphi_2^* - m_{c,a}\int_\Omega \psi_2\varphi_2^* .$$

$$M_{b,d} := \int_\Omega b(x)(m_{d,1}^{-1}\psi_1\varphi_2 + m_{d,1}^{-2}U_2\varphi_1)\varphi_1^* - m_{b,d}\int_\Omega \psi_1\varphi_1^* ,$$

and we have denoted by β_i*,* $i=1,2$*, and* ψ_i*,* $i=1,2$*, the unique solutions of the following linear problems in* Ω *under homogeneous Dirichlet boundary conditions*

$$(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])\beta_1 = m_{a,1}\varphi_1 - a(x)\varphi_1^2 , \qquad \int_\Omega \beta_1\varphi_1 = 0 ,$$

$$(\mathcal{L}_2 - \sigma_1^\Omega[\mathcal{L}_2])\beta_2 = m_{d,1}\varphi_2 - d(x)\varphi_2^2 , \qquad \int_\Omega \beta_2\varphi_2 = 0 ,$$

$$(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])\psi_1 = (-m_{b,d} + m_{d,1}^{-1}b(x)\varphi_2)\varphi_1 , \qquad \int_\Omega \psi_1\varphi_1 = 0 ,$$

$$(\mathcal{L}_2 - \sigma_1^\Omega[\mathcal{L}_2])\psi_2 = (-m_{c,a} + m_{a,1}^{-1}c(x)\varphi_1)\varphi_2 , \qquad \int_\Omega \psi_2\varphi_2 = 0 ,$$

$$U_1 := \beta_1 - \frac{m_{a,2}}{m_{a,1}}\cdot\varphi_1 , \qquad U_2 := \beta_2 - \frac{m_{d,2}}{m_{d,1}}\cdot\varphi_2 ,$$

where

$$m_{a,2} := 2\int_\Omega a\beta_1\varphi_1\varphi_1^* - m_{a,1}\int_\Omega \beta_1\varphi_1^* , \quad m_{d,2} := 2\int_\Omega d\beta_2\varphi_2\varphi_2^* - m_{d,1}\int_\Omega \beta_2\varphi_2^* .$$

Proof. The relations (4.12) follow from the main theorem of [7] applied to (3.1) with $(\mathcal{L},\gamma,f) = (\mathcal{L}_1,\lambda,a)$ and $(\mathcal{L},\gamma,f) = (\mathcal{L}_2,\mu,d)$. Assume $(\mathcal{L},\gamma,f) = (\mathcal{L}_1,\lambda,a)$. For $\lambda \simeq \sigma_1^\Omega[\mathcal{L}_1]$, the semi-trivial branch $(\lambda,\theta_{[\mathcal{L}_1,\lambda,a]})$ may be parametrized by two analytic functions

$$\lambda(s) = \sigma_1^\Omega[\mathcal{L}_1] + \sum_{j=1}^\infty \lambda_j s^j , \qquad \theta_{[\mathcal{L}_1,\lambda,a]}(s) = s\varphi_1 + \sum_{j=1}^\infty u_j s^{j+1} , \qquad s \simeq 0 ,$$

where

$$\int_\Omega u_j\varphi_1 = 0 , \qquad j \geq 1 . \tag{4.14}$$

Substituting these expansions into (3.1) and identifying the terms of order two and three in s yields

$$(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])u_1 = \lambda_1\varphi_1 - a(x)\varphi_1^2 \quad \text{in } \Omega , \qquad u_1|_{\partial\Omega} = 0 , \tag{4.15a}$$

$$(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])u_2 = \lambda_1 u_1 + \lambda_2\varphi_1 - 2a(x)\varphi_1 u_1 \quad \text{in } \Omega , \qquad u_2|_{\partial\Omega} = 0 , \tag{4.15b}$$

respectively. From (4.14) and the Fredholm alternative applied to (4.15) it is easily seen that

$$\lambda_1 = m_{a,1} , \qquad u_1 = \beta_1 , \qquad \lambda_2 = m_{a,2} .$$

To obtain the first relation of (4.12), it suffices calculating s as a function of λ from $\lambda(s)$. Doing so, we obtain that

$$s(\lambda) = m_{a,1}^{-1}(\lambda - \sigma_1^\Omega[\mathcal{L}_1]) - \frac{m_{a,2}}{m_{a,1}^3}(\lambda - \sigma_1^\Omega[\mathcal{L}_1])^2 + O((\lambda - \sigma_1^\Omega[\mathcal{L}_1])^3)\,.$$

Indeed, substituting this expansion into the expansion of $\theta_{[\mathcal{L}_1,\lambda,a]}(s)$, the first relation of (4.12) get shown.

By standard perturbation results (cf. [18]), the principal eigenvalues in the left hand sides of (4.13) vary analytically with λ and μ. Thus, there exist $K_j \in \mathbb{R}$, $j = 1\,, 2$, such that

$$\begin{aligned}\sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}] = \sigma_1^\Omega[\mathcal{L}_2] + K_1(\lambda - \sigma_1^\Omega[\mathcal{L}_1])&\\ + K_2(\lambda - \sigma_1^\Omega[\mathcal{L}_1])^2 + O((\lambda - \sigma_1^\Omega[\mathcal{L}_1])^3)\,.&\end{aligned} \tag{4.16}$$

Moreover, if $\Psi(\lambda) > 0$ stands for the principal eigenfunction of $\sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]$, i.e.

$$\begin{cases} \mathcal{L}_2\Psi(\lambda) - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}\Psi(\lambda) = \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}]\Psi(\lambda) & \text{in } \Omega \\ \Psi(\lambda) = 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{4.17}$$

normalized so that

$$\int_\Omega \Psi(\lambda)^2 = 1\,, \qquad \int_\Omega (\Psi(\lambda) - \varphi_2)\varphi_2 = 0\,, \tag{4.18}$$

then $\Psi(\lambda)$ admits a unique expansion of the form

$$\Psi(\lambda) = \Psi_0 + (\lambda - \sigma_1^\Omega[\mathcal{L}_1])\Psi_1 + (\lambda - \sigma_1^\Omega[\mathcal{L}_1])^2\Psi_2 + O((\lambda - \sigma_1^\Omega[\mathcal{L}_1])^3)\,. \tag{4.19}$$

Using (4.18) gives

$$\Psi_0 = \varphi_2\,, \qquad \int_\Omega \Psi_j\varphi_2 = 0\,, \qquad j \geq 1\,. \tag{4.20}$$

Now, substituting (4.16), (4.19) into (4.17), using (4.12), (4.20) and identifying the terms with the same order in $\lambda - \sigma_1^\Omega[\mathcal{L}_1]$, we find that

$$(\mathcal{L}_2 - \sigma_1^\Omega[\mathcal{L}_2])\Psi_1 = (K_1 + m_{a,1}^{-1}c(x)\varphi_1)\varphi_2\,, \tag{4.21}$$

$$(\mathcal{L}_2 - \sigma_1^\Omega[\mathcal{L}_2])\Psi_2 = c(x)(m_{a,1}^{-1}\varphi_1\Psi_1 + m_{a,1}^{-2}U_1\varphi_2) + K_1\Psi_1 + K_2\varphi_2\,. \tag{4.22}$$

Applying Fredholm's alternative to (4.21) yields

$$K_1 = -m_{a,1}^{-1}\int_\Omega c(x)\varphi_1\varphi_2\varphi_2^* = -m_{c,a}\,, \qquad \Psi_1 = \psi_2\,.$$

Now, substituting these values into (4.22) and applying Fredholm's alternative gives

$$K_2 = -\int_\Omega c(x)(m_{a,1}^{-1}\varphi_1\psi_2 + m_{a,1}^{-2}U_1\varphi_2)\varphi_2^* + m_{c,a}\int_\Omega \psi_2\varphi_2^* = -M_{c,a}\,.$$

By symmetry, $\theta_{[\mathcal{L}_2,\mu,d]}$ and $\sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]$ have the expansions given in the statement. The proof is completed. $\square$

By (4.13), the tangents to the curves of change of stability of the semi-trivial positive solutions (4.3) and (4.4) at the singularity $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$ are given, respectively, by the stright lines

$$\mu = \sigma_1^\Omega[\mathcal{L}_2] - m_{c,a}(\lambda - \sigma_1^\Omega[\mathcal{L}_1])\,, \qquad \lambda = \sigma_1^\Omega[\mathcal{L}_1] - m_{b,d}(\mu - \sigma_1^\Omega[\mathcal{L}_2])\,. \tag{4.23}$$

Close to the singularity $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$ the convexity or concavity of these curves is given by the sign of $M_{c,a}$ and $M_{d,b}$, respectively. Although in general the problem of ascertaining the sign of these quantities might be very difficult to handle with, as they depend upon some unknown solutions of certain homogeneous Dirichlet boundary value problems, there are some special cases where these signs can be easily found out, as the following result shows.

Lemma 4.4. *If $\mathcal{L}_1 = \mathcal{L}_2$ is a selfadjoint operator and the coefficients a and c are constants, then*

$$M_{c,a} > 0\,. \tag{4.24}$$

By symmetry, if b and d are constant, then

$$M_{b,d} > 0\,.$$

Therefore, if a, b, c and d are constant, then the curves of change of stability are concave in a neighborhood of $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$.

Proof. Since $\mathcal{L}_1 = \mathcal{L}_2$ is a selfadjoint operator, we have that

$$\varphi_1 = \varphi_2 = \varphi_1^* = \varphi_2^*\,.$$

Hence,

$$\int_\Omega \psi_2 \varphi_2^* = \int_\Omega \psi_2 \varphi_2 = 0$$

and

$$M_{c,a} := c m_{a,1}^{-1} \int_\Omega \psi_2 \varphi_1^2 + c m_{a,1}^{-2} \int_\Omega U_1 \varphi_1^2\,. \tag{4.25}$$

Moreover,

$$m_{a,1} = a \int_\Omega \varphi_1^3\,, \qquad m_{a,2} = 2a \int_\Omega \beta_1 \varphi_1^2\,, \qquad U_1 = \beta_1 - 2\frac{\int_\Omega \beta_1 \varphi_1^2}{\int_\Omega \varphi_1^3}\,\varphi_1\,, \tag{4.26}$$

and by the uniqueness of the solution of the corresponding boundary value problem in the orthogonal complement of φ_1, we find that

$$\psi_2 = \frac{-c}{a^2 \int_\Omega \varphi_1^3}\,\beta_1\,. \tag{4.27}$$

Thus, substituting (4.26) and (4.27) into (4.25) gives

$$M_{c,a} = -ca^{-2}(\int_\Omega \varphi_1^3)^{-2}(1+c/a)\int_\Omega \beta_1\varphi_1^2\,. \tag{4.28}$$

To complete the proof of (4.24), it remains to show that

$$\int_\Omega \beta_1\varphi_1^2 < 0\,. \tag{4.29}$$

Indeed, from the β_1-equation it is easily seen that

$$\int_\Omega \beta_1(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])\beta_1 = -a\int_\Omega \beta_1\varphi_1^2\,, \tag{4.30}$$

since $\int_\Omega \beta_1\varphi_1 = 0$. Moreover, β_1 changes of sign in Ω, and hence the variational characterization of $\sigma_1^\Omega[\mathcal{L}_1]$ implies that

$$\int_\Omega \beta_1(\mathcal{L}_1 - \sigma_1^\Omega[\mathcal{L}_1])\beta_1 > 0\,.$$

Therefore, (4.30) implies (4.29). This completes the proof. □

In Figure 1 we have represented the curves of change of stability of the semi-trivial positive solutions in the case when a, b, c and d are constant and $\mathcal{L}_1 = \mathcal{L}_2$ is selfadjoint.

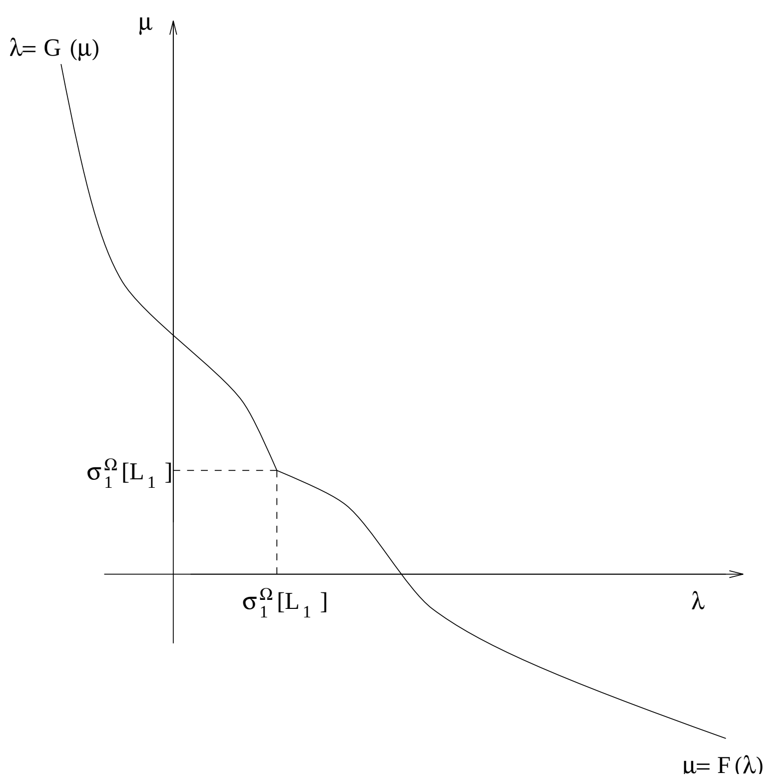

Figure 1: The curves of change of stability.

5. The existence of unbounded continua of coexistence states. Although with less regularity on the several coefficients involved into our setting the abstract theory of [21] applies to (1.1) if the solutions of (1.1) are regarded as fixed points of a compact operator on $(C_0^1(\overline{\Omega}))^2$. This observation provides us with the following result, where the notations introduced in the previous sections will be kept.

Theorem 5.1. *Fix $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ and regard to $\mu \in \mathbb{R}$ as the bifurcation parameter. Then, the point*

$$(\mu, u, v) = (\sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}], \theta_{[\mathcal{L}_1,\lambda,a]}, 0)$$

is the only bifurcation point to coexistence states from the semi-trivial state $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$. Moreover, the maximal component (closed and connected) of coexistence states emanating from $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ at $\mu = F(\lambda)$, say $\mathcal{C}^+_{(\mu,u,0)} \subset \mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, is unbounded.

Now, fix $\mu < \sigma_1^\Omega[\mathcal{L}_2]$ and regard to $\lambda \in \mathbb{R}$ as the bifurcation parameter. By Proposition 4.2 there exists a unique $\lambda_\mu > \sigma_1^\Omega[\mathcal{L}_1]$ such that $\mu = F(\lambda_\mu)$. Then, the point

$$(\lambda, u, v) = (\lambda_\mu, \theta_{[\mathcal{L}_1,\lambda_\mu,a]}, 0)$$

is the only bifurcation point to coexistence states from the curve $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$. Moreover, the maximal component (closed and connected) of coexistence states emanating from $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ at $\lambda = \lambda_\mu$, say $\mathcal{C}^+_{(\lambda,u,0)} \subset \mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, is unbounded.

Similarly, if we fix $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ and regard to $\lambda \in \mathbb{R}$ as the bifurcation parameter, then the point

$$(\lambda, u, v) = (\sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}], 0, \theta_{[\mathcal{L}_2,\mu,d]})$$

is the only bifurcation point to coexistence states from the semi-trivial state $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ and the maximal component (closed and connected) of coexistence states emanating from $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ at $\lambda = G(\mu)$, say $\mathcal{C}^+_{(\lambda,0,v)} \subset \mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, is unbounded.

Finally, fix $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$ and regard to $\mu \in \mathbb{R}$ as the bifurcation parameter. By Proposition 4.2 there exists a unique $\mu_\lambda > \sigma_1^\Omega[\mathcal{L}_2]$ such that $\lambda = G(\mu_\lambda)$. In this case, the point

$$(\mu, u, v) = (\mu_\lambda, 0, \theta_{[\mathcal{L}_2,\mu_\lambda,d]})$$

is the only bifurcation point to coexistence states from the curve $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ and the maximal component (closed and connected) of coexistence states emanating from $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ at $\mu = \mu_\lambda$, say $\mathcal{C}^+_{(\mu,0,v)} \subset \mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, is unbounded.

Proof. The local bifurcations are obtained as an application of the main theorem of [7] using rather standard arguments. It remains to show that each of the continua of coexistence states emanating from the semi-trivial states are unbounded in the phase space. We shall show this for the continuum $\mathcal{C}^+_{(\mu,u,0)}$. The argument can be easily adapted to cover the remaining cases.

By Theorem 4.1 in [21] the continuum $\mathcal{C}^+_{(\mu,u,0)}$ satisfies some of the following alternatives: Either

(i) $\mathcal{C}^+_{(\mu,u,0)}$ is unbounded in $\mathbb{R}\times C^1_0(\overline{\Omega})\times C^1_0(\overline{\Omega})$; or

(ii) there exists $\mu_\infty\in\mathbb{R}$ such that

$$\lambda=\sigma_1^\Omega[\mathcal{L}_1-b\theta_{[\mathcal{L}_2,\mu_\infty,d]}] \tag{5.1}$$

and $(\mu_\infty,0,\theta_{[\mathcal{L}_2,\mu_\infty,d]})\in$ closure $\mathcal{C}^+_{(\mu,u,0)}$; or

(iii) there exists a positive solution $\hat\theta_{[\mathcal{L}_1,\lambda,a]}\neq\theta_{[\mathcal{L}_1,\lambda,a]}$ of

$$\mathcal{L}_1 u=\lambda u-au^2 \qquad \text{in } \ \Omega\,, \qquad u|_{\partial\Omega}=0\,, \tag{5.2}$$

such that $(\sigma_1^\Omega[\mathcal{L}_1-b\hat\theta_{[\mathcal{L}_1,\lambda,a]}],\theta_{[\mathcal{L}_1,\lambda,a]},0)\in$ closure $\mathcal{C}^+_{(\mu,u,0)}$; or

(iv) $\lambda=\sigma_1^\Omega[\mathcal{L}_1]$ and $(\sigma_1^\Omega[\mathcal{L}_2],0,0)\in$ closure $\mathcal{C}^+_{(\mu,u,0)}$.

Since we are assuming that $\lambda>\sigma_1^\Omega[\mathcal{L}_1]$, alternative (iv) is not possible. Moreover, by Theorem 3.1 $\theta_{[\mathcal{L}_1,\lambda,a]}$ is the unique positive solution of (5.2) and hence, alternative (iii) is not possible either. Notice that (5.1) is not possible either, since

$$\sigma_1^\Omega[\mathcal{L}_1-b\theta_{[\mathcal{L}_2,\mu_\infty,d]}]\leq\sigma_1^\Omega[\mathcal{L}_1]\,.$$

Therefore, alternative (i) must occur. This completes the proof. □

6. Coexistence regions for small interaction coefficients.

As an easy consequence from Corollary 3.3 we obtain the following result.

Lemma 6.1. *Assume that*

$$b_Mc_M<a_Ld_L\,, \tag{6.1}$$

and that (1.1) *possesses a coexistence state, say* (u,v). *Then,*

$$\begin{cases} \lambda >(c_1)_L\dfrac{c_Mb_M}{a_Ld_L}+\sigma_1^\Omega[\mathcal{L}_1]\left(1-\dfrac{c_Mb_M}{a_Ld_L}\right)-\dfrac{b_M}{d_L}(\mu-(c_2)_L)\,,\\ \mu >(c_2)_L\dfrac{c_Mb_M}{a_Ld_L}+\sigma_1^\Omega[\mathcal{L}_2]\left(1-\dfrac{c_Mb_M}{a_Ld_L}\right)-\dfrac{c_M}{a_L}(\lambda-(c_1)_L)\,,\end{cases} \tag{6.2}$$

and

$$\begin{cases} u_M\leq\dfrac{(\lambda-(c_1)_L)d_L+(\mu-(c_2)_L)b_M}{a_Ld_L-b_Mc_M}\,,\\ v_M\leq\dfrac{(\mu-(c_2)_L)a_L+(\lambda-(c_1)_L)c_M}{a_Ld_L-b_Mc_M}\,.\end{cases} \tag{6.3}$$

Proof. From (1.1) it is easily seen that

$$u=\theta_{[\mathcal{L}_1,\lambda+bv,a]}\,,\qquad v=\theta_{[\mathcal{L}_2,\mu+cu,d]}\,.$$

Moreover, by Lemma 3.2 and Corollary 3.3 we have

$$\theta_{[\mathcal{L}_1,\lambda+bv,a]} \leq \theta_{[\mathcal{L}_1,\lambda+b_M v_M,a_L]} \leq \frac{\lambda + b_M v_M - (c_1)_L}{a_L}\,.$$

Thus,

$$u_M \leq \frac{\lambda + b_M v_M - (c_1)_L}{a_L}\,. \tag{6.4a}$$

Similarly,

$$v_M \leq \frac{\mu + c_M u_M - (c_2)_L}{d_L}\,. \tag{6.4b}$$

¿From (6.4), relations (6.3) follow readily.

Moreover, the second relation of (6.3) implies

$$\lambda + b_M v_M \leq \frac{\lambda a_L d_L + b_M a_L(\mu - (c_2)_L) - c_M b_M (c_1)_L}{a_L d_L - b_M c_M}\,,$$

and therefore, since $\theta_{[\mathcal{L}_1,\lambda+b_M v_M,a_L]} \geq u > 0$, we find from Theorem 3.1 that

$$\frac{\lambda a_L d_L + b_M a_L(\mu - (c_2)_L) - c_M b_M (c_1)_L}{a_L d_L - b_M c_M} > \sigma_1^\Omega[\mathcal{L}_1]. \tag{6.5a}$$

Similarly,

$$\frac{\mu a_L d_L + c_M d_L(\lambda - (c_1)_L) - c_M b_M (c_2)_L}{a_L d_L - b_M c_M} > \sigma_1^\Omega[\mathcal{L}_2]. \tag{6.5b}$$

Relations (6.2) follow readily from (6.5). This completes the proof. $\square$

Note that if λ and μ satisfy (6.2), then the following relations hold

$$\begin{cases} \lambda > (c_1)_L - \dfrac{b_M}{d_L}(\mu - (c_2)_L)\,, \\ \mu > (c_2)_L - \dfrac{c_M}{a_L}(\lambda - (c_1)_L)\,, \end{cases} \tag{6.6}$$

and therefore, the right hand sides of (6.3) are positive. Indeed, it is easily seen from Theorems 2.2, 2.3 that

$$\sigma_1^\Omega[\mathcal{L}_1] = \sigma_1^\Omega[\mathcal{L}_1 - c_1 + c_1] > \sigma_1^\Omega[\mathcal{L}_1 - c_1] + (c_1)_L > (c_1)_L\,. \tag{6.7}$$

Thus, we find from (6.1) and (6.7) that

$$\frac{c_M b_M}{a_L d_L}(c_1)_L + \sigma_1^\Omega[\mathcal{L}_1](1 - \frac{b_M c_M}{a_L d_L}) > (c_1)_L\,,$$

and hence,

$$(c_1)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_1]\left(1 - \frac{c_M b_M}{a_L d_L}\right) - \frac{b_M}{d_L}(\mu - (c_2)_L) > (c_1)_L - \frac{b_M}{d_L}(\mu - (c_2)_L)\,.$$

Similarly,

$$(c_2)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_2]\left(1 - \frac{c_M b_M}{a_L d_L}\right) - \frac{c_M}{a_L}(\lambda - (c_1)_L) > (c_2)_L - \frac{c_M}{a_L}(\lambda - (c_1)_L)\,.$$

This shows the claim above.

Under assumption (6.1), (6.2) provides us with a simple readily computable necessary condition for the existence of a coexistence state. Moreover, (6.3) shows that we have a priori bounds in $L^\infty(\Omega)$ for the coexistence states of (1.1) uniformly on compact subsets of the parameter space (λ, μ). By the L^p-estimates of Agmon, Douglis and Nirenberg we have uniform a priori bounds in $W^{2,p}(\Omega)$ for all $p \in [2, \infty)$. Notice that the boundary of the non-existence region given by (6.2) consists of the stright lines

$$\lambda = (c_1)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_1](1 - \frac{c_M b_M}{a_L d_L}) - \frac{b_M}{d_L}(\mu - (c_2)_L)\,,$$

$$\mu = (c_2)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_2](1 - \frac{c_M b_M}{a_L d_L}) - \frac{c_M}{a_L}(\lambda - (c_1)_L)\,.$$

In Figure 2 we have represented these lines together with the curves of change of stability of semi-trivial positive solutions.

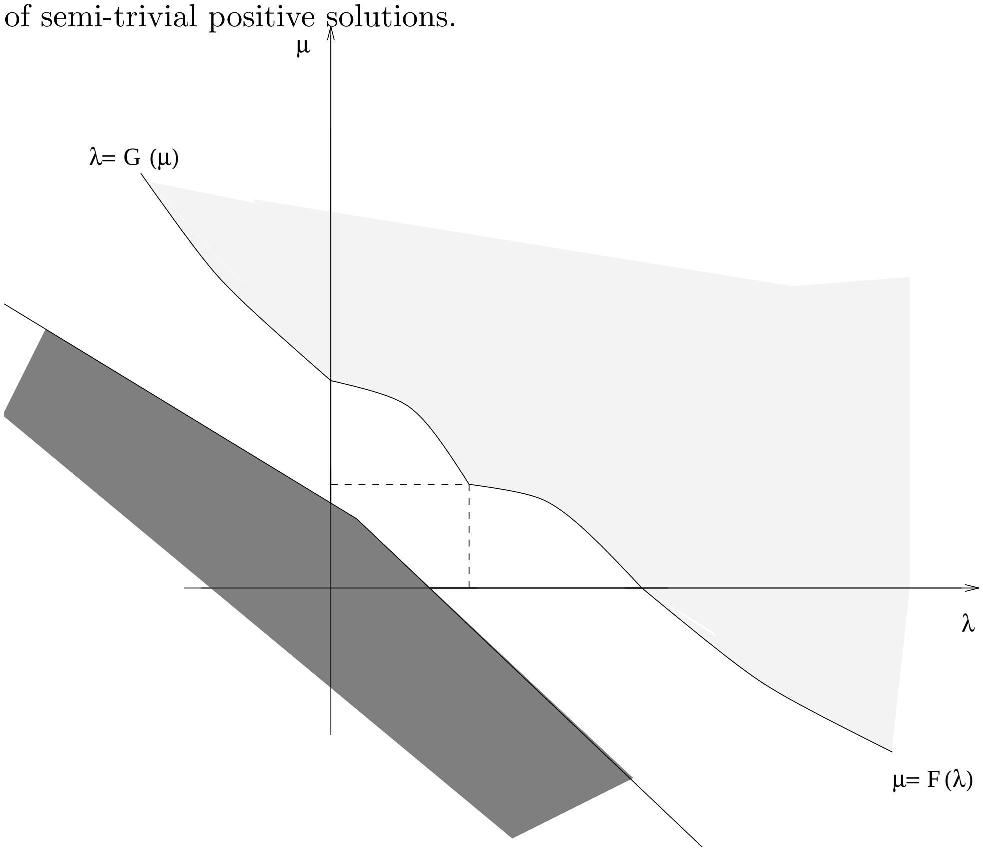

Figure 2: Estimating the coexistence region.

The stright lines in the figure across at the point

$$\left(\sigma_1^\Omega[\mathcal{L}_1] - \frac{b_M}{d_L}(\sigma_1^\Omega[\mathcal{L}_2] - (c_2)_L) \, , \, \sigma_1^\Omega[\mathcal{L}_2] - \frac{c_M}{a_L}(\sigma_1^\Omega[\mathcal{L}_1] - (c_1)_L) \right)$$

and is is easily seen that their relative positions with respects to the curves of change of stability are those shown on it. Indeed, if $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ then we have from Lemma 3.2, Corollary 3.3 and (6.7) that

$$\begin{aligned}\sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}] \geq & \sigma_1^\Omega[\mathcal{L}_1 - \frac{b_M}{d_L}(\mu - (c_2)_L)] = \sigma_1^\Omega[\mathcal{L}_1] - \frac{b_M}{d_L}(\mu - (c_2)_L) \\ > & (c_1)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_1]\left(1 - \frac{c_M b_M}{a_L d_L}\right) - \frac{b_M}{d_L}(\mu - (c_2)_L) \, .\end{aligned}$$

Similarly, if $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ then

$$\sigma_1^\Omega[\mathcal{L}_2 - b\theta_{[\mathcal{L}_1,\lambda,a]}] > (c_2)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_2]\left(1 - \frac{c_M b_M}{a_L d_L}\right) - \frac{c_M}{a_L}(\lambda - (c_1)_L) \, .$$

Therefore, in the dark grey region of Figure 2 (1.1) does not admit a coexistence state.

The next theorem shows that (1.1) possesses a coexistence state in the bright grey region of Figure 2. In the area in between these two regions (1.1) may have or not a coexistence state depending on the size of the coefficients.

Theorem 6.2. *Assume* (6.1) *and*

$$\lambda > \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}] \, , \qquad \mu > \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}] \, . \tag{6.8}$$

Then, (1.1) *possesses a coexistence state.*

Proof. Fix $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ and regard to λ as the main bifurcation parameter. By Lemma 6.1 problem (1.1) does not admit a coexistence state if

$$\lambda \leq (c_1)_L \frac{c_M b_M}{a_L d_L} + \sigma_1^\Omega[\mathcal{L}_1]\left(1 - \frac{c_M b_M}{a_L d_L}\right) - \frac{b_M}{d_L}(\mu - (c_2)_L) \, .$$

Moreover, by Theorem 5.1 the continuum $\mathcal{C}^+_{(\lambda,0,v)}$ of coexistence states emanating from $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ at the value of the parameter $\lambda = \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]$ is unbounded, and thanks to Lemma 6.1 these coexistence states are bounded in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ uniformly on compact subintervals of λ. Therefore, (1.1) possesses a coexistence state for each $\lambda > \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]$. Similarly, if $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$, then (1.1) possesses a coexistence state for each $\mu > \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}]$. This completes the proof. $\square$

7. On the existence of coexistence states for large interaction coefficients in the case $\mathcal{L}_1 = \mathcal{L}_2$. Throughout this section we assume that

$$\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}\,, \tag{7.1}$$

where $\mathcal{L}$ is a differential operator of the form (2.1) with coefficients satisfying (2.2). First we shall obtain some necessary conditions for the existence of a coexistence state. Then, we shall show the existence of uniform a priori bounds for the coexistence states in low space dimensions. Finally, we shall combine these results together with the global bifurcation theorem of Section 5 to get some sufficient conditions for the existence of a coexistence state. All these results will be obtained for the case when the interactions between the species are sufficiently large. How large must be the interactions will be measured in terms of the several coefficients involved in the setting of (1.1).

7.1. Necessary conditions. The following result provides us with some necessary conditions for the existence of coexistence states of (1.1) in the case when the interactions between the two species are sufficiently large.

Theorem 7.1. *Under condition* (7.1)*, the following assertions are true:*

(i) If $\mu \geq \lambda > \sigma_1^\Omega[\mathcal{L}]$ *and*

$$b_L c_L \geq a_M d_M + a_M b_M - a_L b_L\,, \tag{7.2}$$

then (1.1) *does not admit a coexistence state.*

(ii) If $\lambda \geq \mu > \sigma_1^\Omega[\mathcal{L}]$ *and*

$$b_L c_L \geq a_M d_M + d_M c_M - d_L c_L\,, \tag{7.3}$$

then (1.1) *does not admit a coexistence state.*

(iii) If

$$b_L c_L > a_M d_M + a_M b_M - a_L b_L\,, \tag{7.4}$$

then for each $\lambda < \sigma_1^\Omega[\mathcal{L}]$ *there exists* $\mu = \mu(\lambda)$ *such that* $\lambda > \sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu(\lambda),d]}]$ *and* (1.1) *does not admit a coexistence state if* $\mu > \mu(\lambda)$*. Moreover,* $\mu(\lambda)$ *can be chosen to be continuous in* λ*.*

(iv) If

$$b_L c_L > a_M d_M + d_M c_M - d_L c_L\,, \tag{7.5}$$

then for each $\mu < \sigma_1^\Omega[\mathcal{L}]$ *there exists* $\lambda = \lambda(\mu)$ *such that* $\mu > \sigma_1^\Omega[\mathcal{L} - c\theta_{[\mathcal{L},\lambda(\mu),a]}]$ *and* (1.1) *does not admit a coexistence state if* $\lambda > \lambda(\mu)$*. Moreover,* $\lambda(\mu)$ *can be chosen to be continuous in* μ*.*

If the coefficients a, b, c and d of (1.1) are assumed to be constant, then (7.2) and (7.3) become into

$$bc \geq ad \tag{7.6}$$

and it follows from Theorem 7.1 (i), (ii) that under condition (7.6) the problem (1.1) does not admit a coexistence state in the region

$$\lambda > \sigma_1^\Omega[\mathcal{L}]\,, \qquad \mu > \sigma_1^\Omega[\mathcal{L}]\,.$$

Therefore, Parts (i), (ii) of Theorem 7.1 provide us with a substantial extension of Theorem 3.3 in [27] to cover our general setting. Theorem 3.3 of [27] as well as the corresponding non-existence results of [17], [20] and [30] were found for the very special case when $\mathcal{L} = -\Delta$ and $a(x)$, $b(x)$, $c(x)$, $d(x)$ are constants. Parts (iii), (iv) of Theorem 7.1 are new even for this special case and besides their intrinsic interest, they are pivotal to get our existence and multiplicity results from Theorem 5.1. The proof of Theorem 7.1 will follow after a couple of lemmas which are of interest in their own right. The first lemma is an extension of Lemma 3.2 in [27]. The second one is a sharper result showing that the coexistence states of (1.1) must grow to infinity as $\mu \uparrow \infty$ at least linearly in μ, uniformly on compact subsets of Ω.

Lemma 7.2. *(i) Assume* (7.1), $\mu \geq \lambda$, *and let* (u, v) *be any coexistence state of* (1.1). *Then,*

$$u \leq \frac{b_M + d_M}{c_L + a_L}\, v\,. \tag{7.7}$$

(ii) By symmetry, under conditions (7.1) *and* $\lambda \geq \mu$, *we have*

$$v \leq \frac{c_M + a_M}{b_L + d_L}\, u\,, \tag{7.8}$$

for any coexistence state (u, v) *of* (1.1).

Proof. Assume (7.1), $\mu \geq \lambda$, and let (u, v) be any coexistence state of (1.1). Set

$$w = (b_M + d_M)v - (c_L + a_L)u\,.$$

Then, it is easily seen from (1.1) that

$$(\mathcal{L} - \lambda + a_L u + d_M v)w \geq 0\,. \tag{7.9}$$

Moreover, it follows from the second equation of (1.1) that

$$\mu = \sigma_1^\Omega[\mathcal{L} - cu + dv]\,,$$

and hence, by the monotonicity of the principal eigenvalue with respect to the potential we find that

$$\lambda \leq \mu \leq \sigma_1^\Omega[\mathcal{L} - c_L u + d_M v]\,.$$

Thus,

$$\sigma_1^\Omega[\mathcal{L} - \lambda - c_L u + d_M v] \geq 0$$

and

$$\sigma_1^\Omega[\mathcal{L} - \lambda + a_L u + d_M v] > \sigma_1^\Omega[\mathcal{L} - \lambda - c_L u + d_M v] \geq 0\,.$$

Therefore, due to the strong maximum principle, (7.9) implies $w \geq 0$. This completes the proof. $\square$

The following result holds for general differential operators $\mathcal{L}_1$ and $\mathcal{L}_2$, not necessarily equal.

Lemma 7.3. *(i) Fix $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$ and consider $\mu_0(\lambda) > \sigma_1^\Omega[\mathcal{L}_2]$ such that*

$$\lambda > \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}] \qquad \text{for each } \mu > \mu_0(\lambda)\,. \tag{7.10}$$

Assume that there exists a sequence of coexistence states of (1.1)*, say (μ_n, u_n, v_n), $n \geq 1$, such that $\mu_n > \max\{\mu_0(\lambda), 0\}$ for each $n \geq 1$ and $\lim_{n\uparrow\infty} \mu_n = \infty$. Then, for any compact subset $K \subset \Omega$ there exists a positive constant $\alpha = \alpha(K) > 0$ such that for each $n \geq 1$*

$$\frac{v_n}{\mu_n} \geq \alpha \qquad \text{in} \quad K\,. \tag{7.11}$$

(ii) Similarly, if we fix $\mu < \sigma_1^\Omega[\mathcal{L}_2]$, consider $\lambda_0(\mu) > \sigma_1^\Omega[\mathcal{L}_1]$ such that

$$\mu > \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}] \qquad \text{for each } \lambda > \lambda_0(\mu)\,,$$

and assume that there exists a sequence of coexistence states of (1.1)*, say (λ_n, u_n, v_n), $n \geq 1$, such that $\lambda_n > \max\{\lambda_0(\mu), 0\}$ for each $n \geq 1$ and $\lim_{n\uparrow\infty} \lambda_n = \infty$. Then, for any compact subset $K \subset \Omega$ there exists a positive constant $\beta = \beta(K) > 0$ such that for each $n \geq 1$*

$$\frac{u_n}{\lambda_n} \geq \beta \qquad \text{in} \quad K\,. \tag{7.12}$$

Proof. Part (ii) follows by symmetry from Part (i). So, it suffices to prove Part (i). Pick up $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$. The existence of $\mu_0(\lambda)$ satisfying (7.10) is guaranteed from Proposition 4.2. Let (μ_n, u_n, v_n), $n \geq 1$, be a sequence of coexistence states of (1.1) with $\mu_n > \max\{\mu_0(\lambda), 0\}$, $n \geq 1$, and $\lim_{n\uparrow\infty} \mu_n = \infty$. Then, the second equation of (1.1) gives

$$\mathcal{L}_2 v_n = \mu_n v_n - dv_n^2 + cu_n v_n > \mu_n v_n - dv_n^2$$

and hence v_n is a strict positive supersolution of

$$\mathcal{L}_2 w = \mu_n w - dw^2 \quad \text{in } \ \Omega\,, \qquad w|_{\partial\Omega} = 0\,.$$

Thus, thanks to Lemma 3.2,

$$v_n \geq \theta_{[\mathcal{L}_2,\mu_n,d]}\,. \tag{7.13}$$

Substituting (7.13) into the first equation of (1.1) and repeating the previous argument gives

$$u_n \geq \theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu_n,d]},\lambda,a]}\,. \tag{7.14}$$

Note that the function on the right hand side of this inequality is well defined (and strongly positive), because of (7.10). Relation (7.14) yields

$$\liminf_{n\to\infty} \frac{u_n}{\mu_n} \geq \liminf_{n\to\infty} \frac{\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu_n,d]},\lambda,a]}}{\mu_n}\,. \tag{7.15}$$

We now show that

$$\liminf_{n\to\infty} \frac{\theta_{[\mathcal{L}_1-b\theta_{[\mathcal{L}_2,\mu_n,d]},\lambda,a]}}{\mu_n} \geq \frac{b_L}{a_M d_M}\,, \tag{7.16}$$

uniformly on compact subsets of Ω. Let Ω_1, Ω_2 two arbitrary subdomains of Ω such that

$$\overline{\Omega}_1 \subset \Omega_2\,, \qquad \overline{\Omega}_2 \subset \Omega\,. \tag{7.17}$$

Set

$$\Theta_n := \frac{\theta_{[\mathcal{L}_1-b\theta_{[\mathcal{L}_2,\mu_n,d]},\lambda,a]}}{\mu_n}\,.$$

By definition, Θ_n is the unique positive solution of

$$\frac{1}{\mu_n}\mathcal{L}_1 w = \left(\frac{\lambda}{\mu_n} + b\frac{\theta_{[\mathcal{L}_2,\mu_n,d]}}{\mu_n}\right) w - aw^2 \quad \text{in} \quad \Omega\,, \qquad w|_{\partial\Omega} = 0\,. \tag{7.18}$$

By Theorem 3.4,

$$\lim_{n\to\infty} \frac{\theta_{[\mathcal{L}_2,\mu_n,d]}}{\mu_n} = d^{-1} \qquad \text{uniformly in } \overline{\Omega}_2\,.$$

Hence,

$$\lim_{n\to\infty} \left(\frac{\lambda}{\mu_n} + b\frac{\theta_{[\mathcal{L}_2,\mu_n,d]}}{\mu_n}\right) = bd^{-1} \qquad \text{uniformly in } \overline{\Omega}_2\,.$$

Thus, for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for each $n \geq n_0$ we have

$$\frac{\lambda}{\mu_n} + b\frac{\theta_{[\mathcal{L}_2,\mu_n,d]}}{\mu_n} \geq \frac{b_L}{d_M} - \varepsilon \qquad \text{in } \Omega_2\,. \tag{7.19}$$

Now, since Θ_n is the unique positive solution of (7.18), it follows from (7.19) that for each $n \geq n_0$ the function Θ_n is a strict positive supersolution of the following problem

$$\frac{1}{\mu_n}\mathcal{L}_1 w = \left(\frac{b_L}{d_M} - \varepsilon\right) w - aw^2 \quad \text{in } \Omega_2\,, \qquad w|_{\partial\Omega_2} = 0\,. \tag{7.20}$$

Suppose that $\varepsilon > 0$ has been chosen so that $\frac{b_L}{d_M} - \varepsilon > 0$. Then, for n sufficiently large we have that

$$\frac{b_L}{d_M} - \varepsilon > \sigma_1^{\Omega_2}[\frac{1}{\mu_n}\mathcal{L}_1] = \frac{\sigma_1^{\Omega_2}[\mathcal{L}_1]}{\mu_n} \to 0 \qquad \text{as } n \to \infty\,,$$

and hence, it follows from Theorem 3.1 that (7.20) possesses a unique positive solution, say $\Theta_n^{\Omega_2}$. By Lemma 3.2 we have

$$\Theta_n \geq \Theta_n^{\Omega_2} \qquad \text{in } \Omega_2$$

for all n sufficiently large. Moreover, thanks to Theorem 3.4, we find from (7.17) that

$$\lim_{n\to\infty} \Theta_n^{\Omega_2} = \frac{b_L}{ad_M} - \frac{\varepsilon}{a} \qquad \text{uniformly in } \ \Omega_1 .$$

Thus,

$$\liminf_{n\to\infty} \Theta_n \geq \frac{b_L}{a_M d_M} - \frac{\varepsilon}{a_L} \qquad \text{uniformly in } \ \Omega_1 .$$

As this is valid for any $\varepsilon > 0$, (7.16) holds uniformly in Ω_1. As Ω_1 is an arbitrary subdomain of Ω with $\overline{\Omega}_1 \subset \Omega$, it is clear that (7.16) holds uniformly on any compact subset of Ω. Therefore, it follows from (7.15) that

$$\liminf_{n\to\infty} \frac{u_n}{\mu_n} \geq \frac{b_L}{a_M d_M} , \tag{7.21}$$

uniformly on any compact subset of Ω and, in particular, uniformly on $\overline{\Omega}_1$. Now, setting

$$\hat{u}_n := \frac{u_n}{\mu_n} , \qquad \hat{v}_n := \frac{v_n}{\mu_n} ,$$

it follows from the second equation of (1.1) that

$$\frac{1}{\mu_n}\mathcal{L}_2 \hat{v}_n = \hat{v}_n - d\hat{v}_n^2 + c\hat{u}_n\hat{v}_n$$

and hence we find from (7.21) that given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that $\hat{v}_n$ is a strict positive supersolution of

$$\frac{1}{\mu_n}\mathcal{L}_2 w = \left(1 + \frac{c_L b_L}{a_M d_M} - \varepsilon\right) w - dw^2 \quad \text{in } \ \Omega_1 , \qquad w|_{\partial\Omega_1} = 0 \tag{7.22}$$

for each $n \geq n_0$. Suppose that $\varepsilon > 0$ has been chosen sufficiently small so that

$$1 + \frac{c_L b_L}{a_M d_M} > \varepsilon .$$

Then, thanks to Theorem 3.1, for n sufficiently large (7.22) possesses a unique positive solution, denoted by $\Theta_n^{\Omega_1}$, and, due to Lemma 3.2, we find that

$$\hat{v}_n = \frac{v_n}{\mu_n} \geq \Theta_n^{\Omega_1} , \tag{7.23}$$

except at most for a finite number of n's.

Let K be an arbitrary compact subset of Ω and choose Ω_1, Ω_2 satisfying (7.17) and $K \subset \Omega_1$. Then, by Theorem 3.4,

$$\lim_{n\to\infty} \Theta_n^{\Omega_1} = (1 + \frac{c_L b_L}{a_M d_M} - \varepsilon) d^{-1} \qquad \text{uniformly in } \ K ,$$

and since this limit is positive and bounded away from zero, the existence of $\alpha > 0$ satisfying (7.11) is easily obtained from (7.23). This completes the proof. □

Proof of Theorem 7.1. (i) Assume (7.1), (7.2) and pick $\mu \geq \lambda > \sigma_1^\Omega[\mathcal{L}]$. If (1.1) possesses a coexistence state, say (u, v), then we find from the first equation of (1.1) that

$$\lambda = \sigma_1^\Omega[\mathcal{L} + au - bv] \leq \sigma_1^\Omega[\mathcal{L} + a_M u - b_L v]\,. \tag{7.24}$$

Moreover, thanks to Lemma 7.2(i), we find from (7.2) that

$$u \leq \frac{b_M + d_M}{c_L + a_L}\, v \leq \frac{b_L}{a_M}\, v\,.$$

Thus,

$$a_M u - b_L v \leq 0\,,$$

and (7.24) gives $\lambda \leq \sigma_1^\Omega[\mathcal{L}]$, which is impossible. Therefore, (1.1) can not admit a coexistence state. This completes the proof of Part (i). Part (ii) follows by symmetry, interexchanging the roles of λ, a and b by μ, d and c, respectively.

We now prove (iii). Assume (7.1), (7.4) and fix $\lambda < \sigma_1^\Omega[\mathcal{L}]$. We argue by contradiction assuming that there exists a sequence of coexistence states of (1.1), say (μ_n, u_n, v_n), $n \geq 1$, such that $\mu_n > \max\{\mu_0(\lambda), 0\}$, $n \geq 1$, and $\lim_{n\uparrow\infty} \mu_n = \infty$. Without loss of generality we can assume that $\mu_n \geq \lambda$ for each $n \geq 1$. Let $\Omega_1 \subset \Omega$ an arbitrary subdomain of Ω with $\overline{\Omega}_1 \subset \Omega$. By lemma 7.3(i), there exists $\alpha = \alpha(\Omega_1) > 0$ such that for each $n \geq 1$

$$\frac{v_n}{\mu_n} \geq \alpha \quad \text{in } \Omega_1\,.$$

Moreover, by Lemma 7.2(i), we have that for each $n \geq 1$

$$\frac{u_n}{\mu_n} \leq \frac{b_M + d_M}{c_L + a_L}\,\frac{v_n}{\mu_n}\,.$$

Thus, by (7.4) there exists $\varepsilon > 0$ such that for each $n \geq 1$

$$\frac{u_n}{\mu_n} \leq \frac{b_L}{a_M}\,\frac{v_n}{\mu_n} - \varepsilon \quad \text{in } \Omega_1\,.$$

Hence,

$$a_M u_n - b_L v_n \leq -\varepsilon a_M \mu_n \quad \text{in } \Omega_1 \qquad \forall n \geq 1\,. \tag{7.25}$$

On the other hand, we find from the first equation of (1.1) that

$$\lambda = \sigma_1^\Omega[\mathcal{L} + au_n - bv_n] \leq \sigma_1^{\Omega_1}[\mathcal{L} + a_M u_n - b_L v_n]$$

and therefore, (7.25) gives

$$\lambda \leq \sigma_1^{\Omega_1}[\mathcal{L}] - \varepsilon a_M \mu_n \downarrow -\infty \qquad \text{as} \quad n \to \infty\,.$$

This contradiction shows that (1.1) does not admit a coexistence state for μ large and completes the proof of this part. Part (iv) follows by symmetry. □

7.2. A priori bounds for $N \leq 5$. The following result provides us with uniform a priori bounds in L^∞ for the coexistence states of (1.1).

Theorem 7.4. *Under condition* (7.1)*, if $N \leq 5$, $b_L c_L > a_M d_M$ and for some $\alpha > 0$*

$$\max\{\,|\lambda|\,,\,|\mu|\,\} \leq \alpha\,,$$

then there exists a constant $C = C(\alpha, \Omega, a, b, c, d)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\,, \qquad \|v\|_{L^\infty(\Omega)} \leq C\,,$$

for any coexistence state (u, v) of (1.1)*.*

This result is optimal in the sense that if $N > 5$, then there are choices of the several coefficients and of Ω for which the uniform a priori bounds are lost (cf. the final comments in Section 5 of [21] and Theorem 1.4 of [27]). For instance, if a, b, c, d are constants and $\lambda = \mu$, then for any coexistence state (u, v) of (1.1) it is easily seen that

$$(\mathcal{L} - \lambda + au + dv)((b + d)v - (c + a)u) = 0$$

and hence,

$$v = \frac{c + a}{b + d}\, u\,, \tag{7.26}$$

since $\sigma_1^\Omega[\mathcal{L} - \lambda + au + dv] > 0$. Therefore, (u, v) is a coexistence state of (1.1) if, and only if, (7.26) holds and u is a positive solution of

$$\mathcal{L}u = \lambda u + \frac{bc - ad}{b + d}\, u^2 \quad \text{in } \ \Omega\,, \qquad u|_{\partial\Omega} = 0\,. \tag{7.27}$$

If $bc < ad$, then the coefficient of u^2 in (7.27) is negative and hence the positive solutions of (7.27) possesses uniform a priori bounds on compact subintervals of λ. On the contrary, when $bc > ad$ the coefficient of u^2 in (7.27) is positive and therefore (7.27) is a superlinear problem. In this case it is well known that a priori bounds are available if $2 < \frac{N+2}{N-2}$ (cf. [13]), i.e. if $N \leq 5$, while in the case when $N \geq 6$ the a priori bounds are in general lost and the structure of the set of positive solutions can change drastically as either the geometry of Ω changes or the spatial dimension N increases. Being the higher dimensional case outside the scope of this work we send to the interested reader in further details to [4] and [8].

In the special case when $\mathcal{L} = -\Delta$ and a, b, c and d are constants Theorem 7.4 is given by Lemma 4.3 of [27], but the proof of [27] can not be adapted to cover our current situation here, as it will become clear later. The main difficulty coming from the fact that now the coefficients are not constant. To prove Theorem 7.4 we will argue by contradiction using the blowing up argument introduced in [13] for the case of one single equation. It should be noted that our blowing up argument is somewhat different from the corresponding argument used in [27].

Proof of Theorem 7.4. We shall prove the result in case $\lambda \geq \mu$. By symmetry, the result is also true when $\mu \geq \lambda$. If the conclusion of Theorem 7.4 is false, then there exists a sequence of coexistence states $(\lambda_k, \mu_k, u_k, v_k)$, $k \geq 1$, with $-\alpha \leq \mu_k \leq \lambda_k \leq \alpha$, such that

$$\limsup_{k\to\infty}(\|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)}) = \infty \,. \tag{7.28}$$

We claim that

$$\limsup_{k\to\infty} \|u_k\|_{L^\infty(\Omega)} = \limsup_{k\to\infty} \|v_k\|_{L^\infty(\Omega)} = \infty \,. \tag{7.29}$$

Indeed, if $\{\|v_k\|_{L^\infty(\Omega)}\}_{k\geq 1}$ is bounded by some positive constant β, then we find from the first equation of (1.1) that

$$\mathcal{L}u_k \leq (\alpha + b_M\beta)u_k - au_k^2$$

and therefore, it follows from Lemma 3.2 and Corollary 3.3, that $\{\|u_k\|_{L^\infty(\Omega)}\}_{k\geq 1}$ is also bounded. By (7.28) this is impossible. Similarly, if $\{\|u_k\|_{L^\infty(\Omega)}\}_{k\geq 1}$ is bounded, then $\{\|v_k\|_{L^\infty(\Omega)}\}_{k\geq 1}$ is also bounded. Therefore, (7.29) is satisfied. By chosing a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} \|u_k\|_{L^\infty(\Omega)} = \infty \,, \qquad \lim_{k\to\infty} (\lambda_k, \mu_k) = (\lambda_\infty, \mu_\infty) \,, \tag{7.30}$$

for some $(\lambda_\infty, \mu_\infty) \in \mathbb{R}^2$ satisfying $-\alpha \leq \mu_\infty \leq \lambda_\infty \leq \alpha$. Note that thanks to Lemma 7.2(ii) we have that

$$v_k \leq \frac{c_M + a_M}{b_L + d_L}\, u_k \qquad \forall k \geq 1 \,. \tag{7.31}$$

For each $k \geq 1$, pick $x_k \in \Omega$ such that

$$M_k := u_k(x_k) = \|u_k\|_{L^\infty(\Omega)} \,. \tag{7.32}$$

Since Ω is bounded, without loss of generality we can assume that

$$\lim_{k\to\infty} x_k = x_\infty \in \overline{\Omega} \,. \tag{7.33}$$

Now, we consider two different situations, accordingly with whether $x_\infty \in \Omega$ or $x_\infty \in \partial\Omega$.

Assume that $x_\infty \in \Omega$. Then,

$$\delta := d(x_\infty, \partial\Omega)/2 > 0 \,.$$

Moreover, setting

$$\rho_k := M_k^{-1/2} \,, \qquad k \geq 1 \,,$$

we have $\lim_{k\to\infty}\rho_k = 0$, since thanks to (7.30) and (7.32) $\lim_{k\to\infty} M_k = \infty$. Now, it is easily seen that the change of variables

$$y := \frac{x - x_k}{\rho_k}\,, \qquad (z_k, w_k) := \rho_k^2(u_k, v_k)\,, \qquad k \geq 1\,, \tag{7.34}$$

transforms the system of (1.1) into

$$\begin{aligned} \mathcal{A}_k z_k &= \rho_k^2\lambda_k z_k - a(x_k + \rho_k y)z_k^2 + b(x_k + \rho_k y)z_k w_k\,, \\ \mathcal{A}_k w_k &= \rho_k^2\mu_k w_k - d(x_k + \rho_k y)w_k^2 + c(x_k + \rho_k y)z_k w_k\,, \end{aligned} \tag{7.35}$$

where

$$\mathcal{A}_k = -\sum_{i,j=1}^{N} a_{ij}(x_k + \rho_k y)\partial_i\partial_j + \rho_k\sum_{j=1}^{N} b_j(x_k + \rho_k y)\partial_j + \rho_k^2 e(x_k + \rho_k y)\,, \tag{7.36}$$

provided $x_k + \rho_k y \in \Omega$. By definition of δ, for k sufficiently large, $|x - x_k| \leq \delta$ implies $x = x_k + \rho_k y \in \Omega$. Hence, $|y| \leq \frac{\delta}{\rho_k}$ implies $x = x_k + \rho_k y \in \Omega$ and so (7.35) holds. Since $\lim_{k\to\infty}\frac{\delta}{\rho_k} = \infty$, given $R > 0$ arbitrary $B_R \subset B_{\delta/\rho_k}$ for k sufficiently large, where for any $\tau > 0$ B_τ stands for the ball of radius τ centered at the origin. Now, from the definition of ρ_k we have that

$$z_k = \rho_k^2 u_k = \frac{u_k}{M_k}$$

and hence,

$$\|z_k\|_{L^\infty(B_R)} = 1\,, \qquad z_k(0) = 1\,, \qquad \forall k \geq 1\,. \tag{7.37}$$

Moreover, thanks to (7.31) and (7.37), we find that

$$\|w_k\|_{L^\infty(B_R)} \leq \frac{c_M + a_M}{b_L + d_L} \qquad \forall k \geq 1\,. \tag{7.38}$$

Now the same compactness argument of the proof of Theorem 1.1 in [13] shows that given any $p > N$ and passing to a suitable subsequence, again relabeled by k, there exists $(z, w) \geq (0, 0)$ in $W^{2,p}(B_R) \cap C^{1,\nu}(B_R)$, $0 < \nu < 1$, such that

$$\lim_{k\to\infty}(z_k, w_k) = (z, w) \qquad \text{in} \quad (W^{2,p}(B_R) \cap C^{1,\nu}(B_R))^2\,.$$

By Hölder continuity $z(0) = 1$. Moreover, passing to the limit as $k \to \infty$ in (7.35) gives

$$\begin{aligned} -\sum_{i,j=1}^{N} a_{ij}(x_\infty)\partial_i\partial_j z &= -a(x_\infty)z^2 + b(x_\infty)zw\,, \\ -\sum_{i,j=1}^{N} a_{ij}(x_\infty)\partial_i\partial_j w &= -d(x_\infty)w^2 + c(x_\infty)zw\,, \end{aligned} \tag{7.39}$$

in B_R, for any $R > 0$. By a standard diagonal sequence argument it is easily seen that z, $w \in W^{2,p}_{loc}(\mathbb{R}^N)$ and that (7.39) holds true in the whole of $\mathbb{R}^N$. Moreover, standard elliptic regularity theory implies that z, $w \in C^2(\mathbb{R}^N)$. Furthermore, by a linear change of coordinates (cf. [13] pg. 890), (7.39) can be reduced to

$$\begin{aligned} -\Delta z &= -a(x_\infty)z^2 + b(x_\infty)zw \\ -\Delta w &= -d(x_\infty)w^2 + c(x_\infty)zw \end{aligned} \qquad \text{in} \quad \mathbb{R}^N . \tag{7.40}$$

¿From (7.40), it is easily seen that

$$(-\Delta + a(x_\infty)z + d(x_\infty)w)(w - \frac{c(x_\infty) + a(x_\infty)}{b(x_\infty) + d(x_\infty)} z) = 0 .$$

Since $(z, w) \geq (0, 0)$ and $z(0) = 1$, the potential

$$V := a(x_\infty)z + d(x_\infty)w$$

satisfies $V \geq 0$ and $V \neq 0$. Therefore, due to the following lemma, whose proof we postpone up to conclude the proof of Theorem 7.4, we find that

$$w = \frac{c(x_\infty) + a(x_\infty)}{b(x_\infty) + d(x_\infty)} z . \tag{7.41}$$

Lemma 7.5. *Assume that either $D = \mathbb{R}^N$ or $D = \mathbb{R}^N_+$, where*

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \ : \ x_N \geq 0\} .$$

If $V \in L^\infty(D) \cap C^\nu(\overline{D})$, $V \geq 0$, $V \neq 0$, then $\theta = 0$ is the only bounded solution of

$$(-\Delta + V)\theta = 0 \qquad in \quad D . \tag{7.42}$$

Substituting (7.41) into the first equation of (7.40) and rearranging terms gives

$$-\Delta z = \frac{b(x_\infty)c(x_\infty) - a(x_\infty)d(x_\infty)}{b(x_\infty) + d(x_\infty)} z^2 \qquad \text{in } \mathbb{R}^N . \tag{7.43}$$

Since $b_L c_L > a_M d_M$, $b(x_\infty)c(x_\infty) > a(x_\infty)d(x_\infty)$ and hence, thanks to Theorem 1.1 of [13], $z = 0$ is the unique non-negative solution of (7.43), because $N \leq 5$. This is a contradiction with $z(0) = 1$. Therefore, $x_\infty \in \partial\Omega$. Now, the same argument as in Case 2 of the proof of Theorem 1.1 in [13] shows that the problem

$$\begin{aligned} -\Delta z &= -a(x_\infty)z^2 + b(x_\infty)zw \\ -\Delta w &= -d(x_\infty)w^2 + c(x_\infty)zw \end{aligned} \qquad \text{in} \quad \mathbb{R}^N_+ . \tag{7.44}$$

possesses a non-negative solution couple (z, w) with $z(0) = 1$. The same argument as above shows that this is impossible. This contradiction shows the existence of uniform a priori bounds and completes the proof of the theorem. □

We now prove Lemma 7.5, which is a Liouville type result interesting in its own right. In the proof we use the concepts and results in Chapter 4 of [31].

Proof of Lemma 7.5. Thanks to Theorem 3.3(iii) in page 148 of [31], the Schrödinger operator $\Delta - V$ is subcritical on D, i.e. it possesses a Green function $G(x, y)$ on D. Therefore, thanks to Theorem 3.8(i) in page 151 of [31] for each non-negative $p \in C_0^\nu(D)$, $p \neq 0$, there exists positive solutions $u \in C^{2,\nu}(D)$ of

$$(-\Delta + V)u = p\,. \tag{7.45}$$

Moreover, (7.45) possesses a minimal solution u_0, given by

$$u_0(x) = \int_D G(x, y)p(y)\, dy\,,$$

and any other solution of (7.45) must be given by

$$u = u_0 + \theta\,,$$

for some some positive solution θ of (7.42). The minimality of u_0 shows that $\theta = 0$ is the unique solution of (7.42). This completes the proof. □

Remark 7.6. (a) Although (7.31) implies $w \leq \frac{c_M + a_M}{b_L + d_L}\, z$, this does not necessarily entails

$$w \leq \frac{c(x_\infty) + a(x_\infty)}{b(x_\infty) + d(x_\infty)}\, z \tag{7.46}$$

and hence, Lemma 4.5 of [27] can not be applied to show that $(z, w) = (0, 0)$ is the unique solution of (7.40). In fact, our corresponding Liouville type result is substantially sharper than Lemma 4.5 of [27], as we do not need assuming (7.46) to infer $z = w = 0$.

(b) By the L^p estimates of Agmon, Douglis & Nirenberg and Morrey's Theorem, Theorem 7.4 provides us with a uniform a priori bounds in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ for the coexistence states of (1.1) on any compact subset of the (λ, μ)-plane.

7.3. On the existence of coexistence states in case $N \leq 5$. As an immediate consequence, from Theorem 5.1, Theorem 7.1 and Theorem 7.4 we obtain the following result.

Theorem 7.7. *(i) If $N \leq 5$,* (7.4) *and*

$$\lambda < \sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu,d]}]\,, \tag{7.47}$$

are satisfied, then (1.1) *possesses a coexistence state.*

(ii) If $N \leq 5$, (7.5) *and*

$$\mu < \sigma_1^\Omega[\mathcal{L} - c\theta_{[\mathcal{L},\lambda,a]}]\,, \tag{7.48}$$

are satisfied, then (1.1) *possesses a coexistence state.*

(iii) If $N \leq 5$ and either (7.4) *or* (7.5) *is satisfied, then* (1.1) *possesses a coexistence state provided*

$$\lambda < \sigma_1^\Omega[\mathcal{L}]\,, \qquad \mu < \sigma_1^\Omega[\mathcal{L}]\,. \tag{7.49}$$

Proof. We first show Part (i). Fix $\lambda < \sigma_1^\Omega[\mathcal{L}]$ and consider μ as the main bifurcation parameter. By Theorem 7.1 (iii) there exists $\mu = \mu(\lambda)$ such that $\lambda > \sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu(\lambda),d]}]$ and (1.1) does not admit a coexistence state for $\mu > \mu(\lambda)$.

Moreover, by Theorem 5.1 the continuum $\mathcal{C}^+_{(\mu,0,v)}$ of coexistence states emanating from $(0, \theta_{[\mathcal{L},\mu,d]})$ at μ_λ is unbounded, where μ_λ is the unique value of $\mu > \sigma_1^\Omega[\mathcal{L}]$ for which $\lambda = \sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu,d]}]$. Furthermore, (7.4) implies $b_L c_L > a_M d_M$ and hence, we conclude from Theorem 7.4 that (1.1) possesses a coexistence state for each $\mu < \mu_\lambda$. This completes the proof of Part (i). Part (ii) follows by symmetry and Part (iii) is an easy consequence from Parts (i), (ii). ☐

In practice, the verification of conditions (7.47) and (7.48) is far from easy, as each of them involves the evaluation of the principal eigenvalue of a second order elliptic operator whose associated potential is given through by a positive solution of a semilinear elliptic boundary value problem. The next results provide us with some easily computable sufficient conditions in terms of the several coefficients involved in the setting of (1.1) so that (7.47), or (7.48), holds. Our analysis extends to the case of general second order elliptic operators the estimates of Theorem 2.3 (c) in [26], found for the special case of operators in divergence form.

Lemma 7.8. *Assume that $\mathcal{L}$ is a differential operator of the form* (2.1) *whose coefficients satisfy* (2.2). *For $\gamma > \sigma_1^\Omega[\mathcal{L}]$, let $\theta_{[\mathcal{L},\gamma,f]}$ denote the positive solution of* (3.1). *Then, there exists a positive constant*

$$K = K(\mathcal{L}, f, \Omega) \geq \max\left\{\frac{\|\varphi\|_\infty}{m_{f,1}}, \frac{1}{f_L}\right\} \tag{7.50}$$

such that

$$\|\theta_{[\mathcal{L},\gamma,f]}\|_\infty \leq K(\gamma - \sigma_1^\Omega[\mathcal{L}]) \qquad \forall\, \gamma \geq \sigma_1^\Omega[\mathcal{L}]\,,$$

where φ is the principal eigenfunction associated with $\mathcal{L}$, normalized so that

$$\int_\Omega \varphi^2 = 1\,,$$

and $m_{f,1}$ is the constant defined in the statement of Lemma 4.3.

Proof. Thanks to Lemma 4.3

$$\frac{d\theta_{[\mathcal{L},\gamma,f]}}{d\gamma}\rfloor_{\{\gamma=\sigma_1^\Omega[\mathcal{L}]\}} = \frac{\varphi}{m_{f,1}}\,,$$

and hence, there exist $\delta > 0$ and a constant $C > 0$ such that

$$\|\theta_{[\mathcal{L},\gamma,f]}\|_\infty \le C(\gamma - \sigma_1^\Omega[\mathcal{L}]) \tag{7.51}$$

for each $\gamma \in [\sigma_1^\Omega[\mathcal{L}], \sigma_1^\Omega[\mathcal{L}] + \delta]$.

On the other hand, it follows from Corollary 3.3 that

$$\theta_{[\mathcal{L},\gamma,f]} \le \frac{\gamma - e_L}{f_L}\,.$$

Thus, there exists a constant $\hat{C} > 0$ such that

$$\frac{\|\theta_{[\mathcal{L},\gamma,f]}\|_\infty}{\gamma - \sigma_1^\Omega[\mathcal{L}]} \le \frac{\gamma - e_L}{\gamma - \sigma_1^\Omega[\mathcal{L}]} \cdot \frac{1}{f_L} \le \hat{C}\frac{1}{f_L}$$

for each $\gamma \ge \sigma_1^\Omega[\mathcal{L}] + \delta$. This completes the proof. □

Theorem 7.9. *Assume that $\mathcal{L}$ is a differential operator of the form* (2.1) *whose coefficients satisfy* (2.2)*, and let $K_1 := K(\mathcal{L}, a, \Omega)$, $K_2 := K(\mathcal{L}, d, \Omega)$ denote the two constants whose existence was shown by Lemma 7.8. Then, the following assertions are true:*

(i) If $N \le 5$, (7.4) *and*

$$\lambda < \sigma_1^\Omega[\mathcal{L}]\,, \quad \lambda < \min\{\sigma_1^\Omega[\mathcal{L}] - b_M K_2(\mu - \sigma_1^\Omega[\mathcal{L}])\,, \sigma_1^\Omega[\mathcal{L}] - \frac{b_M}{d_L}(\mu - e_L)\}$$

are satisfied, then (1.1) *possesses a coexistence state.*

(ii) If $N \le 5$, (7.5) *and*

$$\mu < \sigma_1^\Omega[\mathcal{L}]\,, \quad \mu < \min\{\sigma_1^\Omega[\mathcal{L}] - c_M K_1(\lambda - \sigma_1^\Omega[\mathcal{L}])\,, \sigma_1^\Omega[\mathcal{L}] - \frac{c_M}{a_L}(\lambda - e_L)\}$$

are satisfied, then (1.1) *possesses a coexistence state.*

Proof. By Lemma 7.8, we have that

$$\|\theta_{[\mathcal{L},\lambda,a]}\|_\infty \le K_1(\lambda - \sigma_1^\Omega[\mathcal{L}])\,, \qquad \|\theta_{[\mathcal{L},\mu,d]}\|_\infty \le K_2(\mu - \sigma_1^\Omega[\mathcal{L}])\,.$$

Thus, it follows from Theorem 2.3 that

$$\begin{aligned}\sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu,d]}] \ge & \sigma_1^\Omega[\mathcal{L} - b_M\|\theta_{[\mathcal{L},\mu,d]}\|_\infty] \ge \sigma_1^\Omega[\mathcal{L} - b_M K_2(\mu - \sigma_1^\Omega[\mathcal{L}])] \\ = & \sigma_1^\Omega[\mathcal{L}] - b_M K_2(\mu - \sigma_1^\Omega[\mathcal{L}])\,.\end{aligned}$$

Similarly,

$$\sigma_1^\Omega[\mathcal{L} - c\theta_{[\mathcal{L},\lambda,a]}] \ge \sigma_1^\Omega[\mathcal{L}] - c_M K_1(\lambda - \sigma_1^\Omega[\mathcal{L}])\,.$$

On the other hand, Corollary 3.3 implies

$$\theta_{[\mathcal{L},\lambda,a]} \le \frac{\lambda - e_L}{a_L}\,, \qquad \theta_{[\mathcal{L},\mu,d]} \le \frac{\mu - e_L}{d_L}\,,$$

and the same argument as above shows that

$$\sigma_1^\Omega[\mathcal{L} - c\theta_{[\mathcal{L},\lambda,a]}] \ge \sigma_1^\Omega[\mathcal{L}] - \frac{c_M}{a_L}(\lambda - e_L)\,,$$

$$\sigma_1^\Omega[\mathcal{L} - b\theta_{[\mathcal{L},\mu,d]}] \ge \sigma_1^\Omega[\mathcal{L}] - \frac{b_M}{d_L}(\mu - e_L)\,.$$

Theorem 7.7 completes the proof. □

8. The maximum principle. Multiplicity results. In this section we use the abstract theory of [2] to show that the method of sub and supersolutions is valid for (1.1). Then, we use it to analyze the structure of the set of λ's (or μ's) for which (1.1) possesses a coexistence state and to get some multiplicity results of coexistence states. The basic technical tool to prove these results is the strong maximum principle for linear cooperative systems. The validity of the strong maximum principle is guaranteed if, for instance, we assume that

$$b(x) > 0\,, \qquad c(x) > 0\,, \qquad \forall x \in \Omega\,. \tag{8.1}$$

So, for the rest of this section we shall assume that this condition is satisfied.

8.1. The strong maximum principle for cooperative systems. If (u_0, v_0) is a coexistence state of (1.1), then its linearized stability is given by the eigenvalues of the linearization of (1.1) at (u_0, v_0), i.e. by the τ's for which the following problem has some solution $(u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$, $(u, v) \neq (0, 0)$, $p > N$,

$$\begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \tau \begin{pmatrix} u \\ v \end{pmatrix}, \tag{8.2}$$

where

$$A = \begin{pmatrix} \lambda - 2au_0 + bv_0 & bu_0 \\ cv_0 & \mu - 2dv_0 + cu_0 \end{pmatrix}. \tag{8.3}$$

Note that thanks to (8.1) the off-diagonal entries of this matrix are positive and so the coupling matrix A is of cooperative type. More generally, we consider the linear cooperative eigenvalue problem (8.2) with $(u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$ for some $p > N$ and

$$A = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \rho(x) \end{pmatrix}, \tag{8.4}$$

where α, β, γ, $\rho \in C(\overline{\Omega})$ and the off-diagonal entries, β and γ, are positive almost everywhere in Ω. In the sequel we set

$$\mathcal{L} := \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} - A \tag{8.5}$$

and suppose that $p > N$. Now, to state the maximum principle we need some of notation. Given $(u, v) \in L^p(\Omega) \times L^p(\Omega)$, it is said that $(u, v) \geq 0$ if $u \geq 0$ and $v \geq 0$. If in addition $u \neq 0$ or $v \neq 0$, then it is said that $(u, v) > 0$. A couple $(u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$ is said to be strongly positive if $u(x) > 0$, $v(x) > 0$ for all $x \in \Omega$ and $\partial_n u(x) < 0$, $\partial_n v(x) < 0$ for all $x \in \partial\Omega$, where n is the outward unit normal at x.

Definition 8.1. *The operator $\mathcal{L}$ defined by* (8.5) *is said to satisfy the strong maximum principle in Ω if $x := (u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$ and $\mathcal{L}x > 0$ imply that x is strongly positive.*

Definition 8.2. *A function* $\overline{x} := (\overline{u}, \overline{v}) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ *is said to be a supersolution of* $\mathcal{L}$ *in* Ω *if* $\overline{x}|_{\partial\Omega} \geq 0$ *and* $\mathcal{L}\overline{x} \geq 0$*. If in addition* $\mathcal{L}\overline{x} > 0$*, or* $\overline{x}|_{\partial\Omega} > 0$*, then it is said that* $\overline{x}$ *is a strict supersolution.*

Now, using Theorems 2.1, 2.2 of Section 2, the proof of Theorem 2.1 in [24] can be easily adapted to cover our general setting providing us with the following general versions of Theorems 2.1, 2.2 of Section 2.

Theorem 8.3. *There exists a least eigenvalue of* (8.2)*, denoted by* $\sigma_1^\Omega[\mathcal{L}]$ *and called principal eigenvalue of* $\mathcal{L}$ *in* Ω*. This eigenvalue is simple and possesses a unique eigenfunction, up to multiplicative constants, which can be taken positive, the so called principal eigenfunction of* $\mathcal{L}$ *in* Ω*. Moreover, the principal eigenfunction is strongly positive and* $\sigma_1^\Omega[\mathcal{L}]$ *is the only eigenvalue of* (8.2) *possessing a positive eigenfunction. Furthermore, any other eigenvalue* σ *of* (8.2) *satisfies*

$$Re\ \sigma > \sigma_1^\Omega[\mathcal{L}]$$

and $(\mathcal{L}+\nu)^{-1} \in \mathcal{L}(L^p(\Omega) \times L^p(\Omega))$ *is positive, compact and irreducible for* $\nu > -\sigma_1^\Omega[\mathcal{L}]$*.*

Theorem 8.4. *The following assertions are equivalent:*

(i) $\sigma_1^\Omega[\mathcal{L}] > 0$*;*

(ii) $\mathcal{L}$ *possesses a positive strict supersolution in* $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$*;*

(iii) $\mathcal{L}$ *satisfies the strong maximum principle.*

Moreover, the following generalized maximum principle holds.

Theorem 8.5. *If* $\mathcal{L}$ *satisfies the strong maximum principle, then any strict supersolution* $\overline{x} := (\overline{u}, \overline{v}) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ *of* $\mathcal{L}$ *is positive in* Ω*. In fact,* $\overline{u}(x) > 0$ *and* $\overline{v}(x) > 0$ *for all* $x \in \Omega$*. It will simply said that* $\mathcal{L}$ *satisfies the generalized maximum principle in* Ω*.*

Proof. It is based upon Theorem A_n of [34]. Thanks to Theorem 8.4, $\sigma_1^\Omega[\mathcal{L}] > 0$. Let $h > 0$ denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L}] > 0$. We have that $\mathcal{L}h > 0$ in Ω. Therefore, thanks to Theorem A_n of [34], some of the following options occurs: Either (i) $\overline{x} > 0$ in Ω, or (ii) $\overline{x} = 0$ in Ω, or (iii) $\overline{x} = \alpha h$ for some $\alpha < 0$. Since, we are assuming that $\overline{x}$ is a strict supersolution, the options (ii) and (iii) are excluded. Therefore, $\overline{x} > 0$ in Ω. Corollary 2 of [34] completes the proof. □

Thanks to these results, for any operator $\mathcal{L}$ of the type (8.5) there exists ω such that $\mathcal{L}+\nu$ satisfies the the generalized maximum principle for all $\nu > \omega$. Therefore, the proof of Theorem 9.4 of [2] carries over mutatis mutandis to our present situation, showing that the method of sub and supersolutions works out for the nonlinear model (1.1). To state our result we need to introduce the concept of sub and supersolution.

Definition 8.6. *A positive function* $\underline{x} = (\underline{u}, \underline{v}) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ *is said to be a subsolution of* (1.1) *if*

$$\begin{aligned} \mathcal{L}_1\underline{u} &\leq \lambda\underline{u} - a(x)\underline{u}^2 + b(x)\underline{u}\,\underline{v} \\ \mathcal{L}_2\underline{v} &\leq \mu\underline{v} - d(x)\underline{v}^2 + c(x)\underline{u}\,\underline{v} \end{aligned} \qquad in\ \ \Omega\,,$$

and $\underline{x}|_{\partial\Omega} \leq 0$. *Similarly, a positive function* $\overline{x} = (\overline{u}, \overline{v}) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ *is said to be a supersolution of* (1.1) *if*

$$\begin{aligned} \mathcal{L}_1 \overline{u} &\geq \lambda \overline{u} - a(x)\overline{u}^2 + b(x)\overline{u}\,\overline{v} \\ \mathcal{L}_2 \overline{v} &\geq \mu \overline{v} - d(x)\overline{v}^2 + c(x)\overline{u}\,\overline{v} \end{aligned} \qquad \text{in } \Omega\,,$$

and $\overline{x}|_{\partial\Omega} \geq 0$.

Theorem 8.7. *Suppose that there exists a subsolution* $\underline{x} = (\underline{u}, \underline{v})$ *and a supersolution* $\overline{x} = (\overline{u}, \overline{v})$ *of* (1.1) *such that* $\underline{x} \leq \overline{x}$. *Then,* (1.1) *possesses a minimal solution* $x_* = (u_*, v_*)$ *and a maximal solution* $x^* = (u^*, v^*)$ *in the order interval* $[\underline{x}, \overline{x}]$. *In particular, if* $\underline{u} > 0$ *and* $\underline{v} > 0$, *then* (1.1) *possesses a coexistence state.*

Notice that this result is valid for any number of symbiotic species as well and therefore it provides us with a substantial generalization of Theorem 3 in [28].

8.2. Structure of the set of λ's for which (1.1) **possesses a coexistence state.** Here, we use Theorem 8.7 to analyze the structure of the set of λ's (resp. μ's) for which (1.1) possesses a coexistence state, denoted by Λ (resp. M). In the case of small interaction coefficients we have the following result.

Theorem 8.8. *Assume* (6.1). *Then, the following assertions are true:*

(i) Assume $\mu > \sigma_1^\Omega[\mathcal{L}_2]$. *Then, either* $\Lambda = (\sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}], \infty)$, *or there exists* $\lambda_* \leq \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]$ *such that* $\Lambda = [\lambda_*, \infty)$.

(ii) Assume $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$. *Then, either* $M = (\sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}], \infty)$ *or there exists* $\mu_* \leq \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda,a]}]$ *such that* $M = [\mu_*, \infty)$.

Proof. We shall prove (i). Part (ii) follows by symmetry. Assume (6.1) and $\mu > \sigma_1^\Omega[\mathcal{L}_2]$. Then, thanks to Theorem 6.2,

$$(\sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}], \infty) \subset \Lambda\,. \tag{8.6}$$

Now, suppose that (1.1) possesses a coexistence state (u_0, v_0) for some $\lambda_0 < \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]$. Then, (u_0, v_0) is a subsolution of (1.1) for each

$$\lambda \in (\lambda_0, \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]]\,. \tag{8.7}$$

On the other hand, by assumption (6.1) it is rather clear that we can choose a couple of positive constants (C_1, C_2) such that for each λ satisfying (8.7)

$$-a_L C_1 + b_M C_2 \leq (c_1)_L - \lambda\,, \qquad c_M C_1 - d_L C_2 \leq (c_2)_L - \mu\,,$$

and $u_0 < C_1$, $v_0 < C_2$ in Ω. Such couple provides us with a supersolution of (1.1). Thus, thanks to Theorem 8.7, for each λ satisfying (8.7) problem (1.1) possesses a coexistence state. Therefore, using (8.6) we find that

$$[\lambda_0, \infty) \subset \Lambda\,.$$

Let λ_* denote the infimum of the set of $\lambda_0 < \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]$ for which (1.1) possesses a coexistence state. We have that $(\lambda_*, \infty) \subset \Lambda$ and that

$$\lambda_* < \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_2,\mu,d]}]\,. \tag{8.8}$$

Moreover, by Lemma 6.1, $\lambda_* > -\infty$ and due to the existence of a priori bounds, there exists a sequence of positive solutions of (1.1), say (λ_n, u_n, v_n), $n \geq 1$, such that

$$\lim_{n\to\infty} (\lambda_n, u_n, v_n) = (\lambda_*, u_*, v_*)\,,$$

for some non-negative solution (u_*, v_*) of (1.1) with $\lambda = \lambda_*$. Necessarily $u_* > 0$ and $v_* > 0$. To show this we argue by contradiction. Indeed, if $u_* = v_* = 0$, then the new sequences $\hat{u}_n = \frac{u_n}{\|u_n\|}$ and $\hat{v}_n = \frac{v_n}{\|v_n\|}$ satisfy

$$\begin{aligned} &\left.\begin{aligned} \mathcal{L}_1 \hat{u}_n &= \lambda_n \hat{u}_n - a(x)\hat{u}_n u_n + b(x)\hat{u}_n v_n \\ \mathcal{L}_2 \hat{v}_n &= \mu \hat{v}_n - d(x)\hat{v}_n v_n + c(x)\hat{v}_n u_n \end{aligned}\right\} \quad \text{in } \Omega\,, \\ &\hat{u}_n = \hat{v}_n = u_n = v_n = 0 \qquad \text{on } \partial\Omega\,, \end{aligned} \tag{8.9}$$

and, since $(\hat{u}_n, \hat{v}_n)$ is uniformly bounded, we can apply a standard bootstrapping argument and extract a convergent subsequence of $(\hat{u}_n, \hat{v}_n)$, again labeled by n, such that $\hat{u}_n \to w$ and $\hat{v}_n \to z$, as $n \to \infty$, for some $w, z \in C_0^1(\bar{\Omega})$. Necessarily $w > 0$, $z > 0$ and passing to the limit in (8.9) we find that

$$\begin{aligned} &\left.\begin{aligned} \mathcal{L}_1 w &= \lambda_* w \\ \mathcal{L}_2 z &= \mu z \end{aligned}\right\} \quad \text{in } \Omega\,, \\ &w = z = 0 \qquad \text{on } \partial\Omega\,. \end{aligned}$$

By the uniqueness of the principal eigenvalue,

$$\lambda_* = \sigma_1^\Omega[\mathcal{L}_1]\,, \qquad \mu = \sigma_1^\Omega[\mathcal{L}_2]\,,$$

and this is impossible, since we are assuming that $\mu > \sigma_1^\Omega[\mathcal{L}_2]$.

If $u_* > 0$ and $v_* = 0$, then we take the sequence $(u_n, \hat{v}_n)$ and the same compactness argument as above shows that $u_* = \theta_{[\mathcal{L}_1,\lambda,a]}$ and that

$$\mu = \sigma_1^\Omega[\mathcal{L}_2 - c\theta_{[\mathcal{L}_1,\lambda_*,a]}] < \sigma_1^\Omega[\mathcal{L}_2]\,,$$

which is not possible either. Finally, if $u_* = 0$ and $v_* > 0$, then $v_* = \theta_{[\mathcal{L}_2,\mu,d]}$ and

$$\lambda_* = \sigma_1^\Omega[\mathcal{L}_1 - c\theta_{[\mathcal{L}_2,\mu,d]}]\,,$$

which contradicts (8.8). Therefore, $u_* > 0$, $v_* > 0$ and

$$\Lambda = [\lambda_*, \infty)\,.$$

This completes the proof. $\square$

Similarly, for the case of large interaction coefficients we have the following result.

Theorem 8.9. *Assume $\mathcal{L}_1 = \mathcal{L}_2$ and $N \leq 5$. Then the following assertions are true:*

(i) Assume (7.4) *and $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$. Then, either $M = (-\infty, \mu_\lambda)$ or $M = (-\infty, \mu^*]$ for some $\mu^* \geq \mu_\lambda$, where μ_λ is the unique value of μ satisfying $\lambda = \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu_\lambda,d]}]$.*

(ii) Assume (7.5) *and $\mu < \sigma_1^\Omega[\mathcal{L}_1]$. Then, either $\Lambda = (-\infty, \lambda_\mu)$ or $\Lambda = (-\infty, \lambda^*]$ for some $\lambda^* \geq \lambda_\mu$, where λ_μ is the unique value of λ satisfying $\mu = \sigma_1^\Omega[\mathcal{L}_1 - c\theta_{[\mathcal{L}_1,\lambda_\mu,a]}]$.*

Proof. We shall prove (i). Part (ii) follows by symmetry. Assume (7.4) and $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$. By Theorem 7.7,

$$(-\infty, \mu_\lambda) \subset M \,. \tag{8.10}$$

Now, suppose that (1.1) possesses a coexistence state (u_0, v_0) for some $\mu_0 > \mu_\lambda$. We now show that (1.1) possesses a coexistence state for each $\mu \in (\mu_\lambda, \mu_0]$. Assume that

$$\mu_\lambda < \mu \leq \mu_0 \,.$$

Then,

$$\lambda > \sigma_1^\Omega[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu,d]}] \,, \tag{8.11}$$

and hence,

$$\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu,d]},\lambda,a]} > 0 \,.$$

Moreover, since $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$, we have $\mu_\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ and hence, for each $\mu \in (\mu_\lambda, \mu_0]$ we find that

$$\theta_{[\mathcal{L}_1,\mu,d]} > 0 \,.$$

Now, observe that the couple

$$(\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu,d]},\lambda,a]}, \theta_{[\mathcal{L}_1,\mu,d]})$$

provides us with a subsolution of (1.1), and that, thanks to Lemma 3.2, for any coexistence state (u, v) of (1.1) we have

$$(\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu,d]},\lambda,a]}, \theta_{[\mathcal{L}_1,\mu,d]}) < (u, v) \,.$$

In particular,

$$(\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu_0,d]},\lambda,a]}, \theta_{[\mathcal{L}_1,\mu_0,d]}) < (u_0, v_0) \,.$$

Thus, thanks again to Lemma 3.2, for each $\mu \in (\mu_\lambda, \mu_0)$ we find that

$$(\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu,d]},\lambda,a]}, \theta_{[\mathcal{L}_1,\mu,d]}) < (\theta_{[\mathcal{L}_1 - b\theta_{[\mathcal{L}_1,\mu_0,d]},\lambda,a]}, \theta_{[\mathcal{L}_1,\mu_0,d]}) < (u_0, v_0) \,,$$

and therefore, it follows from Theorem 8.7 that (1.1) possesses a coexistence state for each $\mu \in (\mu_\lambda, \mu_0]$.

To complete the proof it suffices to show that (1.1) possesses a coexistence state for $\mu = \mu_\lambda$. In the sequel we fix λ and regard to μ as the main bifurcation parameter. By the last part of Theorem 5.1,

$$(\mu, u, v) = (\mu_\lambda, 0, \theta_{[\mathcal{L}_1,\mu_\lambda,d]}) \,,$$

is the only bifurcation point to coexistence states from the semi-trivial curve $(u,v) = (0,\theta_{[\mathcal{L}_1,\mu,d]})$ and the maximal component (closed and connected) of coexistence states emanating from $(0,\theta_{[\mathcal{L}_1,\mu,d]})$ at $\mu=\mu_\lambda$, denoted by $\mathcal{C}^+_{(\mu,0,v)}$, is unbounded in $\mathbb{R}\times C^1_0(\overline{\Omega})\times C^1_0(\overline{\Omega})$. Moreover, by the local bifurcation theorem of [7], there exist a neighborhood $\mathcal{N} := \mathcal{N}(\mu_\lambda,0,\theta_{[\mathcal{L}_1,\mu_\lambda,d]})$ of $(\mu_\lambda,0,\theta_{[\mathcal{L}_1,\mu_\lambda,d]})$ in $\mathbb{R}\times C^1_0(\overline{\Omega})\times C^1_0(\overline{\Omega})$, a real number $s_0>0$ and an analytic mapping

$$(\mu,u,v):(-s_0,s_0)\to\mathbb{R}\times C^1_0(\overline{\Omega})\times C^1_0(\overline{\Omega})$$

such that

$$(\mu(0),u(0),v(0)) = (\mu_\lambda,0,\theta_{[\mathcal{L}_1,\mu_\lambda,d]})$$

and

$$\mathcal{N}\cap\mathcal{C}^+_{(\mu,0,v)} = \{\,(\mu(s),u(s),v(s))\ :\ \ s>0\,\}\,.$$

In fact, the unique coexistence states of (1.1) close to the bifurcation point are those lying on the curve $(\mu(s),u(s),v(s))$. Since $\mu(s)$ is analytic, s_0 can be reduced, if necessary, so that either $\mu(s)<\mu_\lambda$ for each $s\in(0,s_0)$, or $\mu(s)=\mu_\lambda$ for each $s\in(0,s_0)$, or $\mu(s)>\mu_\lambda$ for each $s\in(0,s_0)$. If $\mu(s)=\mu_\lambda$ for each $s\in(0,s_0)$ the proof is completed.

Assume that $\mu(s)<\mu_\lambda$ for each $s\in(0,s_0)$. Since (1.1) possesses a coexistence state for each $\mu\in(\mu_\lambda,\mu_0]$ and thanks to Theorem 7.4 uniform a priori bounds for the coexistence states of (1.1) are available in the range $\mu\in[\mu_\lambda,\mu_0]$, from any sequence of coexistence states of (1.1), say (μ_n,u_n,v_n), with $\mu_n>\mu_\lambda$ and $\mu_n\downarrow\mu_\lambda$, we can substract a convergent subsequence, relabeled by n, such that

$$\lim_{n\to\infty}(u_n,v_n) = (u^*,v^*)$$

for some non-negative solution couple (u^*,v^*) of (1.1) with $\mu=\mu_\lambda$. By the uniqueness obtained from the application of Crandall Rabinowitz theorem [7],

$$(\mu_n,u_n,v_n)\notin\mathcal{N}$$

for n sufficiently large. Hence, $(u^*,v^*)\neq(0,\theta_{[\mathcal{L}_1,\mu_\lambda,d]})$. Moreover, the same compactness argument as in the proof of Theorem 8.8 shows that $(u^*,v^*)\neq(0,0)$. Therefore, (u^*,v^*) must be a coexistence state. This completes the proof in this case.

Finally, assume that $\mu(s)>\mu_\lambda$ for each $s\in(0,s_0)$ and let $\mathcal{C}^+_1$ denote the maximal subcontinuum of $\mathcal{C}^+_{(\mu,0,v)}$ outside $\mathcal{N}$. It is clear that $\mathcal{C}^+_1$ is unbounded. Thanks to Theorem 7.4 uniform a priori bounds on compact intervals of μ are available. Moreover, thanks to Theorem 7.1 (iii), (1.1) does not admit a coexistence state if μ is sufficiently large. Therefore, $\mathcal{C}^+_1$ must go backwards and (1.1) possesses a coexistence state for $\mu=\mu_\lambda$ as well.

The previous analysis shows that

$$(-\infty,\mu_0]\subset M\,. \tag{8.12}$$

Let μ^* denote the supremum of the set of $\mu_0 > \mu_\lambda$ for which (1.1) possesses a coexistence state for each $\mu \in (-\infty, \mu_0]$. By Theorem 7.1, $\mu^* \in \mathbb{R}$. Moreover, $\mu^* > \mu_\lambda$ and due to the existence of a priori bounds, there exists a sequence of positive solutions of (1.1), say (μ_n, u_n, v_n), $n \geq 1$, such that

$$\lim_{n\to\infty} (\mu_n, u_n, v_n) = (\mu^*, u^*, v^*)\,,$$

for some non-negative solution (u^*, v^*) of (1.1) with $\mu = \mu^*$. The same argument as in the proof of Theorem 8.8 shows that $u^* > 0$ and $v^* > 0$. Therefore,

$$M = (-\infty, \mu^*]\,.$$

This completes the proof. □

8.3. Multiplicity results. Here, we use the theory of Section 20 in [2] to give some multiplicity results. Our main result is the following.

Theorem 8.10. *Assume $\mathcal{L}_1 = \mathcal{L}_2$ and $N \leq 5$. Then the following assertions are true:*

(i) Assume (7.4), $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$ *and* $M = (-\infty, \mu^*]$ *with* $\mu^* > \mu_\lambda$. *Then,* (1.1) *possesses at least two coexistence states for each* $\mu \in (\mu_\lambda, \mu^*)$.

(ii) Assume (7.5), $\mu < \sigma_1^\Omega[\mathcal{L}_1]$ *and* $\Lambda = (-\infty, \lambda^*]$ *with* $\lambda^* > \lambda_\mu$. *Then,* (1.1) *possesses at least two coexistence states for each* $\lambda \in (\lambda_\mu, \lambda^*)$.

Proof. To prove this result we use the fixed point index in cones. It suffices to prove Theorem 8.10(i), since Part (ii) follows by symmetry. Notice that thanks to the proof of Theorem 8.9, under the assumptions of Theorem 8.10, Theorem 8.7 guarantees the existence of a minimal coexistence state, which will be denoted by (u_μ, v_μ). If not, from $(0,0)$ or some of the semi-trivial positive solutions should bifurcate a sequence of coexistence states and this is not possible by our assumptions on the coefficients of the model. We now show that (1.1) fits into the abstract setting of [2]. Fix $\alpha < \mu_\lambda$, $\beta > 0$ and consider $I := [\alpha, \mu^* + \beta]$. Since we have uniform a priori bounds for the non-negative solutions of (1.1), there exists $K > 0$ such that

$$au - bv < \lambda + K\,, \qquad dv - cu < \mu + K\,,$$

for each $\mu \in I$ and any non-negative solution (u, v) of (1.1). Enlarge K, if necessary, so that

$$K > -\sigma_1^\Omega[\mathcal{L}_1]\,,$$

and let e denote the unique solution of

$$(\mathcal{L}_1 + K)e = 1 \quad \text{ in } \Omega\,, \qquad e|_{\partial\Omega} = 0\,.$$

We have $e(x) > 0$ for each $x \in \Omega$ and $\partial_n e(x) < 0$ for each $x \in \partial\Omega$, where n stands for the outward unit normal on $\partial\Omega$. Let $C_e(\overline{\Omega})$ denote the ordered Banach space consisting of all functions $u \in C(\overline{\Omega})$ for which there exists a positive constant $\kappa > 0$ such that $-\kappa e \leq u \leq \kappa e$, endowed with the norm

$$\|u\|_e := \inf \{\kappa > 0 \; : \; -\kappa e \leq u \leq \kappa e \}$$

and ordered by its cone of positive functions, P. Then, the operators

$$\mathcal{K}_\mu : C_e(\overline{\Omega}) \times C_e(\overline{\Omega}) \to C_e(\overline{\Omega}) \times C_e(\overline{\Omega})$$

defined by

$$\mathcal{K}_\mu(u,v) = \begin{pmatrix} (\mathcal{L}_1 + K)^{-1}[(\lambda + K)u - au^2 + buv] \\ (\mathcal{L}_1 + K)^{-1}[(\mu + K)v - dv^2 + cuv] \end{pmatrix},$$

for each $\mu \in I$, are compact and strongly order preserving. Moreover, the solutions of (1.1) are the fixed points of $\mathcal{K}_\mu$. Let B_e denote the unit ball of $C_e(\overline{\Omega}) \times C_e(\overline{\Omega})$ and, for each $\rho > 0$, P_ρ the positive part of ρB_e. Since by Theorem 7.4 we have uniform a priori bounds for the non-negative solutions of (1.1), the fixed point index of $\mathcal{K}_\mu$ in P_ρ makes sense for sufficiently large ρ. Moreover, we have the following result.

Lemma 8.11. *Assume $\mu \in (\mu_\lambda, \mu^* + \beta]$. Then, $(0,0)$ and $(0, \theta_{[\mathcal{L}_1,\mu,d]})$ are isolated fixed points of $\mathcal{K}_\mu$ in P^2 and*

$$i(\mathcal{K}_\mu, (0,0)) = i(\mathcal{K}_\mu, (0, \theta_{[\mathcal{L}_1,\mu,d]})) = 0\,. \tag{8.13}$$

Moreover,

$$i(\mathcal{K}_\mu, P_\rho) = 0\,, \tag{8.14}$$

provided ρ is sufficiently large.

Since $\mu > \mu_\lambda$, $(0, \theta_{[\mathcal{L}_1,\mu,d]})$ is linearly unstable by Proposition 4.1 in Section 4, and so $i(\mathcal{K}_\mu, (0, \theta_{[\mathcal{L}_1,\mu,d]})) = 0$ (cf. [23]). On the other hand, from Lemma 13.1(ii) of [2] follows that $i(\mathcal{K}_\mu, (0,0)) = 0$ and therefore (8.13) holds. Relation (8.14) follows by homotopy invariance, taking into account that $(0,0)$ and $(0, \theta_{[\mathcal{L}_1,\mu,d]})$ are the only non-negative solutions of (1.1) for $\mu \in (\mu^*, \mu^* + \beta]$.

Now, we compute the fixed point index of the minimal solution (u_μ, v_μ) of (1.1). To do this computation, we use the following lemmas, which are immediate consequences from Propositions 20.4 and 20.8 of [2], respectively.

Lemma 8.12. *If $\mu \in (\mu_\lambda, \mu^*]$, then the minimal coexistence state (u_μ, v_μ) of (1.1) is weakly stable, i.e.*

$$\sigma_1^\Omega[\mathcal{L}_\mu] \geq 0\,, \tag{8.15}$$

where $\mathcal{L}_\mu$ is the operator defined by (8.5) with $A(x)$ given by (8.3) and $(u_0, v_0) = (u_\mu, v_\mu)$.

Lemma 8.13. *(i) Let $(\mu, u, v) = (\mu_0, u_0, v_0)$ be a coexistence state of (1.1) such that*

$$\sigma_1^\Omega[\mathcal{L}_{\mu_0}] > 0\,, \tag{8.16}$$

where $\mathcal{L}_{\mu_0}$ is the operator defined by (8.5) with $A(x)$ given by (8.3). Then, there exists $\varepsilon > 0$ and a differentiable mapping $(u, v) : (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \to P^2$ such that $(u(\mu_0), v(\mu_0)) = (u_0, v_0)$ and $(\mu, u(\mu), v(\mu))$ is a coexistence state of (1.1) for each $\mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$. Moreover, the mapping $\mu \to (u(\mu), v(\mu))$ is strictly increasing and there exists a neighborhood $\mathcal{Q}$ of (μ_0, u_0, v_0) in $\mathbb{R} \times (C_e(\overline{\Omega}))^2$ such that if $(\mu, u, v) \in \mathcal{Q}$ is a solution of (1.1), then $(u, v) = (u(\mu), v(\mu))$.

(ii) Assume $\sigma_1^\Omega[\mathcal{L}_{\mu_0}] = 0$, instead of (8.15), and let Φ denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L}_{\mu_0}]$. Then, there exists $\varepsilon > 0$ and a differentiable mapping $(\mu, u, v) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times P^2$ such that $(\mu(0), u(0), v(0)) = (\mu_0, u_0, v_0)$ and for each $s \in (-\varepsilon, \varepsilon)$ $(\mu(s), u(s), v(s))$ is a coexistence state of (1.1). Moreover,

$$\mu(s) = \mu_0 + \hat{\mu}(s)\,, \qquad (u(s), v(s)) = (u_0, v_0) + s\Phi + (\hat{u}(s), \hat{v}(s))\,, \tag{8.17}$$

where $\hat{\mu}(s) = 0(s)$, $\hat{u}(s) = o(s)$ and $\hat{v}(s) = o(s)$ as $s \to 0$, and there exists a neighborhood $\mathcal{Q}$ of (μ_0, u_0, v_0) in $\mathbb{R} \times (C_e(\overline{\Omega}))^2$ such that if $(\mu, u, v) \in \mathcal{Q}$ is a solution of (1.1), then

$$(\mu, u, v) = (\mu(s), u(s), v(s))$$

for some $s \in (-\varepsilon, \varepsilon)$. Furthermore,

$$sgn\ \mu'(s) = sgn\ \sigma_1^\Omega[\mathcal{L}_s]\,, \tag{8.18}$$

where

$$\mathcal{L}_s = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} - \begin{pmatrix} \lambda - 2au(s) + bv(s) & bu(s) \\ cv(s) & \mu(s) - 2dv(s) + cu(s) \end{pmatrix}.$$

If $\sigma_1^\Omega[\mathcal{L}_\mu] > 0$, then the Leray-Schauder formula implies that the local index

$$i(\mathcal{K}_\mu, (u_\mu, v_\mu)) = 1$$

and therefore, thanks to Lemma 8.11, (1.1) must have a further coexistence state. Therefore, in this case the proof is completed.

Now, assume that $\sigma_1^\Omega[\mathcal{L}_\mu] = 0$ and let $(\mu(s), u(s), v(s))$ denote the curve of coexistence states through by (μ, u_μ, v_μ), for $s = 0$, whose existence is guaranteed by Lemma 8.13. Since $\Phi > 0$, $(u(s), v(s))$ is strictly increasing and hence, if $\mu(s) = \mu$ for some $s \neq 0$, then (1.1) possesses two coexistence states. Namely, (u_μ, v_μ) and $(u(s), v(s))$. Thus, without loss of generality we can assume that

$$\mu(s) \neq \mu \qquad \forall\ 0 < |s| < \varepsilon\,. \tag{8.19}$$

We claim that

$$\mu(s) < \mu \qquad \forall \ s \in (-\varepsilon, 0)\,. \tag{8.20}$$

Indeed, if there exists $s_1 < 0$ such that $\mu_1 := \mu(s_1) \geq \mu$, then

$$(u(s_1), v(s_1)) < (u(0), v(0)) = (u_\mu, v_\mu) \leq (u_{\mu_1}, v_{\mu_1})\,, \tag{8.21}$$

since $(u(s), v(s))$ is increasing in s and the minimal solution is non-decreasing in μ. Here, (u_{μ_1}, v_{μ_1}) stands for the minimal coexistence state of (1.1) for $\mu = \mu_1$. Relation (8.21) contradicts the minimality of (u_{μ_1}, v_{μ_1}). Thus, (8.20) get shown. Moreover, by (8.19), either $\mu(s) < \mu$ for all $s \in (0, \varepsilon)$, or $\mu(s) > \mu$ for all $s \in (0, \varepsilon)$, so we can distinguish two cases:

Case a: Assume that $\mu(s) < \mu$ for all $s \in (0, \varepsilon)$. Then, since $\mu < \mu^*$ and (1.1) possesses a coexistence state for each value of the parameter in $[\mu, \mu^*]$, there exists a sequence of coexistence states (μ_n, u_n, v_n), $n \geq 1$, such that $\lim_{n\to\infty} \mu_n = \mu$ and $\mu_n > \mu$ for all $n \geq 1$. By the existence of uniform a priori bounds, without loss of generality we can assume that

$$\lim_{n\to\infty} (u_n, v_n) = (u_0, v_0)\,,$$

for some non-negative solution (u_0, v_0) of (1.1). Since $\lambda < \sigma_1^\Omega[\mathcal{L}_1]$ and $\mu > \mu_\lambda$, with a similar argument as in the proof of Theorem 8.8, it is easily seen that (μ, u_0, v_0) is a coexistence state. Moreover, by the uniqueness obtained as an application of Lemma 8.13(ii), $(\mu_n, u_n, v_n) \notin \mathcal{Q}$ for each $n \geq 1$ and hence, $(\mu, u_0, v_0) \notin \mathcal{Q}$. In particular, $(\mu, u_0, v_0) \neq (\mu, u_\mu, v_\mu)$ and therefore, (1.1) possesses at least two coexistence states.

Case b: Now, assume that

$$\mu(s) > \mu \qquad \forall s \in (0, \varepsilon)\,. \tag{8.22}$$

Then, thanks to Lemma 8.13(ii), (μ, u_μ, v_μ) is an isolated solution of (1.1) and so $i(\mathcal{K}_\mu, (u_\mu, v_\mu))$ is well defined. By Lemma 8.11, to complete the proof of Theorem 8.10, it suffices to show that

$$i(\mathcal{K}_\mu, (u_\mu, v_\mu)) = 1\,. \tag{8.23}$$

By (8.22) there exists $s_1 \in (0, \varepsilon)$ for which $\mu'(s_1) > 0$. By (8.18), $\sigma_1^\Omega[\mathcal{L}_{s_1}] > 0$ and therefore, we find from Theorem 8.3 and the linearized stability principle that $(u(s_1), v(s_1))$ is exponentially asymptotically stable. Thus, Leray-Schauder's formula implies

$$i(\mathcal{K}_{\mu(s_1)}, (u(s_1), v(s_1))) = 1\,. \tag{8.24}$$

Since $(\mu(s_1), u(s_1), v(s_1))$ is non-degenerate and $s \to (u(s), v(s))$ is increasing there exists $\delta > 0$ such that if

$$\rho_1 := \|(u(s_1), v(s_1))\|_e - \delta\,, \qquad \rho_2 := \|(u_\mu, v_\mu)\|_e - \delta\,,$$

then (1.1) does not admit a coexistence state in

$$[\mu(s_1), \mu(s_1) + \delta] \times \partial(P_{\rho_1} \setminus \overline{P}_{\rho_2})\,.$$

Moreover, by the uniqueness of Lemma 8.13(ii), $\delta > 0$ can be chosen so that (1.1) does not have a coexistence state in $P_{\rho_1} \setminus \overline{P}_{\rho_2}$ for $\mu = \mu(s_1) + \delta$ either. Thus, the homotopy invariance implies

$$i(\mathcal{K}_{\mu(s_1)}, P_{\rho_1} \setminus \overline{P}_{\rho_2}) = 0\,. \tag{8.25}$$

Now, for $\delta > 0$ sufficiently small set

$$\rho := \|(u(s_1), v(s_1)\|_e + \delta\,.$$

By (8.24), (8.25), we find that

$$i(\mathcal{K}_{\mu(s_1)}, P_\rho \setminus \overline{P}_{\rho_2}) = 1\,.$$

Moreover, by the monotonicity of $(u(s), v(s))$ and the uniqueness given by Lemma 8.13(ii), (1.1) does not admit a coexistence state on

$$[\mu, \mu(s_1)] \times \partial(P_\rho \setminus P_{\rho_2})\,.$$

This implies (8.23) and completes the proof of the theorem. $\square$

Similarly, for the case of small interaction coefficients we have the following result.

Theorem 8.14. *Assume* (6.1). *Then following assertions are true:*

(i) Assume $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ *and* $\Lambda = [\lambda_*, \infty)$ *with* $\lambda_* < \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]$. *Then,* (1.1) *possesses at least two coexistence states for each* $\lambda \in (\lambda_*, \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}])$.

(ii) Assume $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$ *and* $M = [\mu_* \infty)$ *with* $\mu_* < \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]$. *Then,* (1.1) *possesses at least two coexistence states for each* $\mu \in (\mu_*, \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}])$.

Proof. Being the proof rather similar to the proof of Theorem 8.10, we are only to sketch it. By symmetry, it suffices to show Part (ii).

Let (μ_*, u_*, v_*) be a coexistence state of (1.1). Then, it is easily seen that for each $\mu \in (\mu_*, \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}])$

$$\underline{x} = (u_*, v_*)\,, \qquad \overline{x} = (K_1, K_2),$$

is an ordered sub-supersolution pair of (1.1) provided K_1 and K_2 are sufficiently large positive constants. Moreover, thanks to Lemma 6.2, if K_1 and K_2 are sufficiently large, then any coexistence state of (1.1) lies in the order interval $[0, \overline{x}]$. Therefore, (1.1) possesses a maximal coexistence state within the interval $[\underline{x}, \overline{x}]$, denoted by (u^μ, v^μ). Thanks to Proposition 7.8 of [2], (u^μ, v^μ) is weakly stable and so $\sigma_1^\Omega[\mathcal{L}^\mu] \geq 0$ where $\mathcal{L}^\mu$ is the operator defined by (8.5) with $A(x)$ given by (8.3) and $(u_0, v_0) = (u^\mu, v^\mu)$.

If $\sigma_1^\Omega[\mathcal{L}^\mu] > 0$ the same argument of the proof of Theorem 8.10 completes the proof of Theorem 8.11.

If $\sigma_1^\Omega[\mathcal{L}^\mu] = 0$ arguing as in the proof of Theorem 8.10 we find that

$$\mu(s) > \mu \qquad \forall s \in (0, \varepsilon)\,,$$

and two different situations may arise:

Case a. If $\mu(s) > \mu$ for $s \in (-\varepsilon, 0)$, then the same argument of the proof of Theorem 8.10 applies to complete the proof of this one.

Case b. If $\mu(s) < \mu$ for $s \in (-\varepsilon, 0)$, then there exists $s_1 < 0$ such that $\mu'(s_1) > 0$ and hence,

$$i(\mathcal{K}_{\mu(s_1)}, (u(s_1), v(s_1))) = 1\,.$$

Now, setting

$$\rho_1 := \|(u^\mu, v^\mu)\|_e + \delta, \quad \rho_2 := \|(u(s_1), v(s_1))\|_e + \delta, \quad \rho := \|(u(s_1), v(s_1))\|_e - \delta.$$

yields

$$i(\mathcal{K}_{\mu(s_1)}, P_{\rho_1} \setminus \overline{P}_{\rho_2}) = 0\,, \quad i(\mathcal{K}_{\mu(s_1)}, P_{\rho_1} \setminus \overline{P}_{\rho}) = 1\,, \quad i(\mathcal{K}_{\mu}, P_{\rho_1} \setminus \overline{P}_{\rho}) = 1\,,$$

and therefore,

$$i(\mathcal{K}_\mu, (u^\mu, v^\mu)) = 1\,.$$

This completes the proof. □

9. On the uniqueness of the coexistence state. In this section we give a uniqueness result in the case of small interaction coefficients. When the interaction coefficients are large we already know that (1.1) exhibits a superlinear character and so its number of coexistence states might vary drastically when the geometry of the support domain Ω changes, [8]. Our main uniqueness result is the following.

Theorem 9.1. *Assume that* (6.1)*,* (6.8) *and* (8.1) *are satisfied and that for any coexistence state* (u_0, v_0) *of* (1.1)

$$\left(\frac{u_0}{v_0}\right)_M \left(\frac{v_0}{u_0}\right)_M < \left(\frac{a}{b}\right)_L \left(\frac{d}{c}\right)_L\,. \tag{9.1}$$

Then, (1.1) *possesses a unique coexistence coexistence. Moreover, it is exponentially asymptotically stable.*

After the proof of this theorem we shall use Theorem 8.7 to get some upper estimates of the left hand side of (9.1), giving rise to very simple easily computable sufficient conditions, in terms of the several coefficients involved in the model setting, for the uniqueness of the coexistence state.

Proof. Under conditions (6.1) and (6.8) we have uniform a priori bounds for the nonnegative solutions of (1.1) and hence the fixed point index in cones can be used as in Section 8.3. By Proposition 4.1 the semi-trivial positive solutions $(\theta_{[\mathcal{L}_1,\lambda,a]}, 0)$ and $(0, \theta_{[\mathcal{L}_2,\mu,d]})$ are linearly unstable, if they exist, and a rather standard index computation shows that each of them has local index zero (cf. [23] for details). Moreover, the state $(0, 0)$ has index zero and the global index equals one. Therefore, by the principle of linearized stability, it suffices to show that under condition (9.1) any coexistence state is linearly asymptotically stable, since by Leray-Schauder formula any linearly asymptotically stable solution has local index one. Let (u_0, v_0) be a coexistence state of (1.1). Then, the spectrum of the linearization of (1.1) at (u_0, v_0) is given by the τ's for which the following problem has some solution $(u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$, $(u, v) \neq (0, 0)$, $p > N$,

$$\begin{aligned} (\mathcal{L}_1 + 2au_0 - bv_0 - \lambda)u &= bu_0 v + \tau u\,, \\ (\mathcal{L}_2 + 2dv_0 - cu_0 - \mu)v &= cv_0 u + \tau v\,. \end{aligned} \tag{9.2}$$

By Theorem 8.3 if we are able to show that there exist $u > 0$ and $v > 0$ such that

$$(\mathcal{L}_1 + 2au_0 - bv_0 - \lambda)u > bu_0 v\,, \qquad (\mathcal{L}_2 + 2dv_0 - cu_0 - \mu)v > cv_0 u\,, \tag{9.3}$$

then the principal eigenvalue of (9.2) will be positive and therefore, the linearized stability of (u_0, v_0) will follow from Theorem 8.3. Taking $(u, v) = (\alpha u_0, \beta v_0)$, where $\alpha > 0$ and $\beta > 0$ have to be found, (9.3) becomes into

$$\alpha a u_0 > \beta b v_0\,, \qquad \beta d v_0 > \alpha c u_0\,. \tag{9.4}$$

Now, due to (9.1), it is rather clear that there exist $\alpha > 0$ and $\beta > 0$ satisfying (9.4). This completes the proof. $\square$

The following result provides us with a sufficient condition for (9.1) to be hold.

Proposition 9.2. *Assume $\mathcal{L}_1 = \mathcal{L}_2$, $b(x) > 0$ and $c(x) > 0$ for each $x \in \Omega$,*

$$\sigma_1^\Omega[\mathcal{L}_1] > 0\,, \quad b_M c_M < a_L d_L\,, \quad \lambda > \sigma_1^\Omega[\mathcal{L}_1]\,, \quad \mu > \sigma_1^\Omega[\mathcal{L}_1]\,, \tag{9.5}$$

and

$$\frac{a_M d_M}{16 a_L d_L (a_L d_L - b_M c_M)^2} \cdot \frac{(d_L \lambda^2 + b_M \mu^2)(a_L \mu^2 + c_M \lambda^2)}{(\lambda - \sigma_1^\Omega[\mathcal{L}_1])(\mu - \sigma_1^\Omega[\mathcal{L}_1])} \cdot \left(\sup_{\overline{\Omega}} \frac{\psi}{\varphi} \right)^2 < \frac{1}{b_M c_M}\,, \tag{9.6}$$

where $\varphi > 0$ is the principal eigenfunction associated with $\sigma_1^\Omega[\mathcal{L}_1]$, normalized so that $\|\varphi\|_{L^\infty(\Omega)} = 1$ and $\psi > 0$ is the unique solution of

$$\mathcal{L}_1 \psi = 1 \quad \text{in } \Omega\,, \qquad \psi|_{\partial\Omega} = 0\,.$$

Then (1.1) *has exactly one coexistence state.*

Proof. We claim that for each $t > 1$ the couple $(\overline{u}_t, \overline{v}_t)$ defined by

$$\overline{u}_t := \frac{t(d_L \lambda^2 + b_M \mu^2)}{4(a_L d_L - b_M c_M)}\, \psi\,, \qquad \overline{v}_t := \frac{t(a_L \mu^2 + c_M \lambda^2)}{4(a_L d_L - b_M c_M)}\, \psi\,,$$

is a strict supersolution of (1.1). To prove this it suffices to show that

$$\begin{aligned} 1 &\geq \psi \cdot [\lambda - t(a(x)K_1 - b(x)K_2)\psi]\,, \\ 1 &\geq \psi \cdot [\mu - t(d(x)K_2 - c(x)K_1)\psi]\,, \end{aligned} \tag{9.7}$$

where

$$K_1 = \frac{d_L\lambda^2 + b_M\mu^2}{4(a_L d_L - b_M c_M)}\,, \qquad K_2 = \frac{a_L\mu^2 + c_M\lambda^2}{4(a_L d_L - b_M c_M)}\,.$$

Since

$$\sup_{\xi\geq 0}(A - B\xi)\xi = \frac{A^2}{4B}\,,$$

we find that for each $t \geq 1$,

$$\psi \cdot [\lambda - t(a(x)K_1 - b(x)K_2)\psi] \leq \frac{\lambda^2}{4t(a(x)K_1 - b(x)K_2)} \leq \frac{\lambda^2}{4(a_L K_1 - b_M K_2)}\,.$$

Similarly,

$$\psi \cdot [\mu - t(d(x)K_2 - c(x)K_1)\psi] \leq \frac{\mu^2}{4(a_L K_2 - b_M K_1)}\,.$$

Thus, the following conditions imply (9.7)

$$\lambda^2 = 4(a_L K_1 - b_M K_2)\,, \qquad \mu^2 = 4(a_L K_2 - b_M K_1)\,.$$

Since these conditions are satisfied by the choice of K_1 and K_2 itself, the claim above get shown.

Now, we need the following generalized version of the sweeping maximum principle of [28], whose proof is postponed up to the end of the proof of Proposition 9.2.

Lemma 9.3. *Let $x = (u, v) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$, $p > N$, be a solution of the problem*

$$\begin{aligned} \mathcal{L}_1 u &= f(x, u, v) \\ \mathcal{L}_2 v &= g(x, u, v) \end{aligned} \qquad \text{in } \Omega\,,$$

$$u = v = 0 \qquad \text{on } \partial\Omega\,,$$

where f and g are two continuous functions in x and of class C^1 in (u, v), f increasing in v, and g increasing in u. For each $t \in (t_0, t_1]$, let $\overline{x}_t = (\overline{u}_t, \overline{v}_t) \in W_0^{2,p}(\Omega) \times W_0^{2,p}(\Omega)$ be a strict supersolution of this problem. Assume that $\overline{x}_t$ is continuous and strictly increasing in t, that $\overline{x}_{t_1} - x$ is strongly positive, and that $\partial_n \overline{x}_t$ is continuous in t, where n stands for the outward unit normal to Ω. Then,

$$x \leq \overline{x}_{t_0}\,.$$

Thanks to Lemma 9.3, we find that

$$u_0 \leq \frac{d_L\lambda^2 + b_M\mu^2}{4(a_Ld_L - b_Mc_M)}\,\psi\,, \qquad v_0 \leq \frac{a_L\mu^2 + c_M\lambda^2}{4(a_Ld_L - b_Mc_M)}\,\psi\,, \tag{9.8}$$

for any coexistence state (u_0, v_0) of (1.1). Similarly, it follows from Lemma 3.2 that

$$u_0 \geq \theta_{[\mathcal{L}_1,\lambda,a]} \geq \frac{\lambda - \sigma_1^\Omega[\mathcal{L}_1]}{a_M}\,\varphi\,, \qquad v_0 \geq \theta_{[\mathcal{L}_1,\mu,d]} \geq \frac{\mu - \sigma_1^\Omega[\mathcal{L}_1]}{d_M}\,\varphi\,. \tag{9.9}$$

Finally, using (9.8) and (9.9), it is easily seen that (9.6) implies (9.1). Theorem 9.1 completes the proof. □

Proof of Lemma 9.3. Let t_* denote the infimum of the set of $t \in (t_0, t_1)$ for which $x - \overline{x}_t$ is strongly positive. We claim that $t_* = t_0$. On the contrary, assume that $t_* > t_0$. By our assumptions it is rather clear that there exists $K > 0$ such that each of the mappings

$$u \to f(\cdot, u, v) + Ku\,, \qquad v \to g(\cdot, u, v) + Kv\,,$$

is increasing and $K > -\min\{\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2]\}$. Since $\overline{x}_{t_*}$ is a strict supersolution of the problem, some of its components, say $\overline{u}_{t_*}$, satisfies

$$(\mathcal{L}_1 + K)(\overline{u}_{t_*} - u) > f(\cdot, \overline{u}_{t_*}, \overline{v}_{t_*}) + K\overline{u}_{t_*} - f(\cdot, u, v) - Ku > 0\,.$$

Thus, the strong maximum principle implies that $u_{t_*} - u$ is strongly positive. This contradicts the minimality of t_* and completes the proof. □

Note that, thanks to the strong maximum principle, φ and ψ are strongly positive and hence, $\sup_{\overline{\Omega}} \frac{\psi}{\varphi}$ is well defined.

The estimates given by the following result will be used to find out another sufficient condition for (9.1).

Lemma 9.4. *Assume $\mathcal{L}_1 = \mathcal{L}_2$, $b(x) > 0$ and $c(x) > 0$ for each $x \in \Omega$, and*

$$b_Mc_M < a_Ld_L\,, \qquad \lambda \geq \mu > \sigma_1^\Omega[\mathcal{L}_1]\,.$$

Then, for any coexistence state (u, v) of (1.1) the following estimates hold

$$M_1\,\theta_{[\mathcal{L}_1,\mu,d]} \leq u \leq N_1\,\theta_{[\mathcal{L}_1,\lambda,a]}\,, \tag{9.10}$$

$$M_2\,\theta_{[\mathcal{L}_1,\mu,d]} \leq v \leq N_2\,\theta_{[\mathcal{L}_1,\lambda,a]}\,, \tag{9.11}$$

where

$$N_1 = \frac{a_M(d_L + b_M)}{a_Ld_L - c_Mb_M}\,, \qquad N_2 = \frac{a_M(a_L + c_M)}{a_Ld_L - c_Mb_M}\,,$$

$$M_1 = \max\left\{\frac{d_L(b_L + d_M)}{a_Md_M - c_Lb_L}\,, \frac{(b_L + d_L)[d_M(a_M + c_M) - c_Ld_L]}{a_M[d_M(a_M + c_M) - c_L(b_L + d_L)]}\right\}\,,$$

$$M_2 = \max\left\{ \frac{d_L(c_L+a_M)}{a_M d_M - c_L b_L}\,,\ \frac{d_M(a_M+c_M)}{d_M(a_M+c_M)-c_L(b_L+d_L)} \right\}.$$

Proof. Since

$$N_1 a_L - b_M N_2 = a_M\,, \qquad N_2 d_L - N_1 c_M = a_M\,,$$

for each $t \geq 1$ we have that

$$t(N_1 a_L - N_2 b_M) - a_M \geq 0\,, \qquad t(N_2 d_L - N_1 c_M) - a_M \geq 0\,. \tag{9.12}$$

Now, thanks to (9.12) it is easily seen that for each $t > 1$ the couple $(\overline{u}_t, \overline{v}_t)$ defined by

$$(\overline{u}_t, \overline{v}_t) := t\,(N_1 \theta_{[\mathcal{L}_1,\lambda,a]}, N_2 \theta_{[\mathcal{L}_1,\lambda,a]})$$

is a strict supersolution of (1.1). Therefore, thanks to Lemma 9.3, the upper estimates in (9.10) and (9.11) get shown.

Now, in order to prove the validity of the lower estimates in (9.10), (9.11) we will adapt a device coming from [17]. A reiterative application of Lemma 3.2 shows that

$$\alpha_n \theta_{[\mathcal{L}_1,\mu,d]} \leq u\,, \qquad \beta_n \theta_{[\mathcal{L}_1,\mu,d]} \leq v\,, \tag{9.13}$$

for each $n \geq 1$, where

$$\alpha_n = \frac{d_L + b_L \beta_{n-1}}{a_M}\,, \qquad \beta_n = \frac{d_L + c_L \alpha_{n-1}}{d_M}\,, \qquad \alpha_0 = d_L/a_M\,, \qquad \beta_0 = 1\,.$$

Thus, passing to the limit as $n \to \infty$ yields

$$\alpha \theta_{[\mathcal{L}_1,\mu,d]} \leq u\,, \qquad \beta \theta_{[\mathcal{L}_1,\mu,d]} \leq v\,,$$

where

$$\alpha = \frac{d_L(b_L + d_M)}{a_M d_M - b_L c_L}\,, \qquad \beta = \frac{d_L(c_L + a_M)}{a_M d_M - b_L c_L}\,.$$

This provides us with half of the lower estimates in (9.10), (9.11). Now, it follows from Lemma 7.2 (ii) that $v/K \geq \theta_{[\mathcal{L}_1,\mu,d]}$, where

$$K = \frac{d_M(c_M + a_M)}{d_M(a_M + c_M) - c_L(b_L + d_L)}\,.$$

Thus,

$$\mathcal{L}_1 = \lambda u - a(x)u^2 + b(x)uv \geq \mu u - a(x)u^2 + b_L K \theta_{[\mathcal{L}_1,\mu,d]} u$$

and hence, u is a supersolution of

$$\begin{aligned} \mathcal{L}_1 w = (\mu + b_L K \theta_{[\mathcal{L}_1,\mu,d]})w - a(x)w^2 \qquad & \text{in } \Omega\,, \\ w = 0 \qquad \text{on } \ \partial\Omega\,. & \end{aligned} \tag{9.14}$$

Therefore, Theorem 3.1 implies

$$\theta_{[\mathcal{L}_1 - b_L K \theta_{[\mathcal{L}_1,\mu,d]},\mu,a]} \leq u\,.$$

Finally, a further application of Lemma 3.2 shows that

$$B\theta_{[\mathcal{L}_1,\mu,d]} \leq \theta_{[\mathcal{L}_1 - b_L K \theta_{[\mathcal{L}_1,\mu,d]},\mu,a]}\,,$$

where

$$B = \frac{(b_L + d_L)[d_M(a_M + c_M) - c_L d_L]}{a_M[d_M(a_M + c_M) - c_L(b_L + d_L)]}\,.$$

This completes the proof. Note that K and B are positive constants. □

Now, as an immediate consequence from Theorem 9.1 and Lemma 9.4 we obtain the following result.

Corollary 9.5. *Assume* $\mathcal{L}_1 = \mathcal{L}_2$*,* $b(x) > 0$ *and* $c(x) > 0$ *for each* $x \in \Omega$*,*

$$b_M c_M < a_L d_L\,, \qquad \lambda \geq \mu > \sigma_1^\Omega[\mathcal{L}_1]\,,$$

and

$$\frac{N_1}{M_2} \cdot \frac{N_2}{M_1} \left(\sup_{\overline{\Omega}} \frac{\theta_{[\mathcal{L}_1,\lambda,a]}}{\theta_{[\mathcal{L}_1,\mu,d]}} \right)^2 < \frac{a_L d_L}{b_M c_M}\,. \tag{9.15}$$

Then, (1.1) *possesses a unique coexistence state.*

Note that since $\theta_{[\mathcal{L}_1,\lambda,a]}$ and $\theta_{[\mathcal{L}_1,\mu,d]}$ are strongly positive, $\sup_{\overline{\Omega}} \frac{\theta_{[\mathcal{L}_1,\lambda,a]}}{\theta_{[\mathcal{L}_1,\mu,d]}}$ is well defined.

Remark 9.6. (i) If a, b, c and d are assumed to be constant, then

$$M_1 = \frac{d(b+d)}{ad - cb}\,, \qquad M_2 = \frac{d(c+a)}{ad-cb}\,,$$

although in case $a = b = c = 1$ there are choices of $d(x)$ for which some of these relations fails.

(ii) If a, b, c and d are constant, then (9.15) becomes into the condition found in Theorem 3.3 of [17].

(iii) As a consequence from Proposition 9.2 and Corollary 9.5, it follows that if one of the interaction coefficients (b or c) is small, then (1.1) possesses a unique coexistence state. For some special classes of domains and differential operators, how small should be b or c to have uniqueness can be estimated in terms of the several coefficients of the model. For instance, if $\Omega = (0, \pi)$, $\mathcal{L}_1 = \mathcal{L}_2 = -\frac{d^2}{dx^2}$ and $a = d = 1$, then $\sigma_1^\Omega[\mathcal{L}_1] = 1$, $\varphi(x) = \sin(x)$, $\psi(x) = x(\pi - x)/2$, $\sup_{\overline{\Omega}} \frac{\psi}{\varphi} = \pi/2$ and the estimate (9.15) becomes into

$$R(\lambda,\mu) := \sup_{\overline{\Omega}} \frac{\theta_{[\mathcal{L}_1,\lambda,1]}}{\theta_{[\mathcal{L}_1,\mu,1]}} < \frac{1}{\sqrt{bc}}\,. \tag{9.16}$$

Some explicit estimates of $R(\lambda, \mu)$ were found in [17] and [1]. Namely, in [17] it was shown that

$$R^2(\lambda, \mu) \leq \frac{\lambda^3}{(\mu - 1)^2}\,. \tag{9.17}$$

Therefore, thanks to Corollary 9.5, (1.1) possesses a unique coexistence state provided

$$bc < \frac{(\mu - 1)^2}{\lambda^3}\,. \tag{9.18}$$

(iv) In many cases Proposition 9.2 is sharper than Corollary 9.5. Indeed, in the previous example (9.6) becomes into

$$bc < \frac{64}{\pi^2} \cdot \frac{(\lambda - 1)(\mu - 1)(1 - bc)^2}{(\lambda^2 + b\mu^2)(\mu^2 + c\lambda^2)}\,. \tag{9.19}$$

Thus, if $\lambda = 2$, $\mu = 1.5$ and $c = 1$, (9.18) becomes into $b < 1/32 \simeq 0.031$, while (9.19) becomes into $b < b_0$ with $b_0 \simeq 0.099$. Therefore, in this case (9.19) is sharper than (9.18).

Under the assumptions of Theorem 9.1, the problem of the global attractivity of the coexistence state with respect to the cone of positive functions in both components is very difficult to handle with. This is in strong contrast with the competing species counterpart of (1.1), where due to the compressivity of the model (cf [14]) the uniqueness of a stable coexistence state implies its global attractivity as a result from the abstract theory of [9]. Nevertheless, the presence of uniform a priori bounds in the context of Theorem 9.1 allows us to apply the following result of [15] to the parabolic system associated with (1.1).

Theorem 9.7. *Assume that T is a strongly positive monotone continuous dynamical system on X where the cone K has non-empty interior and X is separable. Moreover, assume that $\overline{O(x)}$ (the positive semi-orbit of x) is compact for each $x \in X$. Then, there exists a dense subset A of X such that if $x \in A$, then $\omega(x)$ (the ω-limit of x), is contained in the set of stationary points.*

Using this result we obtain the following one.

Theorem 9.8. *Assume that $b_M c_M < a_L d_L$, $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$, $\mu > \sigma_1^\Omega[\mathcal{L}_2]$, $b(x) > 0$, $c(x) > 0$, for each $x \in \Omega$, and that (1.1) possesses a unique coexistence state, say (u_c, v_c). Consider the following parabolic reaction diffusion problem*

$$\begin{aligned} \partial_t u + \mathcal{L}_1 u &= \lambda u - a u^2 + b u v\,, \\ \partial_t v + \mathcal{L}_2 v &= \mu v - d v^2 + c u v\,, \end{aligned} \qquad \text{in } \Omega \times (0, \infty)\,,$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0\,, \qquad t > 0\,,$$

$$u(x, 0) = u_0(x)\,, \qquad v(x, 0) = v_0(x)\,, \qquad x \in \Omega\,,$$

where $u_0\,, v_0\ \in C(\overline{\Omega})$. *Then, the solution of this problem* $(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$ *is defined for all* $t>0$ *and there exists a dense subset* A *of* $(C(\overline{\Omega}))^2$ *such that if* $u_0>0$, $v_0>0$ *and* $(u_0,v_0)\in A$, *then*

$$\lim_{t\to\infty}\|u(x,t;u_0,v_0)-u_c\|_{L^\infty(\Omega)}=\lim_{t\to\infty}\|v(x,t;u_0,v_0)-v_c\|_{L^\infty(\Omega)}=0\,.$$

Proof. By Theorem 9.7 we know that if $(u_0,v_0)\in A$, then the solution of the previous parabolic problem, denoted by $(u(x,t;u_0,v_0),v(x,t;u_0,v_0))$, converges to some steady state. It suffices to show that if $u_0>0$ and $v_0>0$, then it converges to a coexistence state. Indeed, it follows from the parabolic maximum principle that

$$u(x,t;u_0,v_0)\geq\Phi_{[\mathcal{L}_1,\lambda,a]}(x,t;u_0)\,,\qquad v(x,t;u_0,v_0)\geq\Phi_{[\mathcal{L}_2,\mu,d]}(x,t;v_0)\,,$$

where $\Phi_{[\mathcal{L},\gamma,f]}(x,t;z_0)$ stands for the unique positive solution of

$$\partial_t z+\mathcal{L}=\gamma z-fz^2\qquad\text{in }\ \Omega\times(0,\infty)\,,$$

$$z|_{\partial\Omega}=0\,,\qquad t>0\,,$$

$$z(x,0)=z_0(x)\,,\qquad x\in\Omega\,.$$

Since for $\gamma>\sigma_1^\Omega[\mathcal{L}]$ the positive steady state of this problem, $\theta_{[\mathcal{L},\gamma,f]}$, is a global attractor for any positive solution, we have that

$$\lim_{t\to\infty}\|\Phi_{[\mathcal{L}_1,\lambda,a]}(x,t;u_0)-\theta_{[\mathcal{L}_1,\lambda,a]}\|_{L^\infty(\Omega)}=\lim_{t\to\infty}\|\Phi_{[\mathcal{L}_2,\mu,d]}(x,t;u_0)-\theta_{[\mathcal{L}_2,\mu,d]}\|_{L^\infty(\Omega)}=0\,.$$

Thus,

$$\liminf_{t\to\infty}u(x,t;u_0,v_0)\geq\theta_{[\mathcal{L}_1,\lambda,a]}>0\,,\qquad\liminf_{t\to\infty}v(x,t;u_0,v_0)\geq\theta_{[\mathcal{L}_2,\mu,d]}>0\,,$$

and therefore, the solution must converge to a coexistence state; necessarily (u_c,v_c), since it is unique. □

10. Local bifurcation analysis. In this section we analyze the local structure of the set of positive solutions of (1.1) near

$$(\lambda,\mu,u,v)=(\sigma_1^\Omega[\mathcal{L}_1],\sigma_1^\Omega[\mathcal{L}_2],0,0)\,.$$

To make this analysis we will find out the bifurcation equations at this singularity through by a Lyapunov-Schmidt decomposition of (1.1). In the special case when $\mathcal{L}_1$ and $\mathcal{L}_2$ are perturbations of the Laplacian by some continuous potentials this analysis has been already done in [10], [11], and [12].

10.1. The bifurcation equations. Throughout this section we will consider the Banach spaces

$$\mathcal{U} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,, \qquad \mathcal{V} = L^p(\Omega)\,, \quad p > N\,.$$

Let φ_j, φ_j^* denote the principal eigenfunctions associated with $\mathcal{L}_j$ and $\mathcal{L}_j^*$, respectively, $j = 1, 2$, normalized so that

$$\int_\Omega \varphi_j^2 = 1\,, \qquad \int_\Omega \varphi_j \varphi_j^* = 1\,, \qquad j = 1\,,\, 2\,.$$

Let $L_i\; : \mathcal{U} \to \mathcal{V}$, $i = 1\,,\, 2$, be the differential operators defined by

$$L_1 = -\mathcal{L}_1 + \sigma_1^\Omega[\mathcal{L}_1]\,, \qquad L_2 = -\mathcal{L}_2 + \sigma_1^\Omega[\mathcal{L}_2]\,,$$

and consider the operator $L\; : \mathcal{U}^2 \to \mathcal{V}^2$ defined by

$$L(u,v) = (L_1 u, L_2 v)\,.$$

The null space of L, denoted by $N[L]$, is given by

$$N[L] = \{\, (r\varphi_1, s\varphi_2) \;:\; r\,,\, s \in \mathbb{R}\,\}\,.$$

To short notations, we will denote by $(\lambda_{old}, \mu_{old})$ the original parameters (λ, μ) in (1.1) and introduce the new parameters

$$(\lambda, \mu) = (\lambda_{old}, \mu_{old}) - (\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])\,.$$

Then, the solutions of (1.1) can be regarded as the zeros of the nonlinear mapping $F : \mathcal{U}^2 \times \mathbb{R}^2 \to \mathcal{V}^2$ defined by

$$F(u,v,\lambda,\mu) := \begin{pmatrix} L_1 u + \lambda u - a u^2 + b uv \\ L_2 v + \mu v - d v^2 + c uv \end{pmatrix}.$$

Let $P : \mathcal{V}^2 \to N[L]$ denote the projection

$$P(u,v) := (P_1 u, P_2 v) := \left((\int_\Omega \varphi_1^* u)\varphi_1\,,\, (\int_\Omega \varphi_2^* v)\varphi_2 \right).$$

Then, $Q := I_{\mathcal{V}^2} - P$ is a projection of $\mathcal{V}^2$ onto the complement of $N[L]$, denoted by $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$, where $\mathcal{W}_i$ stands for the complement of $N[L_i]$ in $\mathcal{V}$, $i = 1\,,2$. Once fixed the projection P, each element $(u,v) \in \mathcal{U}^2 \subset \mathcal{V}^2$ admits a unique decomposition of the form $(u,v) = \Phi + w$ with $\Phi \in N[L]$ and $w \in \mathcal{W}$. Namely,

$$\Phi = (\Phi_1, \Phi_2) := P(u,v)\,, \qquad w = (w_1, w_2) := (I - P)(u,v)\,.$$

Since $L(u,v)=Lw$ and for each $w\in\mathcal{W}$ $QLw=Lw$, setting

$$N(u,v,\lambda,\mu)=(\lambda u-au^2+buv,\mu v-dv^2+cuv)$$

it is easily seen that the equation $F(u,v,\lambda,\mu)=0$ is equivalent to the system

$$PLw+PN(\Phi_1+w_1,\Phi_2+w_2,\lambda,\mu)=0\,,\qquad Lw+QN(\Phi_1+w_1,\Phi_2+w_2,\lambda,\mu)=0\,.$$

Since the operator $L:W\cap\mathcal{U}^2\to\mathcal{V}^2$ is a topological isomorphism, the implicit function theorem implies that there exists a neighborhood $\mathcal{N}$ of $0\in N[L]\times\mathbb{R}^2$ and a real analytic function $w:\mathcal{N}\to W$ such that $w(0)=0$ and for each $(\Phi_1,\Phi_2,\lambda,\mu)\in\mathcal{N}$

$$L(w(\Phi_1,\Phi_2,\lambda,\mu))+QN((\Phi_1,\Phi_2)+w(\Phi_1,\Phi_2,\lambda,\mu),\lambda,\mu)=0\,. \tag{10.1}$$

Thus, there exists a neighborhood $\mathcal{M}$ of $0\in\mathcal{U}^2\times\mathbb{R}^2$ such that the solutions of (1.1) within $\mathcal{M}$ are in one-one correspondence with the solutions of the following equation

$$f(r,s,\lambda,\mu):=PN((\Phi_1,\Phi_2)+w(\Phi_1,\Phi_2,\lambda,\mu),\lambda,\mu)=0 \tag{10.2}$$

within $\mathcal{N}\times\mathbb{R}^2$. Notice that $PLw=0$. Equation (10.2) is often referred as the bifurcation equation. Computing the Taylor series of $f:=(f_1,f_2)$ up to third order terms, gives

$$\begin{aligned} f_1(r,s,\lambda,\mu)=&\lambda r+r(-a_1r+a_2s)\\ &+r(-b_1r^2+b_2rs-b_3s^2+d_1\lambda r-d_2\lambda s)\\ &+O(3,(r,s,\lambda,\mu)))\,,\\ f_2(r,s,\lambda,\mu)=&\mu s+s(a_3r-a_4s)\\ &+s(-b_4r^2+b_5rs-b_6s^2-d_3\mu r+d_4\mu s)\\ &+O(3,(r,s,\lambda,\mu)))\,, \end{aligned} \tag{10.3}$$

where $\Phi=(r\varphi_1,s\varphi_2)$, $O(3,(r,s,\lambda,\mu))$ stands for terms of order three in the variables (r,s,λ,μ), as $(r,s,\lambda,\mu)\to 0$, and the several coefficients involved in (10.3) are given by

$$a_1=\int_\Omega a\varphi_1^2\varphi_1^*\,,\quad a_2=\int_\Omega b\varphi_1\varphi_2\varphi_1^*\,,\quad a_3=\int_\Omega c\varphi_1\varphi_2\varphi_2^*\,,\quad a_4=\int_\Omega d\varphi_2^2\varphi_2^*\,,$$

$$b_1=2\int_\Omega a\beta_1\varphi_1\varphi_1^*\,,\quad b_2=\int_\Omega(2a\beta_2\varphi_1+b\beta_1\varphi_2-b\beta_3\varphi_1)\varphi_1^*\,,\quad b_3=\int_\Omega b(\beta_2\varphi_2-\beta_4\varphi_1)\varphi_1^*\,,$$

$$b_4=\int_\Omega c(\beta_3\varphi_1-\beta_1\varphi_2)\varphi_2^*\,,\quad b_5=\int_\Omega(2d\beta_3\varphi_2-c\beta_2\varphi_2+c\beta_4\varphi_1)\varphi_2^*\,,\quad b_6=2\int_\Omega d\varphi_2\beta_4\varphi_2^*\,,$$

$$d_1=\int_\Omega\beta_1\varphi_1^*\,,\quad d_2=\int_\Omega\beta_2\varphi_1^*\,,\quad d_3=\int_\Omega\beta_3\varphi_2^*\,,\quad d_4=\int_\Omega\beta_4\varphi_2^*\,,$$

where the β_i's $i = 1, \dots, 4$ are the unique solutions of the following linear boundary value problems

$$L_1\beta_1 = Q_1 a\varphi_1^2\,, \qquad \beta_1|_{\partial\Omega} = 0\,, \qquad \int_\Omega \beta_1\varphi_1 = 0\,,$$

$$L_1\beta_2 = Q_1 b\varphi_1\varphi_2\,, \qquad \beta_2|_{\partial\Omega} = 0\,, \qquad \int_\Omega \beta_2\varphi_1 = 0\,,$$

$$L_2\beta_3 = Q_2 c\varphi_1\varphi_2\,, \qquad \beta_3|_{\partial\Omega} = 0\,, \qquad \int_\Omega \beta_3\varphi_2 = 0\,,$$

$$L_2\beta_4 = Q_2 d\varphi_2^2\,, \qquad \beta_4|_{\partial\Omega} = 0\,, \qquad \int_\Omega \beta_4\varphi_2 = 0\,.$$

Observe that the bifurcation equation (10.2) is of the form

$$\begin{aligned} \lambda r + rp(r,s,\lambda,\mu) &= 0 \\ \mu s + sq(r,s,\lambda,\mu) &= 0 \end{aligned} \tag{10.4}$$

where p and q are given by (10.3). Thus, as it occurs with (1.1), there are three types of non-negative solutions of (10.4). Those on the manifold of trivial solutions

$$\mathcal{M}_0 := \{(0,0,\lambda,\mu) :\ \lambda,\mu \in (-\varepsilon,\varepsilon)\}\,,$$

where $\varepsilon > 0$ is sufficiently small, those lying on the manifolds of semi-trivial solutions

$$\mathcal{M}_\lambda := \{(t,0,\lambda_1(t,\mu),\mu)\ :\ t \in (0,\sigma)\,,\ \mu \in (-\varepsilon,\varepsilon)\}\,,$$

$$\mathcal{M}_\mu := \{(0,t,\lambda,\mu_2(t,\lambda))\ :\ t \in (0,\sigma)\,,\ \mu \in (-\varepsilon,\varepsilon)\}\,,$$

for some $\sigma > 0$ small enough, where λ_1 and μ_2 are the unique solutions around the origin of

$$\lambda_1(t,\mu) + p(t,0,\lambda_1(t,\mu),\mu) = 0\,, \qquad \mu_2(t,\lambda) + q(0,t,\lambda,\mu_2(t,\lambda)) = 0\,,$$

respectively, and finally, those on the manifold of coexistence solutions

$$\mathcal{M}_{co} := \{(t,\tau,\lambda(t,\tau,\mu),\mu(t,\tau,\lambda))\ :\ t,\,\tau \in (0,\sigma)\}\,,$$

where $\lambda(t,\tau,\mu)$ and $\mu(t,\tau,\lambda)$ are the unique solutions around the origin of the system

$$\lambda(t,\tau,\mu) + p(t,\tau,\lambda(t,\tau,\mu),\mu) = 0\,, \qquad \mu(t,\tau,\lambda) + q(t,\tau,\lambda,\mu(t,\tau,\lambda)) = 0\,.$$

As in the situation described by Theorem 6.2 of [10], $\mathcal{M}_{co}$ bifurcates from the manifolds $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ along the coexistence curves $\Lambda_{1,0}(t) := (\lambda_1(t),\mu_1(t))$ and $\Lambda_{0,1}(t) := (\lambda_2(t),\mu_2(t))$, where μ_1 and λ_2 are the local solutions of

$$\mu_1(t) + q(t,0,\lambda_1(t,\mu_1(t)),\mu_1(t)) = 0\,, \qquad \lambda_2(t) + p(0,t,\lambda_2(t),\mu_2(t,\lambda_2(t))) = 0\,.$$

In fact, $\Lambda_{1,0}$ and $\Lambda_{0,1}$ provide us with a local parametrization at $(\sigma_1^\Omega[\mathcal{L}_1],\sigma_1^\Omega[\mathcal{L}_2])$ of the curves of change of stability of the semi-trivial positive solutions $\mu = \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1,\lambda,a]}]$ and $\lambda = \sigma_1^\Omega[\mathcal{L}_1 - b(x)\theta_{[\mathcal{L}_2,\mu,d]}]$, respectively.

10.2. Finding out the bifurcation directions. As in [10], [11] and [12] we can use some techniques from singularity theory to analyze the bifurcation equation (10.2). We point out that in our current situation we are not interested in the problem of ascertaining whether or not the curve of change of stability of the semi-trivial states meet or not, since except at $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2]))$ when we deal with (1.1) these curves never meet, in strong contrast with the case of competing species. Now, our interest will be focused toward the problem of analyzing the bifurcation directions to coexistence states from the manifolds of semi-trivial states.

The following result is an immediate consequence from our previous analysis and Theorem 5.1 (i) of [11], where we refer for any further detail.

Theorem 10.1. *If*

$$A := \frac{a_2 a_3}{a_1 a_4} \neq 1 \,, \tag{10.5}$$

then, f is $\mathcal{K}$-equivalent in the sense of [11] *to*

$$(r(\lambda - r + As), s(\mu + r - s)) \,. \tag{10.6}$$

Moreover, f is its own universal unfolding and A is a modal parameter.

Under condition (10.5), the curves $\Lambda_{1,0}(t)$ and $\Lambda_{0,1}(t)$ can be easily calculated. Moreover, the bifurcation directions to coexistence states from the semi-trivial states can be easily found. As in [10], it suffices finding out the signs of the Jacobian of the mapping (p,q) along each of the curves of change of stability $\Lambda_{1,0}(t)$ and $\Lambda_{0,1}(t)$. These Jacobians have the values

$$(p_r q_s - q_r p_s)(t, 0, \lambda_1(t), \mu_1(s)) = 1 - A \,,$$

$$(p_r q_s - q_r p_s)(0, t, \lambda_2(t), \mu_2(t)) = 1 - A \,,$$

and therefore, the bifurcation is supercritical if $A < 1$, while it is subcritical if $A > 1$.

In Figure 3 we describe each of the possible local bifurcation μ-diagrams for a given $\lambda > \sigma_1^\Omega[\mathcal{L}_1]$; the horizontal line represents the semitrivial states $\mathcal{M}_\lambda$, and the other curve is filled in by coexistence states $\mathcal{M}_{co}$. The value of μ where $\mathcal{M}_{co}$ bifurcates from $\mathcal{M}_\lambda$ is given by $\mu_{old} = \sigma_1^\Omega[\mathcal{L}_2 - c(x)\theta_{[\mathcal{L}_1, \lambda_{old}, a]}]$. Stable (resp. unstable) solutions are represented by solid (resp. dashed) lines. The local qualitative behavior of the solutions in $\mathcal{M}_\lambda$ is given by Proposition 4.1. Figures 3 (a), (b) show the generic bifurcation diagrams for $A < 1$ and $A > 1$ respectively. By the symmetry of the problem, the corresponding results are true by fixing $\mu > \sigma_1^\Omega[\mathcal{L}_2]$ and using λ as the main bifurcation parameter.

A priori, the global behavior of $\mathcal{M}_{co}$ is unknown, being strongly dependent on the size of the interaction coefficients, the geometry of the domain Ω and the spatial dimension N. Thanks to Theorem 6.2, if $b_M c_M < a_L d_L$, then $\mathcal{M}_{co}$ goes to the right and it is defined for all values of μ above the bifurcation value, while thanks to Theorem 7.7, if $N \leq 5$ and (7.5) holds, then $\mathcal{M}_{co}$ goes back, being defined for all parameter values below the critical one. If in the latest case we assume that $N > 5$, instead of $N \leq 5$,

$\mathcal{M}_{co}$ might not be defined for all these values of the parameter. This is the case if we make the choice $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, $\lambda = \mu$, and assume all the coefficients to be constant. In this example it is easily seen that the lack of a priori bounds forces $\mathcal{M}_{co}$ to blow up at a finite value of the parameter.

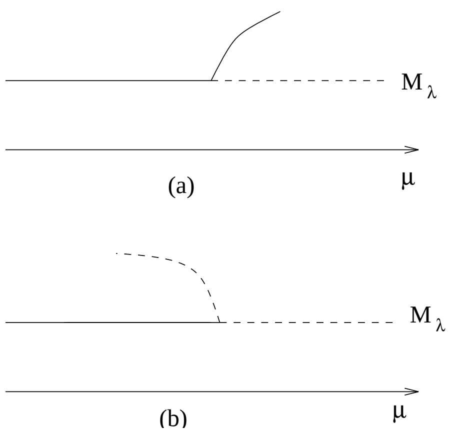

Figure 3: Local bifurcation μ-diagrams.

The following result provides us with a sufficient condition so that $A < 1$ for all values of (λ, μ) on the curves of change of stability of the semi-trivial states.

Lemma 10.2. *Assume that*

$$b_M c_M < a_L d_L, \tag{10.7}$$

and that either $\mathcal{L}_1 = \mathcal{L}_2$, or both, $\mathcal{L}_1$ and $\mathcal{L}_2$, are selfadjoint. Then,

$$A < 1\,.$$

Proof. In case $\mathcal{L}_1 = \mathcal{L}_2$ the result follows readily and so we omit the details. Now, assume that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are selfadjoint operators and that (10.7) holds. Then, $\varphi_j = \varphi_j^*$, $j = 1, 2$, and hence,

$$\begin{aligned} a_2 a_3 = \left(\int_\Omega b\varphi_1^2\varphi_2\right)\left(\int_\Omega c\varphi_1\varphi_2^2\right) &\leq b_M c_M \left(\int_\Omega \varphi_1^2\varphi_2\right)\left(\int_\Omega \varphi_1\varphi_2^2\right) \\ < a_L d_L \left(\int_\Omega \varphi_1^3\right)^{2/3}\left(\int_\Omega \varphi_2^3\right)^{1/3} &\left(\int_\Omega \varphi_2^3\right)^{2/3}\left(\int_\Omega \varphi_1^3\right)^{1/3} \\ = a_L d_L \left(\int_\Omega \varphi_1^3\right)\left(\int_\Omega \varphi_2^3\right) &\leq \left(\int_\Omega a\varphi_1^3\right)\left(\int_\Omega d\varphi_2^3\right) = a_1 a_4\,. \end{aligned}$$

This completes the proof. $\square$

Similarly, if instead of (10.7), we assume (7.5) and $\mathcal{L}_1 = \mathcal{L}_2$, then $A > 1$.

We should point out that all the previous information is of local nature, i.e. it provides us with the bifurcation directions to coexistence states from the semi-trivial states for values of the parameters close to the co-dimension two singularity $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$. Being the problem of finding out global information about the nature of these local bifurcations very difficult to handle with in our general setting, in the next section we will restrict ourselves to the consider the very special case when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and all the coefficients are constant. In particular, it will be shown that there are ranges of the parameters for which there is a change of the bifurcation direction to coexistence states provided $ad - bc > 0$ is sufficiently small. This will provide us with some sufficient conditions so that the model exhibits at least two coexistence states accordingly to the multiplicity results of Section 8.

If $A = 1$, then Theorem 10.1 can not be applied and the complexity of the bifurcation diagrams increase. In this case, Theorem 5.1 (ii) of [11] gives the following result.

Theorem 10.3. *Assume $A = 1$, and set*

$$\varepsilon = \textit{sign } a', \qquad \bar{c} = -a_3^2 a_2 c' |a'|^{-1},$$

where

$$a' = -\frac{1}{2}\frac{a_4}{a_3} b_2 + b_3 - \frac{1}{2}\frac{a_2}{a_3} b_5 + \frac{a_2}{a_4} b_6 + \frac{1}{2}\frac{a_2 a_4}{a_3}(d_1 + d_3) - \frac{1}{2} a_2 (d_2 + d_4),$$

$$c' = \frac{a_4}{a_2} b_1 - \frac{1}{2}\frac{a_3}{a_2} b_2 + b_4 - \frac{1}{2}\frac{a_3}{a_4} b_5 - \frac{1}{2} a_3 (d_1 + d_3) + \frac{1}{2}\frac{a_3^2}{a_4}(d_2 - d_4).$$

Then, if $a'c'((c')^2 - (a')^2) \neq 0$, f is $\mathcal{K}$-equivalent to

$$\begin{pmatrix} r(\lambda - r + s - \varepsilon s^2) \\ s(\mu + r - s - \bar{c} r^2) \end{pmatrix}.$$

Moreover, the universal unfolding of f is given by

$$\begin{pmatrix} r(\lambda - r + (1+\beta)s - \varepsilon s^2) \\ s(\mu + r - s - \bar{c} r^2) \end{pmatrix} \tag{10.8}$$

and $\bar{c}$ is a modal parameter. Here, $\beta \simeq 0$ is an unfolding parameter.

From (10.8), the bifurcation directions to coexistence states can be very easily found out. In our present situation, the signs of the $p_r q_s - q_r p_s$ depend on the parameter t, as shown by the following identities

$$(p_r q_s - q_r p_s)(\Lambda_{1,0}(t)) = -\beta + 2(1+\beta)\bar{c}t + \dots, \qquad (p_r q_s - q_r p_s)(\Lambda_{0,1}(t)) = -\beta + 2\varepsilon t + \dots.$$

Notice that since $\Lambda_{1,0}(0) = \Lambda_{0,1}(0) = 0$, when t grows $\Lambda_{1,0}(t)$ and $\Lambda_{0,1}(t)$ separate from $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$.

The list bellow provides us with all the bifurcation directions as s grows from zero. Without lost of generality, we can assume that $\varepsilon = 1$.

1. Bifurcation directions along $\Lambda_{0,1}$

1.1- If $\beta > 0$, then for values of the parameters sufficiently close to $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$ the bifurcation to coexistence states is subcritical, up to some value of the parameter where it becomes into supercritical.

1.2- If $\beta < 0$, then the bifurcation is always supercritical.

2. Bifurcation directions along $\Lambda_{1,0}$

2.1- If $\overline{c} > 0$ and $\beta > 0$, then the situation described in case 1.1 occurs.

2.2- If $\overline{c} > 0$ and $\beta < 0$, then the bifurcation direction is supercritical.

2.3- $\overline{c} < 0$ and $\beta > 0$, then the bifurcation direction is subcritical.

2.4- If $\overline{c} < 0$ and $\beta < 0$, then for values of the parameters sufficiently close to $(\sigma_1^\Omega[\mathcal{L}_1], \sigma_1^\Omega[\mathcal{L}_2])$ the bifurcation is supercritical, while after some critical value becomes subcritical.

We should point out that, due to the symmetry of the problem, if $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and $a' = c' = 0$, then f is much more degenerate than (10.8). To treat these degenerate situations we refer to the Appendix of [10].

11. The special case $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ with constant coefficients. Throughout this section we assume that $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and that a, b, c and d are constant. After a change of variables we can assume that

$$a = d = 1\,.$$

In the sequel we use the notation

$$\sigma_1[q] := \sigma_1^\Omega[-\Delta + q]\,, \qquad \sigma_1 := \sigma_1[0]\,, \qquad \theta_\gamma := \theta_{[-\Delta,\gamma,1]}\,,$$

and extend the definition of θ_γ taking $\theta_\gamma := 0$ for $\gamma \leq \sigma_1$. As an immediate consequence from the results in the previous sections we obtain the following global theorem, which is a substantial improvement of all the previous results in the references.

Theorem 11.1. *(i) Assume $bc < 1$. Then, the following assertions are true:*

(i.1) If any of the semi-trivial positive solutions is linearly unstable, then (1.1) *possesses a coexistence state. If in addition $\lambda > \sigma_1$, $\mu > \sigma_1$, then there exists $I_0 > 0$ such that if either $b < I_0$ or $c < I_0$, then the coexistence state is unique and exponentially asymptotically stable.*

(i.2) If for $(\lambda, \mu) = (\lambda_0, \mu_0)$ some of the semi-trivial positive solutions is linearly stable and (1.1) *possesses a coexistence state, then it possesses a coexistence state for each (λ, μ) satisfying $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and at least two coexistence states if $\lambda > \lambda_0$, $\mu > \mu_0$ and some of the semi-trivial positive solutions is linearly stable.*

(i.3) For each $\lambda \in \mathbb{R}$*, there exists* $\mu_{ext}(\lambda) \in \mathbb{R}$ *such that* (1.1) *does not admit a coexistence state if* $\mu \leq \mu_{ext}(\lambda)$*. Similarly, for each* $\mu \in \mathbb{R}$*, there exists* $\lambda_{ext}(\mu) \in \mathbb{R}$ *such that* (1.1) *does not admit a coexistence state if* $\lambda \leq \lambda_{ext}(\mu)$*. Moreover, thanks to Lemma 6.2,*

$$\mu_{ext}(\lambda) \geq (1-bc)\sigma_1 - c\lambda\,, \qquad \lambda_{ext}(\mu) \geq (1-bc)\sigma_1 - b\mu\,. \tag{11.1}$$

(ii) Assume $bc > 1$*. Then, the following assertions are true:*

(ii.1) If $N \leq 5$ *and some of the semi-trivial positive solutions is linearly stable, then* (1.1) *possesses a coexistence state.*

(ii.2) If $N \leq 5$ *and there exists* $(\lambda, \mu) = (\lambda_0, \mu_0)$ *for which* (1.1) *possesses a coexistence state being any of the semi-trivial states linearly unstable, then* (1.1) *possesses a coexistence state for each* (λ, μ) *satisfying* $\lambda \leq \lambda_0$ *and* $\mu \leq \mu_0$*, and at least two coexistence states if* $\lambda < \lambda_0$ *and* $\mu < \mu_0$ *and any of the semi-trivial states is linearly unstable.*

(ii.3) For each $\lambda \in \mathbb{R}$*, there exists* $\mu_{ext}(\lambda) \in \mathbb{R}$ *such that* (1.1) *does not admit a coexistence state if* $\mu \geq \mu_{ext}(\lambda)$*. Similarly, for each* $\mu \in \mathbb{R}$*, there exists* $\lambda_{ext}(\mu) \in \mathbb{R}$ *such that* (1.1) *does not admit a coexistence state if* $\lambda \geq \lambda_{ext}(\mu)$*.*

The first goal of this section is finding out sharper estimates than (11.1) for the values of $\lambda_{ext}(\mu)$ and $\mu_{ext}(\lambda)$ in the case $bc < 1$. Our main result in this direction reads as follows:

Theorem 11.2. *Assume* $bc < 1$ *and*

$$\lambda > \sigma_1\,, \qquad \lambda \geq \mu > \sigma_1[-c\frac{1+b}{1-bc}\theta_\lambda]\,. \tag{11.2}$$

Then,

$$u \leq \frac{1+b}{1-bc}\theta_\lambda\,, \qquad v \leq \theta_{[-\Delta - c\frac{1+b}{1-bc}\theta_\lambda, \mu, 1]}\,, \tag{11.3}$$

for any coexistence state (u, v) *of* (1.1)*. Therefore, if* $\lambda > \sigma_1$ *and*

$$\mu \leq \max\{\sigma_1[-c\frac{1+b}{1-bc}\theta_\lambda]\,,\, \sigma_1(1-bc) - c\lambda\,\} \tag{11.4}$$

then (1.1) *does not admit a coexistence state. By symmetry, the same result holds if* $\mu > \sigma_1$ *and*

$$\lambda \leq \max\{\sigma_1[-b\frac{1+c}{1-bc}\theta_\mu]\,,\, \sigma_1(1-bc) - b\mu\,\}.$$

Proof. Thanks to Lemma 7.2, $(1+b)v \leq (1+c)u$ and hence, we find from the u-equation of the system that

$$-\Delta u \leq \lambda u - \frac{1-bc}{1+b}u^2\,.$$

Thus, Lemma 3.2 implies the first upper estimate of (11.3). Substituting this estimate into the v-equation of the system gives

$$(-\Delta - c\,\frac{1+b}{1-bc}\,\theta_\lambda)v \le \mu v - v^2\,,$$

and Lemma 3.2 completes the proof of (11.3). The remaining assertions follow readily from Theorem 3.1 and Theorem 11.1 (i.3). ☐

Remark 11.3. The curve defined by the right hand side of (11.4) meets (σ_1, σ_1) at the value $\lambda = \sigma_1$, since $\lim_{\lambda\downarrow\sigma_1}\theta_\lambda = 0$ and hence,

$$\lim_{\lambda\downarrow\sigma_1}\max\{\sigma_1[-c\frac{1+b}{1-bc}\theta_\lambda]\,,\,\sigma_1(1-bc)-c\lambda\,\} = \lim_{\lambda\downarrow\sigma_1}\sigma_1[-c\frac{1+b}{1-bc}\theta_\lambda] = \sigma_1\,,$$

thanks to the continuous dependence of the principal eigenvalue with respect to the potential. Therefore, the estimate of the extinction region given by (11.4) is optimal for values of $\lambda \simeq \sigma_1$.

Moreover, (11.4) is also optimal for values of λ varying on compact subintervals of $[\sigma_1, \infty)$ provided b is sufficiently small, as the following result shows.

Theorem 11.4. *Assume $bc < 1$, $\lambda > \sigma_1$ and $\mu < \sigma_1[-c\theta_\lambda]$. Then, there exists $b_0 = b(\lambda) > 0$ such that* (1.1) *does not admit a coexistence state if $b \in [0, b_0]$. Moreover, $b(\lambda)$ varies continously with λ.*

Proof. The function

$$h(b) := -c\,\frac{1+b}{1-bc}\,,$$

is decreasing and it satisfies

$$h(0) = -c\,, \qquad \lim_{b\uparrow c^{-1}} h(b) = -\infty\,.$$

Thus, there exists a unique $b_0 = b(\lambda) > 0$ such that

$$\mu = \sigma_1[-c\,\frac{1+b_0}{1-b_0c}\,\theta_\lambda] < \sigma_1[-c\theta_\lambda]\,.$$

Therefore, for $b \in [0, b_0]$ we have that

$$\mu \le \sigma_1[-c\,\frac{1+b}{1-bc}\,\theta_\lambda] \le \sigma_1[-c\theta_\lambda]$$

and Theorem 11.2 completes the proof. ☐

Remark 11.5. Thanks to the estimate (4.10) in the proof of Theorem 4.1 in [25], we find that

$$\sigma_1[-c\frac{1+b}{1-bc}\theta_\lambda] \le \sigma_1 - c\,\frac{1+b}{1-bc}\,(\lambda - \sigma_1)$$

and therefore, the following estimate for $\mu_{ext}(\lambda)$ is obtained

$$\mu_{ext}(\lambda) \geq \begin{cases} \sigma_1 - c\,\frac{1+b}{1-bc}\,(\lambda - \sigma_1) & \text{if } \lambda \leq \sigma_1\,\frac{b(2-bc)+1}{b(c+1)}\,, \\ \sigma_1(1-bc) - c\lambda & \text{if } \lambda > \sigma_1\,\frac{b(2-bc)+1}{b(c+1)}\,. \end{cases}$$

This estimate provides us with some very readily computable sufficent condition in terms of the several coefficients involved in the model setting for the extinction of the species v.

In Figure 4 we have represented the curve of change of stability of $(\theta_\lambda, 0)$ together with the boundary of the extinction region given by the estimate (11.4); for values of (λ, μ) in the bright grey region the model possesses a coexistence state, while for the values of (λ, μ) in the darker region the species v is driven to extinction by u.

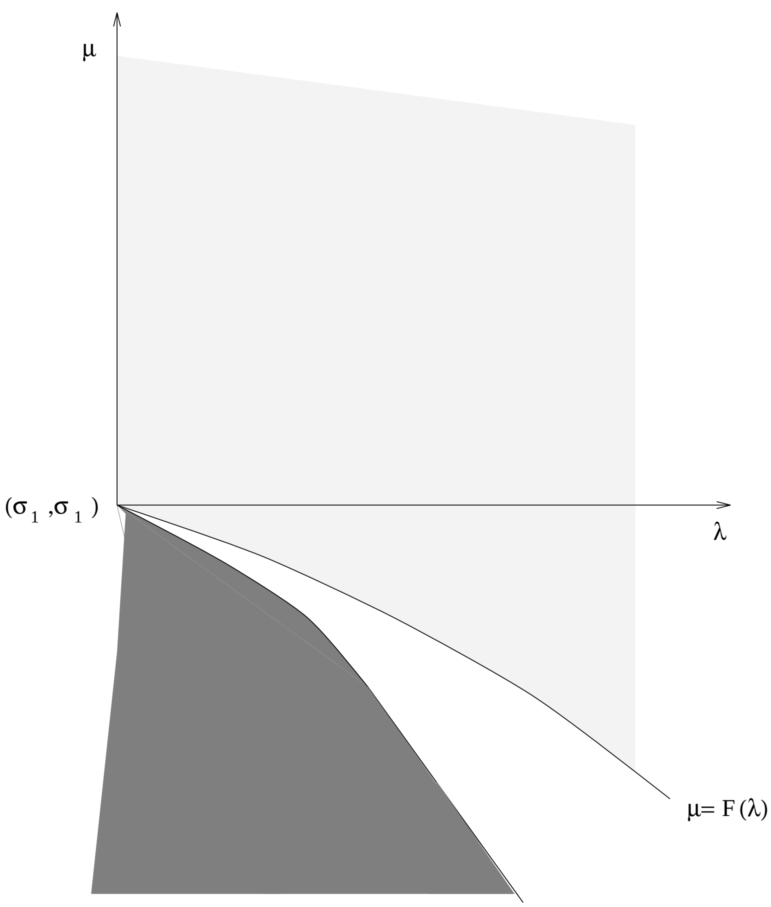

Figure 4: The coexistence and extinction regions.

In the next result we complete the local analysis of Section 10 by giving some sufficient conditions for completely ascertaining the bifurcation directions to coexistence states in the case $bc > 1$.

Theorem 11.6. *Assume $bc > 1$, $bc \geq 2 + c$ and fix $\lambda > \sigma_1$. Then the bifurcation direction to coexistence states from $(\mu, u, v) = (\sigma_1[-c\theta_\lambda], \theta_\lambda, 0)$ is subcritical. By symmetry, if $bc > 1$, $bc \geq 2 + b$ and we fix $\mu > \sigma_1$, then the bifurcation direction from $(\lambda, u, v) = (\sigma_1[-b\theta_\mu], 0, \theta_\mu)$ is subcritical.*

Proof. Let $(\mu(s), u(s), v(s))$ denote the local curve of coexistence states emanating from $(\theta_\lambda, 0)$ at $\mu = \sigma_1[-c\theta_\lambda]$. The main theorem of [7] guarantees that $\mu(s)$ is real analytic in s and hence it possesses an expansion of the form

$$\mu(s) = \sigma_1[-c\theta_\lambda] + s\mu_1(\lambda) + O(s^2)\,, \qquad \text{as } \; s \to 0\,,$$

for some $\mu_1(\lambda) \in \mathbb{R}$. A rather standard calculation shows that (cf. [6] and [10] for details)

$$\mu_1(\lambda) = (2+c)^{-1}[(2+c-bc)\int_\Omega \varphi_\lambda^3 - bc(\lambda - \sigma_1[-c\theta_\lambda])\int_\Omega \varphi_\lambda^2 \mathcal{R}(\lambda)\varphi_\lambda]\,, \tag{11.5}$$

where $\mathcal{R}(\lambda) := (-\Delta + 2\theta_\lambda - \lambda)^{-1}$ and $\varphi_\lambda > 0$ is the principal eigenfunction associated with $\sigma_1[-c\theta_\lambda]$ normalized so that $\|\varphi_\lambda\|_2 = 1$. This completes the proof. $\square$

Modulo the change of b and c by $-b$ and $-c$, respectively, the formula (5.2) of [10] provides us with the sign of $\mu_1(\lambda)$ for $\lambda \simeq \sigma_1$.

Lemma 11.7. *(i) If λ is sufficiently close to σ_1, then*

$$\text{sign} \;\; \mu_1(\lambda) = \text{sign} \;\; (1 - bc)\,.$$

(ii) Similarly, for $\mu \simeq \sigma_1$,

$$\text{sign} \;\; \lambda_1(\mu) = \text{sign} \;\; (1 - bc)\,,$$

where $\lambda_1(\mu) = \frac{d\lambda}{ds}|_{s=0}$. Here, $\lambda(s)$ stands for the λ-component of the curve of coexistence states emanating from $(\lambda, u, v) = (\sigma_1[-b\theta_\mu], 0, \theta_\mu)$, whose existence is guaranteed by Theorem 5.1.

We now show how change the bifurcation directions to coexistence states along the semi-trivial branches as bc grows from the critical value 1, so completing the results of Section 10. For this we will use the local bifurcation analysis already done in Section 10. Interexchanging the roles of b and c in [10] by $-b$ by $-c$ here, we obtain the bifurcation equation

$$\lambda r - rp(r, s, \lambda, \mu, b, c) = 0\,, \qquad \mu s - sq(r, s, \lambda, \mu, b, c) = 0\,, \tag{11.6}$$

where $q(r, s, \lambda, \mu, b, c) = p(s, r, \mu, \lambda, c, b)$ and

$$\begin{aligned} p(r, s, \lambda, \mu, b, c) &= M(r - bs) + N[2r^2 - b(3-c)rs - b(1-b)s^2] \\ &\quad + K\{5r^3 - b(c^2 - 4c + 10)r^2 s - 3b[(1-b)(1-c) - b]rs^2 \\ &\quad - b(b^2 - 2b + 2)s^3\} \\ &\quad + L[2\lambda r^2 - b(3\lambda - c\mu)rs - b(\mu - b\lambda)s^2] \\ &\quad + O(4, (r, s, \lambda, \mu))\,, \end{aligned}$$

where M, N, K, L are the constants defined by (3.6) in [10]. We should point out that if $bc = 1$, then the constants a' and c' of the statement of Theorem 10.2 equal zero, and so Theorem 10.2 does not cover this case. This is why to analyze the change of criticality of the local bifurcations from the semi-trivial branches third order terms are needed. Our main result in this direction is the following, where the notations introduced in Section 10 are kept.

Theorem 11.8. *If $bc - 1 > 0$ is sufficiently small, then there exists a unique change of criticality in a neighborhood of the origin along each of the curves $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$.*

Proof. After some strightforward manipulations, we find that

$$\lambda_1(t) = Mt + 2Nt^2 + (5K + 2LM)t^3 + O(t^4)\,, \tag{11.7a}$$

$$\mu_1(t) = -Mct - Nc(1-c)t^2 - (Kc(c^2 - 2c + 2) + LMc(1 + c^2))t^3 + O(t^4)\,. \tag{11.7b}$$

Thus, setting

$$\mathrm{Jac}_1(t) = (p_r q_s - p_s q_r)(t, 0, \lambda_1(t), \mu_1(t))\,,$$

and substituting (11.7) in it gives

$$\mathrm{Jac}_1(t) = \varepsilon M^2 + \varepsilon(4 - 3c)NMt + [2(c+1)^2(KM - N^2) + \varepsilon \mathcal{F}_c]t^2 + O(t^3)\,,$$

where

$$\varepsilon := 1 - bc\,, \qquad \mathcal{F}_c = M^2 L(3c^2 + 4) + KM(4c^2 - 7c + 13) + N^2(2c^2 - 8c + 2)\,.$$

Making the change of variables

$$\varepsilon = -\tau^2\,, \qquad s = s_0\tau\,,$$

and setting

$$\mathrm{Jac}_1(\tau, s_0) := \frac{\mathrm{Jac}_1(-\tau^2, s_0\tau)}{\tau^2}\,, \qquad P := 2(c+1)^2(KM - N^2)\,,$$

it is easily seen that

$$\mathrm{Jac}_1(\tau, s_0) = -M^2 - (4 - 3c)NMs_0\tau + Ps_0^2 - \tau^2 s_0^2 \mathcal{F}_c + O(s_0^3\tau)\,,$$

We already know that $P > 0$ (cf. [10], pg. 109). Moreover, we have that

$$\mathrm{Jac}_1(0, \frac{M}{\sqrt{P}}) = 0\,, \qquad D_{s_0}\mathrm{Jac}_1(0, \frac{M}{\sqrt{P}}) = 2\sqrt{P}M \neq 0\,.$$

Thus, thanks to the implicit function theorem, there exists a unique function $\overline{s_0}$ such that for each $\tau \simeq 0$

$$\overline{s_0}(0) = M(P)^{-1/2}\,, \qquad \mathrm{Jac}_1(\tau, \overline{s_0}(\tau)) = 0\,.$$

Henceforth,

$$\mathrm{Jac}_1(-\tau^2, \overline{s_0}(\tau)\tau) = 0\,.$$

Therefore, there exists a unique $t(\varepsilon) > 0$ such that

$$\mathrm{Jac}_1(t(\varepsilon)) = 0\,.$$

By symmetry, the remaining assertions get shown. This completes the proof. □

Some further discussion. We now summarize the information given by the results in the last two sections. For this, it is convenient regarding b and c as the main parameters of the model. More precisely, we will fix $c > 0$ and vary b. Thanks to Lemma 10.2, if $b < c^{-1}$, then the bifurcation directions to coexistence states are supercritical. Thanks to Theorem 11.8, there exists $\varepsilon_0 = \varepsilon_0(c) > 0$ such that if $c^{-1} < b < (1+\varepsilon_0)c^{-1}$ then the bifurcation directions are subcritical for (λ, μ) close to (σ_1, σ_1), in fact this holds in a $\sqrt{bc-1}$-neighborhood of (σ_1, σ_1), while they become supercritical outside this neighborhood, within another slightly larger neighborhood of (σ_1, σ_1). Now, since the curves $bc = 2+b$ and $bc = 2+c$ in the statement of Theorem 11.6 meet at $(b, c) = (2, 2)$, changing their relative positions as c acrosses 2, two different cases must be considered. If $c < 2$, then we find from Theorem 11.6 that $(1+\varepsilon_0)c^{-1} < 1 + \frac{2}{c}$, since for $bc \geq c+2$ all bifurcation directions from $(\theta_\lambda, 0)$ became subcritical. If $c < 1$, then our results do not provide us with any further global information about the bifurcation directions along $(0, \theta_\mu)$, while in case $1 < c < 2$ it follows from Theorem 11.6 that if b increases up to acrossing some critical value, necessarily less than $\frac{2}{c-1}$, then all bifurcations to coexistence states from $(0, \theta_\mu)$ will change to subcritical either. In case $c > 2$ these global changes in the nature of the bifurcations occur in the converse order. Now, any bifurcation direction from $(\theta_\lambda, 0)$ is subcritical if $b > \frac{2}{c+1}$ and moreover all bifurcation directions from any of the semi-trivial states are subcritical if $b > 1 + \frac{2}{c}$.

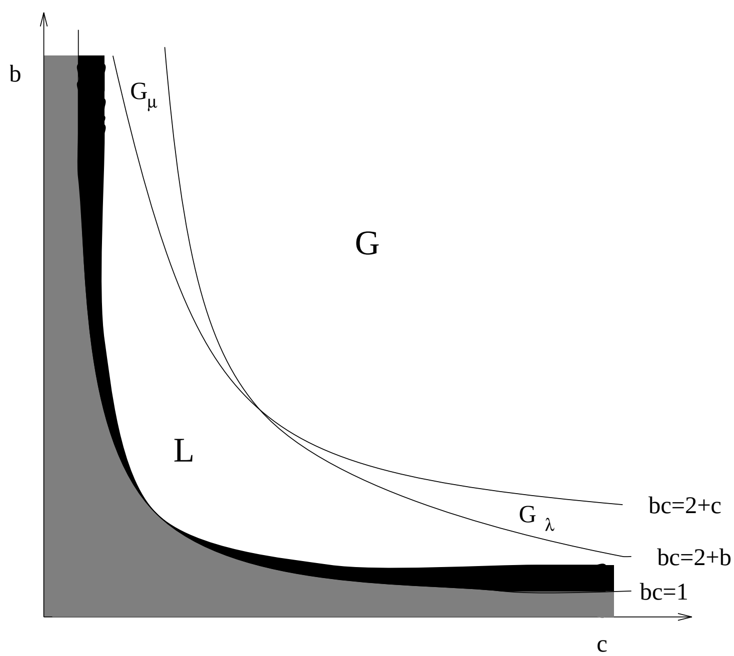

Figure 5: Varying b and c.

In Figure 5 we have summarized all the previous information. The first quadrant is divided into four regions. The bright grey region stands for $bc < 1$, where we only have local information; the black region, which is a thin streep above $bc > 1$, where we know that the local change of criticality occurs; the regions G_λ and G_μ, in between $bc = 2+c$ and $bc = 2+b$, where we know that the bifurcation direction from one of the semi-trivial branches, respectively $(\theta_\lambda, 0)$ and $(0, \theta_\mu)$, is always subcritical but no global information about the nature of the bifurcation along the remaining semi-trivial branch is available; in the region G, thanks to Theorem 11.6 all bifurcation directions are subcritical, and finally the region L, where only local information is supplied by our analysis. By the continuous dependence of the bifurcation directions with respect to (λ, μ, b, c), if we move away from L towards $G_\lambda \cup G_\mu$ (or the region G), any point of change of criticality on any of the semi-trivial branches should vary along this branch up to either meet with another point of change of criticality or grow up to infinity. In the first case, both points of change of criticality shrink at the meeting value and then dismiss. To complete our discussion, in Figure 6 we have represented a typical bifurcation diagram for a value of (b, c) lying the black area of Figure 5; a value of (b, c) where the points of change of criticality are still close to the co-dimension two singularity (σ_1, σ_1).

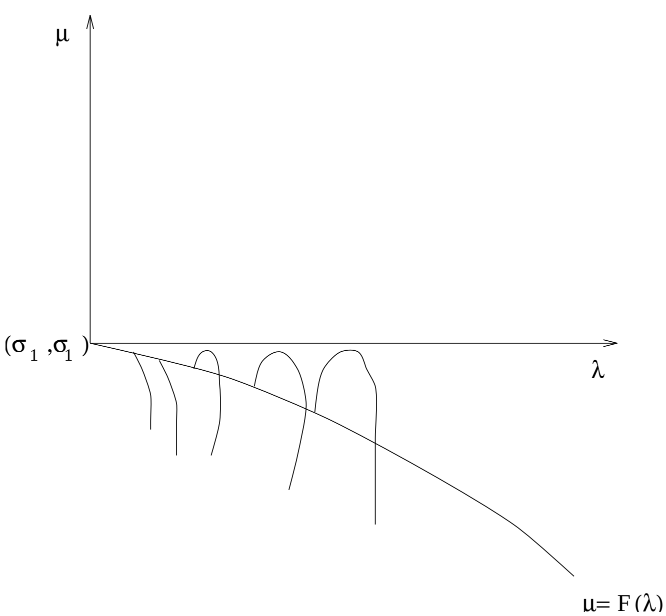

Figure 6: Local bifurcation diagrams along the curve of change of stability.

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