On the uniqueness of positive solution of an elliptic equation[1]

M. Delgado and A. Suárez[2]

Dpto. Ecuaciones Diferenciales y Análisis Numérico

Fac. Matemáticas, C/ Tarfia s/n

C.P. 41012, Univ. Sevilla, Spain

e-mail: madelgado@us.es and suarez@us.es

Abstract

This work deals with the uniqueness of positive solution for an elliptic equation whose nonlinearity satisfies an specific monotony property. In order to prove the main result, we employ a change of variable used in previous papers and the maximum principle.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a regular domain and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ a measurable function. We are interested in the classical and positive solutions of the elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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[2]Author who receives correspondence and proofs of correction

One of the more difficult problem related to (1.1) is proving the uniqueness of solution of (1.1). It is well known that if f is decreasing in u then there exists at most one solution of (1.1), see for instance [1] and [2]. When for a. e. $x \in \Omega$ the map

$$u \mapsto \frac{f(x,u)}{u} \quad \text{is decreasing in } (0,\infty) \tag{1.2}$$

then, there exists at most one positive solution of (1.1), see [3] and [4].

In this note, we employ an appropriate change of variable (yet used in [5], [6], [7] and [8]) and the strong maximum principle to prove that if there exists a regular, positive and concave function g (see Theorem 2.1 and Proposition 2.2 for the exact conditions on g) such that

$$u \mapsto \frac{f(x,u)}{g(u)} \quad \text{is non-increasing in } (0,\infty) \text{ for a. e. } x \in \Omega \tag{1.3}$$

then, there exists a unique positive solution.

When $f(x,u) = a(x)g(u)$ with $a \in L^\infty(\Omega)$, the uniqueness was studied in [5], [6], [7] and [8]. We refer to [6] where a review of the uniqueness question is made. We would like to remark that although the conditions (1.2) and (1.3) seem rather similar, the techniques for the proofs of uniqueness are quite different. In fact, the proofs of the uniqueness result under (1.2) use the monotonicity of the quotient between $f(x,t)$ and exactly the linear function $g(t) = t$. Our proof, which allows us to use the monotonicity of the quotient between $f(x,t)$ and a concave function $g(t)$, does not reach the linear function; whereas $f(x,t)/g(t)$ is not necessarily decreasing.

In the following section we prove the main result of this work. In the last section we employ a specific example from population dynamics that shows that our result improves and complements that obtained under the condition (1.2).

2 Main result

Our main result reads as follows:

Theorem 2.1 *Assume that there exists a function $g \in C^1(0,+\infty) \cap C^0([0,+\infty))$, $g(t) > 0$ for $t > 0$, such that*

a) g' is non-increasing and

$$\int_0^r \frac{1}{g(t)}dt < \infty, \qquad \textit{for } r > 0. \tag{2.1}$$

b) The map

$$u \mapsto \frac{f(x,u)}{g(u)} \quad \textit{is non-increasing in } (0,\infty) \textit{ for a. e. } x \in \Omega. \tag{2.2}$$

Then, there exists at most one positive solution of (1.1)*.*

Proof: Consider the change of variable

$$v = \int_0^u \frac{1}{g(t)} dt \tag{2.3}$$

which transforms (1.1) into

$$\begin{cases} -\Delta v = g'(h(v))|\nabla v|^2 + \dfrac{f(x,h(v))}{g(h(v))} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$u = h(v), \tag{2.5}$$

and h satisfies, from (2.3), $h'(t) = g(h(t))$.

Assume that there exists two positive solutions $u_1 \neq u_2$ of (1.1). Let $\Omega_1 := \{x \in \Omega : u_1(x) > u_2(x)\}$. Assume that Ω_1 is not empty. It is clear that $u_1 = u_2$ on $\partial\Omega_1$. Thanks to monotonicity of h, $v_1 > v_2$ in Ω_1 and $v_1 = v_2$ on $\partial\Omega_1$, where $u_i = h(v_i)$ $i = 1,2$. Consider the function

$$\Phi := v_1 - v_2,$$

which is positive in Ω_1 and $\Phi = 0$ on $\partial\Omega_1$. After some calculation, we obtain that Φ verifies

$$-\Delta\Phi - g'(h(v_1))|\nabla v_1|^2 + g'(h(v_2))|\nabla v_2|^2 = \left(\frac{f(x,h(v_1))}{g(h(v_1))} - \frac{f(x,h(v_2))}{g(h(v_2))}\right). \tag{2.6}$$

Since g' is non-increasing, $g'(h(v_1)) \leq g'(h(v_2))$; and by (2.2), we get that

$$-\Delta\Phi - g'(h(v_1))\nabla(v_1 + v_2) \cdot \nabla\Phi \leq 0,$$

which is a contradiction by the maximum principle. This completes the proof. □

If we look for positive solutions in a more restrictive set, we can weaken the condition (2.1). Let define

$$P := \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0,\ u \neq 0 \quad \text{in } \Omega\},$$

whose interior is

$$int(P) = \{u \in P : u(x) > 0 \quad \text{for all } x \in \Omega,\ \partial u/\partial n < 0 \quad \text{on } \partial\Omega\},$$

where n denotes the outward normal direction.

Proposition 2.2 *Assume that there exists g as in Theorem 2.1 but verifying*

$$\lim_{s\to 0} \frac{s}{g(s)} = 0, \tag{2.7}$$

instead of (2.1)*. Then, there exists a unique solution in $int(P)$ of* (1.1)*.*

Proof: Observe first that if $u \in int(P)$, there exist positive constants $0 < k_1 \le k_2$ such that

$$k_1 dist(x) \le u(x) \le k_2 dist(x), \tag{2.8}$$

where $dist(x) := dist(x, \partial\Omega)$. Assume that there exists two positive $u_1 \neq u_2$ of (1.1) with $u_i \in int(P)$, $i = 1, 2$. Let $\Omega_1 := \{x \in \Omega : u_1(x) > u_2(x)\}$. We define now for $x \in \Omega_1$

$$\Phi(x) := \int_{u_2(x)}^{u_1(x)} \frac{1}{g(t)} dt.$$

First, observe that function Φ is continuous in $\overline{\Omega}_1$ and

$$\Phi = 0 \quad \text{on } \partial\Omega_1.$$

Indeed, for $x \in \partial\Omega_1 \cap \Omega$ it is clear that $\Phi(x) = 0$. For each $x \in \Omega_1$ there exists $\xi(x)$ with $u_2(x) \le \xi(x) \le u_1(x)$ such that

$$\Phi(x) = \frac{u_1(x) - u_2(x)}{g(\xi(x))} \le \frac{C dist(x)}{g(\xi(x))} \to 0, \quad \text{as } dist(x) \to 0,$$

where we have used (2.7) and (2.8).

On the other hand, as in the proof of Theorem 2.1, we get that

$$-\Delta\Phi - g'(u_1)\left(\frac{\nabla u_1}{g(u_1)} + \frac{\nabla u_2}{g(u_2)}\right) \cdot \nabla\Phi \le 0.$$

This last inequality leads to a contradiction to the maximum principle in the same way as in the proof of Theorem 2.1. □

Remark 2.3 *a) Observe that, for example, $g(s) = s\log^2(s)$ verifies* (2.7) *but not* (2.1)*.*

b) *Conditions on f can be imposed in order that every non-negative and non-trivial solution of* (1.1) *belongs to* $int(P)$, *see for instance [9].*

c) *The same results hold for second order uniformly elliptic operator of the form*

$$\mathcal{L} := -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i}$$

with $a_{ij} \in C^0(\Omega)$, $b_i \in C^0(\Omega)$, $a_{ij} = a_{ji}$, *see [12].*

d) *If g is positive only in* $(0, R)$ *for some* $R > 0$, *and*

$$\int_0^R \frac{1}{g(t)} dt < +\infty \tag{2.9}$$

then, we deduce a uniqueness result for positive solutions, u, such that $\|u\|_\infty \leq R$.

3 Example and comparison

In this section we apply our result to the nonlinearity

$$f(x, u) = a(x)u^q + b(x)u^p$$

with different values of q and p, and $a, b \in L^\infty(\Omega)$. This nonlinearity arises from the study of the population density of a species whose mobility depends upon its density, see [10] and [11]. Some uniqueness results were obtained in [12] and [13]. For this function, the condition (1.2) is equivalent to

$$(q-1)a(x) + (p-1)b(x)u^{p-q} < 0. \tag{3.1}$$

Now, we distinguish between the different cases:

Case $q = 1$, $p < 1$: In this case, (3.1) holds if $b > 0$. Theorem 2.1 complements this result. Indeed, taking $g(u) = u^p$ we obtain uniqueness of positive solution for $a \leq 0$ and any function b.

Case $q < 1$, $p > 1$: (3.1) holds, for example, if a is positive and $b \leq 0$; a positive and b positive or changes sign and $\|u\|_\infty$ small, see [14] and [11]. By Theorem 2.1, there exist at most one positive solution if $b \leq 0$ and any function a.

Case $q < 1$, $p < 1$: In this case (3.1) is satisfied if, for example, a and b are both positive. In the particular case $p = q$, (3.1) is equivalent to $a + b > 0$.

By Theorem 2.1 we consider three cases:

a) If $p < q$, then we have uniqueness of positive solution for any function a and $b \geq 0$ (taking $g(u) = u^q$) and for any function b and $a \leq 0$ (taking $g(u) = u^p$).

b) If $p > q$, then the result is similar to case a) changing a by b and b by a.

c) If $p = q$, then there exists at most one positive solution if $a + b$ is non-negative or changes sign. Observe that if $a+b$ is non-positive, (1.1) does not posses non-negative solution.

In the cases $p = 1$, $q < 1$ and $p < 1$, $q > 1$ similar results to the first and third cases respectively can be obtained interchanging the roles of a and b.

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References

[1] H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.*, **21**, 125–146 (1971).

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18**, 620–709 (1976).

[3] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal.*, **10**, 55–64 (1986).

[4] P. Hess, On uniqueness of positive solutions of nonlinear elliptic boundary value problems, *Math. Z.*, **154**, 17–18 (1977).

[5] C. Bandle, M. A. Pozio and A. Tesei, The asymptotic behaviour of the solutions of degenerate parabolic equations, *Trans. Amer. Math. Soc.*, **303**, 487–501 (1987).

[6] H. Brezis and S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, *Manuscripta Math.*, **74**, 87–106 (1992).

[7] T. Laetsch, Uniqueness for sublinear boundary value problems, *J. Differential Equations*, **13**, 13–23 (1973).

[8] J. Spruck, Uniqueness in a diffusion model of population biology, *Comm. in Partial Diff. Eqns.*, **15**, 1605–1620 (1983).

[9] P. Pucci and J. Serrin, The strong maximum principle revisited, *J. Differential Equations*, **196**, 1–66 (2004).

[10] M. E. Gurtin and R. C. MacCamy, On the diffusion of biological populations, *Math. Biosci.*, **33**, 35–49 (1977).

[11] M. Delgado and A. Suárez, Positive solutions for the degenerate logistic indefinite superlinear problem: the slow diffusion case, *Houston J. of Math.*, **29**, 801–820 (2003).

[12] M. Delgado and A. Suárez, On the structure of the positive solutions of the logistic equation with nonlinear diffusion, *J. Math. Anal. Appl.*, **268**, 200–216 (2002).

[13] M. Delgado, J. López-Gómez and A. Suárez, Characterizing the existence of large solutions for a class of sublinear problems with nonlinear diffusion, *Adv. Differential Eqns.*, **7**, 1235–1256 (2002).

[14] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, **122**, 519–543 (1994).