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Towards a new framework for domination[☆]

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A B S T R A C T

Dominating concepts constitute a cornerstone in Graph Theory. Part of the efforts in the field have been focused in finding different mathematical frameworks where domination notions naturally arise, providing new points of view about the matter. In this paper, we introduce one of these frameworks based in convexity. The main idea consists of defining a convexity in a graph, already used in image processing, for which the usual parameters of convexity are closely related to domination parameters. Moreover, the Helly number of this convexity may be viewed as a new domination parameter whose study would be of interest.

Keywords:
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Helly number
Domination chain

1. Introduction

Whenever a problem is modeled as a graph in which every vertex controls, supervises, corrects, supplies or assists its neighbors, the notion of domination is obviously present. Such ideas appear so naturally that domination theory has become, to no surprise, a cornerstone of Graph Theory. Currently, a myriad of variations has been developed in order to adapt domination to specific situations, and around 2400 papers on the topic have been published. An excellent and exhaustive study on the many aspects of domination can be found in two books of Haynes et al. [1,2], in the first of which the authors have suggested to study logical structures or *frameworks* where the concept of domination arises naturally. They pointed out that:

> "*Each framework provides a unifying theory and a generalized viewpoint that enables one to identify and define new parameters, see relationships among these parameters and develop insight into the computational problems involving these parameters.*"

They also introduced ten of these mathematical frameworks ranging from hypergraphs to integer programming and claimed that:

> "*...this is but the beginning, that is, other frameworks for the domination number undoubtedly exist, each of which provides a rich understanding of the concept of domination in graphs.*"

Since then, other frameworks have been encountered as the one based in stratified graphs [3]. In this paper, we introduced a new framework based on a certain convexity in graphs which has been previously studied for processing images.

The paper is organized as follows: in Section 2 we recall basic notations on domination parameters. In Section 3 basic notions on graph convexity are introduced along with the new framework, and we additionally show the relationship between convexity number and rank with domination parameters. Section 4 is devoted to the study of a new parameter that appears in our framework. Finally, we conclude by summarizing our results and showing the lines of future work.

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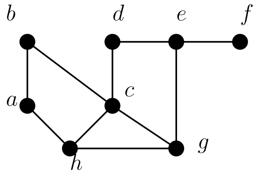

Fig. 1. A graph with $\gamma = 3$.

2. Domination parameters

We begin fixing some basic notations. All the graphs considered here are simple, undirected and connected. The *order* of a graph G is usually denoted as n, and the *minimum degree* of its vertices as $\delta(G)$. Given a vertex a of G, its *closed neighborhood* $N[a]$ consists of a and its adjacent vertices. This concept is extended naturally to a set of vertices A: the closed neighborhood of A, denoted $N[A]$, consists of the union of the closed neighborhoods of vertices in A. For undefined graph-theoretical terminology, we refer the reader to [4].

A subset of vertices A in a graph $G = (V, E)$ is called *dominating* [5,6] if $N[A] = V$, i.e. a vertex "dominates" all its neighborhood and itself. The *domination number* $\gamma(G)$ of the graph G is the cardinality of a smallest dominating set in G. In Fig. 1, the sets $A = \{a, c, e\}$ and $A' = \{a, d, f, h\}$ are dominating sets. Since there exists no dominating set with two vertices, A is a minimum dominating set of G and hence $\gamma(G) = 3$.

Clearly, domination is not hereditary. The set A will be *minimal dominating* if it is dominating and contains no proper dominating subset. Thus $\gamma(G)$ may be defined as the minimum cardinality of a minimal dominating set of vertices in G, whereas a new domination parameter $\Gamma(G)$, called the *upper domination number*, arises as the maximum cardinality of a minimal dominating set. Computing the domination number is a well-known NP-complete problem [7], however there are polynomial domination algorithms for worthwhile classes of graphs, for example see [8] for trees and [9] for other algorithmic aspects of domination.

The set $A' = \{a, d, f, h\}$ is minimal dominating in Fig. 1 and there is no minimal dominating set with five vertices, so $\Gamma(G) = 4$.

Other classical ideas closely related to domination are *independence* and *irredundance*. A subset A of vertices in the graph G is called *independent* if no two vertices in A are adjacent, or equivalently every vertex in A is *isolated*. In Fig. 1, the sets $\{b, d, f, g\}$ and $\{a, c, e\}$ are independent. Two new parameters are associated with this concept in a graph G: its *independent domination number* denoted as $i(G)$ which is the minimum cardinality of a maximal independent set, and the *independent number* $\beta_0(G)$ defined to be the maximum cardinality of an independent set.

Finally, a vertex subset A is called *irredundant* if for any $a \in A$, then $N[a] \setminus N[A \setminus \{a\}] \neq \emptyset$, that is, a has a neighbor b outside A and b is not adjacent to any other vertex in A. In this situation, b is said to be a *private neighbor* of a with respect to A (the set will be omitted when is clear from the context). For example, the set $\{a, g, h\}$ in Fig. 1 is not irredundant. The *irredundance number*, $ir(G)$ of a graph G is the minimum cardinality of a maximal irredundant set, and the *upper irredundance number* $IR(G)$ is the cardinality of a biggest irredundant set in G.

The relationship between these parameters is known as the *domination chain* which first appeared in [10] except for the first and last inequalities (respectively obtained in [5,11]), and has been thoroughly studied in the literature.

$$\frac{\gamma}{2} \leq ir \leq \gamma \leq i \leq \beta_0 \leq \Gamma \leq IR \leq n - \delta.$$

3. The new framework for domination

The notion of a convexity in a discrete metric space follows from the Euclidean convexity and it has been widely studied in the literature (see [12] for general reference). Although no convexity is completely conformal with the Euclidean one, it is possible to isolate some of its features and replicate them in a graph which have led to an enormous number of known graph convexities [13]. Some of them are of special importance for image processing where certain classical operations have been defined in terms of discrete convexities [14,15]. Recently, the field has received a new impetus from the study of convexity as a means of reconstructing a vertex subset in a graph [16,17] and the computational aspects of that reconstruction [18,19].

The new framework is constructed by means of a certain convexity in a graph. From an abstract point of view [12], a *convex hull operator* over a graph $G = (V, E)$ is a map $CH: 2^V \to 2^V$ which satisfies the following four conditions over any two vertex subsets A and B:

1. $CH(\emptyset) = \emptyset$,
2. $A \subseteq CH(A)$,
3. $A \subseteq B \Rightarrow CH(A) \subseteq CH(B)$,
4. $CH(CH(A)) = CH(A)$.

The set A is said to be *convex* if $A = CH(A)$, it will be *independent convex* if there does not exist $a \in A$ such that $CH(A) = CH(A \setminus \{a\})$, and it is *Helly independent* if $\bigcap_{a \in A} CH(A \setminus \{a\}) = \emptyset$.

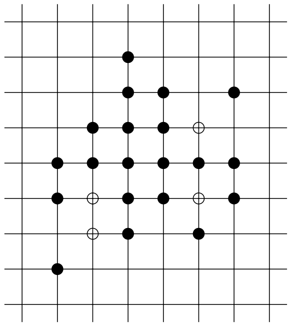

Fig. 2. The d-convex hull $CH_d(A)$ of the set A of black vertices.

The general behavior and properties of a certain convexity are commonly described with a number of associated parameters. Among others, one can find in the literature the *convexity number* as the cardinality of the biggest proper convex set, the *rank* as the maximum cardinality of an independent convex set, and the *Helly number* as the minimum cardinality of a maximal Helly independent set.

Although several notions of convexity can be defined over a graph [13], we are interested in one that comes from the field of image processing [15], mainly used to filter digital images [14]. Given a graph G, the *digital convex hull* operator, or *d-convex hull* $CH_d(A)$ of a set of vertices A, is defined as $CH_d(A) = \{a \in V : N[a] \subseteq N[A]\}$ (see Fig. 2). This operator defines a convexity in the graph, or *digital convexity*, whose convex sets are called *d-convex*. Following the same convention, we call *d-independent convex* and *Helly d-independent* sets to the corresponding sets in the digital convexity.

At this point, we show how two parameters of the digital convexity in a graph are related with the domination chain. For the sake of simplicity, we will use the same notation for the parameter of the digital convexity as its abstract counterpart, since its meaning will be clear from the context. The next result gives the value of the convexity number, denoted by $con(G)$, depending on the order and minimum degree of the graph.

Theorem 1. *Let $G = (V, E)$ be a graph. Then:*

1. *Given $a \in V$, the set $V \setminus N[a]$ is d-convex.*
2. *$con(G) \geq n - k - 1$ if and only if $\delta(G) \leq k$.*
3. *$con(G) = n - \delta(G) - 1$.*

Proof. 1. On the contrary suppose that $CH(V \setminus N[a]) \not\subseteq V \setminus N[a]$, in other words there exists $b \in CH(V \setminus N[a])$ such that $b \in N[a]$. If $a = b$, the vertex a would have a neighbor in $V \setminus N[a]$ which is not possible. If $a \neq b$ and $b \in CH(V \setminus N[a])$, this means that any neighbor of b has a neighbor in $V \setminus N[a]$ and in particular, a would have a neighbor in $V \setminus N[a]$, which leads to a contradiction.

2. First, suppose $\delta(G) \leq k$. Then there exists a vertex a for which the cardinality of $N[a]$ is $\delta(G) + 1 \leq k + 1$. By 1, $V \setminus N[a]$ is d-convex and $con(G) \geq |V \setminus N[a]| \geq n - (k + 1)$.
Conversely, suppose that $con(G) = n - (r + 1) \geq n - (k + 1)$ and that C is the biggest proper d-convex set in G. Thus $C = V \setminus \{b_0, \dots, b_r\}$ where $b_0, \dots, b_r \in V$. If for all b_i with $0 \leq i \leq r$ has a neighbor in C, then $\{b_0, \dots, b_r\} \subseteq N[C]$ so $CH(C) = V$ which leads to a contradiction. Thus, there exists a certain b_j with no neighbors in C, and therefore $N[b_j] \subseteq \{b_0, \dots, b_r\}$. Thus $\delta(G) \leq \delta(b_j) \leq r \leq k$.
3. By 2, $con(G) \geq n - \delta(G) - 1$. Suppose that $con(G) > n - \delta(G) - 1$. Then $con(G) \geq n - (\delta(G) - 1) - 1$, however again by 2 $\delta(G) \leq \delta(G) - 1$, which is an obvious contradiction. Therefore $con(G) = n - \delta(G) - 1$. □

Hence, the digital convexity number plus one turns out to be the natural upper bound for all the domination parameters, and we can place it at the end of the chain:

$$\frac{\gamma}{2} \leq ir \leq \gamma \leq i \leq \beta_0 \leq \Gamma \leq IR \leq n - \delta = con(G) + 1.$$

Other parameters of interest are the cardinality of the smallest and biggest maximal independent d-convex set in G which are respectively its *rank* $r(G)$ and *upper rank* $R(G)$.

Theorem 2. *Given a graph $G = (V, E)$, a set of vertices $A \subseteq V$ is d-independent convex if and only if A is irredundant.*

Proof. Let A be a d-dependent convex set, that is, there is an $a \in A$ such that $CH(A) = CH(A \setminus \{a\})$ which in terms of the digital convexity is equivalent to $N[a] \subseteq N[A \setminus \{a\}]$, i.e. a has no private neighbor, and thus A is not irredundant. □

Consequently, $ir(G) = r(G)$ and $IR(G) = R(G)$, that is irredundance parameters agree with rank parameters in the digital convexity. These are two new pieces of our framework.

4. A dual relationship between domination and digital convexity

In the previous section, it has been established that the convexity number, the rank and the upper rank of the digital convexity in a graph have their place in the domination chain. In this section, we study the relationship of Helly d-independent sets with domination. The minimum and maximum cardinality of a maximal Helly independent set in any convexity are called its *Helly number* $hi(G)$, and its *upper Helly number* $HI(G)$. For the sake of simplicity, we will use the same notation for these parameters when dealing with the digital convexity.

It is remarkable that there is a dual relationship between dominating sets and Helly d-independent sets, which leads to the equality $HI(G) = \Gamma(G)$. However, no such relationship holds for γ and hi.

The next result is a local characterization of Helly d-independent sets, and it will result in an important tool for the rest of the section.

Theorem 3. *The vertex subset A of a graph $G = (V, E)$ is Helly d-independent if and only if the following conditions hold:*

1. *Any vertex $a \in A$ satisfies at least one of the following properties:*
   - ***P1**. a is isolated in A.*
   - ***P2**. a has a private neighbor b with respect to A.*
2. *Each $b \in N[A] \setminus A$, verifies at least one of the following properties:*
   - ***Q1**. b has an isolated neighbor in A;*
   - ***Q2**. b is the private neighbor of a vertex in A;*
   - *b has at least two neighbors in A and satisfies at least one of the following conditions:*
     - ***Q3**. b is adjacent to b' which is the private neighbor of a vertex in A;*
     - ***Q4**. b has a neighbor c such that $d(c, A) = 2$.*

Proof. Let $A \subseteq V$ satisfying the conditions above. We will show that $\bigcap_{a\in A} CH(A \setminus \{a\}) = \emptyset$:

On the contrary, suppose that $\bigcap_{a\in A} CH(A \setminus \{a\}) \neq \emptyset$, thus there exists $b \in CH(A \setminus \{a\})$, for any $a \in A$, or, equivalently, $N[b] \subseteq N[A \setminus \{a\}]$, for any $a \in A$. In the following, we show that b does not verify the conditions of the hypothesis.

If $b \in A$, then $N[b] \subseteq N[A \setminus \{b\}]$, but this is impossible since b should satisfy P1 or P2. If $b \notin A$ then $d(b, A) = 1$, so b has to verify either Q1, Q2, Q3 or Q4. Because of $N[b] \subseteq N[A \setminus \{a\}]$, $\forall a \in A$, the vertex b has at least two neighbors in A, and thus it will not satisfy Q2.

If b has an isolated neighbor $a' \in A$, then $a' \in N[b] \subseteq N[A \setminus \{a'\}]$ which contradicts our hypothesis so b does not satisfy Q1. By a similar reason, if b verifies Q3, then there will be a vertex $b' \in N[b]$ which is the private neighbor of c in A. Then $b' \notin N[A \setminus \{c\}]$, which contradicts the assumption that $b' \in N[b] \subseteq N[A \setminus \{c\}]$.

Finally, since $N[b] \subseteq N[A \setminus \{a\}]$, $\forall a \in A$, each neighbor of b is in $N[A] \setminus A$, and b will not satisfy Q4 neither.

Therefore $\bigcap_{a\in A} CH(A \setminus \{a\}) = \emptyset$ and A is Helly d-independent.

Conversely, let $A \subseteq V$ be Helly d-independent, then it is shown that at least one of the conditions of the theorem holds true. On the contrary, suppose a vertex $a \in A$ which does not verify P1 nor P2. Therefore $N[a] \subseteq N[A \setminus \{a\}]$. Hence $a \in CH(A \setminus \{a\})$ and $a \in \bigcap_{x\in A} CH(A \setminus \{x\})$ so A is not Helly d-independent.

Let b be a vertex at distance one from A and let $A_b = A \cap N(b)$. If $|A_b| = 1$ then b satisfies Q2. So from now on, assume that $|A_b| \geq 2$. Moreover, if some vertex in A_b is isolated, then b will satisfy Q1, so we can suppose also that every vertex in A_b is not isolated.

Since A is Helly d-independent, there exists $a \in A$ such that $N[b] \nsubseteq N[A \setminus \{a\}]$, i.e., there is a certain c in $N[b] \setminus N[A \setminus \{a\}]$. If $a \in A_b$ and $c = b$, then $|A_b| = 1$ which is not possible.

If $a \in A_b$ and $c \neq b$, then it has a private neighbor outside A, since a is not isolated. If such a neighbor is c, then it will verify Q3; otherwise, c cannot be a neighbor of another vertex in A, hence $d(c, A) = 2$ and b satisfies Q4. □

With this characterization, the vertices of a Helly d-independent set will be classified into P1 and P2 vertices in the rest of the section. Similarly, we will denote the vertices at distance one as Q1, Q2, Q3 or Q4 vertices. Note that this classification is not a partition of vertices since a vertex may fulfill more that one property.

The next two results provide a first approximation to the role that Helly d-independent sets play in domination.

Proposition 4. *Let A be a Helly d-independent subset of vertices in the graph $G = (V, E)$. If A is dominating then it is maximal Helly d-independent.*

Proof. Let $A \subseteq V$ be Helly d-independent and dominating.

Suppose that A is not a maximal Helly d-independent set, then there exits $b \in V \setminus A$ such that $A' = A \cup \{b\}$ is Helly d-independent, but since A is dominating, there exists $a \in A$ such that $b \in N[a]$. Therefore b is not a P1 vertex.

Also b is not P2, otherwise there exists $c \in N[b] \setminus A'$ with $N[c] \cap A' = \{b\}$, which means that c is not dominated by A contradicting our hypothesis.

Hence, A is maximal Helly d-independent. □

Note that if A is a Helly d-independent and dominating set then any vertex is P1 or P2, so by Ore's Theorem (see [20]) A will be a minimal dominating set.

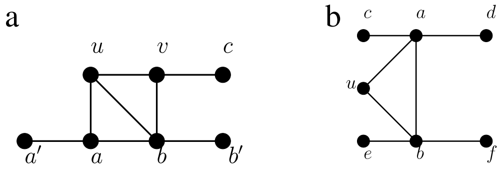

Fig. 3. Two examples where $hi(G) = 2$, $ir(G) = \gamma(G) = 3$; and $hi(G) = 3$, $ir(G) = \gamma(G) = 2$.

Proposition 5. *In a graph $G = (V, E)$, every maximal independent vertex set is maximal Helly d-independent. Thus $hi(G) \le i(G)$.*

Proof. Let A be a maximal independent vertex set, thus every a in A is a P1 vertex. Since A is also minimal dominating (see for example [21]), any vertex not in A is a Q1 vertex. Thus A is Helly d-independent and, by Proposition 4, it is a maximal Helly d-independent set. □

However, it is not possible to establish an equivalent lower bound for hi. Although every Helly d-independent set is irredundant, this relation is false for maximal sets. In the graph (a) of Fig. 3, $\{a, b\}$ is a maximal Helly d-independent set (indeed $hi(G) = 2$) and irredundant but not a maximal irredundant set, because $\{a, b, c\}$ is also irredundant. Furthermore, the set $\{a, b, c\}$ is minimal dominating and $\gamma(G) = ir(G) = 3$ because there is no dominating set with two vertices.

On the graph (b), $\{a, b\}$ is a minimal dominating set and maximal irredundant, so $\gamma(G) = ir(G) = 2$. Nevertheless, it is not a Helly d-independent set. Moreover, $hi(G) = 3$ because $\{b, c, d\}$ is a maximal Helly d-independent set and there is no Helly d-independent set with cardinality 2. Thus, both examples show that hi is independent of γ and ir.

Fortunately, as we mentioned above, it is possible to find a dual relation between minimal dominating sets and maximal Helly d-independent sets which leads us to obtain an equality for the upper parameters HI and Γ. In order to describe this relationship, we need the following technical result.

Lemma 6. *Let $G = (V, E)$ be a graph and let $A = S \cup T$ be a maximal Helly d-independent set of vertices, with $S = \{s_1, \ldots, s_n\}$ the set of isolated vertices in A with no private neighbors (*P1 *vertices which are not* P2*) and $T = \{t_1, \ldots, t_m\}$, the set of vertices in A with a private neighbor (P2 vertices). Then the following conditions hold:*

1. *Every $b \in A_1 = N[A] \setminus A$ satisfies property* Q1 *or* Q2 *or* Q3 *(no vertex is only a* Q4 *vertex).*
2. *$d(b, A) \le 2$, for every vertex b.*

Proof. 1. Suppose, on the contrary, that there exists $b \in A_1$ which only satisfies property Q4, so it has a neighbor c such that $d(c, A) = 2$. Then consider $A' = A \cup \{b\}$. It is clear that all $s_i \in S$ are (P1)' vertices (that is, P1 in A'). Also all $t_j \in T$ are (P2)' because b is not Q3. Vertex b is (P2)' having c as its private neighbor. Now, for all $b' \in A'_1 = N[A'] \setminus A'$, there are the following cases:

- if $d(b', b) = 1$ and $d(b', A) = 2$, then b' is a private neighbor of b and so b' is (Q2)',
- if $b' \in A_1$ is Q1, then it is also (Q1)',
- if $b' \in A_1$ is Q2, then b' is not a neighbor of b, because b is not Q3, so b' is (Q2)',
- if $b' \in A_1$ is Q3, then b' is (Q3)',
- if $b' \in A_1$ is Q4, let d be the neighbor of b' such that $d(d, A) = 2$. If d and b are neighbors, then b' is (Q3)' and b' is (Q4)' in the other case.

So A' is a Helly d-independent set greater than A, which is not possible by hypothesis.

2. Suppose, on the contrary that there exists $b \in V$ such that $d(b, A) = 3$ and then consider $A' = A \cup \{b\}$. It is clear that all $s_i \in S$ and b are (P1)' vertices and additionally all $t_j \in T$ are (P2)'. Suppose $b' \in A'_1$. Then if $b' \in A_1$ and it is Q1, Q2 or Q3 then it is respectively (Q1)', (Q2)' or (Q3)'. On the other hand, if $d(b', b) = 1$, it is clear that b' is (Q1)'. So A' is a Helly d-independent set greater than A, which is not possible by hypothesis. □

The following two results show how to obtain a minimal dominating set from a maximal Helly d-independent set and vice versa. Since both processes are the "inverse" of the other, there is a certain duality between these two types of sets.

Proposition 7. *Let $G = (V, E)$ be a graph and let $A = S \cup T$ be a maximal Helly d-independent set in it, where $S = \{s_1, \ldots, s_n\}$ is the set of isolated vertices in A with no private neighbors, and $T = \{t_1, \ldots, t_m\}$ is the set of vertices in A with private neighbors. We denote by T_j the set of such private neighbors of each t_j. Consider $D' \subseteq \bigcup_{j=1}^{m} T_j$ a minimal set of vertices which dominates T and let D be the union of D' with all* Q2 *and* Q3 *vertices of the graph. Then $S \cup D$ is a minimal dominating set of G.*

Proof. By construction, $S \cup D$ dominates $S \cup T = A$ and also vertices which are Q1, Q2 and Q3, so by Lemma 6.1, it dominates A_1. Note that D must contain at least one vertex in each T_j in order to dominate T.

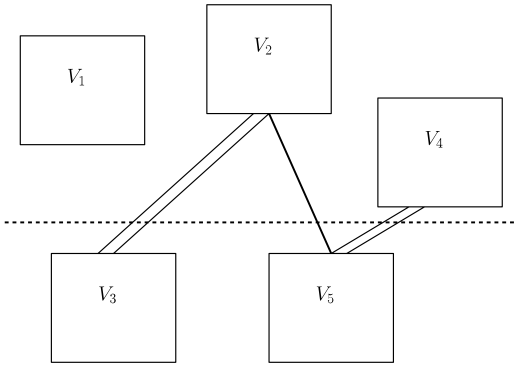

Fig. 4. Given $2 < s < t$, the subsets V_1, V_2 and V_3 have s vertices and V_4 and V_5 contain t vertices. A double line between two sets means that every vertex in one set is adjacent with exactly one vertex in the other set. The single line stands for any vertex in one set is connected with any vertex in the other set. The subgraphs induced by the vertices on either side of the dotted line are complete (see [22]).

Let us see that D dominates the set of vertices $A_2 = \{c \in V(G): d(c, A) = 2\}$. Suppose, on the contrary that there exists $c \in A_2$ such that $N[c] \cap D = \emptyset$, then define $A' = A \cup \{c\}$. It is clear that every $s_i \in S$ and c are (P1)' vertices. Also every $t_j \in T$ is a (P2)' vertex, since the condition $N[c] \cap D = \emptyset$ implies that c is not a neighbor of, at least, one vertex in each T_j. Consider now $b \in A'_1$, then there exist the following cases:

- if $b \in A_1$ is Q1, then it is also (Q1)',
- if $b \in A_1$ is Q2 and $d(b, c) \geq 2$ then it is (Q2)', and if $d(b, c) = 1$ then it is (Q1)',
- if $b \in A_1$ is Q3 and $d(b, c) = 1$, then it is (Q1)'. If $d(b, c) \geq 2$ then, using that D dominates b but it does not dominate c, b has a neighbor in D, so in some T_j, which is not a neighbor if c, so b is (Q3)',
- if $b \in A_2$ with $d(b, c) = 1$, then b is (Q1)'.

So A' is a Helly d-independent set greater than A, which is not possible by hypothesis. Thus $S \cup D$ dominates $A \cup A_1 \cup A_2 = V$, and by construction, it is minimal. □

Proposition 8. *In a graph $G = (V, E)$, let $D = S \cup T$ be a minimal dominating set of V, where $S = \{s_1, \ldots, s_n\}$ is the set of isolated vertices in D with no private neighbors, and $T = \{t_1, \ldots, t_m\}$ is the vertex subset of D having a private neighbor. Let $A = \{a_1, \ldots, a_m\}$ be such that each a_j is a private neighbor of t_j. Then $S \cup A$ is a Helly d-independent set in G.*

Proof. It is clear that all vertices in S are P1 in $S \cup A$ as well as each $a_j \in A$ is P2, where t_j is its private neighbor. It only remains to prove that each $b \in N[S \cup A] \setminus (S \cup A)$ verifies the conditions of Theorem 3.

In addition, if $d(b, S) = 1$ then it is Q1, and if $b = t_j \in T$ then it is the private neighbor of a_j, so it is Q2. Suppose that $b \notin T$ and it is not Q1 or Q2. Since D is dominating, there exists $d \in D$ such that it is a neighbor of b. That vertex d is not in S, otherwise b will be Q1. Thus $d \in T$ and it is a private neighbor of a vertex in A, which implies that b is Q3. □

Finally, our dual relationship provides the desired equality.

Theorem 9. *Let $G = (V, E)$ be a graph. Then $HI = \Gamma$.*

Proof. Let $A = S \cup T$ be a maximum Helly d-independent set of V as in Proposition 7. Then there exists a minimal dominating set $S \cup D$ such that D contains at least one vertex in each T_j, so $|T| \leq |D|$ and we obtain

$$HI = |A| = |S \cup T| \leq |S \cup D| \leq \Gamma.$$

On the other hand, let $D = S \cup T$ be a minimal dominating set of V as in Proposition 8 with maximum size $\Gamma = |D|$. Then there exists a Helly d-independent set $S \cup A$ such that $|T| = |A|$, so we obtain

$$\Gamma = |D| = |S \cup T| = |S \cup A| \leq HI. \quad \square$$

Hence, the two parameters always agree although the maximum sets may be very different. The previous results have been inspired by the following graph which was first studied in [22] as an example that the parameter Γ may be different from β_0 and IR. Let $2 < s < t$ and G be the graph whose vertex set consists of the union of three set V_1, V_2 and V_3 with s vertices, and V_4 and V_5 with t vertices. These sets are represented as rectangles in Fig. 4, where two rectangles are joined with a double line if every vertex in one set is one-to-one adjacent with the vertices in the other set, and with a single line when every vertex in one set is joined with any vertex in the other rectangle. Finally, the subgraph induced by $V_1 \cup V_2 \cup V_4$ is complete as well as the subgraph induced by $V_3 \cup V_5$. The unique minimal dominating set with maximum cardinality is V_2, and the maximum Helly d-independent set is V_3. Note that both sets are disjoint and their vertices are neighbors in a one-to-one correspondence.

5. Conclusions and future work

In this work, we have introduced a new way to build a domination theory based in digital convexity. We have shown that convexity number, rank and upper rank in this convexity can be related with other parameters of domination, in the following way:

$$\frac{\gamma}{2} \leq ir = r \leq \gamma \leq i \leq \beta_0 \leq \Gamma = HI \leq IR = R \leq n - \delta = conv + 1.$$

It has been also established that $hi \leq i$ and it is independent of ir and γ. Moreover $HI = \Gamma$, although the sets where those parameters are attained may be different or even disjoint. However there exists a certain duality between both types of sets.

Those are not the only parameters of a convexity that can be found in the literature. Others, such as Caratheodory and exchange number have not been studied yet from this point of view, and they could be subjects of a future study line.

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