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On the intersection of the classes of doubly diagonally dominant matrices and S-strictly diagonally dominant matrices

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Abstract

We denote by H_0 the subclass of H-matrices consisting of all the matrices that lay simultaneously on the classes of doubly diagonally dominant (DDD) matrices ($A = [a_{ij}] \in \mathbb{C}^{n\times n} : |a_{ii}||a_{jj}| \geq \sum_{k\neq i} |a_{ik}| \sum_{k\neq j} |a_{jk}|, i \neq j$) and S-strictly diagonally dominant (S-SDD) matrices. Notice that strictly doubly diagonally dominant matrices (also called Ostrowsky matrices) are a subclass of H_0. Strictly diagonally dominant matrices (SDD) are also a subclass of H_0. In this paper we analyze some properties of the class $H_0 = \text{DDD} \cap S\text{-SDD}$.

1 Introduction

In this paper we analyze some properties of the matrices that lay simultaneously on the classes of doubly diagonally dominant (DDD) matrices, see [11], and S-strictly diagonally dominant (S-SDD) matrices; see [4], [15]. This class, that we denote here by $H_0 = \text{DDD} \cap S\text{-SDD}$ is a subclass of H-matrices. In several practical applications H-matrices play a key role; e.g., in the numerical solution of Euler equations in fluid dynamics [7], in nonlinear boundary problems and in the Lyapounov stability analysis for large scale evolution systems (see [14] and the references therein, for more details). H-matrices were defined by Ostrowsky in [13] as a generalization of M-Matrices. H-matrices and M-matrices are called this way in homage to Hadamard and Minkowsky, respectively [15].

We recall that a nonsingular matrix A having all non-positive off-diagonal entries is called an M-matrix if the inverse is (entry-wise) nonnegative, i.e., $A^{-1} \geq O$; see, e.g.,

[1] for more characterizations. For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, its *comparison* matrix $\langle A\rangle = (\alpha_{ij})$ can be defined by

$$\alpha_{ii} = |a_{ii}|, \quad \alpha_{ij} = -|a_{ij}|, \quad i \neq j.$$

A matrix A is said to be an H-matrix if $\langle A\rangle$ is a nonsingular M-matrix. In particular, A is a nonsingular H-matrix if and only if it is (strictly) generalized (row) diagonally dominant, i.e.,

$$|a_{ii}|w_i > \sum_{i\neq j} |a_{ij}|w_j, \quad i = 1, \ldots, n, \tag{1}$$

for some positive vector $w = (w_1, \ldots, w_n)^T$. This is equivalent to say that A is an H-matrix if and only if there exists a positive diagonal matrix $W = diag(w_1, w_2, \ldots, w_n)$ such that AW is an strictly (row) diagonally dominant (SDD) matrix. Some useful characterizations of H-matrices (see, for example, [10], [8], [14], [9], [5]) are based on devising adequate scaling matrices W. A different strategy to the problem of finding classes of H-matrices resides in describing subclasses of H-matrices which are easily characterizable. Following this approach some new subclasses of H-matrices were introduced in [4]. In this paper we focus on the subclass of H_0-matrices. It is also interesting to note that SDD matrices are the simplest case for this class; these ideas are depicted in Figure 1 below.

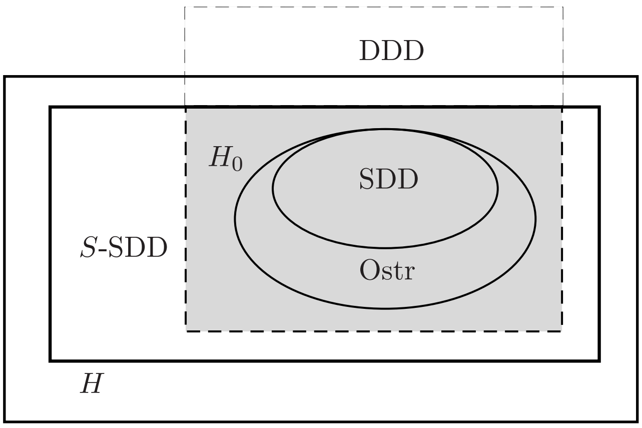

Figure 1: DDD matrices and some subclasses of H-matrices

2 S-SDD matrices

We begin with some definitions which can be found, e.g., in [2], [4], [6], [15].

Definition 1 *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, let us define the ith deleted absolute row sum as*

$$r_i(A) = \sum_{j\neq i, j=1}^{n} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n,$$

and the ith deleted absolute row-sum with columns in the set of indices $S = \{i_1, i_2, \ldots\} \subseteq N := \{1, 2, \ldots n\}$ *as*

$$r_i^S(A) = \sum_{j \neq i, j \in S} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n.$$

Given any nonempty set of indices $S \subseteq N$ we denote its complement in N by $\bar{S} := N \backslash S$. Note that for any $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ we have that $r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A)$.

Definition 2 *Given a matrix* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$*,* $n \geq 2$ *and given a nonempty subset* S *of* $\{1, 2, \ldots, n\}$*, then* A *is an* S*-strictly diagonally dominant matrix if the following two conditions hold*

$$\left.\begin{array}{lll} i) & |a_{ii}| > r_i^S(A) & \forall i \in S, \\ ii) & (|a_{ii}| - r_i^S(A))\,(|a_{jj}| - r_j^{\bar{S}}(A)) > r_i^{\bar{S}}(A)\, r_j^S(A) & \forall i \in S, \forall j \in \bar{S}. \end{array}\right\} \quad (2)$$

It was shown in [6] that an S-strictly diagonally dominant matrix (S-SDD) is a nonsingular H-matrix. In particular, when $S = \{1, 2, \ldots n\}$, then $A = (a_{ij}) \in C^{n \times n}$ is a strictly diagonally dominant matrix (SDD). It is easy to show that an SDD matrix is an S-SDD matrix for any proper subset S, but the converse is not always true [3].

Notice that condition 1) of definition 2 implies that the diagonal of any S-SDD matrix is nonzero. We also note that condition 1) can be substituted for $|a_{ii}| > r_i^S(A)$, for some $i \in S$, since the condition 2) ensures that 1) will be satisfied for all $i \in S$; see [4].

The class of S-SDD can be expressed equivalently in the following way. For arbitrary nonempty proper set of indices S let us define the interval $J_A(S)$ as

$$J_A(S) := (\mu_1^S(A), \mu_2^S(A)), \quad (3)$$

where

$$\mu_1^S(A) := \max_{i \in S} \frac{r_i^{\overline{S}}(A)}{|a_{ii}| - r_i^S(A)} \quad \text{and} \quad \mu_2^S(A) := \min_{j \in \overline{S}, r_j^S(A) \neq 0} \frac{|a_{jj}| - r_j^{\overline{S}}(A)}{r_j^S(A)}. \quad (4)$$

By convention, when $S = \emptyset$ or $S = N$ we define $J_A(S) = (0, +\infty)$. Furthermore, when $r_j^S(A) = 0$, $\forall\, j \in \overline{S}$ then we take $\mu_2^S(A) = +\infty$.

The next lemma, which is proved in [2], shows another characterization of S-SDD matrices. Here we denote by $A[S]$ the principal submatrix of A with indices from the set S.

Lemma 1 *Given* $S \in N$*, let* $A[S]$ *and* $A[\overline{S}]$ *be strictly diagonally dominant matrices. Then* $A \in C^{n \times n}$ *is an S-SDD matrix if and only if the interval* $J_A(S)$ *given by (3) is nonempty.*

3 Doubly diagonally dominant matrices

The class of DDD matrices, see [11], is defined as follows.

$$\{\, A = [a_{ij}] \in \mathbb{C}^{n \times n} : \; |a_{ii}||a_{jj}| \geq r_i(A)\, r_j(A), \quad i \neq j\} \quad (5)$$

Example 1 *The matrices* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ *are DDD matrices.*

Example 2 *The matrix*

$$\begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

is DDD but it is not into the class H_0*, i.e., is not an* S*-SDD matrix for any* S*. But it is a nonsingular* H *matrix.*

We remark that we are interested in DDD matrices with at least one equality in (5). Otherwise, we would have SDDD (Ostrowsky) matrices or simply SDD matrices, which are known classes.

4 H_0-matrices

In order to study the class $H_0 = \text{DDD} \cap S\text{-SDD}$ we can adopt three points of view: a) we can stay in the DDD class and look for conditions to be in the class S-SDD, b) we can stay in the class S-SDD and look for conditions to be in the DDD class and c) we can impose all the conditions to be in the class $\text{DDD} \cap S\text{-SDD}$ and try to simplify the derived relations.

In this communication we explore the options a) and c).

Before giving sufficient conditions for a DDD matrix to be an S-SDD matrix we establish the following result.

Lemma 2 *Let* $A \in \mathbb{C}^{n\times n}$ *and* $S \subseteq N := \{1, 2, \dots, n\}$*. If*

1) $A[S]$ *and* $A[\overline{S}]$ *are SDD matrices*

2) $r_i^S(A)\, r_j(A) > r_j^{\overline{S}}(A)\, |a_{ii}|, \quad \forall i \in S, \forall j \in \overline{S}$

3) $r_i(A)\, r_j^{\overline{S}}(A) > r_i^S(A)\, |a_{jj}|, \quad \forall i \in S, \forall j \in \overline{S}$

then A *is an* S*-SDD matrix.*

Proof We first note that 1) implies: $|a_{ii}| > r_i^S(A), \forall i \in S$ and $|a_{jj}| > r_j^{\overline{S}}(A), \forall j \in \overline{S}$. According to Lemma 1, we only have to show that the interval $J_A(S)$ given by equation (3) is nonempty. Note that condition 2) can be written as

$$r_i^S(A)\, r_j^S(A) > r_j^{\overline{S}}(A) \left[|a_{ii}| - r_i^S(A)\right], \quad \forall i \in S, \forall j \in \overline{S} \tag{6}$$

and since $A[S]$ is SDD, equation (6) implies that $r_i^S(A)\, r_j^S(A) > 0, \quad \forall i \in S, \forall j \in \overline{S}$.

Now, from (6) and the definition of $\mu_1^S(A)$, see equation (4), we conclude that

$$\mu_1^S(A) > \frac{r_i^{\overline{S}}(A)\, r_j^{\overline{S}}(A)}{r_i^S(A)\, r_j^S(A)}, \quad \forall i \in S, \forall j \in \overline{S}$$

In a similar way, condition 3) yields to

$$r_i^{\overline{S}}(A)\, r_j^{\overline{S}}(A) > r_i^S(A) \left[|a_{jj}| - r_j^{\overline{S}}(A)\right], \quad \forall i \in S, \forall j \in \overline{S} \tag{7}$$

and this equation jointly with the definition of $\mu_2^S(A)$, equation (4), leads to

$$\mu_2^S(A) < \frac{r_i^{\overline{S}}(A)\, r_j^{\overline{S}}(A)}{r_i^S(A)\, r_j^S(A)}, \quad \forall i \in S, \forall j \in \overline{S}$$

and the proof follows.

In the following result we show that when A is a DDD matrix then we can replace the condition 1) of Lemma 2 by the simple condition $|a_{ii}| > r_i^S(A)$ for some $i \in S$.

Proposition 1 *Let $A \in \mathbb{C}^{n\times n}$ be a DDD matrix. Let $S \subseteq N := \{1, 2, \ldots, n\}$. If*

1) $|a_{ii}| > r_i^S(A)$ *for some* $i \in S$

2) $r_i^S(A)\, r_j(A) > r_j^{\overline{S}}(A)\, |a_{ii}|, \quad \forall i \in S, \forall j \in \overline{S}$

3) $r_i(A)\, r_j^{\overline{S}}(A) > r_i^S(A)\, |a_{jj}|, \quad \forall i \in S, \forall j \in \overline{S}$

then A is an S-SDD matrix.

Proof Since A is a DDD matrix we have that

$$|a_{ii}||a_{jj}| \geq r_i(A)\, r_j(A), \quad i \neq j$$

Note that

$$\begin{aligned}
|a_{ii}||a_{jj}| &\geq r_i(A)\, r_j(A) \\
&= [r_i^S(A) + r_i^{\bar{S}}(A)]\, [r_j^S(A) + r_j^{\bar{S}}(A)] \\
&= r_i^S(A) r_j^S(A) + r_i^S(A) r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A) r_j^S(A) + r_i^{\bar{S}}(A) r_j^{\bar{S}}(A) \\
&= r_i^S(A) r_j(A) + r_i^S(A) r_j^{\bar{S}}(A) - r_i^S(A) r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A) r_j^S(A) + r_i^{\bar{S}}(A) r_j^{\bar{S}}(A) \\
&= r_i^S(A) r_j(A) + r_i(A) r_j^{\bar{S}}(A) - r_i^S(A) r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A) r_j^S(A)
\end{aligned} \tag{8}$$

and using conditions 2) and 3) we conclude

$$|a_{ii}||a_{jj}| > r_j^{\bar{S}}(A)|a_{ii}| + r_i^S(A)|a_{jj}| - r_i^S(A) r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A) r_j^S(A)$$

from which we obtain

$$(|a_{ii}| - r_i^S(A))\, (|a_{jj}| - r_j^{\bar{S}}(A)) > r_i^{\bar{S}}(A) r_j^S(A)$$

and this holds $\forall i \in S$, $\forall j \in \overline{S}$. In conclusion, we have that A is an S-SDD matrix.

In the next section we will show some properties of the matrices that lay on the class H_0.

4.1 Set of pairs of indices

In this section we consider $N := \{1, 2, \dots, n\}$, such that $n \geq 2$. Let us define the set $N_2 = \{(i, j) \, : \, i, j \in N, i \neq j\}$. Obviously, $card(N_2) = \frac{n(n-1)}{2}$.

Definition 3 *Let $A \in \mathbb{C}^{n \times n}$ be a DDD matrix such that $n \geq 2$. We define the set of pairs of indices*

$$E(A) = \{ \, (i, j) \in N_2 : |a_{ii}||a_{jj}| = r_i(A) \, r_j(A) \, \} \, .$$

We denote its complement by $\overline{E(A)} = N_2 \backslash E(A)$.

Example 3 *Given the following DDD matrix*

$$A = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0 & 1 & 2 \end{bmatrix}$$

we have $N_2 = \{(1, 2), (1, 3), (2, 3)\}$ and $E(A) = \{(1, 2)\}$.

Definition 4 *We define the class of matrices $H_0(S)$ which is formed by square matrices A of order n such that they are simultaneously DDD matrices and S-SDD matrices for some proper subset $S \subseteq N$.*

Example 4 *The matrix given by example 3 is DDD and $\{1, 2\}$-SDD, therefore it belongs to the class $H_0(\{1, 2\})$.*

Lemma 3 *Let $A \in \mathbb{C}^{n \times n}$ such that $A \in H_0(S)$ for some proper subset S and such that there exists $i \in N : r_i(A) = 0$. Then $(i, j) \in \overline{E(A)}$, $\forall j \in N$.*

Proof Let us suppose that there exists $j \in N \, : \, (i, j) \in E(A)$. Therefore $|a_{ii}||a_{jj}| = r_i(A) \, r_j(A) \, = 0$ which implies $a_{ii} = 0$ or $a_{jj} = 0$. But this is a contradiction because A is a nonsingular H-matrix.

Remark 1 *Note that this lemma still holds when A is a DDD matrix whose diagonal entries are nonzero.*

Lemma 4 *Let $A \in \mathbb{C}^{n \times n}$ such that $A \in H_0(S)$ for some proper subset S and such that there exists $i \in S : |a_{ii}| = r_i(A)$. Then $(i, j) \in \overline{E(A)}$, $\forall j \in \overline{S}$.*

Proof Let us suppose that there exists $j \in \overline{S} \, : \, (i, j) \in E(A)$. Therefore $|a_{ii}||a_{jj}| = r_i(A) \, r_j(A)$ and using the hypothesis $|a_{ii}| = r_i(A)$ we conclude that $|a_{jj}| = r_j(A)$. Therefore we have

$$(|a_{ii}| - r_i^S(A)) \, (|a_{jj}| - r_j^{\bar{S}}(A)) = r_i^{\bar{S}}(A) \, r_j^S(A), \quad \text{with} \quad i \in S, j \in \overline{S}$$

and the condition *ii*) of the definition of S-SDD matrices is not satisfied. Therefore A does not belong to $H_0(S)$, which is a contradiction.

The counterpart of the previous lemma is the following.

Lemma 5 *Let $A \in \mathbb{C}^{n\times n}$ such that $A \in H_0(S)$ for some proper subset S and such that there exists $i \in \overline{S} : |a_{ii}| = r_i(A)$. Then $(i,j) \in \overline{E(A)}$, $\forall j \in S$.*

As a consequence of the two previous results we have the following.

Proposition 2 *Let $A \in \mathbb{C}^{n\times n}$ such that $A \in H_0(S)$ and let T be the set of indices*

$$T = \{i \in N : |a_{ii}| = r_i(A)\}.$$

Then $T \subseteq S$ or $T \subseteq \overline{S}$.

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