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The numerical analysis of higher-order nonlinear FE method for advection dominated problems

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Resumen

The numerical analysis of nonlinear discontinuity-capturing methods applied to advection dominated problems has not been completely established yet. Some particular results give very good contribution towards the existence of discrete solutions although no uniqueness results are demonstrated. This paper studies the conditions for the uniqueness of the solution when using to the CAU (Consistent Approximate Upwind) Petrov-Galerkin for solving advection dominated problems. The main issue in this analysis is that it relies on the optimality property of the CAU solution which does not depend on any restrictions of the approximation spaces.

1. Introduction

It is well known that numerical simulations of advection-dominated problems present numerical difficulties related to the lack of stability: because advection dominates diffusion, classical Galerkin finite element (FE) methods generate unstable approximations, which usually exhibit spurious oscillations. The SUPG (Streamline Upwind Petrov-Galekin) method, proposed by Brooks and Hughes [1], was the first variationally consistent, stable and accurate finite element model for advection dominated problems. For regular solutions this method presents quasi-optimal rates of convergence for the streamline derivative and was first analyzed by Johnson et al. [4]. Nevertheless, for nonregular solutions, localized oscillations are still observed in the neighborhood of steep gradients meaning that the streamline is not always the appropriate upwind direction. To overcome this lack of monotonicity many discontinuity capturing terms were designed to enhance stability either in a linear [5] or nonlinear way [2, 3]. Quite promising results were obtained using the nonlinear CAU (Consistent Approximate Upwind) finite element method proposed in [2]. The systematic way of updating the upwind direction in the CAU method results in adding to the SUPG formulation a nonlinear discontinuity-capturing term in a consistent way, engendering an additional stability in the direction of the approximate gradient. The theory has been refined over the years in several directions.

The numerical analysis of the CAU method, even to higher-order elements, has been discussed in a recent paper [3]. The stability analysis was shown based on a linearized iterative scheme, which uses the solution of the SUPG method as an initial guess in order to solve the CAU (nonlinear) method. However, this analysis has some open ends [6] concerning the solvability of the iterative scheme. Therefore, our main goal is to improve the developed analysis. In particular, we address the problem associated to the convergence of the linearized iterative scheme as well as the uniqueness of the solution for the nonlinear approximate method.

2. Mathematical Model

We are interested in steady-state solutions of convection-dominated reaction-diffusion scalar problems of the form

$$\begin{aligned} \mathcal{L}\phi := \gamma\phi + \mathbf{u}\cdot\nabla\phi - \epsilon\Delta\phi &= f \quad \text{in} \quad \Omega \\ \phi &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{1}$$

where $\mathbf{u}$ denotes a given velocity field, ϵ is the given (small) positive diffusion coefficient, γ is the given non-negative reaction coefficient and f is a source term. The problem is defined in the domain $\Omega \subset \mathcal{R}^n$ with boundary Γ.

The Galerkin formulation for problem (1) reads: find $\phi_h \in V_h^p \subset H_0^1(\Omega)$ such that,

$$B(\phi_h, \eta_h) = (f, \eta_h), \quad \forall \eta_h \in V_h^p,$$

with

$$B(\phi_h, \eta_h) := \epsilon(\nabla\phi_h, \nabla\eta_h) + (\mathbf{u}\cdot\nabla\phi_h, \eta_h) + \gamma(\phi_h, \eta_h)$$

and $V_h^p = \{\eta_h \in C^0(\Omega); \eta_h|_{\Omega_e} \in P^p \ \forall\Omega_e, \eta_h|_\Gamma = 0\}$, where P^p is the set of interpolation polynomials of degree less or equal p defined in each finite element Ω_e with characteristic element length denoted by h.

For advection-dominated problems, when ϵ is taken very small in problem (1), it is well-known that classical Galerkin and stabilized finite element methods generate global and local unstable approximations, respectively. As a remedy, discontinuity-capturing variants are considered as an additional (but nonlinear) stabilization. Quite promising results have been obtained using the CAU (Consistent-Upwind Petrov-Galerkin) [2, 3] formulation, which reads as: find $\phi_h \in V_h^p$ such that

$$D(\phi_h, \eta_h) + \sum_{e=1}^{N_e} (\mathcal{L}(\phi_h) - f, \tau_c[\mathbf{u} - \mathbf{v}]\cdot\nabla\eta_h)|_{\Omega_e} = F(\eta_h), \quad \forall \eta_h \in V_h^p, \tag{2}$$

with

$$D(\phi_h, \eta_h) := B(\phi, \eta) + \sum_{e=1}^{N_e} (\mathcal{L}\phi_h, \tau_s \mathbf{u} \cdot \nabla \eta_h)|_{\Omega_e} \ ;$$
$$F(\eta_h) := (f, \eta_h) + \sum_{e=1}^{N_e} (f, \tau_s \mathbf{u} \cdot \nabla \eta_h)|_{\Omega_e},$$

where τ_s and τ_c are the stabilization parameters from the SUPG [1, 4] and CAU [2] methods, respectively. There are a variety of possible designs for these parameters in the literature [1, 2, 3, 4, 6, 5]. The auxiliary vector field $\mathbf{v}$ in equation (2) is a modified velocity field such that it satisfies the original partial differential equation (1) for the approximate solution $\phi_h \in V_h^p$ in each element. This requirement implies

$$[\mathbf{u} - \mathbf{v}(\phi_h)] \cdot \nabla \phi_h = \mathcal{L}(\phi_h) - f =: R(\phi). \tag{3}$$

Then, the vector field $\mathbf{v}$ is determined such that it is the velocity field closest to the real velocity field $\mathbf{u}$ in the L^2 sense. Hence, it is obtained by solving the following local minimization problem: find $\mathbf{u} \in L^\infty(\Omega)^d$ such that

$$\|\mathbf{u} - \mathbf{v}\|_{L^2(\Omega_e)} \leq \|\mathbf{u} - \mathbf{m}\|_{L^2(\Omega_e)}, \quad \forall \mathbf{m} \in Q_{\mathbf{m}}, \tag{4}$$

with

$$Q_{\mathbf{m}} = \{\mathbf{m};\ \mathbf{m} \cdot \nabla \phi_h - \epsilon \Delta \phi_h + \gamma \phi_h - f = 0 \quad \text{in} \ \ \Omega_e,\ e = 1, \ldots, N_e\}.$$

The solution of (4) together with (3) leads to

$$\begin{cases} \mathbf{v} - \mathbf{u} = 0\ , & \text{if} \quad |\nabla \phi_h| = 0\ ; \\ \mathbf{u} - \mathbf{v} = \frac{R(\phi_h)}{|\nabla \phi_h|^2} \nabla \phi_h\ , & \text{otherwise.} \end{cases} \tag{5}$$

Thus, if $|\nabla \phi_h| = 0$, the linear SUPG method is recovered. Otherwise, $|\mathbf{u} - \mathbf{v}| = \frac{|R(\phi_h)|}{|\nabla \phi_h|} =: \beta(\phi_h)$, which ensures that, in each element Ω_e, $\lim_{h \mapsto 0} \mathbf{v}(\phi_h) = \mathbf{u}$, when $\lim_{h \mapsto 0} \phi_h = \phi$. Besides, as $\mathbf{v}$ is selected minimizing $\|\mathbf{u} - \mathbf{v}\|_{L^2(\Omega_e)}$, the following inequality is verified

$$\beta(\phi_h) = \frac{|R(\phi_h)|}{|\nabla \phi_h|} \leq \frac{|R(\psi_h)|}{|\nabla \psi_h|} = \beta(\psi_h), \quad \forall\ \psi_h \in V_h^p. \tag{6}$$

Due to this property, one may conclude that the approximate solution of (2) using (5) is *optimal*. This is a remarkable feature of this method. In fact, the property (6) is crucial to yield the uniqueness result presented in next section.

3. Uniqueness of solution

Let the CAU method (2) be re-written as: find $\phi_h \in V_h^p$ such that

$$a(\phi_h; \phi_h, \eta_h) = F(\eta_h), \quad \forall \eta_h \in U_h^p, \tag{7}$$

where

$$a(\phi_h; \phi_h, \eta_h) := D(\phi_h, \eta_h) + \sum_{e=1}^{N_e} c(\phi_h; \phi_h, \eta_h)|_{\Omega_e}, \tag{8}$$

with

$$c(\phi_h;\phi_h,\eta_h)|_{\Omega_e} := \left(\tau_c \frac{|R(\phi_h)|^2}{|\nabla\phi_h|^2}\nabla\phi_h, \nabla\eta_h\right)|_{\Omega_e}.$$

The non-symmetric bilinear form $D(\cdot,\cdot)$ is coercive (see, for example, [3]), which implies the following SUPG-stability property

$$D(w_h,w_h) \geq C(\theta)|||w_h|||^2, \quad \forall w_h \in V_h^p, \tag{9}$$

with $C(\theta) = 1 - 1/2\sqrt{1+\theta}$, where θ is defined in [5]. The bilinear form $D(\cdot,\cdot)$ is also continuous in the following stabilized mesh dependent norm:

$$|||\eta_h|||^2 :\equiv \epsilon\|\nabla\eta_h\|^2 + \gamma\|\eta_h\|^2 + \sum_{e=1}^{N_e}\tau_s\|\mathbf{u}\cdot\nabla\eta_h\|^2_{\Omega_e}. \tag{10}$$

yielding uniqueness and existence solution for the SUPG method [4], taking $\tau_c = 0$ in problem (7). Besides, providing that $\tau_c \geq 0$, $c(\eta_h;\eta_h,\eta_h)|_{\Omega_e} \geq 0$ for all $\eta_h \in V_h^p$, the stability of the CAU method follows immediately from the previous SUPG stability result,

$$a(\eta_h;\eta_h,\eta_h) \geq C'(\theta)|||\eta_h|||^2. \tag{11}$$

For a quite similar method, it was assumed in [7] that the quantity $|\beta(\cdot)|$ satisfies

$$q_0 \leq |\beta(\phi_h)| \leq q_1, \tag{12}$$

for $q_0, q_1 > 0$, implying coercivity and continuity for the bilinear form $a(\eta_h;\cdot,\cdot)$ and a unique solution that comes from the Lax-Milgram theorem [5, 6, 7]. However, a proof of the lower bounded is not exhibited in [7]. Latter on, it was applied a variant of Brouwer's fixed point theorem to prove the existence of a discrete solution for the CAU nonlinear problem (7) but a uniqueness result remained still open [5]. Therefore, using only the optimal-CAU property (6) we will show this fact in the following theorem.

Theorem 1: Assuming that the minimization property (6) holds, the CAU method (7) has a unique solution $\phi_h \in V_h^p$.

Proof: *Consider two solutions ϕ_h and $\tilde{\phi_h}$ belong to V_h^p of problem (7). Using the definition of function $\beta(\cdot)$, we have*

$$\begin{aligned} D(\phi_h - \tilde{\phi_h},\eta_h) + \sum_{e=1}^{N_e} c(\phi_h;\phi_h,\eta_h) - \sum_{e=1}^{N_e} c(\tilde{\phi_h},\tilde{\phi_h},\eta_h) = D(\phi_h - \tilde{\phi_h},\eta_h) \\ + \sum_{e=1}^{N_e}\Big[(\tau_c[\beta(\phi_h)]^2\nabla\phi_h,\nabla\eta_h)|_{\Omega_e} - (\tau_c[\beta(\tilde{\phi_h})]^2\nabla\tilde{\phi_h},\nabla\eta_h)|_{\Omega_e}\Big] = 0. \end{aligned} \tag{13}$$

From the optimal property of the CAU solution, $\beta(\phi_h) \leq \beta(\psi_h)$, $\forall\psi_h \in V_h^p$, that is, $\beta(\phi_h) \leq \beta(\tilde{\phi_h})$, since $\tilde{\phi_h} \in V_h^p$. Similarly, $\beta(\tilde{\phi_h}) \leq \beta(\phi_h)$ as well. Hence,

$$\beta(\phi_h) = \beta(\tilde{\phi_h}) = \alpha, \tag{14}$$

where α is a non-negative constant. Notice that $\alpha = 0$ only when $R(\phi_h) = 0$, that is, ϕ_h satisfies the residual equation and the non-linear operator vanishes. Now, substituting (14) into (13) yields

$$D(\phi_h - \tilde{\phi_h}, \eta_h) + \sum_{e=1}^{N_e} \Big(\tau_c \alpha^2 (\nabla\phi_h - \nabla\tilde{\phi_h}), \nabla\eta_h \Big) |_{\Omega_e} = 0, \quad \forall\, \eta_h \in V_h^p. \tag{15}$$

Next, taking $\eta_h = \phi_h - \tilde{\phi_h} \in V_h^p$ and using the SUPG-stability property (9), the equation (15) is re-written as

$$C(\theta)|||\phi_h - \tilde{\phi_h}|||^2 + \alpha^2 \sum_{e=1}^{N_e} \Big(\tau_c (\nabla\phi_h - \nabla\tilde{\phi_h}), \nabla\phi_h - \nabla\tilde{\phi_h} \Big) |_{\Omega_e} \leq$$
$$C_1 \left\{ |||\phi_h - \tilde{\phi_h}|||^2 + \sum_{e=1}^{N_e} \tau_c |\nabla\phi_h - \nabla\tilde{\phi_h}|^2_{\Omega_e} \right\} \leq 0,$$

where $C_1 = \max\{C(\theta), \alpha^2\}$ is a positive constant. Therefore, since $\tau_c \geq 0$, the above inequality yields $\phi_h = \tilde{\phi_h}$. $\square$

4. Linearized problem

To solve the nonlinear problem (7) it is necessary to consider iterative methods which are intended to keep the main properties of the CAU method at each iteration. This can be achieved by using the following simple method: from the previous computed iterate solution ϕ_h^n, $n \in \mathcal{N}$, we get

$$c(\phi_h^n; \phi_h^{n+1}, \eta_h)|_{\Omega_e} = (R(\phi_h^{n+1}), \tau_c^n [\mathbf{u} - \mathbf{v}(\phi_h^n)] \cdot \nabla\eta_h)|_{\Omega_e} = \big(\tau_c |\beta(\phi_h^n)|^2 \nabla\phi_h^{n+1}, \nabla\eta_h \big) |_{\Omega_e}, \tag{16}$$

where

$$\tau_c^n = \tau_c(\mathbf{u} - \mathbf{v}(\phi_h^n)) \quad \text{and} \quad (\mathbf{u} - \mathbf{v}(\phi_h^n)) = \frac{R(\phi_h^n)}{|\nabla\phi_h^n|^2} \nabla\phi_h^n.$$

Hence, the iterative algorithm consists in: given ϕ_h^n and $[\mathbf{u} - \mathbf{v}(\phi_h^n)]$, find $\{\phi_h^{n+1}\} \in V_h^p$ such that

$$a(\phi_h^n; \phi_h^{n+1}, \eta_h) = F(\eta_h), \quad \forall\eta_h \in V_h^p, \tag{17}$$

where

$$a(\phi_h^n; \phi_h^{n+1}, \eta_h) = D(\phi_h^{n+1}, \eta_h) + \sum_{e=1}^{N_e} c(\phi_h^n; \phi_h^{n+1}, \eta_h)|_{\Omega_e}$$

is a non-symmetric bilinear form defined on $V_h^p \times V_h^p$, for a given $\phi_h^n \in V_h^p$. It is coercive and continuous [3, 5, 6] in the norm (10). Consequently, the linearized problem (17) is uniquely solvable via the Lax-Milgram theorem. The zero-th iterative solution ϕ_h^0 is the SUPG solution, which can be seen as the zero order residual correction for each $\tau_c^0 = 0$. Due to the property (6), any iterative solution $\{\phi_h^{n+1}\}$ of (17) satisfies

$$\beta(\phi_h) = \frac{|R(\phi_h)|}{|\nabla\phi_h|} \leq \frac{|R(\phi_h^n)|}{|\nabla\phi_h^n|} = \beta(\phi_h^n). \tag{18}$$

In addition, the sequence of solutions $\{\phi^{n+1}\}$ is also upper bounded [3], that is, there is a positive constant C, independent of n, such that,

$$|||\phi_h^{n+1}||| \leq C\|f\|, \quad \forall n. \tag{19}$$

From this fact and assuming that $\{\phi_h^{n+1}\}$, solution of the linearized problem (17), converges to ϕ_h, solution of the nonlinear CAU method (7), when $n \mapsto \infty$, we may prove that the additional term $c(\phi_h, \phi_h, \eta_h)$ does not degrade the rates of convergence as long as regular solutions are concerned [3]. This result is presented in the following theorem.

Theorem 2: Considering $\epsilon = O(h)$ and $\tau_s = O(h/p)$ we arrive at the following *a priori* error estimate:

$$|||\phi - \phi_h|||^2 \leq C \sum_{e=1}^{N_e} \left(\frac{h}{p}\right)^{2s+1} |\phi|^2_{s+1,\Omega_e}, \tag{20}$$

where $\phi|_{\Omega_e} \in H^{k+1}(\Omega_e)$ for some $k \geq 1$, for any $0 \leq s \leq \text{mín}(p,k)$, $p \geq 1$.

The proof of the previous result is based on an important open problem: to prove that the sequence $\{\phi_h^{n+1}\}$ converges to ϕ_h, solution of the nonlinear CAU method (7), when $n \mapsto \infty$. Work in this subject is ongoing and will be addressed in a forthcoming paper.

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