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The Restricted 3-Body Problem on S^1: regularization and a particular solution.

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Resumen

We study a special case of the two body problem when the particles are restricted to move in the space S^1. We regularize all collisions using a symplectic transformation and classify the trajectories in four families. After that, we add a third infinitesimal body, getting four restricted three body problems on S^1, corresponding to each one of the previous families. Then, we apply only one symplectic transformation that regularizes all the singularities due to binary collisions between the infinitesimal body with each primary. We show the global dynamics in one of the restricted problems, when the primaries are fixed at the poles of S^1. We exhibit a particular set of solutions which takes place when the primaries perform hyperbolic motions only.

1. Introduction.

One of the most famous problems in Mathematics is the well known three-body problem. Newton himself, after he explained the Kepler's laws, turned his attention into the three-body problem (Sun-Earth-Moon), but he faced some problems that remained unsolved. Almost 100 years later, by 1772, Euler proposed a simpler formulation (see Sebehely [3]). It was the first time when the restricted three-body problem was stated. This formulation is a limit case of the three-body problem. Namely, two bodies, which are called primaries, are moving under influence of mutual gravitational attraction in an Euclidean space (two-body problem), while a third body, with a negligible mass, is attracted by the previous two, but not influencing their motion.

Despite the efforts of many people, nowadays there is no more than partial results about the restricted three-body problem. Most of these results have been obtained either by adding restrictions or studying special cases. Here, we propose the restricted three-body problem in a compact space. We restricted the motion to S^1. As we will see later, the restriction of movement to the compact space endows the restricted problems with a different dynamics, unlike what take place in the restricted problems defined in infinite spaces.

A complete and well understand study about the restricted three-body problem on S^1, will be useful for the comprehension for others restricted problems and studies in Celestial Mechanics. To our knowledge, there are not reports about restricted problems in compact spaces, so this work could be the beginning of the study about this kind of problems in compact spaces.

2. Two-body problem on S^1.

Let m_1 and m_2 be two punctual masses restricted to move on the space S^1 (centered at the origin of the Euclidean plane), interacting among themselves through no other forces than their mutual gravitational attraction according to Newton's Law. As in the classical two-body problem we study the above formulation as a central force problem. According to second Newton's Law and using polar coordinates, with the metric given by the arc length, we write the equation of motion for this problem:

$$\ddot{\theta} = -\frac{M}{\theta^2} + \frac{M}{(2\pi - \theta)^2}, \tag{1}$$

where $M = m_1 + m_2$. It is natural to call this formulation the Kepler problem on S^1 (see figure 1).

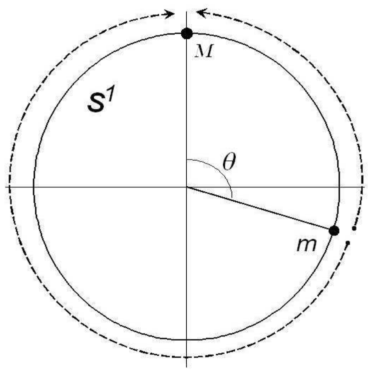

Figura 1: *the Kepler problem on S^1.*

The body m_1 stays at the origin in this coordinate system, θ sweeps clockwise the interval $(0, 2\pi)$ and we choose unit of mass in such a way that the universal gravitational constant equals 1. The lines of attraction force lie on S^1 at both sides of each body, as they are drawn for m_2 in figure 1; $(\dot{})$ is the derivative with respect to t.

Let $q = \theta$ and $p = \dot{\theta}$, so equation (1) becomes an autonomous Hamiltonian system with one degree of freedom with its respective Hamiltonian:

$$H(q,p) = \frac{1}{2}p^2 - \frac{M}{q} - \frac{M}{2\pi - q}. \tag{2}$$

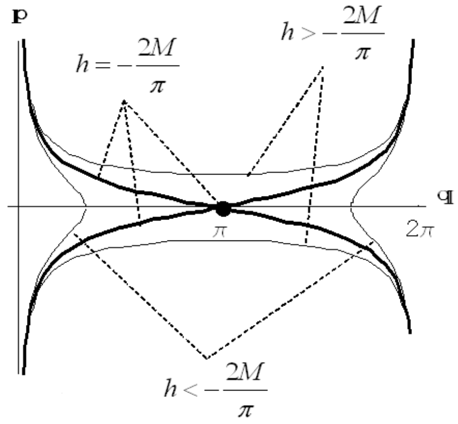

Figura 2: *phase portrait for the Kepler problem on S^1.*

There are other ways of drawing the attraction force lines, but is not difficult to show that the associated flow generated by their respective vector fields are topologically conjugated (see for example Franco [2, chapter 2]).

2.1. Dynamics for the Kepler problem on S^1.

Since the Hamiltonian (2) is a constant of motion, it is easy to get the phase space drawn in figure 2. Here, the configuration space corresponds to $(0, 2\pi)$. We can see that for the energy level $h = -2M/\pi$ there is only one equilibrium point, it is denoted by $(q_o, p_o) = (\pi, 0)$, and there are separatrices, namely orbits that are in the border between two qualitative different sets of orbits. Thus we can classify the orbits according to the levels of energy. In order to preserve the classic notation used for the Kepler problem, we define:

$$\begin{array}{lcl} h < -2M/\pi & \longrightarrow & \text{elliptic orbits,} \\ h = -2M/\pi & \longrightarrow & \text{parabolic orbits and the equilibrium point,} \\ h > -2M/\pi & \longrightarrow & \text{hyperbolic orbits.} \end{array}$$

For $h < -2M/\pi$ the orbits are elliptic, meaning that the orbit of the particle m_2 is a collision-ejection orbit, this orbit never reaches the equilibrium point (the antipodal point of the origin). For $h = -2M/\pi$, the no-equilibrium orbits are parabolic, meaning that m_2 comes from collision and reaches the equilibrium point at infinite time. The last case, when $h > -2M/\pi$ (the hyperbolic orbits) also correspond to a collision-ejection but now the body m_2 comes from ejection, pass through the equilibrium point and dies in collision again.

However, in the three cases the no-equilibrium orbits present a common feature, $p \to \pm\infty$ as either $q \to 0^+$ or $q \to 2\pi^-$. This common behavior means that the Hamiltonian (2) is singular at two points, $q = 0$ and $q = 2\pi$.

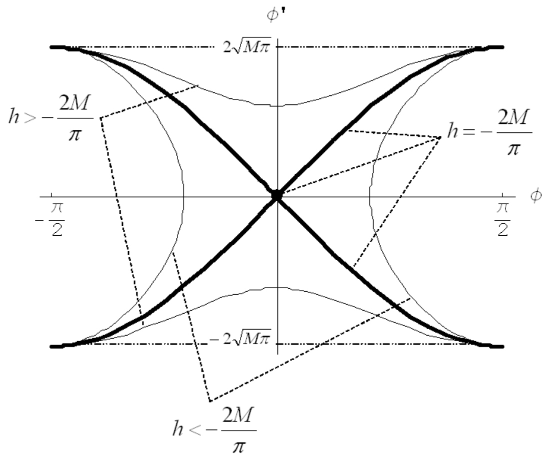

Figura 3: *regularized phase space for the Kepler problem on S^1.*

2.2. Regularization.

Since there are two singularities in the Hamiltonian (2), corresponding to $q = 0$ and $q = 2\pi$, we have to regularize them in order to pass across the singularity. We shall turn the singularities into elastic bounce using a transformation of coordinates and a re-scaling of time. Using the ideas of Érdi [1], we define

$$q = g(\phi), \qquad \frac{dq}{dt} f(q) = \frac{dq}{d\tau},$$

where ϕ is the new position variable and τ the new time.

Lemma 1 . *The transformation T is symplectic and it is a global regularization for the Hamiltonian (2) if and only if $g(\phi) = \pi \sin\phi + \pi$ and $f(q) = q(2\pi - q)$.*

By applying the functions of this lemma to the Hamiltonian H, we obtain a new one on the zero level energy,

$$0 \equiv \tilde{H}(\phi, \phi') = \frac{1}{2}(\phi')^2 - 2\pi M - h\pi^2 \cos^2\phi. \tag{3}$$

After the regularization, the change of coordinates places the origin at the antipodal point of the old origin of coordinates and ϕ sweeps clockwise the interval $[-\pi/2, \pi/2]$.

By using the Hamiltonian (3), for every fix h, we can draw the orbits to get the regularized phase portrait depicted in the figure 3. If we analyze the phase portrait, we can see that the singularities became elastic bounce at $\phi = \frac{-\pi}{2}$ and $\phi = \frac{\pi}{2}$; in this way we obtain well-defined solutions for all τ. All the above is summarized in the following result.

Theorem 1 . *The Kepler problem on S^1 has the following solutions, depending on the* *energy levels as follows:*

$$
\begin{array}{lll}
-h > -\frac{2M}{\pi} & \longrightarrow & \textit{hyperbolic solutions} \\
 & & \quad \phi(\tau) = \sin^{-1}(sn(\sqrt{2\pi}\sqrt{2M+h\pi\tau})) \quad \textit{for } h > 0, \\
 & & \quad \phi(\tau) = \pm 2\sqrt{\pi M}\tau \quad\quad\quad\quad\quad\quad\quad\quad\quad \textit{for } h = 0, \\
-h = -\frac{2M}{\pi} & \longrightarrow & \begin{cases} \textit{parabolic solutions} \quad \phi(\tau) = 2\arctan\Big(-e^{\pm 2\sqrt{\pi M}\tau}\Big), \\ \textit{equilibrium solution} \quad \phi(\tau) = 0, \end{cases} \\
-h < -\frac{2M}{\pi} & \longrightarrow & \textit{elliptic solutions.}
\end{array}
$$

defined for all $\tau \in \mathbb{R}$.

3. Restricted three-body problem on S^1 with one fix center.

3.1. Statement and classification.

Once we have studied the two-body problem on S^1, we can state the restricted three-body problem with one fix center, when the massless particle is also restricted to move on the compact space S^1. Let μ and $1-\mu$ be the masses of two bodies, which we call primaries, with $\mu \in (0,1)$. The third body has a negligible mass which we denote as m.

Since the primaries constitute a two-body problem on S^1, we fix the primary body μ at $(0,1) \in S^1$, so the primaries are in the Kepler problem on S^1 (see figure 4). According to the motion of the primary $1-\mu$ stated in theorem 1, we obtain four restricted three-body problems on S^1 with one fix center:

- restricted with two fixed centers, if $1-\mu$ is at the equilibrium point,
- parabolic restricted, if $1-\mu$ follows the parabolic solutions,
- elliptic restricted, if $1-\mu$ moves like the elliptic solutions and
- hyperbolic restricted, if $1-\mu$ follows the hyperbolic solutions.

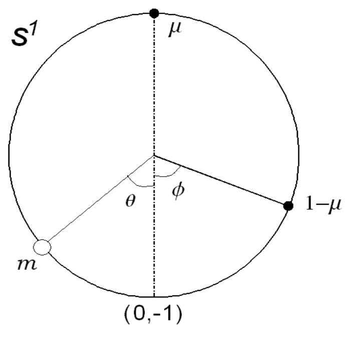

Figura 4: *the restricted three-body problem on S^1.*

The equations of motion for the massless particle m constitutes a Hamiltonian system with its respective Hamiltonian:

$$
K(x,y,\tau) \quad = \quad \frac{y^2}{2} - \frac{\mu}{\frac{\pi}{2}-x} - \frac{\mu}{\frac{\pi}{2}+x} - \frac{1-\mu}{x-\phi(\tau)} - \frac{1-\mu}{\pi-(x-\phi(\tau))}, \tag{4}
$$

where x is the position coordinate and ($'$) stands for $d/d\tau$; ϕ is the position for the $(1-\mu)$-body. Let us remember that the origin is at $(0,-1) \in S^1$ and the variables x and ϕ sweep clockwise the interval $(-\pi/2, \pi/2)$. The lines of attraction force lie over S^1, as we explained in the previous section.

Thus, we have a non-autonomous Hamiltonian system with one degree of freedom. Moreover, this Hamiltonian system is reversible, namely this system satisfies the reversed involution $(q, p, t) \to (q, -p, -t)$.

On the other hand, as we can see, the Hamiltonian (4) is not well defined when its denominators vanish. It is because of the collisions between the infinitesimal body with one of the primaries or because of total collision. From here on, we shall study the dynamics of the negligible mass m while there is not total collision. In order to do that, we shall regularize all the singularities due to binary collisions.

3.2. Regularization.

Let us work on the configuration of masses outlined in figure 4. The other case is obtained by symmetry. The set of singularities due to binary collisions between the infinitesimal mass with each primary is $\Delta_{bin} = \{q | q = \frac{\pi}{2}, q = \phi(\tau)\}$. This set depends on the position of the primary $1-\mu$, so it depends on τ. The regularization that we developed de-singularizes the Hamiltonian (4) by means of only one transformation of coordinates and only one time re-scaling. Moreover, the transformation is symplectic and it depends on τ. In accordance with what we did in the last section, we shall replace the singularities by elastic bounces.

After we apply the global regularization to the Hamiltonian (4), we obtain a new one on the zero level energy, which describes the dynamics of the negligible mass for all time while there is not total collision in the restricted three-body problem on S^1 with one fix center. In addition, this Hamiltonian has sense for the four restricted problems that we scheme at the beginning of this section. We don't write here this resultant Hamiltonian because of its size.

4. Restricted three-body problem on S^1 with two fix centers.

Now we are in the case when the mass $1-\mu$ remains at the equilibrium point. Thus $\phi(\tau) = 0$ for all τ, and by substituting this in the regularized Hamiltonian obtained from (4), we obtain the energy equation for the negligible mass in the restricted three-body problem on S^1 with two fix centers:

$$0 = \frac{1}{2}p^2 - \frac{\pi^2}{16}\bar{K}\cos^2 q - \frac{\pi(\sin q + 1)}{\sin q + 3} - \mu\frac{4\pi \sin q}{\sin^2 q - 9}. \tag{5}$$

The expression (5) is an autonomous Hamiltonian of one degree of freedom, so we shall depict the qualitative dynamics for the infinitesimal body.

From the change of coordinates introduced through the last regularization, the position coordinate q sweeps counterclockwise the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, matching the points $(0, 1)$ and $(0, -1)$ on S^1. The origin is at the point $(-1, 0)$ on S^1 (see figure 5).

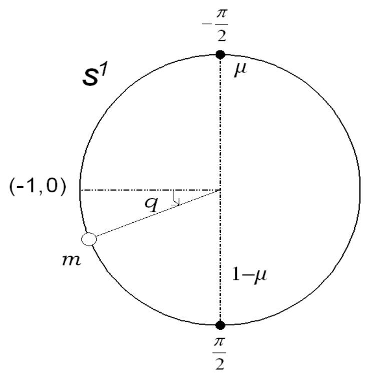

Figura 5: *the restricted three-body problem on S^1 with two fixed centers.*

4.1. Dynamics.

In order to describe the dynamics for the negligible mass m, we begin looking for possible equilibrium points. First, we write the zero velocity set, namely those points where m presents velocity equals zero,

$$\left\{ q \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \;\middle|\; \bar{K} = -\frac{16 \sec^2 q (\sin^2 q + 4\mu \sin q - 2 \sin q - 3)}{\pi(\sin^2 q - 9)} \right\}. \tag{6}$$

This set is obtained by putting $p = 0$ in the equation (5).

Proposition 1 . *The restricted three-body problem on S^1 with two fix centers has only one equilibrium point for each $\mu \in (0,1)$; and for each $q_o \in (-\frac{\pi}{2}, \frac{\pi}{2})$, exists only one μ_o for which q_o defines an equilibrium point.*

Following with the qualitative study of the motion for m, we state the next result.

Proposition 2 . *For every $\mu \in (0,1)$ given, the zero velocity set for the restricted three-body problem on S^1 with two fixed centers contains two elements if $\bar{K} < \bar{K}_{eq}$, one element (equilibrium point) if $\bar{K} = \bar{K}_{eq}$ and is empty if $\bar{K} > \bar{K}_{eq}$.*

Using the reversibility property of this problem and the two previous propositions we can depict the phase space for m. We distinguish three great families of orbits according to energy levels (see figure 6). We do a classification of the motions as we did for the two-body problem on S^1 and so, we obtain the next theorem.

Theorem 2 . *The restricted three-body problem on S^1 with two fixed centers has the following solutions according to energy level:*

- *solutions of elliptic type for $\bar{K} < \bar{K}_{eq}$,*
- *solutions of parabolic type and a equilibrium point for $\bar{K} = \bar{K}_{eq}$ and*
- *solutions of hyperbolic type for $\bar{K} > \bar{K}_{eq}$,*

for every $\mu \in (0,1)$.

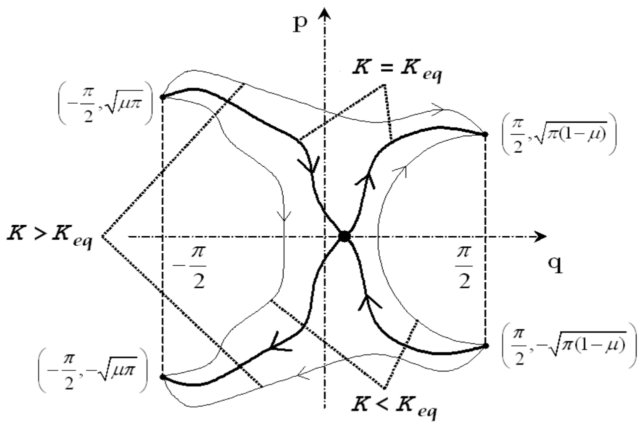

Figura 6: *regularized phase space for the restricted three-body problem on S^1 with two fixed centers.*

5. A particular set of solutions.

Let us begin with the definition of the type of solutions we exhibit further ahead.

Definition 1 . *The solution $(\hat{q}(\tau), \hat{p}(\tau))$ for (6) is an apparent equilibrium solution if $\hat{p}'(\tau) = 0$.*

These kind of solutions mean that the infinitesimal body is attracted by both primaries with equal forces for every τ. We show in the next theorem the existence of these solutions and it is truth only for the symmetric linear hyperbolic case. Namely, when the primary $1-\mu$ is performing a linear motion and for the value $\mu = 1/2$.

Theorem 3 . *The restricted three-body problem on S^1 has apparent equilibrium solutions if and only if $\phi(\tau) = \sqrt{2\pi\tau}$ and $\mu = \frac{1}{2}$. And this solution is given by $\hat{q}(\tau) = \frac{1}{4}(\pi + \sqrt{8\pi\tau})$.*

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Referencias

[1] Bálint Érdi, *Global Regularization of the Restricted Problem of Three Bodies*, Celestial Mechanics and Dynamical Astronomy **90**: 35-42, 2004.

[2] Luis Franco P., *Problema Restringido de 3-Cuerpos en S^1*, Universidad Autónoma Metropolitana, México, 2006. Available on: http://tesiuami.izt.uam.mx/uam/default2.php

[3] Victor Szebehely, *Theory of Orbits, the Restricted Problem of Three Bodies*, Academic Press, USA, 1967.