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Efficient resolution of singularly perturbed coupled systems: Equations of reaction-diffusion

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Resumen

In this communication we consider a class of singularly perturbed linear system of reaction-diffusion type coupled in the reaction terms. To approximate its solution, in [3] the backward Euler method and the central difference scheme on a layer–adapted mesh of Shishkin type was used. We propose a new semi-implicit method which decouples the linear system to be solved at each time level and we prove that it is a uniformly convergent scheme (with respect to the diffusion parameters) in the discrete maximum norm. We display some numerical experiments illustrating in practice the theoretical results. From these examples we can see both the uniform convergence of the numerical method and also its efficiency to approximate the solution of the reaction–diffusion system.

1. Introduction

We consider reaction–diffusion singularly perturbed problems

$$\left\{\begin{array}{l} L_{\vec{\varepsilon}}\vec{u} \equiv \dfrac{\partial \vec{u}}{\partial t} + L_{x,\vec{\varepsilon}}\vec{u} = \vec{f}, \qquad (x,t) \in Q = \Omega \times (0,T] = (0,1) \times (0,T], \\ \vec{u}(0,t) = \vec{g}_0(t),\ \vec{u}(1,t) = \vec{g}_1(t), \quad \forall t \in [0,T], \\ \vec{u}(x,0) = \vec{0}, \quad \forall x \in \Omega, \end{array}\right. \tag{1}$$

where

$$L_{x,\vec{\varepsilon}} \equiv \left(\begin{array}{cc} -\varepsilon_1 \frac{\partial^2}{\partial x^2} & \\ & -\varepsilon_2 \frac{\partial^2}{\partial x^2} \end{array} \right) + A, \qquad A = \left(\begin{array}{cc} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{22}(x,t) \end{array} \right).$$

We denote $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$, with $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$, the vectorial singular perturbation parameter. The coupling matrix A satisfies

$$a_{ij} \leq 0 \quad \text{if} \quad i \neq j, \tag{2}$$

and also

$$a_{ii} > 0, \qquad \min_{(x,t)\in\bar{Q}} a_{ii} \geq \max_{(x,t)\in\bar{Q}} |a_{ij}|, \quad i \neq j, \quad i,j = 1,2. \tag{3}$$

If condition (3) does not hold we consider the transformation $\vec{v}(x,t) = \vec{u}(x,t)e^{-\alpha_0 t}$, with $\alpha_0 > 0$ sufficiently large, in order to transform diagonal entries. These hypothesis guarantee that the maximum principle holds.

Also, we assume that enough regularity and compatibility conditions hold for data of problem (1) in order that $\vec{u} \in C^{4,2}(\bar{Q})$, i.e., the spatial partial derivatives of the solution are continuous up to fourth order and the time partial derivatives are continuous up to second order. For instance, we will suppose the conditions

$$\begin{aligned} \vec{g}_i^{(k)}(0) = \vec{0}, \quad i = 0,1, \ k = 0,1,2, \\ \frac{\partial^{k+k_0}\vec{f}}{\partial x^k \partial t^{k_0}}(0,0) = \frac{\partial^{k+k_0}\vec{f}}{\partial x^k \partial t^{k_0}}(1,0) = \vec{0}, \quad 0 \leq k + 2k_0 \leq 2, \end{aligned} \tag{4}$$

which are an extension of the compatibility conditions for the scalar case (see [4]).

These small parameters cause a multiscale character of the solution and, depending on the values of the boundary conditions and the values of the singular perturbation parameters, it can appear two overlapping boundary layers of a width $\mathcal{O}(\varepsilon_i^{1/2})$, $i = 1,2$, in both sides $x = 0$ and $x = 1$ of the domain. This behaviour was proven in [5] and more recently in [3]. Examples of this type of problems appear in some areas, by instance, in the study of the flow in fractured porous media (Barenblatt system), in the modelling of diffusion process in bones, considered as a multiple porosity medium, and in diffusion process in electroanalytic chemistry.

To approximate efficiently the solution of the system it is convenient to have robust methods in order to achieve a prescribed accuracy using discretization parameters independent of the diffusion coefficients. This type of methods are called uniformly convergent methods. In recent years fitted mesh methods has been used extensively where a classical operator is defined on a layer–adapted mesh. The meshes introduced by Shiskhin have received a special attention and they are piecewise uniform condensing the grid points in the boundary layers. In [3] the authors consider a Shishkin mesh to approximate the solution of problem (1) and on this mesh the backward Euler scheme for the time stepping and the classical central difference scheme for the space discretization are used. In [2] the Crank–Nicolson scheme is used instead of the Euler scheme with the purpose of improving the order of the resulting finite difference scheme.

In order to be more efficient in the numerical resolution, in this communication we will use a Jacobi additive scheme for the time discretization which decouples the unknowns and it is well suited to parallel implementation. This is a desirable property, specially for the multidimensional case and also when the system has a larger number of unknowns (for instance in chemical reactions with n reactants). In the section devoted to the numerical experiments we display the corresponding to a system with three equations. Also, in this communication the uniform convergence, with respect to the singular perturbation

parameters, in the discrete maximum norm of this new scheme is showed. Finally, some numerical experiments that corroborate the order of uniform convergence of this scheme are given. In addition, the computational advantages over the scheme given in [3] are shown.

Henceforth, any positive constant is independent of the diffusion parameters ε_1, ε_2 and the discretization parameters N and Δt.

2. Numerical scheme

To approximate the solution of (1) we consider the backward Euler and the central difference schemes to discretize the time and spatial variables respectively. The numerical solution is defined on the mesh

$$\bar{Q}^N = \bar{\Omega}^N \times \bar{\omega}^N,$$

where, for simplicity, we consider a uniform mesh in the time discretization

$$\bar{\omega}^N = \{k\Delta t,\ 0 \leq k \leq M,\ \Delta t = T/M\},$$

and a piecewise uniform mesh $\bar{\Omega}^N$ in the spatial discretization, condensing the mesh points in the boundary layers. The structure of the solution was analyzed in [3] and the authors proved that the solution has two overlapping boundary layers at both sides $x = 0$ and $x = 1$ of the domain of a width $\mathcal{O}(\varepsilon_i^{1/2})$, $i = 1, 2$. The asymptotic analysis of the error leads to choose the transition points in the mesh as follows

$$\tau_{\varepsilon_2} = \min\{1/4, \sqrt{\varepsilon_2}\ln N\}, \quad \tau_{\varepsilon_1} = \min\{\tau_{\varepsilon_2}/2, \sqrt{\varepsilon_1}\ln N\}. \tag{5}$$

In the subintervals $[0, \tau_{\varepsilon_1}]$, $[\tau_{\varepsilon_1}, \tau_{\varepsilon_2}]$, $[\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_2}]$, $[1-\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_1}]$ and $[1-\tau_{\varepsilon_1}, 1]$ we distribute uniformly $N/8+1$, $N/8+1$, $N/2+1$, $N/8+1$ and $N/8+1$ mesh points respectively (see Figure 1).

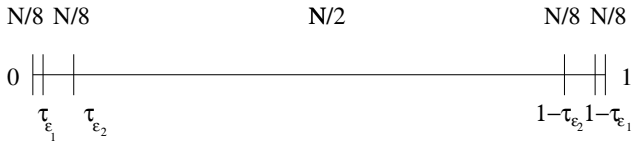

Figura 1: Shishkin mesh and number of intervals.

So the mesh points are given by

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\ x_{N/8} + (j - N/8)h_{\varepsilon_2}, & j = N/8+1, \ldots, N/4, \\ x_{N/4} + (j - N/4)H, & j = N/4+1, \ldots, 3N/4, \\ x_{3N/4} + (j - 3N/4)h_{\varepsilon_2}, & j = 3N/4+1, \ldots, 7N/8, \\ x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8+1, \ldots, N, \end{cases}$$

where

$$h_{\varepsilon_1} = \frac{8\tau_{\varepsilon_1}}{N}, \qquad h_{\varepsilon_2} = \frac{8(\tau_{\varepsilon_2} - \tau_{\varepsilon_1})}{N}, \qquad H = \frac{2(1 - 2\tau_{\varepsilon_2})}{N}.$$

On this mesh, we define the following finite difference scheme

$$(I + \Delta t L^N_{x,\vec{\varepsilon}})\vec{U}_j^{n+1} = \vec{U}_j^n + \Delta t \vec{f}_j^{n+1}, \quad 0 < j < N,\ n = 0, \ldots, M-1, \tag{6}$$

where

$$L^N_{x,\vec{\varepsilon}} \equiv \begin{pmatrix} -\varepsilon_1\delta^2 & \\ & -\varepsilon_2\delta^2 \end{pmatrix} + A_j^{n+1}, \qquad \delta^2 Z_j = \frac{2}{h_j + h_{j+1}}\left(\frac{Z_{j+1} - Z_j}{h_{j+1}} - \frac{Z_j - Z_{j-1}}{h_j}\right),$$

with $h_j = x_j - x_{j-1},\ j = 1, \ldots, N$, $A_j^{n+1} = (a(x_j, t_{n+1}))$, $\vec{f}_j^{n+1} = \vec{f}(x_j, t_{n+1})$ and $\vec{U}_j^n$ denotes the approximation of the value $\vec{u}(x_j, t_n)$. So, $\vec{U}_j^0 = \vec{0}$, $0 \le j \le N$ and $\vec{U}_0^{n+1} = \vec{g}_0(t_{n+1})$, $\vec{U}_N^{n+1} = \vec{g}_1(t_{n+1})$.

In [3] the following result of convergence was proved.

Theorem 1 *Let $\vec{u}(x,t)$ be the solution of (1) and $\{\vec{U}_i^{n+1}\}$ the solution of (6). If the coefficients of matrix A satisfy the positivity conditions (2) and (3), then*

$$\|\vec{u}(x_j, t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{Q}^N} \le C(N^{-2+q}\ln^2 N + \Delta t), \quad 0 < q < 1, \tag{7}$$

where $N, \Delta t$ and q are such that $N^{-q} \le C\Delta t$.

In scheme (6), a coupled system must be solved at each time level. For a large number of variables the resolution of the linear systems can be very expensive and for this reason we propose a new scheme. We begin written the scheme (6) as follows

$$\begin{array}{l} \vec{U}_j^0 = \vec{0},\ 0 \le j \le N, \\ \text{For } n = 0, \ldots, M-1, \\ \begin{pmatrix} I + \Delta t T_1 & \Delta t D_1 \\ \Delta t D_2 & I + \Delta t T_2 \end{pmatrix} \vec{U}^{n+1} = \Delta t \vec{f}^{n+1} + \vec{U}^n, \end{array} \tag{8}$$

where I is the $(N-1)$–identity matrix, the matrices D_1 and D_2 are diagonal

$$D_1 = diag[a_{12}(x_i, t_{n+1})], \quad D_2 = diag[a_{21}(x_i, t_{n+1})], \quad 1 \le i \le N-1,$$

and T_1 and T_2 are $(N-1 \times N-1)$ tridiagonal matrices

$$T_k = \left[\frac{-2\varepsilon_k}{h_j(h_j + h_{j+1})}, \frac{2\varepsilon_k}{h_j h_{j+1}} + a_{kk}(x_j, t_{n+1}), \frac{-2\varepsilon_k}{h_{j+1}(h_j + h_{j+1})}\right],\ 1 < j < N-1,\ k = 1, 2.$$

We note that the first and last rows of these matrices are special because the boundary conditions must be considered. An important characteristic, that it is fundamental in the analysis of the convergence, is that T_1 and T_2 (and also $I + \Delta t T_1$ and $I + \Delta t T_2$) are M–matrices.

We propose the following additive scheme to approximate the solution of problem (1)

$$\begin{array}{l} \vec{V}_j^0 = \vec{0},\ 0 \le j \le N, \\ \text{For } n = 0, \ldots, M-1, \\ \begin{pmatrix} I + \Delta t T_1 & \\ & I + \Delta t T_2 \end{pmatrix} \vec{V}^{n+1} = \Delta t \vec{f}^{n+1} + \begin{pmatrix} I & -\Delta t D_1 \\ -\Delta t D_2 & I \end{pmatrix} \vec{V}^n. \end{array} \tag{9}$$

For this scheme we have the following result of convergence.

Theorem 2 *Let $\vec{u}(x,t)$ be the solution of (1) and $\{\vec{V}_i^{n+1}\}$ the solution of (9). If the coefficients of matrix A satisfy the positivity conditions (2) and (3), then*

$$\|\vec{u}(x_j,t_{n+1}) - \vec{V}_j^{n+1}\|_{\bar{Q}^N} \leq C(N^{-2+q}\ln^2 N + \Delta t), \quad 0 < q < 1, \tag{10}$$

where $N, \Delta t$ and q are such that $N^{-q} \leq C\Delta t$.

Proof. Firstly we determine an estimation for $\vec{E}^{n+1} = \vec{U}^{n+1} - \vec{V}^{n+1}$, where $\vec{U}$ is the solution of problem (8) and $\vec{V}$ is the solution of problem (9). We consider the following decomposition

$$\vec{E}^{n+1} = \vec{e}^{n+1} + \vec{F}^{n+1},$$

where $\vec{e}^{n+1} = \vec{U}^{n+1} - \vec{V}_{aux}^{n+1}$ and $\vec{F}^{n+1} = \vec{V}_{aux}^{n+1} - \vec{V}^{n+1}$, and $\vec{V}_{aux}^{n+1}$ is the solution of the following auxiliary problem

$$\begin{pmatrix} I + \Delta t T_1 & \\ & I + \Delta t T_2 \end{pmatrix} \vec{V}_{aux}^{n+1} = \Delta t \vec{f}^{n+1} + \begin{pmatrix} I & -\Delta t D_1 \\ -\Delta t D_2 & I \end{pmatrix} \vec{U}^n. \tag{11}$$

From (8) and (11), we have that $\vec{e}^{n+1}$ is solution of

$$\begin{pmatrix} I + \Delta t T_1 & \\ & I + \Delta t T_2 \end{pmatrix} \vec{e}^{n+1} = \begin{pmatrix} & -\Delta t D_1 \\ -\Delta t D_2 & \end{pmatrix} (\vec{U}^{n+1} - \vec{U}^n).$$

Using that $(I + \Delta t T_1)$ and $(I + \Delta t T_2)$ are M–matrices, we have that

$$\|\vec{e}^{n+1}\|_{\bar{\Omega}^N} \leq C\Delta t \|\vec{U}^{n+1} - \vec{U}^n\|_{\bar{\Omega}^N}.$$

Taking into account that $\|\vec{u}_t\| \leq C$ (see [3]) and Theorem 1, it is straightforward to deduce that

$$\|\vec{e}^{n+1}\|_{\bar{\Omega}^N} \leq C\Delta t(N^{-2+q}\ln^2 N + \Delta t).$$

On the other hand, $\vec{F}^{n+1}$ is solution of the following problem

$$\begin{pmatrix} I + \Delta t T_1 & \\ & I + \Delta t T_2 \end{pmatrix} \vec{F}^{n+1} = \begin{pmatrix} I & -\Delta t D_1 \\ -\Delta t D_2 & I \end{pmatrix} \vec{E}^n.$$

Using that $\|(I + \Delta t T_k)^{-1}\|_{\bar{\Omega}^N} < 1$, $\|\Delta t (I + \Delta t T_k)^{-1} D_k\|_{\bar{\Omega}^N} < 1$, $k = 1, 2$ and (3), we deduce that

$$\|\vec{F}^{n+1}\|_{\bar{\Omega}^N} \leq \|\vec{E}^n\|_{\bar{\Omega}^N}.$$

Hence, a recursive argument proves

$$\|\vec{E}^{n+1}\|_{\bar{\Omega}^N} \leq \sum_{i=1}^{n+1} \|\vec{e}^{i}\|_{\bar{\Omega}^N} \leq C\Delta t M(N^{-2+q}\ln^2 N + \Delta t) = C(N^{-2+q}\ln^2 N + \Delta t). \tag{12}$$

The result follows from Theorem 1, the bound (12) and the triangular inequality

$$\|\vec{u}(x_j,t_{n+1}) - \vec{V}_j^{n+1}\|_{\bar{Q}^N} \leq \|\vec{u}(x_j,t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{Q}^N} + \|\vec{E}^{n+1}\|_{\bar{Q}^N} \leq C(N^{-2+q}\ln^2 N + \Delta t).$$

A similar result of convergence to Theorem 2 can be established for the additive scheme

$$\begin{array}{l} \vec{V}_j^0 = \vec{0},\ 0 \leq j \leq N, \\ \text{For } n = 0, \ldots, M-1, \\ \begin{pmatrix} I + \Delta t T_1 & \\ \Delta t D_2 & I + \Delta t T_2 \end{pmatrix} \vec{V}^{n+1} = \Delta t \vec{f}^{n+1} + \begin{pmatrix} I & -\Delta t D_1 \\ & I \end{pmatrix} \vec{V}^n. \end{array} \tag{13}$$

3. Numerical experiments

We consider the following test problem

$$\left.\begin{array}{l} \dfrac{\partial u_1}{\partial t} - \varepsilon_1 \dfrac{\partial^2 u_1}{\partial x^2} + \kappa(u_1 - u_2) = 1, \\ \dfrac{\partial u_2}{\partial t} - \varepsilon_2 \dfrac{\partial^2 u_2}{\partial x^2} + \kappa(u_2 - u_1) = 1, \end{array}\right\} \quad (x,t) \in (0,1) \times (0,1], \\ \vec{u}(0,t) = \vec{u}(1,t) = 0, \quad t \in [0,T], \quad \vec{u}(x,0) = 0, \quad x \in [0,1]. \tag{14}$$

This coupled system is used to model the flow in fractured porous media. The first equation of the system models the flow in the fracture system and the second equation models the flow in the porous matrix structure (see [1]). In these equations u_i are the fluid pressures, ε_i the permeabilities and κ is the coefficient that control the exchange of fluid between the pores and the fractures. In the numerical experiments we take $\kappa = 1$.

We use a variant of the double mesh principle to estimate the pointwise errors $|\vec{U}_i^n - \vec{u}(x_i,t_n)|$ in the mesh points $\{(x_i,t_n)\}$. We calculate a new approximation $\{\hat{\vec{U}}_i^n\}$ on the mesh $\{(\hat{x}_i,\hat{t}_n)\}$ that contains the mesh points of the original mesh and their midpoints

$$\begin{array}{llll} \hat{x}_{2i} = x_i, & i = 0,\ldots,N, & \hat{x}_{2i+1} = (x_i + x_{i+1})/2, & i = 0,\ldots,N-1, \\ \hat{t}_{2n} = t_n, & n = 0,\ldots,M, & \hat{t}_{2n+1} = (t_n + t_{n+1})/2, & n = 0,\ldots,M-1. \end{array}$$

At the mesh points of the coarse mesh we calculate the maximum errors and the uniform errors by

$$\vec{d}_{\vec{\varepsilon},N,\Delta t} = \max_{0\le n\le M} \max_{0\le i\le N} |\vec{U}_i^n - \hat{\vec{U}}_{2i}^{2n}|, \qquad \vec{d}_{N,\Delta t} = \max_{S} d_{\vec{\varepsilon},N,\Delta t}, \tag{15}$$

where the singular perturbation parameters take values on the set

$$S = \{(\varepsilon_1,\varepsilon_2) \ | \varepsilon_2 = 2^0, 2^{-2}, \ldots, 2^{-30}, \ \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-58}, 2^{-60}\},$$

to permit that the maximum errors stabilize. From values (15) we determinate the corresponding orders of convergence in a standard way

$$\vec{p} = \frac{\log(\vec{d}_{\vec{\varepsilon},N,\Delta t}/\vec{d}_{\vec{\varepsilon},2N,\Delta t/2})}{\log 2}, \quad \vec{p}_{uni} = \frac{\log(\vec{d}_{N,\Delta t}/\vec{d}_{2N,\Delta t/2})}{\log 2}.$$

In Tables 1 and 2 we display the numerical results for the schemes (8) and (9) respectively. The first row corresponds to the first variable and the second one to the second variable. The spatial discretization parameter takes the values $N = 64, 128, 256, 512, 1024$ and the time discretization parameter $\Delta t = 0{,}1, 0{,}1/2, 0{,}1/2^2, 0{,}1/2^3, 0{,}1/2^4$. From these tables we observe that for this problem the scheme (8) gives better errors than the scheme (9) but the errors are reduced in the same proportion as far as N increases.

To show the advantages of the additive scheme we compare the CPU time of both schemes (8) and (9). It is clear that scheme (9) become more efficient when the number of equations increases. For this reason, we introduce the following test problem

$$\left.\begin{array}{l} \dfrac{\partial u_1}{\partial t} - \varepsilon_1 \dfrac{\partial^2 u_1}{\partial x^2} + \kappa(2u_1 - u_2 - u_3) = 1, \\ \dfrac{\partial u_2}{\partial t} - \varepsilon_2 \dfrac{\partial^2 u_2}{\partial x^2} + \kappa(2u_2 - u_1 - u_3) = 1, \\ \dfrac{\partial u_3}{\partial t} - \varepsilon_3 \dfrac{\partial^2 u_3}{\partial x^2} + \kappa(2u_3 - u_1 - u_2) = 1, \end{array}\right\} \quad (x,t) \in (0,1) \times (0,1], \\ \vec{u}(0,t) = \vec{u}(1,t) = 0, \quad t \in [0,T], \quad \vec{u}(x,0) = 0, \quad x \in [0,1], \tag{16}$$

Tabla 1: Scheme (8): Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{uni}$ for problem (14).

$\varepsilon_1, \varepsilon_2 \in S$	N=64 $\Delta t = 0{,}1$	N=128 $\Delta t = 0{,}1/2$	N=256 $\Delta t = 0{,}1/2^2$	N=512 $\Delta t = 0{,}1/2^3$	N=1024 $\Delta t = 0{,}1/2^4$
$[\vec{d}_{N,\Delta t}]_1$	0.123E-1	0.638E-2	0.326E-2	0.165E-2	0.827E-3
$[\vec{p}_{uni}]_1$	0.950	0.969	0.985	0.993	
$[\vec{d}_{N,\Delta t}]_2$	0.101E-1	0.542E-2	0.281E-2	0.143E-2	0.724E-3
$[\vec{p}_{uni}]_2$	0.893	0.945	0.972	0.986	

Tabla 2: Scheme (9): Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{uni}$ for problem (14).

$\varepsilon_1, \varepsilon_2 \in S$	N=64 $\Delta t = 0{,}1$	N=128 $\Delta t = 0{,}1/2$	N=256 $\Delta t = 0{,}1/2^2$	N=512 $\Delta t = 0{,}1/2^3$	N=1024 $\Delta t = 0{,}1/2^4$
$[\vec{d}_{N,\Delta t}]_1$	0.438E-1	0.224E-1	0.119E-1	0.621E-2	0.313E-2
$[\vec{p}_{uni}]_1$	0.968	0.912	0.939	0.986	
$[\vec{d}_{N,\Delta t}]_2$	0.456E-1	0.230E-1	0.121E-1	0.627E-2	0.315E-2
$[\vec{p}_{uni}]_2$	0.985	0.929	0.948	0.992	

and we use similar schemes to (8) and (9) to approximate its solution. Now three overlapping boundary layers can appear at each end of the spatial domain and then we consider six transition points τ_{ε_i} and $1 - \tau_{\varepsilon_i}$ with $i = 1, 2, 3$, where

$$\tau_{\varepsilon_3} = \min\{1/4, \sqrt{\varepsilon_3}\ln N\}, \quad \tau_{\varepsilon_2} = \min\{\tau_{\varepsilon_3}/2, \sqrt{\varepsilon_2}\ln N\}, \quad \tau_{\varepsilon_1} = \min\{\tau_{\varepsilon_2}/2, \sqrt{\varepsilon_1}\ln N\}, \tag{17}$$

and now we distribute uniformly $N/12+1$ mesh in the fine meshes and $N/2+1$ points in the coarse mesh. In the numerical experiments the singular perturbation parameters take values on the set

$$\hat{S} = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_3 = 2^0, 2^{-2}, \ldots, 2^{-30},\ \varepsilon_2 = \varepsilon_3, 2^{-2}\varepsilon_2, \ldots, 2^{-38}, 2^{-40},$$
$$\varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-58}, 2^{-60}\}.$$

In Tables 3 and 4 we show the numerical results for problem (16) observing the same behaviour than in the previous test problem (14).

Tabla 3: Scheme (8): Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{uni}$ for problem (16).

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \hat{S}$	N=64 $\Delta t = 0{,}1$	N=128 $\Delta t = 0{,}1/2$	N=256 $\Delta t = 0{,}1/2^2$	N=512 $\Delta t = 0{,}1/2^3$	N=1024 $\Delta t = 0{,}1/2^4$
$[\vec{d}_{N,\Delta t}]_1$	0.135E-01	0.658E-02	0.338E-02	0.171E-02	0.861E-03
$[\vec{p}_{uni}]_1$	1.036	0.962	0.981	0.990	
$[\vec{d}_{N,\Delta t}]_2$	0.153E-01	0.686E-02	0.311E-02	0.157E-02	0.790E-03
$[\vec{p}_{uni}]_2$	1.161	1.141	0.984	0.994	
$[\vec{d}_{N,\Delta t}]_3$	0.182E-01	0.695E-02	0.281E-02	0.143E-02	0.724E-03
$[\vec{p}_{uni}]_3$	1.391	1.304	0.973	0.986	

Finally, in Table 5 we show the CPU time (in seconds) to complete the previous Tables 1–4. The PC used for the timing results is a Pentium IV with 2,6Mhz. From Table

Tabla 4: Scheme (9): Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{uni}$ for problem (16).

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \hat{S}$	N=64 $\Delta t = 0{,}1$	N=128 $\Delta t = 0{,}1/2$	N=256 $\Delta t = 0{,}1/2^2$	N=512 $\Delta t = 0{,}1/2^3$	N=1024 $\Delta t = 0{,}1/2^4$
$[\vec{d}_{N,\Delta t}]_1$ $[\vec{p}_{uni}]_1$	0.762E-01 0.884	0.413E-01 0.864	0.227E-01 0.913	0.121E-01 0.973	0.614E-02
$[\vec{d}_{N,\Delta t}]_2$ $[\vec{p}_{uni}]_2$	0.762E-01 0.884	0.413E-01 0.864	0.227E-01 0.912	0.121E-01 0.973	0.614E-02
$[\vec{d}_{N,\Delta t}]_3$ $[\vec{p}_{uni}]_3$	0.784E-01 0.898	0.421E-01 0.878	0.229E-01 0.919	0.121E-01 0.977	0.615E-02

5 we can deduce the computational advantages of the additive scheme proposed in this communication.

Tabla 5: Schemes (8) and (9): CPU times for problems (14) and (16).

	Scheme (8)	Scheme (9)
Test problem (14) $(\varepsilon_1, \varepsilon_2 \in S)$	348.219"	228.125"
Test problem (16) $(\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \hat{S})$	34749.2340"	4601.6880"

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