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# Control of the 1-d wave equation from an interior moving point

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**Resumen**

We consider the one-dimensional linear wave equation with Dirichlet boundary conditions in a bounded interval, and with a control acting on a single point which moves following a regular trajectory in time. We analyze the exact controllability problem.

## 1. Introduction

We consider the one-dimensional linear wave equation, on a finite interval domain $(0, L)$, with an interior control $f$ which acts on a single moving (in time) point $x = \gamma(t)$,

$$\begin{cases} u_{tt} - u_{xx} = f(t)\delta_{\gamma(t)}(x), & \text{in} \quad 0 < x < L, \quad 0 < t < T, \\ u(0,t) = u(L,t) = 0, & \text{in} \quad 0 < t < T, \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x) & \text{in} \quad 0 < x < L. \end{cases} \tag{1}$$

Here, $\delta_{\gamma(t)}$ represents the Dirac measure on $x = \gamma(t)$ and the function $\gamma$ describes the trajectory in time of the location of the control. We assume that the function $\gamma : [0,T] \to (0,L)$ belongs to the class $\gamma \in C^1([0,T])$.

We are interested in the following exact controllability problem: *Given $T > 0$, some initial data $(u^0, u^1)$ and final data $(v^0, v^1)$, find a control $f$ such that the solution $u$ of (1) satisfies*

$$u(x,T) = v^0(x), \qquad u_t(x,T) = v^1(x), \qquad \forall x \in (0,L). \tag{2}$$

Let us briefly describe some related results and the main motivation of this problem.

When the control acts in an interior open set $\omega$ or one of the extremes of the domain $(0, L)$, it is well known that the corresponding exact controllability property of the wave equation holds, for some sufficiently large time $T$ (see [9]). On the other hand, in most practical situations, the support of the control is required to be very small compared to the total size of the domain $(0, L)$ and therefore it is very natural to consider a limit situation where the subinterval $\omega$ is reduced to a single point $\gamma \in (0, L)$. It turns out that the controllability property of system (1) depends on the location of $\gamma$. Indeed, it can be shown that this property holds if and only if the only eigenfunction of the Laplacian with homogeneous Dirichlet boundary conditions and vanishing on $x = \gamma$ is the identically zero one (see [10], [11], [1] or the more recent reference [5], for example). In the sequel, the points $\gamma$ for which this spectral property is satisfied will be referred to as *strategic points*.

The property of $\gamma$ being strategic is difficult to establish in practice since it is extremely unstable. In fact $\gamma \in (0, L)$ is strategic if and only if it is irrational with respect to the length of the interval $L$. Consequently, controllability properties over points are hard to use in practice.

To overcome this difficulty one may consider controls supported on moving points $\{\gamma(t)\}_{0 \leq t \leq T}$, as suggested in [11]. The main advantage of moving controls is that it is easy to construct trajectories $\{\gamma(t)\}_{0 \leq t \leq T}$ for which the strategic property holds for $\gamma(t) \in (0, L)$ a.e. in $t \in [0, T]$. For example, this is the case when we assume that the control is located at a point that moves in time with constant velocity. In this case, $\gamma(t)$ is irrational, and therefore strategic, a.e. in $t \in [0, T]$. Therefore, the exact controllability is likely to hold for such moving controls. The aim of this work is to show that this is indeed the case under suitable conditions on the function $\gamma(t)$.

It is worth noting that in the context of parabolic equations a similar situation appears. We refer to [3], [7], [1], [11] and the references therein for a detailed analysis of this related problem.

The rest of this paper is divided as follows: in section 2 we state the main results, namely the existence of solutions for system (1) in suitable functional spaces and the exact controllability property. Both results can be reduced, by classical duality arguments, to some suitable regularity and observability estimates for the uncontrolled wave equation respectively. In section 3 we give the proof of these estimates.

## 2. Main results

We assume that the function $\gamma : [0, T] \to (0, L)$ belongs to $C^1$ and satisfies the following hypothesis: There exist constants $c_1, c_2$ such that

$$0 < c_1 < |\gamma'(t)| \leq c_2 < 1 \text{ for all } t \in (0, T). \tag{3}$$

The control $f(t)$ in (1) is assumed to belong to $H^{-1}(0, T)$ and the initial data $(u^0, u^1)$ in the class

$$(u^0, u^1) \in L^2 \times H^{-1}(0, L).$$

We define the weak solutions of system (1) by transposition (see [8]). To do that let $\psi \in L^1(0, T; L^2(0, L))$ be a function and consider the non-homogeneous adjoint wave equation

$$\left\{ \begin{array}{lll} \varphi_{tt} - \varphi_{xx} = \psi(x, t) & \text{in } 0 < x < L, & 0 < t < T, \\ \varphi(0, t) = \varphi(L, t) = 0 & \text{in} & 0 < t < T, \\ \varphi(x, T) = \varphi_t(x, T) = 0, & \text{in } 0 < x < L, & 0 < t < T. \end{array} \right. \tag{4}$$

It is well known that system (4) admits a unique solution $\varphi$ of (4) in the class

$$\varphi \in C([0,T]; H_0^1(0,L)) \cap C^1([0,T]; L^2(0,L)). \tag{5}$$

Multiplying the equations in (1) by $\varphi$ and integrating by parts we easily obtain, at least formally, the following identity:

$$\begin{aligned}\int_0^L &< u^1(x), \varphi(x,0) >_1 dx - \int_0^L u^0(x)\varphi_t(x,0)\, dx + < f, \varphi(\gamma(t),t) >_1^t \; dt \\ &= \int_0^T \int_0^L \psi(x,t)u(x,t)\, dx\, dt, \text{ for all } \psi \in L^1(0,T; L^2(0,L)),\end{aligned} \tag{6}$$

where $< \cdot, \cdot >_1$ and $< \cdot, \cdot >_1^t$ denote the duality products between $H_0^1(0,L)$ and its dual, and between $H_0^1(0,T)$ and its dual respectively.

We adopt identity (6) as the definition of solutions of (1), in the sense of transposition.

The following result establishes the existence of solutions for system (1).

**Theorem 2.1** *Assume that $\gamma : [0,T] \to (0,L)$ is in the class $\gamma \in C^1([0,T])$ and satisfies the hypothesis (3). Given any initial data $(u^0, u^1) \in L^2 \times H^{-1}(0,L)$ and $f \in H^{-1}(0,T)$, there exists an unique solution $u$ of (1), in the sense of transposition, in the class*

$$u \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-1}(0,L)).$$

*Moreover, there exists a one-to-one correspondence between the data and the solution in the given spaces.*

Concerning the exact controllability problem of system (1) the following result holds:

**Theorem 2.2** *Let $T > 2L$ and $\gamma : [0,T] \to (0,L)$ be a function in the class $\gamma \in C^1([0,T])$ satisfying the hypothesis (3). Then, system (1) is exactly controllable, i.e. for any initial data $(u^0, u^1) \in L^2 \times H^{-1}(0,L)$ and final data $(v^0, v^1) \in L^2 \times H^{-1}(0,L)$, there exists a control $f \in H^{-1}(0,T)$ such that the solution $u$ of (1) satisfies (2).*

The proof of the existence result above (Theorem 2.1) can be obtained from a suitable regularity property stated below (estimate (8)) by a straightforward duality argument. We refer to [8] for a general description, and [6] or [2] where this is done for very similar problems.

The proof of the exact controllability property (Theorem 2.2) is also a straightforward consequence of the observability inequality (9) below and the Hilbert Uniqueness Method introduced by J.-L. Lions in [9]. We also refer to [6] and [2] where this method is applied for similar problems.

## 3. Observability

As we have said, the main results in this paper can be obtained from some inequalities for the uncontrolled wave equation. In this section we prove these inequalities.

Consider the system

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & \text{in} \quad 0 < x < L, \quad 0 < t < T, \\ \varphi(0,t) = \varphi(L,t) = 0, & \text{in} \quad \quad 0 < t < T, \\ \varphi(x,0) = \varphi^0(x), \ \varphi_t(x,0) = \varphi^1(x), & \text{in} \quad 0 < x < L. \end{cases} \tag{7}$$

We assume that $(\varphi^0, \varphi^1) \in H_0^1 \times L^2(0,L)$. The following holds:

**Proposition 3.1** *Assume that $\gamma \in C^1$ satisfies the hypothesis (3). Then, there exists a constant $c(\gamma) > 0$ such that the solution $\varphi$ of (7) satisfies*

$$\int_0^T \left| \frac{d}{dt} \varphi(\gamma(t), t) \right|^2 dt \leq c(\gamma) \left\| (\varphi^0, \varphi^1) \right\|^2_{H_0^1 \times L^2}. \tag{8}$$

*Moreover, if $T > 2L$, then there exists a constant $C(\gamma) > 0$ such that*

$$\left\| (\varphi^0, \varphi^1) \right\|^2_{H_0^1 \times L^2} \leq C(\gamma) \int_0^T \left| \frac{d}{dt} \varphi(\gamma(t), t) \right|^2 dt. \tag{9}$$

**Remark 3.1** *Estimate (8) is a regularity result for the trace of the solution of the wave equation $\varphi(x,t)$ on the curve defined by the trajectory $\gamma$. This result cannot be obtained from classical arguments or semigroup theory.*

*Estimate (9) is an observability inequality which establishes that the total energy of the solutions of the wave equation can be estimated from the value of the solution $\varphi$ at $\gamma(t)$ for a large enough time interval $t \in (0,T)$.*

**Proof.** Note that it is enough to consider smooth solutions since for other solutions we can argue by density. We first prove the estimate (8).

We observe that in the one-dimensional wave equation one can change the variables $x$ by $t$ and $t$ by $x$ without altering the equation. Thus, D'Alambert formula can be used to obtain the solution $\varphi(x,t)$ in terms of the solution at one extreme, say $x = 0$, instead of the data at $t = 0$ as usual. Indeed, we have

$$\varphi(x,t) = \frac{1}{2} \left[ \varphi(0, t-x) + \varphi(0, t+x) \right] + \frac{1}{2} \int_{t-x}^{t+x} \varphi_x(0,s) \ ds. \tag{10}$$

If $\varphi$ is defined on $(x,t) \in [0,L] \times [0,T]$ this formula holds only for those values $(x,t)$ for which $0 \leq t - x \leq t + x \leq T$. However, we can extend the solution of the wave equation $\varphi$ to $(x,t) \in (0,L) \times (-\infty, \infty)$ and formula (10) is still valid for the whole domain $(x,t) \in (0,L) \times (0,T)$. This is always posible because the wave equation with Cauchy data at $t = 0$ is well-posed for $t \geq 0$ and $t \leq 0$.

In particular, taking into account the homogeneous Dirichlet boundary conditions in (7) we have

$$\varphi(\gamma(t), t) = \frac{1}{2} \int_{t-\gamma(t)}^{t+\gamma(t)} \varphi_x(0,s) \ ds.$$

Therefore,

$$2 \frac{d}{dt} \varphi(\gamma(t), t) = (1 + \gamma'(t)) \varphi_x(0, t + \gamma(t)) - (1 - \gamma'(t)) \varphi_x(0, t - \gamma(t)), \tag{11}$$

and

$$\begin{aligned}
& 4\int_0^T \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt \\
\leq\ & 2\int_0^T (1+\gamma'(t))^2 \left|\varphi_x(0,t+\gamma(t))\right|^2\ dt + 2\int_0^T (1-\gamma'(t))^2 \left|\varphi_x(0,t-\gamma(t))\right|^2\ dt \\
\leq\ & 2\sup_{t\in[0,T]}(1+\gamma'(t))^2 \int_{\gamma(0)}^{T+\gamma(T)} \left|\varphi_x(0,s)\right|^2\ ds + \\
& 2\sup_{t\in[0,T]}(1-\gamma'(t))^2 \int_{-\gamma(0)}^{T-\gamma(T)} \left|\varphi_x(0,s)\right|^2\ ds \\
\leq\ & 4\int_{\gamma(0)}^{T+\gamma(T)} \left|\varphi_x(0,s)\right|^2\ ds + 4\int_{-\gamma(0)}^{T-\gamma(T)} \left|\varphi_x(0,s)\right|^2\ ds \leq 8\int_{-\gamma(0)}^{T+\gamma(T)} \left|\varphi_x(0,s)\right|^2\ ds \\
\leq\ & C\left\|(\varphi^0,\varphi^1)\right\|^2_{H_0^1\times L^2(0,L)}.
\end{aligned}$$

Here, the last inequality can be obtained by classical multipliers techniques (see [9]).

Now, we prove the estimate (9). We divide the analysis in two cases depending on the sign of $\gamma'$.

**Case A:** Assume that $-\infty < -c_2 \leq \gamma'(t) \leq -c_1 < 0$ for all $t \in (0,T)$. From identity (11), we can estimate $|\varphi_x(0,t-\gamma(t))|$ as follows:

$$\begin{aligned}
& \left|\varphi_x(0,t-\gamma(t))\right|^2 \\
=\ & \left(\frac{1+\gamma'(t)}{1-\gamma'(t)}\varphi_x(0,t+\gamma(t)) - \frac{2}{1-\gamma'(t)}\frac{d}{dt}\varphi(\gamma(t),t)\right)^2 \\
=\ & \left(\frac{1+\gamma'(t)}{1-\gamma'(t)}\right)^2 \left|\varphi_x(0,t+\gamma(t))\right|^2 + \left(\frac{2}{1-\gamma'(t)}\right)^2 \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 \\
& -2\frac{1+\gamma'(t)}{1-\gamma'(t)}\varphi_x(0,t+\gamma(t))\frac{2}{1-\gamma'(t)}\frac{d}{dt}\varphi(\gamma(t),t) \\
\leq\ & (1+a)\left(\frac{1+\gamma'(t)}{1-\gamma'(t)}\right)^2 \left|\varphi_x(0,t+\gamma(t))\right|^2 + \left(1+\frac{1}{a}\right)\left(\frac{2}{1-\gamma'(t)}\right)^2 \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2
\end{aligned}$$

for any $a > 0$ to be chosen later. Here we have used Young's inequality.

Multiplying by $1-\gamma'(t)$ and integrating in $t \in (0,T)$ we obtain,

$$\begin{aligned}
& \int_0^T \left|\varphi_x(0,t-\gamma(t))\right|^2 \left(1-\gamma'(t)\right) dt \\
\leq\ & (1+a)\int_0^T \frac{1+\gamma'(t)}{1-\gamma'(t)} \left|\varphi_x(0,t+\gamma(t))\right|^2 \left(1+\gamma'(t)\right)\ dt \\
& + \left(1+\frac{1}{a}\right)\int_0^T \frac{4}{1-\gamma'(t)}\left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt \\
\leq\ & (1+a)\frac{1-c_1}{1+c_1}\int_{\gamma(0)}^{T+\gamma(T)} \left|\varphi_x(0,s)\right|^2\ dt \\
& + \left(1+\frac{1}{a}\right)\frac{4}{1+c_1}\int_0^T \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt.
\end{aligned} \tag{12}$$

Therefore,

$$\int_{-\gamma(0)}^{T-\gamma(T)} |\varphi_x(0,t)|^2\,dt - (1+a)\frac{1-c_1}{1+c_1}\int_{\gamma(0)}^{T+\gamma(T)} |\varphi_x(0,t)|^2\;dt$$
$$\leq \left(1+\frac{1}{a}\right)\frac{4}{1+c_1}\int_0^T \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt. \tag{13}$$

Now we take the constants $T_0, a$ such that $T_0-\gamma(T_0)+\gamma(0)=2L$ and $0<a<\frac{1+c_1}{1-c_1}-1$ respectively. Then, from the $2L$-periodicity of the solutions of the wave equation (7) and the fact that $\gamma$ is decreasing we can estimate the left hand side of (13) with $T=T_0$ as follows

$$\int_{-\gamma(0)}^{T_0-\gamma(T_0)} |\varphi_x(0,t)|^2\,dt - (1+a)\frac{1-c_1}{1+c_1}\int_{\gamma(0)}^{T_0+\gamma(T_0)} |\varphi_x(0,t)|^2\;dt$$
$$\geq \left(1-(1+a)\frac{1-c_1}{1+c_1}\right)\int_0^{2L} |\varphi_x(0,t)|^2\;dt. \tag{14}$$

Combining this last inequality with (13) we obtain that there exist constant $C>0$ such that

$$\int_0^{T_0}\left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt \geq C\int_0^{2L} |\varphi_x(0,t)|^2\;dt \geq C'\|(\varphi^0,\varphi^1)\|_{H_0^1\times L^2(0,L)},$$

where the last inequality is the classical boundary observability inequality for the one-dimensional wave equation (see, for example, [4]). Finally, inequality (9) follows for any $T>T_0$ and, in particular, for $T>2L$.

**Case B:** Assume now that $0<c_1<\gamma'(t)<c_2<1$ for all $t\in(0,T)$. From identity (11), we estimate now $|\varphi_x(0,t+\gamma(t))|$ as follows,

$$\begin{aligned}
&|\varphi_x(0,t+\gamma(t))|^2\\
=&\left(\frac{1-\gamma'(t)}{1+\gamma'(t)}\varphi_x(0,t-\gamma(t)) - \frac{2}{1+\gamma'(t)}\frac{d}{dt}\varphi(\gamma(t),t)\right)^2\\
=&\left(\frac{1-\gamma'(t)}{1+\gamma'(t)}\right)^2|\varphi_x(0,t-\gamma(t))|^2 + \left(\frac{2}{1+\gamma'(t)}\right)^2\left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2\\
&-2\frac{1-\gamma'(t)}{1+\gamma'(t)}\varphi_x(0,t-\gamma(t))\frac{2}{1+\gamma'(t)}\frac{d}{dt}\varphi(\gamma(t),t)\\
\leq&\ (1+a)\left(\frac{1-\gamma'(t)}{1+\gamma'(t)}\right)^2|\varphi_x(0,t-\gamma(t))|^2 + \left(1+\frac{1}{a}\right)\left(\frac{2}{1+\gamma'(t)}\right)^2\left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2
\end{aligned}$$

for any $a>0$ to be chosen later.

Multiplying by $1+\gamma'(t)$ and integrating in $t\in(0,T)$ we obtain,

$$\begin{aligned}
&\int_0^T |\varphi_x(0,t+\gamma(t))|^2\left(1+\gamma'(t)\right)dt\\
\leq&\ (1+a)\int_0^T \frac{1-\gamma'(t)}{1+\gamma'(t)}|\varphi_x(0,t-\gamma(t))|^2\left(1-\gamma'(t)\right)\;dt\\
&+\left(1+\frac{1}{a}\right)\int_0^T\frac{4}{1+\gamma'(t)}\left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt.
\end{aligned} \tag{15}$$

Therefore,

$$\int_{\gamma(0)}^{T+\gamma(T)} |\varphi_x(0,t)|^2\,dt - (1+a)\frac{1-c_1}{1+c_1}\int_{-\gamma(0)}^{T-\gamma(T)} |\varphi_x(0,t)|^2\,dt$$
$$\leq \left(1+\frac{1}{a}\right)\frac{4}{1+c_1}\int_0^T \left|\frac{d}{dt}\varphi(\gamma(t),t)\right|^2 dt.$$

Now we take the constants $T_0, a$ such that $T_0-\gamma(T_0)+\gamma(0)=2L$ and $0<a<\frac{1+c_1}{1-c_1}-1$ respectively. Then, from the $2L$-periodicity of the solutions of the wave equation (7) and the fact that $\gamma$ is increasing we can estimate the left hand side of (13) with $T=T_0$ as in (14). Then, we can argue as in the previous case. This concludes the proof.

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### Referencias

[1] S. Avdonin and S. Ivanov, *Families of exponentials: The method of moments in controllability problems for distributed paramenter systems,* Cambridge University Press, 1995.

[2] C. Castro, *Boundary controllability of the one-dimensional wave equation with rapidly oscillating density.* Asymptotic Analysis, **20**, 317-350, 1999.

[3] C. Castro y E. Zuazua, *Unique continuation and control for the heat equation from a lower dimensional manifold*, SIAM J. Cont. Optim, **42** (4), 1400-1434.

[4] S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end,* Indiana University Mathematics Journal, **44** (2) (1995), 545-573.

[5] R. Dáger and E. Zuazua, *Wave propagation observation and control in 1-d flexible multi-structures*, Mathématiques et Applications, Vol. 50, Springer Verlag, 2006.

[6] S. Hansen y E. Zuazua, *Exact Controllability and Stabilization of a Vibrating String with an interior Point Mass*, SIAM J. Control Optim., **33**, 5 (1995), 1357-1391.

[7] A. Khapalov, *Mobile point controls versus locally distributed ones for the controllability of the semilinear parabolic equation*, SIAM J. Cont. Optim., **40** (1), (2001) 231-252.

[8] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications I,* Springer-Verlag, 1972.

[9] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués*. Tomes 1 & 2. Masson, RMA **8** & **9**, Paris, 1988.

[10] J.-L. Lions, *Some methods in the mathematical analysis of systems and their control*, Gordon and Breach, 1981.

[11] J.-L. Lions, *Pointwise control for distributed systems*, in Control and estimation in distributed parameter systems, edited by H.T. Banks, SIAM, 1992.